Published for SISSA by Springer

Received: November 26, 2013
Accepted: February 12, 2014
Published: March 7, 2014

The $\mathcal{N} = 1$ Chiral Multiplet on $T^2 \times S^2$ and Supersymmetric Localization

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Abstract: We compute the supersymmetric partition function of an $\mathcal{N} = 1$ chiral multiplet coupled to an external Abelian gauge field on complex manifolds with $T^2 \times S^2$ topology. The result is locally holomorphic in the complex structure moduli of $T^2 \times S^2$. This computation illustrates in a simple example some recently obtained constraints on the parameter dependence of supersymmetric partition functions.

We also devise a simple method to compute the chiral multiplet partition function on any four-manifold $\mathcal{M}_4$ preserving two supercharges of opposite chiralities, via supersymmetric localization. In the case of $\mathcal{M}_4 = S^3 \times S^1$, we provide a path integral derivation of the previously known result, the elliptic gamma function, which emphasizes its dependence on the $S^3 \times S^1$ complex structure moduli.

Keywords: Topological Field Theories, Supergravity Models, Field Theories in Lower Dimensions

ArXiv ePrint: 1311.2430

doi:10.1007/JHEP03(2014)040

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1 Introduction

$\mathcal{N} = 1$ supersymmetric field theories with an R-symmetry can be defined on a compact manifold $\mathcal{M}_4$ while preserving at least one supercharge, if and only if $\mathcal{M}_4$ is Hermitian [1, 2]. It was shown recently [3] that the partition function $Z_{\mathcal{M}_4}$ of such theories is independent of the Hermitian metric on $\mathcal{M}_4$ and that it depends holomorphically on the complex structure parameters of the underlying complex manifold.

We initiate the study of supersymmetric partition functions on $T^2 \times S^2$ by computing the partition function $Z^{\Phi}_{T^2\times S^2}$ of an $\mathcal{N} = 1$ chiral multiplet Φ coupled to an external vector multiplet. (The inclusion of dynamical gauge fields will be discussed elsewhere.) Our result provides a new concrete example to illustrate the general properties of supersymmetric partition functions obtained in [3]. In particular, the partition function $Z^{\Phi}_{T^2\times S^2}$ is locally holomorphic in the complex structure moduli of $T^2 \times S^2$.

We will work with a two-dimensional moduli space of complex structures on $T^2 \times S^2$. Consider $\mathbb{C} \times S^2$ with coordinates[1] w, z, with the identifications

$$(w, z) \sim (w + 2\pi, e^{2\pi i\alpha} z) \sim (w + 2\pi\tau, e^{2\pi i\beta} z)\,. \tag{1.1}$$

Here $\tau = \tau_1 + i\tau_2$ with $\tau_2 > 0$ is the standard modular parameter of T^2, while α, β are two real parameters which rotate the S^2 as it goes around the periods of the torus. The quotient space is diffeomorphic to $T^2 \times S^2$. The two complex parameters

$$\tau = \tau_1 + i\tau_2\,, \qquad \sigma = \tau\alpha - \beta\,, \tag{1.2}$$

are the complex structure moduli. There exist additional families of complex structures on $T^2 \times S^2$ [4]. The family that we consider here is complete, in the sense that for generic values of the moduli (1.2) the allowed deformations still lie in the same family.

On $T^2 \times S^2$ with the above complex structure, it is possible to preserve (at most) two supercharges of opposite R-charge for any value of (1.2).[2] An important feature of the corresponding supergravity backgrounds [1, 3, 6] is that the R-symmetry background gauge field has one unit of magnetic flux through the S^2. A consequence of this R-symmetry monopole is that the supercharges commute with the isometries of the sphere. Another consequence is that we are only allowed to consider fields of integer R-charges.[3]

[1]The coordinate z covers the sphere except for the south pole, where we change coordinates to $z' = 1/z$.

[2]For theories with a Ferrara-Zumino (FZ) supercurrent multiplet, this point was mentioned in [5].

[3]Yet another consequence is that the dimensional reduction of the $T^2 \times S^2$ background to $S^1 \times S^2$ does not correspond to the 3d $\mathcal{N} = 2$ "superconformal index" background of [7], which can preserve four supercharges, but rather to a different $S^1 \times S^2$ background with one unit of magnetic flux. See [3, 8] for related discussions of $S^1 \times S^2$ backgrounds.

