RECEIVED: November 24, 2014

REVISED: January 31, 2015

ACCEPTED: February 3, 2015

PUBLISHED: March 5, 2015

Determination of boundary contributions in recursion relation

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ABSTRACT: In this paper, we propose a new algorithm to systematically determine the missing boundary contributions, when one uses the BCFW on-shell recursion relation to calculate tree amplitudes for general quantum field theories. After an instruction of the algorithm, we will use several examples to demonstrate its application, including amplitudes of color-ordered ϕ^4 theory, Yang-Mills theory, Einstein-Maxwell theory and color-ordered Yukawa theory with ϕ^4 interaction.

KEYWORDS: Scattering Amplitudes, Gauge Symmetry

ARXIV EPRINT: [1411.0452](https://arxiv.org/abs/1411.0452)

¹The unconventional order of authors is merely to satisfy the outdated requirement for Phy. Degree of the school.

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1 Introduction

Inspired by Witten’s twistor program [1], a powerful approach to calculate tree amplitudes is developed in [2, 3].¹ When applying this newly discovered on-shell recursion relation, the large z behavior of amplitudes under a deformation parameterized by z is crucial. For an amplitude A , if $\lim_{z \rightarrow \infty} A(z) = 0$, it can be nicely reconstructed by sewing lower-point on-shell amplitudes. However, if $\lim_{z \rightarrow \infty} A(z) \neq 0$, nontrivial boundary contributions arise which, in general, cannot be reconstructed recursively. The analysis of large z behavior is a nontrivial issue since naive power counting of z based on Feynman diagrams may lead to wrong conclusions in many cases. A nice way to tackle this by applying the background field method is presented in [7] by Arkani-Hamed and Kaplan. In this way, it has been shown [7, 8] that when the amplitude contains at least one gluon or graviton, there is at least one deformation with convergent (which means good) large z behavior.² However, for theories involving only scalars and fermions, or some effective theories, boundary contributions are

¹For more details, see reviews [4–6] and corresponding citations.

²More accurately, one needs other conditions, such as two derivative theories and spins of other particles should be less than one for the case of gluon, and two for the case of graviton.

unavoidable. For example, the Yukawa theory, is part of the Standard Model. Thus it is necessary to generalize the on-shell recursion relation to deal with these terms.

Several proposals have been made to handle this difficult task. The first [9, 10] is to introduce auxiliary fields so that in the enlarged theory, there are no boundary contributions. After working out the parent amplitudes, by proper reduction one gets the desired derivative amplitudes. But there are two problems: Firstly it is unknown in general whether the enlarged theory exists, or how to construct it if it exists. Secondly the parent amplitudes could be far more complicated than expected, thus this way is not quite efficient. The second [11–13] is to carefully analyze Feynman diagrams and then isolate their boundary contributions, which can be evaluated directly or recursively afterwards. However, this approach is useful only when boundary contributions are located on merely a few Feynman diagrams. The third [14–16] is to express boundary contributions in terms of roots of amplitudes, which is a fascinating idea, but to find roots is an extremely challenging work.

In this paper, we introduce a systematic algorithm to determine boundary contributions for general quantum field theories. The key point is simple: Similar to tree amplitudes, the boundary contributions are also rational functions of external momenta. Thus, after carefully analyzing their pole structure, one can capture these elusory quantities by applying exactly the same idea used to derive the on-shell recursion relation. At this point, it is important to restrict the algorithm with certain conditions. Since it is based on pole structure, boundary contributions that do not contain any poles cannot be determined. In general the mass dimension of n -point amplitudes is $(4 - n)$, thus for theories without coupling constants of negative mass dimension, there is at least one pole for $n > 4$. But for some effective theories, it is imaginable that such counterexamples could appear and then one needs alternative approaches.

The paper is organized as follows. In section 2, we present the general framework of this algorithm. In section 3, we use several examples to demonstrate its versatility in applications. In section 4, we give a brief summary and several further directions. In appendix A, the new algorithm is reinterpreted in a more concise algebraic language.

2 The new algorithm

In this section, we will present the new algorithm in detail. Assume that all external legs and internal propagators are massless, then one can use the spinor formalism to simplify the calculation, although this requirement is not essential. From now on, let's adopt the QCD convention, i.e., $s_{ij} = \langle i|j \rangle [j|i]$. Furthermore, the i 's in usual definitions of amplitude and propagator are dropped, i.e., $\frac{i}{P^2} \rightarrow \frac{1}{P^2}$. Thus the familiar BCFW recursion relation reads as $-A \sim \sum \frac{A_L A_R}{P^2}$.

Now let's recall the on-shell recursion relation, starting by deforming a pair of spinors, e.g., $0 \equiv \langle i_0 | j_0 \rangle$ with $|i_0\rangle \rightarrow |i_0\rangle - z_0 |j_0\rangle$ and $|j_0\rangle \rightarrow |j_0\rangle + z_0 |i_0\rangle$.³ Under deformation 0 ,

³Since multiple deformations will be used, to simplify symbols we adopt the following notations: $\underline{s} = \langle i_s | j_s \rangle$ represents the s -th deformation and its parameter $z_{\underline{s}}$. The location of its pole associated with P^2 reads as $z_{\underline{s}, P^2}$.

all physical propagator⁴ P_i^2 's are divided into two categories: the *detectable propagators* which depend on z_0 , and the *undetectable propagators* which do not. These two sets are denoted as $\mathcal{D}^0$ and $\mathcal{U}^0$ respectively, where the superscript indicates the deformation. There will be also spurious poles and their set is denoted as $\mathcal{S}^0$. As a rational function of z_0 , the amplitude obtained by Feynman rules can be decomposed as

$$-A^0(z_0) = \frac{N(z_0)}{\prod P_t^2(z_0)} = \sum_{P_t^2 \in \mathcal{D}^0} \frac{A_{t;L}(\widehat{z}_{0,t})A_{t;R}(\widehat{z}_{0,t})}{P_t^2(z_0)} + C_0^0 + \sum C_i^0 z_0^i. \quad (2.1)$$

For later convenience, we define the *recursive part*⁵ as

$$\mathcal{R}^0(z_0) = \sum_{P_t^2 \in \mathcal{D}^0} \frac{A_{t;L}(\widehat{z}_{0,t})A_{t;R}(\widehat{z}_{0,t})}{P_t^2(z_0)}, \quad (2.2)$$

and the *boundary part*⁶ as

$$\mathcal{B}^0(z_0) = C_0^0 + \sum C_i^0 z_0^i. \quad (2.3)$$

Setting $z_0 = 0$ to conceal the deformation after decomposition, let's simply denote the 0th recursive and boundary parts as $\mathcal{R}^0$ and $\mathcal{B}^0$ respectively. It is obvious that the numerator of $\mathcal{R}^0$ is a product of left and right on-shell amplitudes evaluated at $P_t^2(z_0) = 0$, which can be calculated recursively, while $\mathcal{B}^0$ is nothing but the boundary contribution we seek for. Detailed analysis on expression (2.1) informs us the following important facts:

- (A-1) Coefficients C_0^0 , C_i^0 and $A_{t;L}(\widehat{z}_{0,t})A_{t;R}(\widehat{z}_{0,t})$ are all simply rational functions of spinors $\lambda_i, \widetilde{\lambda}_i$. To understand these coefficients, it is crucial to determine their pole structure.
- (A-2) It is well known that term by term, there may be *spurious poles* in $A_{t;L}(\widehat{z}_{0,t})A_{t;R}(\widehat{z}_{0,t})$. Some spurious poles will cancel each other when we sum over t , but others will still remain. To cancel them for a physical amplitude, $\mathcal{B}^0$ must also depend on the same spurious poles. But if all of them are canceled in $\mathcal{R}^0$, $\mathcal{B}^0$ does not need to have this dependence any more.
- (A-3) Now a key observation is that by such formulation, physical poles $P_t^2 \in \mathcal{D}^0$ will appear once and only once with power one in $\mathcal{R}^0$. In other words, *they cannot be the poles of boundary contribution $\mathcal{B}^0$ and sub-amplitudes $A_{t;L}(\widehat{z}_{0,t})A_{t;R}(\widehat{z}_{0,t})$.*
- (A-4) Having excluded $P_t^2 \in \mathcal{D}^0$ from $\mathcal{B}^0$, now we have a clear picture of its pole structure: (1) It must be either a physical or spurious pole which belongs to $\mathcal{U}^0$ or $\mathcal{S}^0$; (2) The powers of spurious poles in $\mathcal{B}^0$ may be larger than one. In fact, their degrees are determined by the corresponding degrees in coefficients $A_{t;L}(\widehat{z}_{0,t})A_{t;R}(\widehat{z}_{0,t})$.

⁴More strictly, physical poles are defined to have non-vanishing factorization limits. And all other poles are spurious.

⁵Since for the recursive part, z_0 appears in all poles, so it is called the ‘pole part’ as well.

⁶The boundary part is called the ‘regular part’ as well.

Understanding the pole structure of $\mathcal{B}^0$, it is natural to determine it by using other deformations. To proceed, let's perform a new deformation $\underline{1} \equiv \langle i_1 | j_1 \rangle$. The only condition for the new deformation is⁷ that it can detect at least one pole in $(\mathcal{U}^0 \cup \mathcal{S}^0)$ (which means one pole in $\mathcal{B}^0$). Obviously, in practice it is better to choose a new deformation which maximizes the number of detected poles in $(\mathcal{U}^0 \cup \mathcal{S}^0)$.

Under deformation $\underline{1}$, one can write the full amplitude as a function of $z_{\underline{1}}$ by two different ways. The first is to use the expression given by Feynman rules directly as (2.1), namely

$$-A^1(z_{\underline{1}}) = \frac{N(z_{\underline{1}})}{\prod_r P_r^2(z_{\underline{1}})} = \sum_{P_r \in \mathcal{D}^1} \frac{A_{r;L}(\hat{z}_{\underline{1},r}) A_{r;R}(\hat{z}_{\underline{1},r})}{P_r^2(z_{\underline{1}})} + C_0^1 + \sum C_i^1 z_{\underline{1}}^i = \mathcal{R}^1(z_{\underline{1}}) + \mathcal{B}^1(z_{\underline{1}}). \quad (2.4)$$

The second is to use (2.1), (2.2) and (2.3) to perform deformation $\underline{1}$, then

$$-A^1(z_{\underline{1}}) = \mathcal{R}^0(z_{\underline{1}}) + \mathcal{B}^0(z_{\underline{1}}). \quad (2.5)$$

Obviously, as a rational function of $z_{\underline{1}}$, expression (2.4) must equal to (2.5). Although there are unknown terms in both (2.4) and (2.5), their recursive parts $\mathcal{R}^0(z_{\underline{1}})$ and $\mathcal{R}^1(z_{\underline{1}})$ are known, thus we can determine part of $\mathcal{B}^0(z_{\underline{1}})$ as the following. Since $\mathcal{B}^0(z_{\underline{1}})$ is a rational function of $z_{\underline{1}}$, it can be decomposed into the recursive part (i.e., the pole part) and boundary part as⁸

$$\mathcal{B}^0(z_{\underline{1}}) = \mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}}) + \mathcal{B}^{01}(z_{\underline{1}}), \quad (2.6)$$

where

$$\begin{aligned} \mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}}) &= \sum_{P_t^2 \in \mathcal{U}^0 \cap \mathcal{D}^1} \sum_{a=1}^{n_{P_t^2}} \frac{c_{t,a}}{(P_t^2(z_{\underline{1}}))^a} + \sum_{S_t \in \mathcal{S}^0 \cap \mathcal{D}^1} \sum_{b=1}^{n_{S_t}} \frac{d_{t,b}}{(S_t(z_{\underline{1}}))^b}, \\ \mathcal{B}^{01}(z_{\underline{1}}) &= \mathcal{B}^{01} + \sum C_{0i}^{01} z_{\underline{1}}^i, \end{aligned} \quad (2.7)$$

where a, b are the degrees of corresponding poles. In decomposition (2.6), note that $\mathcal{B}^{01}$ no longer contains poles in $(\mathcal{U}^0 \cup \mathcal{S}^0) \cap \mathcal{D}^{01}$, although it may produce new spurious poles. We now use $\mathcal{U}^{01}$ to denote the remaining undetectable physical poles, and $\mathcal{S}^{01}$ to denote the remaining undetectable spurious poles in $\mathcal{S}^0$ as well as newly generated spurious poles.

Similarly, one can decompose $\mathcal{R}^0(z_{\underline{1}})$ in (2.5) into the recursive part and boundary part as

$$\mathcal{R}^0(z_{\underline{1}}) = \mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}}) + \mathcal{R}\mathcal{B}^{0,1}(z_{\underline{1}}), \quad (2.8)$$

where both $\mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}})$ and $\mathcal{R}\mathcal{B}^{0,1}(z_{\underline{1}})$ are known. Now compare the recursive parts of (2.4) and (2.5), we trivially reach an identity

$$\mathcal{R}^1(z_{\underline{1}}) = \mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}}) + \mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}}). \quad (2.9)$$

⁷While keeping $\mathcal{D}^0$ to denote only physical poles detected by deformation $\langle i_0 | j_0 \rangle$, we now use $\mathcal{D}^{01}$ to denote poles that have been detected by two consequent deformations (thus including spurious poles in $\mathcal{S}^0$ detected by deformation $\langle i_1 | j_1 \rangle$).

⁸The order in symbol $\mathcal{B}\mathcal{R}^{0,1}$ makes a difference. It means the 1st recursive part of the 0th boundary part. In contrast, $\mathcal{R}\mathcal{B}^{0,1}$ means the 1st boundary part of the 0th recursive part.

From this one can determine coefficients $c_{t,a}$ and $d_{t,b}$. Explicitly, we have the following observations:

- (B-1) Firstly, since spurious poles $S_t(z_{\underline{1}})$ do not appear in $\mathcal{R}^1(z_{\underline{1}})$, they must be canceled by the identical terms in $\mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}})$ and $\mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}})$ so that we can determine coefficients $d_{t,b}$ in $\mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}})$ from the known $\mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}})$. Similarly we can determine coefficients $c_{t,a}$ by subtracting corresponding contributions by $\mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}})$ for poles $P_t^2 \in \mathcal{U}^0 \cap \mathcal{D}^1$ in $\mathcal{R}^1(z_{\underline{1}})$.
- (B-2) Secondly, for physical poles $P_t^2 \in \mathcal{D}^0 \cap \mathcal{D}^1$, since $\mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}})$ does not contribute, contribution from $\mathcal{R}^1(z_{\underline{1}})$ must equal to the one from $\mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}})$. This serves as an important consistency check of the algorithm.

In practice, there is no need to determine coefficients $c_{t,a}, d_{t,b}$ separately, instead one can directly write

$$\mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}} = 0) = \mathcal{R}^1(z_{\underline{1}} = 0) - \mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}} = 0). \quad (2.10)$$

Plugging this back, the full amplitude (2.1) becomes

$$-A = \mathcal{R}^{01} + \mathcal{B}^{01}, \quad \mathcal{R}^{01} = \mathcal{R}^0(z_{\underline{0}} = 0) + \mathcal{R}^1(z_{\underline{1}} = 0) - \mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}} = 0), \quad (2.11)$$

where the unknown $\mathcal{B}^{01}$ can only contain poles in $(\mathcal{U}^{01} \cup \mathcal{S}^{01})$. And $\mathcal{B}\mathcal{R}^{0,1}$ can be further simplified, as will be explained in the end of this section.

What do we have achieved after performing the new deformation $\underline{1}$? Before this, we know that the unknown $\mathcal{B}^0$ can only contain poles in $\mathcal{U}^0 \cup \mathcal{S}^0$. After this, we find the $\mathcal{B}\mathcal{R}^{0,1}$ part while the remaining unknown part $\mathcal{B}^{01}$ can only contain poles in $(\mathcal{U}^{01} \cup \mathcal{S}^{01})$, where $\mathcal{U}^{01}$ is a subset of $\mathcal{U}^0$.

Now one sees the recursive pattern: Treating $\mathcal{R}^{01}, \mathcal{B}^{01}$ in (2.11) as $\mathcal{R}^0, \mathcal{B}^0$ in (2.5), we can repeat the entire procedure again. Each time performing a new deformation, we get part of the boundary contribution, while the remaining unknown part contains less and less physical poles. If at each time the new deformation can detect at least one physical pole, after finite steps, the unknown part will contain no physical poles at all, although it can depend on spurious poles. Then one needs to check whether all spurious poles are canceled out without the unknown part. If this holds, we can safely drop the unknown part. If this fails, we need to use new deformations to detect the uncanceled spurious poles in order to determine their corresponding terms. Repeat same procedures until one has excluded all dependence on spurious poles. Since all pole parts have been found, the remaining part must be zero (up to possible polynomial terms), then the boundary contribution is fully determined. The use of auxiliary deformations has also appeared in the study of one-loop rational parts in [17–19]. In some sense, the auxiliary deformations bring poles at infinity to finite locations, so that they can be calculated recursively.⁹

Before ending this section, as advertised before, we now demonstrate how to simplify $\mathcal{B}\mathcal{R}^{0,1}$ in (2.10). The first trick is: From item (B-2), one can see that for $\mathcal{R}^1$ in (2.10)

⁹We would like to thank David Kosower for suggesting this viewpoint.

we don't need to calculate all poles in $\mathcal{D}^1$, but only P_r^2 's that belong to $\mathcal{D}^1$ and not $\mathcal{D}^0$. Similarly, when to calculate $\mathcal{R}\mathcal{R}^{0,1}$ in (2.10), one should neglect the pole part that belongs to $\mathcal{D}^0 \cap \mathcal{D}^1$. For complicated cases, this can save considerable amount of calculation.