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Consider an $\mathcal{N}=1$ chiral multiplet Φ of R-charge r and Q_f-charge q_f, where Q_f is an Abelian flavor symmetry $\mathrm{U}(1)_f$. We couple Φ to a background $\mathrm{U}(1)_f$ real vector multiplet in the most general way that preserves the two supercharges [1, 3]. The $\mathrm{U}(1)_f$ background gauge field has flux $g \in \mathbb{Z}$ through the S^2 and flat connections a_x, a_y along the one-cycles of the torus. Let us define the shifted R-charge $\mathbf{r} = r + q_f\, g$ and the fugacities

$$q = e^{2\pi i\tau}\,, \qquad x = e^{2\pi i(\tau\alpha-\beta)}\,, \qquad t = e^{2\pi i(\tau a_x - a_y)}\,. \tag{1.3}$$

The shifted R-charge $\mathbf{r}$ must also be integer. For $\mathbf{r}>1$, we find

$$Z^{\Phi}_{\mathbf{r}>1}(q,x,t) = \left(\frac{q^{\frac{1}{12}}}{\sqrt{t^{q_f}}}\right)^{\mathbf{r}-1} \prod_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1} \prod_{k\geq 0} \left(1-q^{k+1}x^{-m}t^{-q_f}\right)\left(1-q^k x^m t^{q_f}\right). \tag{1.4}$$

Similarly, for $\mathbf{r}\leq 1$:

$$\begin{aligned} Z^{\Phi}_{\mathbf{r}=1}(q,x,t) &= 1\,, \\ Z^{\Phi}_{\mathbf{r}<1}(q,x,t) &= \left(\frac{\sqrt{t^{q_f}}}{q^{\frac{1}{12}}}\right)^{|\mathbf{r}|+1} \prod_{m=-\frac{|\mathbf{r}|}{2}}^{\frac{|\mathbf{r}|}{2}} \prod_{k\geq 0} \frac{1}{(1-q^{k+1}x^{-m}t^{-q_f})\,(1-q^k x^m t^{q_f})}\,. \end{aligned} \tag{1.5}$$

We will present two independent derivations of (1.4), (1.5). The first method is to compute the partition function by canonical quantization. We define a supersymmetric index for states on $S^1\times S^2$,

$$\mathcal{I}_{T^2\times S^2} = \mathrm{Tr}\left((-1)^F e^{-2\pi H}\right). \tag{1.6}$$

Here F is the fermion number and H is the Hamiltonian with respect to the "time" direction along the remaining S^1. The Hamiltonian in (1.6) is computed at a generic point on the moduli space of complex structures. On states that contribute to the index, it takes the form

$$H = -i\left(\tau P + \sigma J_3 + (\tau a_x - a_y)Q_f\right), \tag{1.7}$$

with P the momentum along the spatial S^1 and J_3 the Cartan of the SO(3) isometry of the S^2. Such states are thus weighted by $q^P x^{J_3} t^{Q_f}$ in (1.6). One could also take the more traditional point of view of computing the index with the "round" metric, corresponding to the special point $\tau=i, \sigma=0$ in the complex structure moduli space, while q,x,t would be introduced as fugacities for operators commuting with the supercharges. The advantage of our approach is that the geometrical meaning of the fugacities q,x,t is clear from the onset.

The index (1.6) coincides with the supersymmetric partition function on $T^2\times S^2$ up to local terms. To compute it, we dimensionally reduce the theory on the sphere. This leads to an infinite tower of decoupled $\mathcal{N}=(0,2)$ supersymmetric theories on the torus. Only states in short $\mathcal{N}=(0,2)$ multiplets contribute to (1.6), and the effective Hamiltonian (1.7) acting on those states encodes the dependence on the complex structure moduli τ,σ of

$T^2 \times S^2$. The index (1.6) reduces to a Witten index on T^2, also known as an $\mathcal{N} = (0, 2)$ elliptic genus [9, 10]. Our results for the index overlap with some recent work on elliptic genera [11, 12].

Our second derivation of (1.4), (1.5) is a path integral computation. We take advantage of the two Killing spinors present on our geometric background to rewrite the $\mathcal{N} = 1$ chiral multiplet in terms of more convenient variables. We then present a general method, based on supersymmetric localization (see for instance [13–15]), to compute the chiral multiplet one-loop determinant on *any* four-manifold $\mathcal{M}_4$ preserving two supercharges. In particular, in the case $\mathcal{M}_4 = T^2 \times S^2$ we confirm the result (1.4), (1.5).