The second trick is to note that sometimes $\mathcal{R}\mathcal{R}^{0,1}(z_1)$ in (2.8) is quite difficult to directly extract from $\mathcal{R}^0(z_1)$, while to extract $\mathcal{R}\mathcal{B}^{0,1}(z_1)$ is easy, thus one can use

$$\mathcal{R}\mathcal{R}^{0,1}(z_1) = \mathcal{R}^0(z_1) - \mathcal{R}\mathcal{B}^{0,1}(z_1). \quad (2.12)$$

Plugging this back into (2.11), we obtain

$$\begin{aligned} -A &= \mathcal{R}^{01} + \mathcal{B}^{01}, \\ \mathcal{R}^{01} &= \mathcal{R}^0(z_0=0) - (\mathcal{R}^0(z_1=0) + \mathcal{R}\mathcal{B}^{0,1}(z_1=0)) + \mathcal{R}^1(z_1=0) \\ &= \mathcal{R}\mathcal{B}^{0,1} + \mathcal{R}^1. \end{aligned} \quad (2.13)$$

Between (2.11) and (2.13), which choice is simpler depends on which of $\mathcal{R}\mathcal{R}^{0,1}(z_1)$ and $\mathcal{R}\mathcal{B}^{0,1}(z_1)$ is easier to calculate.

Let's give a brief remark. Although we have assumed the calculation will stop at finite steps, we don't have a rigorous proof yet. Also, there are many different choices of BCFW deformations, and among those how many of them will be minimally enough to completely determine the boundary contribution, and how to arrange their order will work most efficiently? In this paper we will not try to answer these questions, but a deeper systematic study is certainly needed in the future.

In appendix A, we will reinterpret the new algorithm in a more abstract but concise language in terms of operators.

3 Examples

In this section, we present various applications to demonstrate the new algorithm.

3.1 Six-point amplitude of color-ordered ϕ^4 theory

The first example is a simple six-point amplitude of color-ordered ϕ^4 theory [11]. For this case, all possible physical poles are $P_{i(i+1)(i+2)}$ with $i = 1, 2, 3$. Let's start with bad deformation $\underline{0} = \langle 6|1 \rangle$ (i.e., it has nonzero boundary contribution), then the amplitude is¹⁰

$$-A_6 = \mathcal{R}^0 + \mathcal{B}^0, \quad \mathcal{R}^0 = \frac{\lambda^2}{P_{123}^2}, \quad (3.1)$$

with the corresponding sets $\mathcal{D}^0 = \{P_{123}^2\}$, $\mathcal{U}^0 = \{P_{234}^2, P_{345}^2\}$, $\mathcal{S}^0 = \emptyset$. Next, we perform deformation $\underline{1} = \langle 4|6 \rangle$ since it can detect both poles in $\mathcal{U}^0$, and its recursive part is

$$\mathcal{R}^1(z_1) = \frac{\lambda^2}{P_{234}^2(z_1)} + \frac{\lambda^2}{P_{345}^2(z_1)}, \quad (3.2)$$

¹⁰All i 's have been removed from coupling constants for brevity.

which is simply the l.h.s. of (2.9). For its r.h.s., from $\mathcal{R}^0$ in (3.1) we find $\mathcal{R}\mathcal{R}^{0,1} = 0$, thus

$$\mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}}) = \mathcal{R}^1(z_{\underline{1}}) = \frac{\lambda^2}{P_{234}^2(z_{\underline{1}})} + \frac{\lambda^2}{P_{345}^2(z_{\underline{1}})}. \quad (3.3)$$

Applying (2.11), we get

$$-A_6 = \frac{\lambda^2}{P_{123}^2} + \frac{\lambda^2}{P_{234}^2} + \frac{\lambda^2}{P_{345}^2} + \mathcal{B}^{01}. \quad (3.4)$$

In above expression, the known part satisfies three criteria: (1) It does not contain any spurious poles. (2) It contains all physical poles. (3) It satisfies all factorization limits. Thus we can finally set $\mathcal{B}^{01} = 0$. In other words, we have determined the boundary contribution $\mathcal{B}^0$.

3.2 Five-point gluon amplitude $A_5(1^-, 2^+, 3^+, 4^-, 5^+)$

The second example is a well known MHV gluon amplitude $A_5(1^-, 2^+, 3^+, 4^-, 5^+)$, so it is easy to check its answer by the new algorithm. For this case, all physical poles are $P_{i(i+1)}^2$ with $i = 1, 2, 3, 4, 5$. However, for a two-particle pole, one must factorize it into the spinor and anti-spinor parts. By factorization limits only holomorphic poles exist, thus all possible physical poles are $\langle i|i+1 \rangle$ with $i = 1, 2, 3, 4, 5$. We start with bad deformation $\underline{0} = \langle 1|5 \rangle$ which does have boundary contribution, then the amplitude is

$$-A_5^0(1^-, 2^+, 3^+, 4^-, 5^+) = \mathcal{R}^0 + \mathcal{B}^0, \quad \mathcal{R}^0 = \frac{-\langle 5|1 \rangle^3 \langle 4|2 \rangle^4}{\langle 1|2 \rangle \langle 2|3 \rangle \langle 3|4 \rangle \langle 4|5 \rangle \langle 5|2 \rangle^4}, \quad (3.5)$$

with the corresponding sets

$$\mathcal{D}^0 = \{\langle 1|2 \rangle\}, \quad \mathcal{U}^0 = \{\langle 2|3 \rangle, \langle 3|4 \rangle, \langle 4|5 \rangle, \langle 5|1 \rangle\}, \quad \mathcal{S}^0 = \{\langle 2|5 \rangle^4\}. \quad (3.6)$$

Note that the power of above spurious pole is four.

Now we try to perform a new deformation to detect as many poles as possible in $\mathcal{U}^0$. One such choice is bad deformation $\underline{1} = \langle 4|2 \rangle$, and its recursive part is directly calculated as

$$\mathcal{R}^1(z_{\underline{1}}) = \frac{\langle 1|5 \rangle^3 \langle 4|2 \rangle^3}{\langle 1|2 \rangle \langle 2|3 \rangle \langle 3|5 \rangle \langle 5|2 \rangle^3 \langle 4|5 \rangle - z_{\underline{1}} \langle 2|5 \rangle} + \frac{\langle 4|2 \rangle^3 \langle 3|1 \rangle^4}{\langle 1|2 \rangle \langle 3|5 \rangle \langle 5|1 \rangle \langle 3|2 \rangle^4 \langle 3|4 \rangle - z_{\underline{1}} \langle 3|2 \rangle} \quad (3.7)$$

while the pole part of $\mathcal{R}^0$ from (3.5) under $\underline{1}$ is given by

$$\mathcal{R}\mathcal{R}^{0,1}(z_{\underline{1}}) = \frac{\langle 5|1 \rangle^3 \langle 4|2 \rangle^3}{\langle 1|2 \rangle \langle 2|3 \rangle \langle 5|2 \rangle^4 \langle 3|5 \rangle} \left(\frac{\langle 2|5 \rangle}{\langle 4|5 \rangle - z_{\underline{1}} \langle 2|5 \rangle} - \frac{\langle 3|2 \rangle}{\langle 3|4 \rangle - z_{\underline{1}} \langle 3|2 \rangle} \right). \quad (3.8)$$

Thus applying (2.10), we find

$$\mathcal{B}\mathcal{R}^{0,1}(z_{\underline{1}}) = \frac{\langle 4|2 \rangle^3}{\langle 1|2 \rangle \langle 3|5 \rangle} \left(\frac{\langle 3|1 \rangle^4}{\langle 5|1 \rangle \langle 3|2 \rangle^4} - \frac{\langle 5|1 \rangle^3}{\langle 5|2 \rangle^4} \right) \frac{1}{\langle 3|4 \rangle - z_{\underline{1}} \langle 3|2 \rangle}. \quad (3.9)$$

Note that although $\langle 1|2 \rangle$ appears in the denominator of $\mathcal{B}\mathcal{R}^{0,1}$, it is actually excluded by the vanishing factor $\frac{\langle 3|1 \rangle^4}{\langle 5|1 \rangle \langle 3|2 \rangle^4} - \frac{\langle 5|1 \rangle^3}{\langle 5|2 \rangle^4}$ when $\langle 1|2 \rangle$ tends to zero in the collinear limit,

which satisfies the claim that $\mathcal{B}^0$ cannot contain any poles in $\mathcal{D}^0$. Plugging this back, (3.5) becomes

$$-A_5^0(1^-, 2^+, 3^+, 4^-, 5^+) = \mathcal{R}^{01} + \mathcal{B}^{01},$$

$$\mathcal{R}^{01} = \frac{-\langle 5|1\rangle^3 \langle 4|2\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|2\rangle^4} + \frac{\langle 4|2\rangle^3}{\langle 1|2\rangle \langle 3|5\rangle} \left(\frac{\langle 3|1\rangle^4}{\langle 5|1\rangle \langle 3|2\rangle^4} - \frac{\langle 5|1\rangle^3}{\langle 5|2\rangle^4} \right) \frac{1}{\langle 3|4\rangle}, \quad (3.10)$$

where the unknown $\mathcal{B}^{01}$ can only contain poles in

$$\mathcal{U}^{01} = \mathcal{U}^0 \cap \mathcal{U}^1 = \{\langle 2|3\rangle, \langle 5|1\rangle\}, \quad \mathcal{S}^{01} = \mathcal{S}^0 \cup \mathcal{S}^1 = \{\langle 2|5\rangle^4, \langle 2|3\rangle^4, \langle 3|5\rangle\}. \quad (3.11)$$

Next, let's perform another bad deformation $\underline{2} = \langle 1|2]$, and its recursive part is

$$\mathcal{R}^2(z_{\underline{2}}) = \frac{-\langle 4|5\rangle^3 \langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|4\rangle \langle 5|2\rangle^4} \frac{1}{\langle 5|1\rangle - z_{\underline{2}} \langle 5|2\rangle}, \quad (3.12)$$

while the pole part of $\mathcal{R}^{01}$ from (3.10) gives

$$\mathcal{R}\mathcal{R}^{01,2}(z_{\underline{2}}) = \frac{\langle 4|2\rangle^3}{\langle 3|5\rangle \langle 3|4\rangle} \frac{\langle 5|3\rangle^4 \langle 1|2\rangle^3}{\langle 5|2\rangle^4 \langle 3|2\rangle^4} \frac{1}{\langle 5|1\rangle - z_{\underline{2}} \langle 5|2\rangle}. \quad (3.13)$$

Then we get

$$\mathcal{B}\mathcal{R}^{01,2}(z_{\underline{2}}) = \frac{\langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|4\rangle \langle 5|2\rangle^4} \left(-\langle 4|5\rangle^3 - \frac{\langle 4|2\rangle^3 \langle 3|5\rangle^3}{\langle 2|3\rangle^3} \right) \frac{1}{\langle 5|1\rangle - z_{\underline{2}} \langle 5|2\rangle}. \quad (3.14)$$

Again it is easy to see that although the denominator contains $\langle 3|4\rangle$, it is excluded by the vanishing factor $-\langle 4|5\rangle^3 - \frac{\langle 4|2\rangle^3 \langle 3|5\rangle^3}{\langle 2|3\rangle^3}$, which again satisfies the claim that only poles in (3.11) can appear. Plugging this back, (3.5) becomes

$$-A_5^0(1^-, 2^+, 3^+, 4^-, 5^+) = \mathcal{R}^{012} + \mathcal{B}^{012},$$

$$\mathcal{R}^{012} = \frac{-\langle 5|1\rangle^3 \langle 4|2\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|2\rangle^4} + \frac{\langle 4|2\rangle^3}{\langle 1|2\rangle \langle 3|5\rangle \langle 3|4\rangle} \left(\frac{\langle 3|1\rangle^4}{\langle 5|1\rangle \langle 3|2\rangle^4} - \frac{\langle 5|1\rangle^3}{\langle 5|2\rangle^4} \right)$$

$$+ \frac{\langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|4\rangle \langle 5|2\rangle^4 \langle 5|1\rangle} \left(-\langle 4|5\rangle^3 - \frac{\langle 4|2\rangle^3 \langle 3|5\rangle^3}{\langle 2|3\rangle^3} \right), \quad (3.15)$$

where the unknown $\mathcal{B}^{012}$ can only contain poles in

$$\mathcal{U}^{012} = \{\langle 2|3\rangle\}, \quad \mathcal{S}^{012} = \{\langle 2|5\rangle^4, \langle 2|3\rangle^4, \langle 3|5\rangle\}. \quad (3.16)$$

Finally, there is only one physical pole $\langle 2|3\rangle$ left, and we need to choose a deformation to detect it then produce a simplest term from (3.16). One choice is $\underline{3} = \langle 3|4]$ (there is no bad deformation for the last pole $\langle 2|3\rangle$), then its recursive part is

$$\mathcal{R}^3(z_{\underline{3}}) = \frac{-\langle 1|4\rangle^4}{\langle 1|2\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|1\rangle} \frac{1}{\langle 2|3\rangle - z_{\underline{3}} \langle 3|4\rangle}, \quad (3.17)$$

while the pole part of $\mathcal{R}^{012}$ in (3.15) is simply

$$\left\{ \frac{-\langle 5|1\rangle^3 \langle 4|2\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|2\rangle^4} + \frac{\langle 4|2\rangle^3}{\langle 1|2\rangle \langle 3|5\rangle \langle 3|4\rangle} \left(\frac{\langle 3|1\rangle^4}{\langle 5|1\rangle \langle 3|2\rangle^4} - \frac{\langle 5|1\rangle^3}{\langle 5|2\rangle^4} \right) + \frac{\langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|4\rangle \langle 5|2\rangle^4 \langle 5|1\rangle} \left(-\langle 4|5\rangle^3 - \frac{\langle 4|2\rangle^3 \langle 3|5\rangle^3}{\langle 2|3\rangle^3} \right) \right\}_{|3\rangle \rightarrow |3\rangle - z_3|4\rangle}. \quad (3.18)$$

From above one finds

$$\mathcal{BR}^{012,3} = \frac{-\langle 1|4\rangle^4}{\langle 1|2\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|1\rangle \langle 2|3\rangle} - \left\{ \frac{-\langle 5|1\rangle^3 \langle 4|2\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|2\rangle^4} + \frac{\langle 4|2\rangle^3}{\langle 1|2\rangle \langle 3|5\rangle \langle 3|4\rangle} \left(\frac{\langle 3|1\rangle^4}{\langle 5|1\rangle \langle 3|2\rangle^4} - \frac{\langle 5|1\rangle^3}{\langle 5|2\rangle^4} \right) + \frac{\langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|4\rangle \langle 5|2\rangle^4 \langle 5|1\rangle} \left(-\langle 4|5\rangle^3 - \frac{\langle 4|2\rangle^3 \langle 3|5\rangle^3}{\langle 2|3\rangle^3} \right) \right\}. \quad (3.19)$$

Plugging this back, (3.5) becomes

$$-A_5^0(1^-, 2^+, 3^+, 4^-, 5^+) = \frac{-\langle 1|4\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|4\rangle \langle 4|5\rangle \langle 5|1\rangle} + \mathcal{B}^{0123}, \quad (3.20)$$

where the unknown $\mathcal{B}^{0123}$ can only contain poles in (no poles, actually)

$$\mathcal{U}^{0123} = \emptyset, \quad \mathcal{S}^{0123} = \emptyset. \quad (3.21)$$

Since there is no spurious pole in the known part of (3.20), we can conclude that $\mathcal{B}^{0123} = 0$, and this gives the correct answer.

Let's give a summary of this example. Starting from bad deformation $\langle 1|5]$, we choose another three deformations $\langle 4|2]$, $\langle 1|2]$, $\langle 3|4]$ to determine the unknown boundary contributions. Among these, the first two are intentionally chosen since they are both bad deformations. After each step, the number of physical poles in $\mathcal{U}$ on which remaining boundary contributions can depend, is reduced. We are forced to perform the last good deformation by demanding that it can detect the last pole $\langle 2|3]$.

3.3 Einstein-Maxwell theory

In this subsection, we present the amplitudes in Einstein-Maxwell theory, which dictates interaction between photons and gravitons, and among gravitons themselves. This case has been studied in [14, 20], where various relevant expressions can be found.

The recursion relation starts with the primitive three-point amplitudes below, i.e., the photon-photon-graviton amplitudes

$$A_3(1_\gamma^{-1}, 2_\gamma^{+1}, 3_g^{-2}) = \kappa \frac{\langle 1|3\rangle^4}{\langle 1|2\rangle^2}, \quad A_3(1_\gamma^{-1}, 2_\gamma^{+1}, 3_g^{+2}) = \kappa \frac{[2|3]^4}{[1|2]^2}, \quad (3.22)$$

and the three-graviton amplitudes

$$A_3(1_g^{-2}, 2_g^{-2}, 3_g^{+2}) = \kappa \frac{\langle 1|2\rangle^6}{\langle 2|3\rangle^2 \langle 3|1\rangle^2}, \quad A_3(1_g^{+2}, 2_g^{+2}, 3_g^{-2}) = \kappa \frac{[1|2]^6}{[2|3]^2 [3|1]^2}. \quad (3.23)$$

These are the building blocks of all higher-point amplitudes.

3.3.1 Four-point amplitude $A_4(1_\gamma^{-1}, 2_\gamma^{+1}, 3_\gamma^{-1}, 4_\gamma^{+1})$

For this four-point amplitude,¹¹ analysis on factorization limits shows that for its helicity configuration $(1_\gamma^{-1}, 2_\gamma^{+1}, 3_\gamma^{-1}, 4_\gamma^{+1})$, all possible physical poles are $\{\langle 1|2\rangle, [1|2], \langle 1|4\rangle, [1|4]\}$. Let's start with deformation $\underline{0} = \langle 2|1\rangle$, then its recursive part is

$$\mathcal{R}^{\underline{0}} = \kappa^2 \frac{\langle 1|4\rangle [2|4]^4}{[1|4] [2|3]^2}, \quad (3.24)$$

with the corresponding sets $\mathcal{D}^{\underline{0}} = \{[1|4]\}$, $\mathcal{U}^{\underline{0}} = \{\langle 1|2\rangle, [1|2], \langle 1|4\rangle\}$, $\mathcal{S}^{\underline{0}} = \{[2|3]^2\}$.