We also study the partition function of a chiral multiplet on $S^3 \times S^1$ using that same localization method. This provides a new derivation of a well-known result. $Z^{\Phi}_{S^3\times S^1}$ is given by an elliptic gamma function [16, 17], whose natural domain of definition is the complex structure moduli space of $S^3 \times S^1$ [3].[4]

This paper is organized as follows. In section 2, we review some necessary background about supersymmetry on curved spaces, while we study the $T^2 \times S^2$ background more thoroughly in section 3. We derive the result (1.4), (1.5) in section 4 by computing the supersymmetric index (1.6) for an $\mathcal{N} = 1$ chiral multiplet, and we briefly describe some of its interesting properties. In section 5, we propose a general method to compute the chiral multiplet partition function on any background with two supercharges, and we apply this method to the cases of $T^2 \times S^2$ and $S^3 \times S^1$. Some useful additional material is presented in appendices A, B and C.

2 Curved space supersymmetry and the $\mathcal{N} = 1$ chiral multiplet

In this section, we review curved space rigid supersymmetry for $\mathcal{N} = 1$ supersymmetric theories with an R-symmetry, focussing on the case of two supercharges of opposite chiralities. We discuss in detail the $\mathcal{N} = 1$ chiral multiplet coupled to an external gauge multiplet.

2.1 Background supergravity multiplet

Curved space supersymmetry on compact manifolds is best understood as a rigid limit of off-shell supergravity [6]. Four-dimensional $\mathcal{N} = 1$ supersymmetric theories with an exact R-symmetry possess a supercurrent multiplet, called the $\mathcal{R}$-multipet, whose bottom component is the conserved R-symmetry current [19–21]. The $\mathcal{R}$-multiplet couples to the new minimal supergravity multiplet of [19], which contains the metric $g_{\mu\nu}$, a gravitino, and two auxiliary fields A_μ and V_μ. The field A_μ is an R-symmetry gauge field while V_μ is a vector which satisfies $\nabla_\mu V^\mu = 0$. Rigid supersymmetry on a compact four-manifold $\mathcal{M}_4$ corresponds to a choice of supersymmetric background values for $g_{\mu\nu}, A_\mu, V_\mu$ [6], such that the (generalized) Killing spinor equations

$$\begin{aligned}(\nabla_\mu - iA_\mu)\zeta &= -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta\,,\\ (\nabla_\mu + iA_\mu)\widetilde{\zeta} &= iV_\mu\widetilde{\zeta} + iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta}\,,\end{aligned} \tag{2.1}$$

[4] The computation of $Z^{\Phi}_{S^3\times S^1}$ was also reconsidered recently in [18] from the canonical quantization point of view.

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admit at least one non-trivial solution, which we can take to be ζ. The Killing spinors ζ_α and $\widetilde{\zeta}^{\dot{\alpha}}$ are spinors of R-charge $+1$ and -1, respectively, of opposite chiralities. Any non-trivial solution is nowhere vanishing.

In [1, 2], it was shown that the existence of a single Killing spinor ζ is equivalent to the existence of a complex structure on $\mathcal{M}_4$ compatible with the metric. Given ζ, the metric-compatible complex structure is given explicitly by[5]

$$J^\mu{}_\nu = -\frac{2i}{|\zeta|^2}\zeta^\dagger\sigma^\mu{}_\nu\zeta\,. \tag{2.2}$$

Conversely, given a complex manifold with a choice of a Hermitian metric, one can solve for the auxiliary fields A_μ, V_μ and for ζ explicitly [1]. In this paper we will focus on backgrounds preserving at least *two supercharges of opposite chiralities*, δ_ζ and $\delta_{\widetilde{\zeta}}$. Given two Killing spinors ζ and $\widetilde{\zeta}$, there exists a complex Killing vector [1]

$$K = \zeta\sigma^\mu\widetilde{\zeta}\,\partial_\mu\,. \tag{2.3}$$

This vector is anti-holomorphic with respect to (2.2). Moreover, we will assume that K commutes with its complex conjugate, $[K, K^\dagger] = 0$.[6] Such backgrounds are torus fibrations over a Riemann surface Σ. We can pick complex coordinates w, z adapted to the complex structure (2.2) such that $K = \partial_{\bar{w}}$. The Hermitian metric is locally given by

$$ds^2 = \Omega^2(z,\bar{z})\Big((dw + h(z,\bar{z})dz)(d\bar{w} + \bar{h}(z,\bar{z})d\bar{z}) + c^2(z,\bar{z})dzd\bar{z}\Big)\,. \tag{2.4}$$