Next, we perform deformation $\underline{1} = \langle 4|1\rangle$ to detect $[1|2]$. Its recursive part can be obtained by exchanging labels 2 and 4 of $\mathcal{R}^{\underline{0}}$, then

$$\mathcal{R}^{\underline{1}} = \kappa^2 \frac{\langle 1|2\rangle [2|4]^4}{[1|2] [3|4]^2}. \quad (3.25)$$

On the other hand, $\mathcal{R}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) = 0$ under $\underline{1}$ so $\mathcal{R}\mathcal{B}^{\underline{0},\underline{1}} = \mathcal{R}^{\underline{0}}$. According to (2.13), we get

$$\mathcal{R}^{\underline{01}} = \mathcal{R}^{\underline{0}} + \mathcal{R}^{\underline{1}} = \kappa^2 \frac{\langle 1|4\rangle [2|4]^4}{[1|4] [2|3]^2} + \kappa^2 \frac{\langle 1|2\rangle [2|4]^4}{[1|2] [3|4]^2}. \quad (3.26)$$

This is already the correct answer, as one can check its factorization limits. But strictly speaking, we still have undetected poles in $\mathcal{D}^{\underline{01}} = \{[1|2], [1|4]\}$, $\mathcal{U}^{\underline{01}} = \{\langle 1|2\rangle, \langle 1|4\rangle\}$, $\mathcal{S}^{\underline{01}} = \{[2|3]^2, [3|4]^2\}$. Then let's perform deformation $\underline{2} = \langle 1|2\rangle$ to detect $\langle 1|4\rangle$, and its recursive part is

$$\mathcal{R}^{\underline{2}} = -\kappa^2 \frac{[1|4] \langle 1|2\rangle^4 \langle 3|4\rangle^4}{\langle 1|4\rangle \langle 2|3\rangle^2 \langle 2|4\rangle^4}. \quad (3.27)$$

To continue, there are two equivalent ways, as given in section 2. The first is to write

$$\mathcal{R}\mathcal{R}^{\underline{01},\underline{2}}(z_{\underline{2}}) = \kappa^2 \frac{\langle 2|4\rangle [3|4]^4 [2|1]^4}{[1|4] [1|3]^5 ([3|2] + z_{\underline{2}} [3|1])}, \quad (3.28)$$

compare this with $\mathcal{R}^{\underline{2}}$ and use (2.10), we find $\mathcal{B}\mathcal{R}^{\underline{01},\underline{2}} = 0$, thus $\mathcal{R}^{\underline{012}} = \mathcal{R}^{\underline{01}}$.

The second way as by (2.13), requires the constant term of $\mathcal{R}^{\underline{01}}(z_{\underline{2}})$, which is

$$\mathcal{R}\mathcal{B}^{\underline{01},\underline{2}} = \kappa^2 \frac{4P_{13}^2 [2|1]^2 [3|4]^2 - 2P_{14}^2 [3|1]^2 [4|2]^2 + P_{14}^2 [3|2]^2 [4|1]^2}{[3|1]^4} + \kappa^2 \frac{\langle 1|2\rangle [2|4]^4}{[1|2] [3|4]^2} \quad (3.29)$$

thus we get

$$\mathcal{R}^{\underline{012}} = \mathcal{R}\mathcal{B}^{\underline{01},\underline{2}} + \mathcal{R}^{\underline{2}} = \mathcal{R}^{\underline{01}}. \quad (3.30)$$

Similarly, one can perform another deformation to detect $\langle 1|2\rangle$ and then confirm that there is no further boundary contribution. Let's simplify the result into a more compact form, since we need it to construct higher-point amplitudes, as

$$A_4(1_\gamma^{-1}, 2_\gamma^{+1}, 3_\gamma^{-1}, 4_\gamma^{+1}) = \kappa^2 \frac{P_{13}^2 \langle 1|3\rangle^2 [2|4]^2}{P_{12}^2 P_{14}^2}. \quad (3.31)$$

It is worth mentioning that in above we have only used deformations $\langle 2|1\rangle$ and $\langle 4|1\rangle$. Each deformation has nonzero boundary contribution, as can be checked in (3.31) directly.

¹¹For four-point amplitudes, pole $\langle i|j\rangle$ is equivalent to $[k|l]$ ($i \neq j \neq k \neq l$) by momentum conservation.

3.3.2 Four-point amplitude $A_4(1_\gamma^{-1}, 2_\gamma^{+1}, 3_g^{-2}, 4_g^{+2})$

For this case, analysis on factorization limits shows that all possible physical poles are $\{\langle 1|2\rangle, [1|2], [1|3], \langle 1|4\rangle\}$. And the good deformations are $\langle 4|1\rangle$ and $\langle 1|3\rangle$, but to demonstrate the new algorithm, only bad deformations are used here.

Starting with deformation $\underline{0} = \langle 2|1\rangle$, we get

$$-A_4 = -\kappa^2 \frac{\langle 1|3\rangle^3 [2|4]^2}{\langle 1|4\rangle^2 [1|3]} + \mathcal{B}^0, \quad (3.32)$$

with the corresponding sets $\mathcal{D}^0 = \{[1|3]\}$, $\mathcal{U}^0 = \{\langle 1|2\rangle, [1|2], \langle 1|4\rangle\}$, $\mathcal{S}^0 = \{\langle 1|4\rangle^2\}$.

Next, we perform deformation $\underline{1} = \langle 3|1\rangle$, under which the pole part of $\mathcal{R}^0$ is zero. And the recursive part of $\underline{1}$ is

$$\mathcal{R}^1(z_{\underline{1}}) = \kappa^2 \frac{\langle 1|3\rangle^2 [2|4]^4}{\langle 1|2\rangle [2|3]^3 \left(z_{\underline{1}} + \frac{[1|2]}{[3|2]}\right)}. \quad (3.33)$$

From these we get

$$\mathcal{B}\mathcal{R}^{0,1} = \mathcal{R}^1 - \mathcal{R}\mathcal{R}^{0,1} = \kappa^2 \frac{\langle 1|3\rangle^2 [2|4]^4}{\langle 1|2\rangle [2|3]^3 \frac{[1|2]}{[3|2]}}, \quad (3.34)$$

then the amplitude becomes

$$-A_4 = \mathcal{R}^0 + \mathcal{B}\mathcal{R}^{0,1} + \mathcal{B}^{01} = -\kappa^2 \frac{\langle 1|3\rangle^2 \langle 2|3\rangle^2 [2|4]^4}{P_{12}^2 P_{13}^2 P_{14}^2} + \mathcal{B}^{01}, \quad (3.35)$$

with the corresponding sets $\mathcal{D}^{01} = \{[1|2], [1|3]\}$, $\mathcal{U}^{01} = \{\langle 1|2\rangle, \langle 1|4\rangle\}$, $\mathcal{S}^{01} = \emptyset$. Now the unknown $\mathcal{B}^{01}$ may depend on $\langle 1|2\rangle$ and $\langle 1|4\rangle$, so one should perform further deformations to detect these two. However, since $\mathcal{R}^{01}$ already satisfies all factorization limits, we can conclude that $\mathcal{B}^{01} = 0$, and it gives the correct answer.

To check this, one can perform deformation $\underline{2} = \langle 1|4\rangle$ to detect $\langle 1|2\rangle$, and its recursive part is

$$\mathcal{R}^2(z_{\underline{2}}) = \kappa^2 \frac{\langle 2|3\rangle^6 \langle 1|4\rangle^4 [2|1]}{\langle 2|4\rangle^6 \langle 3|4\rangle^2 \langle 2|4\rangle \left(z_{\underline{2}} - \frac{\langle 1|2\rangle}{\langle 4|2\rangle}\right)}, \quad (3.36)$$

then the pole part of $\mathcal{B}^{01}$ is given by

$$\mathcal{B}\mathcal{R}^{01,2} = \mathcal{R}^2 - \mathcal{R}\mathcal{R}^{01,2} = 0. \quad (3.37)$$

From this we get

$$-A_4 = -\kappa^2 \frac{\langle 1|3\rangle^2 \langle 2|3\rangle^2 [2|4]^4}{P_{12}^2 P_{13}^2 P_{14}^2} + \mathcal{B}^{012}, \quad (3.38)$$

where the unknown $\mathcal{B}^{012}$ does not contain any poles in $\mathcal{D}^{012} = \{\langle 1|2\rangle, [1|2], [1|3]\}$. Similarly, one can perform deformation $\underline{3} = \langle 1|2\rangle$ to detect $\langle 1|4\rangle$ and confirm that there is no further boundary contribution from $\mathcal{B}^{012}$.

Thus, the correct answer is

$$A_4(1_\gamma^{-1}, 2_\gamma^{+1}, 3_g^{-2}, 4_g^{+2}) = \kappa^2 \frac{\langle 1|3 \rangle^2 \langle 2|3 \rangle^2 [2|4]^4}{P_{12}^2 P_{13}^2 P_{14}^2} . \quad (3.39)$$

Again in above only bad deformations $\langle 2|1 \rangle$ and $\langle 3|1 \rangle$ have been used.

3.3.3 Five-point amplitude $A_5(1_\gamma^{-1}, 2_\gamma^{+1}, 3_\gamma^{-1}, 4_\gamma^{+1}, 5_g^{-2})$

For this case, analysis on factorization limits shows that all possible physical poles are $\{[1|2], [1|4], [1|5], [2|3], [2|5], [3|4], [3|5], [4|5]\}$. Starting with deformation $\underline{0} = \langle 2|1 \rangle$, we find

$$\mathcal{R}^{\underline{0}} = -\kappa^3 \frac{\langle 1|4 \rangle \langle 3|5 \rangle [3|4] [2|4]^4}{[1|4] [2|3] [2|5] [3|5] [4|5]} + \kappa^3 \frac{\langle 1|5 \rangle \langle 3|4 \rangle [3|5] [2|4]^5}{[1|5] [2|3] [3|4] [4|5] [2|5]^2}, \quad (3.40)$$

with the corresponding sets $\mathcal{D}^{\underline{0}} = \{[1|4], [1|5]\}$, $\mathcal{U}^{\underline{0}} = \{[1|2], [2|3], [2|5], [3|4], [3|5], [4|5]\}$, $\mathcal{S}^{\underline{0}} = \{[2|5]^2\}$.

Next, we perform deformation $\underline{1} = \langle 3|2 \rangle$ to detect $[1|2]$ and $[2|5]$, and its recursive part is

$$\mathcal{R}^{\underline{1}} = \kappa^3 \frac{\langle 1|2 \rangle \langle 4|5 \rangle [2|3]^4 [1|4]^5}{[1|2] [1|5] [3|4] [3|5] [4|5] [1|3]^4} - \kappa^3 \frac{\langle 2|5 \rangle \langle 1|4 \rangle [1|3] [2|3]^4 [4|5]^5}{[2|5] [1|4] [3|4] [1|5] [3|5]^6}. \quad (3.41)$$

On the other hand, $\mathcal{RB}^{\underline{0}, \underline{1}}$ is given by

$$\begin{aligned} \mathcal{RB}^{\underline{0}, \underline{1}} = & \kappa^3 \frac{\langle 1|4|3 \rangle (\langle 2|5 \rangle [2|3]^4 [4|5]^4 + \langle 3|5 \rangle [4|3] [5|3] ([4|5] [3|2] - [4|2] [5|3]) ([4|5]^2 [3|2]^2 + [4|2]^2 [5|3]^2))}{[3|2] [4|1] [5|4] [5|3]^6} \\ & + \kappa^3 \frac{\langle 1|5 \rangle (-5 \langle 2|4 \rangle [3|2]^4 [5|4]^4 + \langle 3|4 \rangle [4|3] [5|3] ([4|3]^3 [5|2]^3 + 5 [3|2]^3 [5|4]^3 + 5 [3|2] [5|4] [4|2]^2 [5|3]^2))}{[3|2] [5|1] [5|4] [5|3]^5}. \end{aligned} \quad (3.42)$$

Then we get

$$\begin{aligned} \mathcal{R}^{\underline{01}} = & \mathcal{RB}^{\underline{0}, \underline{1}} + \mathcal{R}^{\underline{1}} \\ = & \kappa^3 \frac{\langle 1|4|3 \rangle (\langle 2|5 \rangle [2|3]^4 [4|5]^4 + \langle 3|5 \rangle [4|3] [5|3] ([4|5] [3|2] - [4|2] [5|3]) ([4|5]^2 [3|2]^2 + [4|2]^2 [5|3]^2))}{[3|2] [4|1] [5|4] [5|3]^6} \\ & + \kappa^3 \frac{\langle 1|5 \rangle (-5 \langle 2|4 \rangle [3|2]^4 [5|4]^4 + \langle 3|4 \rangle [4|3] [5|3] ([4|3]^3 [5|2]^3 + 5 [3|2]^3 [5|4]^3 + 5 [3|2] [5|4] [4|2]^2 [5|3]^2))}{[3|2] [5|1] [5|4] [5|3]^5} \\ & + \kappa^3 \frac{\langle 1|2 \rangle \langle 4|5 \rangle [2|3]^4 [1|4]^5}{[1|2] [1|5] [3|4] [3|5] [4|5] [1|3]^4} - \kappa^3 \frac{\langle 2|5 \rangle \langle 1|4 \rangle [1|3] [2|3]^4 [4|5]^5}{[2|5] [1|4] [3|4] [1|5] [3|5]^6}, \end{aligned} \quad (3.43)$$

with the corresponding sets $\mathcal{D}^{\underline{01}} = \{[1|2], [1|4], [1|5], [2|5]\}$, $\mathcal{U}^{\underline{01}} = \{[2|3], [3|4], [3|5], [4|5]\}$, $\mathcal{S}^{\underline{01}} = \{[1|3]^4, [3|5]^5, [3|5]^6\}$. Next, let's perform deformation $\underline{2} = \langle 4|3 \rangle$ to detect $[2|3]$ and $[3|5]$, and its recursive part is

$$\mathcal{R}^{\underline{2}} = -\kappa^3 \frac{\langle 3|2 \rangle \langle 1|5 \rangle [1|2] [4|2]^4}{[3|2] [4|1] [4|5] [1|5] [2|5]} + \kappa^3 \frac{\langle 3|5 \rangle \langle 1|2 \rangle [1|5] [4|2]^5}{[3|5] [4|1] [1|2] [2|5] [4|5]^2}, \quad (3.44)$$

while $\mathcal{RB}^{01,2}$ is given by

$$\mathcal{RB}^{01,2} = \kappa^3 \frac{(\langle 1|3\rangle \langle 2|5\rangle [2|1] [5|4] - \langle 1|2\rangle \langle 3|5\rangle [4|1] [5|2]) [2|4]^4}{[2|1] [4|3] [5|1] [5|2] [5|4]^2}. \quad (3.45)$$

Summing these, we get

$$\begin{aligned} \mathcal{R}^{012} &= \mathcal{RB}^{01,2} + \mathcal{R}^2 \\ &= \kappa^3 \left\{ \frac{(\langle 1|3\rangle \langle 2|5\rangle [2|1] [5|4] - \langle 1|2\rangle \langle 3|5\rangle [4|1] [5|2]) [2|4]^4}{[2|1] [4|3] [5|1] [5|2] [5|4]^2} - \frac{\langle 3|2\rangle \langle 1|5\rangle [1|2] [4|2]^4}{[3|2] [4|1] [4|5] [1|5] [2|5]} \right. \\ &\quad \left. + \frac{\langle 3|5\rangle \langle 1|2\rangle [1|5] [4|2]^5}{[3|5] [4|1] [1|2] [2|5] [4|5]^2} \right\}. \end{aligned} \quad (3.46)$$

with the corresponding sets $\mathcal{D}^{012} = \{[1|2], [1|4], [1|5], [2|3], [2|5], [3|5]\}$, $\mathcal{U}^{012} = \{[3|4], [4|5]\}$, $\mathcal{S}^{012} = \{[4|5]^2\}$, which requires further detections. However, $\mathcal{R}^{012}$ is already the correct answer. To verify this, one can perform deformation $\underline{3} = \langle 1|4$ to detect $[3|4]$ and $[4|5]$. After some simplification the pole part of $\mathcal{R}^{012}(z_{\underline{3}})$ under $\underline{3}$ becomes

$$\begin{aligned} \mathcal{R}\mathcal{R}^{012,3}(z_{\underline{3}}) &= \kappa^3 \frac{1}{z_{\underline{3}} + \frac{[3|4]}{[3|1]}} \frac{\langle 3|4\rangle \langle 2|5\rangle [4|1]^4 [3|2]^5}{[3|1] [3|5] [1|2] [1|5] [2|5] [3|1]^4} \\ &\quad - \kappa^3 \frac{1}{z_{\underline{3}} + \frac{[4|5]}{[1|5]}} \frac{\langle 4|5\rangle \langle 3|2\rangle [3|1] [4|1]^4 [2|5]^5}{[1|5] [3|2] [1|2] [3|5] [1|5]^6}, \end{aligned} \quad (3.47)$$

which is the same as the recursive part of $\underline{3}$, namely

$$\mathcal{R}^{\underline{3}} = \kappa^3 \frac{\langle 3|4\rangle \langle 2|5\rangle [4|1]^4 [3|2]^5}{[3|4] [3|5] [1|2] [1|5] [2|5] [3|1]^4} - \kappa^3 \frac{\langle 4|5\rangle \langle 3|2\rangle [3|1] [4|1]^4 [2|5]^5}{[4|5] [3|2] [1|2] [3|5] [1|5]^6}. \quad (3.48)$$

This observation implies $\mathcal{R}^{0123} = \mathcal{R}^{012}$. One can check that (3.46) is equivalent to the relevant expression in [14, 20]. Again only bad deformations have been used here.