Note that, in general, the real and imaginary parts of K do not have closed orbits, but they are instead part of a larger $\mathrm{U}(1)^3$ isometry. $\mathcal{M}_4$ is therefore a T^2 fibration (by choosing a T^2 out of the $\mathrm{U}(1)^3$ isometry orbits), but not necessarily a holomorphic one. (The fibration is holomorphic only when the orbits of K are bona fide tori.)[7] The background supergravity fields A_μ, V_μ read

$$\begin{aligned} V_\mu &= \frac{1}{2}\nabla_\nu J^\nu{}_\mu + \kappa K_\mu\,,\\ A_\mu &= -\frac{1}{4}J_\mu{}^\nu\partial_\nu\log\sqrt{g} + \frac{1}{4}(\delta_\mu{}^\nu + iJ_\mu{}^\nu)\nabla_\sigma J^\sigma{}_\nu + \frac{3}{2}\kappa K_\mu - \frac{i}{2}\partial_\mu\log s\,, \end{aligned} \tag{2.5}$$

The expression for A_μ is only valid in complex coordinates —in particular, $\sqrt{g} = \frac{1}{4}\Omega^4 c^2$ is the determinant of the Hermitian metric (2.4). The function κ is such that $K^\mu\partial_\mu\kappa = 0$, but otherwise arbitrary. In the holomorphic frame

$$e^1 = \Omega(z,\bar{z})(dw + h(z,\bar{z})dz)\,, \qquad e^2 = \Omega(z,\bar{z})c(z,\bar{z})dz\,, \tag{2.6}$$

the Killing spinors read

$$\zeta_\alpha = \sqrt{\frac{s}{2}}\begin{pmatrix}1\\0\end{pmatrix}\,, \qquad \widetilde{\zeta}^{\dot{\alpha}} = \frac{\Omega}{\sqrt{2s}}\begin{pmatrix}-1\\0\end{pmatrix} \tag{2.7}$$

[5] We are following the conventions of [3], which differ from the ones of [1] by some signs. However, our R-symmetry gauge field A_μ is defined as in [1], with $A_\mu^{(R)} = A_\mu - \frac{3}{2}V_\mu$ the gauge field used in [3].

[6] This is what we mean by “two supercharges” throughout this paper. The case $[K, K^\dagger] \neq 0$ was analysed in [1].

[7] See section 5.1 of [22] for a similar discussion in three dimensions.

From the Killing spinor ζ_α, we can also construct a holomorphic two-form

$$P = \zeta\sigma_{\mu\nu}\zeta\, dx^\mu \wedge dx^\nu = s\, g^{\frac{1}{4}}\, dw \wedge dz\,. \tag{2.8}$$

Note that s is a nowhere-vanishing global section of $\mathcal{K} \otimes L^2$, with $\mathcal{K}$ the canonical line bundle and L the R-symmetry line bundle. Therefore, the line bundle $\mathcal{K} \otimes L^2$ is trivial and we can identify $L \cong \mathcal{K}^{-\frac{1}{2}}$ up to a trivial line bundle [1].

2.2 Background vector multiplet

In addition to the supergravity background (2.4), (2.5), it is natural to consider background gauge fields coupling to conserved currents (whenever the field theory has any global symmetry). For simplicity, consider a background vector multiplet for an Abelian symmetry $\mathrm{U}(1)_f$. It has bosonic components v_μ and D, with v_μ the $\mathrm{U}(1)_f$ gauge field. Let us define its field strength

$$f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu\,. \tag{2.9}$$

In order to preserve the same Killing spinors (2.7) as the supergravity background (2.4), (2.5), the background fields v_μ, D have to satisfy [3]

$$f_{\bar{w}\bar{z}} = f_{w\bar{w}} = f_{z\bar{w}} = 0\,, \qquad D = -\frac{2i}{\Omega^2 c^2}(f_{z\bar{z}} - h f_{w\bar{z}})\,, \tag{2.10}$$

using our adapted coordinates w, z and the metric (2.4). For v_μ real, $f_{z\bar{z}}$ is the only component of the field strength that can be turned on while preserving two supercharges.