3.4 Color-ordered Yukawa theory

In this subsection, we present the color-ordered amplitudes of fermions and scalars in Yukawa theory, where among scalars there is also ϕ^4 coupling. Here we focus on one type of amplitudes, namely $A_n(1_f, 2_s, \dots, (n-1)_s, n_f)$ with only one pair of fermions $(1_f, n_f)$. This case has been studied in [11]. For checking convenience, we summarize all relevant results calculated by Feynman diagrams up to six points as below: The three-point amplitudes are

$$A_3(1^+, 2, 3^+) = g [1|3], \quad A_3(1^-, 2, 3^-) = g \langle 1|3\rangle, \quad A_3(1^+, 2, 3^-) = 0, \quad A_3(1^-, 2, 3^+) = 0, \quad (3.49)$$

the four-point amplitudes are

$$\begin{aligned} A_4(1^-, 2, 3, 4^-) &= 0, \quad A_4(1^+, 2, 3, 4^+) = 0, \\ A_4(1^-, 2, 3, 4^+) &= g^2 \frac{\langle 1|2\rangle [2|4]}{P_{12}^2} = -g^2 \frac{\langle 1|3\rangle}{\langle 4|3\rangle}, \\ A_4(1^+, 2, 3, 4^-) &= g^2 \frac{[1|2] \langle 2|4\rangle}{P_{12}^2} = -g^2 \frac{[1|3]}{[4|3]}, \end{aligned} \quad (3.50)$$

the five-point amplitudes are

$$\begin{aligned}
 A_5(1^-, 2, 3, 4, 5^-) &= -g^3 \frac{[2|4]}{[2|1][5|4]} + g\lambda \frac{1}{[5|1]}, \\
 A_5(1^+, 2, 3, 4, 5^+) &= -g^3 \frac{\langle 2|4 \rangle}{\langle 2|1 \rangle \langle 5|4 \rangle} + g\lambda \frac{1}{\langle 5|1 \rangle}, \\
 A_5(1^+, 2, 3, 4, 5^-) &= 0, \quad A(1^-, 2, 3, 4, 5^+) = 0,
 \end{aligned} \tag{3.51}$$

and the six-point amplitudes are

$$\begin{aligned}
 A_6(1^-, 2, 3, 4, 5, 6^-) &= 0, \quad A_6(1^+, 2, 3, 4, 5, 6^+) = 0, \\
 A_6(1^-, 2, 3, 4, 5, 6^+) &= g^4 \frac{[2|4 + 6|5]}{[2|1] \langle 6|5 \rangle P_{456}^2} - g^2 \lambda \frac{\langle 1|5 \rangle}{\langle 6|5 \rangle P_{234}^2} + g^2 \lambda \frac{[2|6]}{[2|1] P_{345}^2}, \\
 A_6(1^+, 2, 3, 4, 5, 6^-) &= g^4 \frac{\langle 2|4 + 6|5]}{\langle 2|1 \rangle [6|5] P_{456}^2} - g^2 \lambda \frac{[1|5]}{[6|5] P_{234}^2} + g^2 \lambda \frac{\langle 2|6 \rangle}{\langle 2|1 \rangle P_{345}^2}.
 \end{aligned} \tag{3.52}$$

Before proceeding, several matters need to be emphasized. Since a fermion propagator is $\frac{P}{P^2}$, its choice of momentum direction makes a difference, so we must insist one consistently. Such a subtlety has been studied in [26–28]. Moreover, the following conventions are adopted: Propagators of both fermions and scalars are without i 's. For null momentum $-P$, we choose $|-P\rangle = |P\rangle$ and $|-P] = -|P]$.

3.4.1 Four-point amplitude with two fermions

The first example is amplitude $A_4(1^-, 2, 3, 4^+)$. All possible physical poles are $\{[1|2] = \langle 3|4 \rangle\}$. From its expression in (3.50), among $2C_4^2 = 12$ choices only $\langle 4|1 \rangle, \langle 4|2 \rangle, \langle 3|1 \rangle$ are good deformations.

Let's start with bad deformation $\underline{0} = \langle 3|2 \rangle$, then the amplitude is

$$-A_4 = g^2 \frac{\langle 1|4 \rangle [1|4]}{\langle 2|4 \rangle [2|1]} + \mathcal{B}^{\underline{0}}, \tag{3.53}$$

with the corresponding sets $\mathcal{D}^{\underline{0}} = \{[1|2]\}$, $\mathcal{U}^{\underline{0}} = \emptyset$, $\mathcal{S}^{\underline{0}} = \{\langle 2|4 \rangle\}$.

Although up to (3.53) all physical poles are detected, the spurious pole $\langle 2|4 \rangle$ has not been eliminated, thus we cannot set $\mathcal{B}^{\underline{0}} = 0$ yet. Instead, let's perform deformation $\underline{1} = \langle 4|3 \rangle$ to detect $\langle 2|4 \rangle$. Since its recursive part is zero, from (3.53) we find

$$\mathcal{BR}^{\underline{0}, \underline{1}} = -g^2 \frac{\langle 1|2 \rangle \langle 4|3 \rangle [1|4]}{\langle 2|3 \rangle \langle 2|4 \rangle [2|1]}. \tag{3.54}$$

It is worth mentioning that, although $[2|1]$ and $\langle 2|3 \rangle$ appear in the denominator above, the special combinations $\frac{[1|4]}{\langle 2|3 \rangle}$ and $\frac{\langle 4|3 \rangle}{[2|1]}$ prevent them from being actual poles, which is again a safety check. Plugging (3.54) back, we get

$$-A_4 = -g^2 \frac{[2|4]}{[2|1]} + \mathcal{B}^{\underline{0}, \underline{1}}. \tag{3.55}$$

Now above known term does not contain any spurious poles and all physical poles have been detected, we can safely set $\mathcal{B}^{\underline{0}, \underline{1}} = 0$, and reach the correct answer (3.50).

3.4.2 Five-point amplitude with two fermions

The second example is amplitude $A_5(1^+, 2, 3, 4, 5^+)$. Analysis of its factorization limits shows that all possible physical poles are $\{\langle 1|2\rangle, \langle 4|5\rangle, \langle 5|1\rangle\}$. Here among $2C_5^2 = 20$ choices, only $\langle 1|3\rangle, \langle 1|4\rangle, \langle 5|2\rangle, \langle 5|3\rangle$ are good deformations.

Let's start with bad deformation $\underline{0} = \langle 1|5\rangle$, then the amplitude is

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} + \mathcal{B}^0, \quad (3.56)$$

with the corresponding sets $\mathcal{D}^0 = \{\langle 1|2\rangle\}$, $\mathcal{U}^0 = \{\langle 4|5\rangle, \langle 5|1\rangle\}$, $\mathcal{S}^0 = \emptyset$. Next, we perform deformation $\underline{1} = \langle 5|4\rangle$, and its recursive part is

$$\mathcal{R}^1(z_{\underline{1}}) = g\lambda \frac{1}{\langle 1|5\rangle - z_{\underline{1}} \langle 1|4\rangle}, \quad (3.57)$$

compare this with (3.56), the pole part gives

$$\mathcal{B}\mathcal{R}^{0,1} = g\lambda \frac{1}{\langle 1|5\rangle}. \quad (3.58)$$

Plugging this back, we get

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} - g\lambda \frac{1}{\langle 5|1\rangle} + \mathcal{B}^{01}, \quad (3.59)$$

with the corresponding sets

$$\mathcal{D}^{01} = \{\langle 1|2\rangle, \langle 5|1\rangle\}, \quad \mathcal{U}^{01} = \{\langle 4|5\rangle\}, \quad \mathcal{S}^{01} = \emptyset. \quad (3.60)$$

To continue, we need to perform another deformation, e.g., $\underline{2} = \langle 5|1\rangle$ to detect $\langle 4|5\rangle$. However, it can be checked that under $\underline{2}$ the pole part of $\mathcal{B}^{01}$ is zero. Since all physical poles have been detected, we can conclude that $\mathcal{B}^{01} = 0$, and the correct answer is

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} - g\lambda \frac{1}{\langle 5|1\rangle}, \quad (3.61)$$

which matches (3.51). Again only bad deformations have been used here.

3.4.3 Six-point amplitude with two fermions

The last example is amplitude $A_6(1^+, 2, 3, 4, 5, 6^-)$. Careful analysis of its factorization limits shows that all possible physical poles are $\{\langle 1|2\rangle, [5|6], P_{123}^2, P_{234}^2, P_{345}^2\}$. And among $2C_6^2 = 30$ choices, the only good deformations are $\langle 1|i\rangle$ with $i = 3, 4, 5, 6$, $\langle 2|j\rangle$ with $j = 5, 6$, and $\langle k|6\rangle$ with $k = 3, 4$.

Let's start with bad deformation $\underline{0} = \langle 4|1\rangle$, then the amplitude is

$$-A_6 = -g^2\lambda \frac{\langle 6|2\rangle}{\langle 1|2\rangle P_{345}^2} + g^2\lambda \frac{\langle 6|1+5|4\rangle}{\langle 1|5+6|4\rangle P_{234}^2} - g^4 \left(\frac{\langle 2|4+6|5\rangle}{\langle 2|1\rangle [6|5] P_{123}^2} + \frac{[4|5]}{[6|5] \langle 1|2+3|4\rangle} \right) + \mathcal{B}^0, \quad (3.62)$$

with the corresponding sets

$$\mathcal{D}^0 = \{P_{123}^2, P_{234}^2, P_{345}^2\}, \quad \mathcal{U}^0 = \{\langle 1|2\rangle, [5|6]\}, \quad \mathcal{S}^0 = \{\langle 1|5+6|4\rangle, \langle 1|2+3|4\rangle\}. \quad (3.63)$$

Next, we perform deformation $\underline{1} = \langle 5|2\rangle$ to detect $[5|6]$, $\langle 1|5+6|4\rangle$ and $\langle 1|2+3|4\rangle$, and its recursive part is

$$\mathcal{R}^1(z_{\underline{1}}) = -g^4 \frac{\langle 2|4+6|5\rangle}{\langle 2|1\rangle [6|5] P_{123}^2(z_{\underline{1}})} + g^2 \lambda \frac{[1|5]}{[6|5] P_{234}^2(z_{\underline{1}})} - g^2 \lambda \frac{\langle 6|2\rangle}{\langle 1|2\rangle P_{345}^2(z_{\underline{1}})}, \quad (3.64)$$

compare this with the pole part of (3.62) (in fact, all known terms are pole parts), which gives

$$\mathcal{BR}^{0,\underline{1}} = g^2 \lambda \frac{[1|5]}{[6|5] P_{234}^2} - g^2 \lambda \frac{\langle 6|1+5|4\rangle}{\langle 1|5+6|4\rangle P_{234}^2} + g^4 \frac{[4|5]}{[6|5] \langle 1|2+3|4\rangle}. \quad (3.65)$$

Thus we get

$$-A_6 = -g^2 \lambda \frac{\langle 6|2\rangle}{\langle 1|2\rangle P_{345}^2} + g^2 \lambda \frac{[1|5]}{[6|5] P_{234}^2} - g^4 \frac{\langle 2|4+6|5\rangle}{\langle 2|1\rangle [6|5] P_{123}^2} + \mathcal{B}^{01}, \quad (3.66)$$

with the corresponding sets

$$\mathcal{D}^{01} = \{P_{123}^2, P_{234}^2, P_{345}^2, [5|6]\}, \quad \mathcal{U}^{01} = \{\langle 1|2\rangle\}, \quad \mathcal{S}^{01} = \emptyset. \quad (3.67)$$

We can continue with deformation $\underline{2} = \langle 2|3\rangle$ to detect the last pole $\langle 1|2\rangle$. It is easy to check that under $\underline{2}$ the pole part of $\mathcal{B}^{01}$ is zero. Since all physical poles have been detected, we conclude that $\mathcal{B}^{01} = 0$, and reach the correct answer by (3.66).

4 Conclusion

After presenting the new algorithm and several examples, we will give some remarks below. Since this approach is based on pole structure of boundary contributions, and factorization limits of amplitudes, which are general properties of quantum field theories, we expect that it can be applied to theories with massive external and internal particles, as well as theories in other dimensions. It will be interesting to explore these directions. As mentioned before, if the boundary term does not contain any poles, this approach can not be applied, which can happen for some effective theories. More explicitly, if an effective theory has a primary interaction term as Φ^m in its Lagrangian, an m -point amplitude will have a pure polynomial contribution for boundary terms. Then to determine n -point amplitudes with $n > m$, we do need to know all these primary interaction vertices.

This algorithm is self-contained to calculate amplitudes without beforehand analysis of the large z behavior of an amplitude under a given deformation, although knowing this will make calculations much easier. In practical calculations, if the analysis of large z behavior is difficult, we can boldly use the on-shell recursion relation with arbitrary choices of deformations. Then we need to judge whether the result obtained is correct. There are three criteria: (1) All spurious poles must be canceled out. (2) The power of any physical

pole must be at most one. (3) It must have correct factorization limits for all physical poles. If the result satisfies above all, it is very likely to be correct (up to possible polynomial terms). If it fails to satisfy at least one, there must be missing boundary contributions, then one needs to continue the algorithm to amend it until the correct answer is found.

At this point, we would like to discuss an important issue of the algorithm: its efficiency. This paper concerns mostly with the feasibility of the algorithm, as demonstrated by the examples. However, among all allowed choices, how to choose deformations to maximally simplify the calculation is still not clear. As shown by our cases, although naively the algorithm requires to detect all possible physical poles (maybe plus some spurious poles), in practice quite often fewer steps than necessary can reach the complete answer. Thus, how to optimize the choices is an important future project.

The idea behind this algorithm is quite simple and general, so it can be generalized to many other cases. For example, so far we have only considered the BCFW deformation, but there are other types such as the Risager deformation [29]. It can also be applied to calculate the rational parts of one-loop amplitudes [17–19]. In this case double poles exist, for which the general physical picture is not yet fully understood. Having determined the full tree amplitudes, and after a combined use of unitarity cut [21–24],¹² it is possible to reconstruct loop amplitudes without recourse to Feynman diagrams. Furthermore, it is also intriguing to apply it to generalize the recursion relation for loop integrands of $\mathcal{N} = 4$ SYM [25], to general quantum field theories.

Acknowledgments

The authors would like to thank David Kosower and Qingjun Jin for valuable discussions. This work is supported, in part, by Qiu-Shi¹³ funding and Chinese NSF funding under contracts No.11031005, No.11135006 and No.11125523.

A More abstract operation

In this appendix, we introduce a concise algebraic language to reinterpret the new algorithm. Assume that A is a rational function of external kinematic variables $\lambda_i, \tilde{\lambda}_i$, its poles can be *physical or spurious* and we treat them on the same footing. Consider its deformed version $A(z_s)$ with the deformed spinor pair $\langle i_s | j_s]$, as a rational function of z_s , $A(z_s)$ is decomposed in the same way as (2.1), namely

$$A(z_s) = \sum_{P_t^2 \in \mathcal{D}^s} \frac{a_t}{P_t^2(z_s)} + C_0^s + \sum C_i^s z_s^i, \quad (\text{A.1})$$

¹²When combining tree amplitudes together along cuts, there are a few ambiguities to be fixed. The similar idea was used in [20] recently, where tree amplitudes are reconstructed by combining all factorization channels.

¹³Qiu-Shi in Chinese means ‘to explore the truth’.

where again $\mathcal{D}^{\underline{s}}$ and $\mathcal{U}^{\underline{s}}$ denote the detectable and undetectable poles of deformation $\underline{s}$ respectively. After the decomposition, one can define two operators as below

$$\mathcal{P}^{\underline{s}}[A] \equiv \sum_{P_t^2 \in \mathcal{D}^{\underline{s}}} \frac{a_t}{P_t^2(z_{\underline{s}} = 0)}, \quad \mathcal{C}^{\underline{s}}[A] \equiv C_0. \quad (\text{A.2})$$

By this definition the identity operator can be trivially written as

$$\mathcal{I} = \mathcal{P}^{\underline{s}} + \mathcal{C}^{\underline{s}}. \quad (\text{A.3})$$

Now the new algorithm can be rewritten as the following: Starting from (A.3), one can insert the identity operator into any place as pleased. One way is

$$\begin{aligned} \mathcal{I} &= \mathcal{P}^0 + \mathcal{C}^0 = \mathcal{P}^0 + \mathcal{I}\mathcal{C}^0 = \mathcal{P}^0 + (\mathcal{P}^1 + \mathcal{C}^1)\mathcal{C}^0 \\ &= \mathcal{P}^0 + \mathcal{P}^1\mathcal{C}^0 + \mathcal{C}^1\mathcal{C}^0, \end{aligned} \quad (\text{A.4})$$

which exactly matches (2.11), where $\mathcal{B}\mathcal{R}^{0,1} = \mathcal{P}^1\mathcal{C}^0[A]$ and $\mathcal{B}^{01} = \mathcal{C}^1\mathcal{C}^0[A]$. In the other way, (2.13) can be recovered by

$$\begin{aligned} \mathcal{I} &= \mathcal{P}^1 + \mathcal{C}^1 = \mathcal{P}^1 + \mathcal{C}^1\mathcal{I} = \mathcal{P}^1 + \mathcal{C}^1(\mathcal{P}^0 + \mathcal{C}^0) \\ &= \mathcal{P}^1 + \mathcal{C}^1\mathcal{P}^0 + \mathcal{C}^1\mathcal{C}^0, \end{aligned} \quad (\text{A.5})$$

where $\mathcal{R}\mathcal{B}^{0,1} = \mathcal{C}^1\mathcal{P}^0[A]$ and $\mathcal{B}^{01} = \mathcal{C}^1\mathcal{C}^0[A]$. The pattern of the new algorithm is then clear: After inserting identity operators repeatedly into terms with $\mathcal{C}$'s only, one can expand them to specify the calculation at each step. For the $\mathcal{P}$'s at rightmost, we know how to calculate by using the recursion relation. For the $\mathcal{C}$'s following after $\mathcal{P}$'s, the constant extraction is also known. And for the last term with $\mathcal{C}$'s only, namely $\mathcal{C}^n\mathcal{C}^{n-1}\dots\mathcal{C}^1\mathcal{C}^0$, when the n -th step covers all possible physical poles and there is no spurious pole in all known terms, one can finally set it to zero (up to possible polynomial terms).

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