2.3 Supersymmetry transformations and supersymmetric Lagrangian

Let δ_ζ and $\delta_{\widetilde{\zeta}}$ be the two supersymmetries associated to the Killing spinors ζ and $\widetilde{\zeta}$. They satisfy the supersymmetry algebra [1]

$$\delta_\zeta^2 = 0\,, \qquad \delta_{\widetilde{\zeta}}^2 = 0\,, \qquad \{\delta_\zeta, \delta_{\widetilde{\zeta}}\} = 2i\hat{\mathcal{L}}_K\,. \tag{2.11}$$

Here K is the Killing vector (2.3). The twisted Lie derivative $\hat{\mathcal{L}}_K$ along K acts as

$$\hat{\mathcal{L}}_K = \mathcal{L}_K - iK^\mu(rA_\mu + q_f v_\mu) \tag{2.12}$$

on any field of R-charge r and Q_f-charge q_f, with $\mathcal{L}_K$ the ordinary Lie derivative.[8]

We consider a chiral multiplet $\Phi = (\phi, \psi, F)$ of R-charge r and Q_f-charge q_f, in the supersymmetric background (2.4), (2.5), (2.10). Let us define the covariant derivative

$$D_\mu = \nabla_\mu - irA_\mu - iq_f v_\mu\,, \tag{2.13}$$

acting on fields of charges r and q_f. The curved space supersymmetry transformations read [6]

$$\begin{aligned} \delta\phi &= \sqrt{2}\zeta\psi\,, \\ \delta\psi_\alpha &= \sqrt{2}\zeta_\alpha F + \sqrt{2}i(\sigma^\mu\widetilde{\zeta})_\alpha\, D_\mu\phi\,, \\ \delta F &= \sqrt{2}iD_\mu(\widetilde{\zeta}\widetilde{\sigma}^\mu\psi). \end{aligned} \tag{2.14}$$

[8]Note that the background gauge field v_μ appears in the supersymmetry algebra (2.11). Technically, this is because we work with the Wess-Zumino gauge.

Similarly, for an anti-chiral multiplet $\widetilde{\Phi} = (\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ of R-charge $-r$ and Q_f-charge $-q_f$, we have

$$\begin{aligned} \delta\widetilde{\phi} &= \sqrt{2}\widetilde{\zeta}\widetilde{\psi}\,, \\ \delta\widetilde{\psi}^{\dot\alpha} &= \sqrt{2}\widetilde{\zeta}^{\dot\alpha}\widetilde{F} + \sqrt{2}i(\widetilde{\sigma}^\mu\zeta)^{\dot\alpha}\, D_\mu\widetilde{\phi}\,, \\ \delta\widetilde{F} &= \sqrt{2}iD_\mu(\zeta\sigma^\mu\widetilde{\psi}). \end{aligned} \tag{2.15}$$

One can check that these supersymmetry transformations realize the supersymmetry algebra (2.11) if and only if ζ, $\widetilde{\zeta}$ are solutions of (2.1). One can construct the curved-space generalization of the canonical kinetic term for a chiral multiplet coupled to a background gauge field. It is given by the following supersymmetric Lagrangian:

$$\begin{aligned} \mathscr{L}_{\Phi\widetilde{\Phi}} = \;& D_\mu\widetilde{\phi}D^\mu\phi + i\widetilde{\psi}\widetilde{\sigma}^\mu D_\mu\psi - F\widetilde{F} + V^\mu\left(iD_\mu\tilde{\phi}\,\phi - i\tilde{\phi}D_\mu\phi + \frac{1}{2}\tilde{\psi}\tilde{\sigma}_\mu\psi\right) \\ & - \frac{r}{4}(R - 6V^\mu V_\mu)\,\widetilde{\phi}\phi + q_f D\widetilde{\phi}\phi\,. \end{aligned} \tag{2.16}$$

Here R is the Ricci scalar on $\mathcal{M}_4$,[9] and the various background fields take their supersymmetric values.

2.4 A comment on global anomalies

The theory of a single chiral multiplet that we are considering has cubic and mixed U(1)-gravitional anomalies for its $\mathrm{U}(1)_f$ and $\mathrm{U}(1)_R$ symmetries:

$$\mathrm{Tr}(R^3) = (r-1)^3\,, \quad \mathrm{Tr}(Q_f^3) = q_f^3\,, \quad \mathrm{Tr}(R) = r-1\,, \quad \mathrm{Tr}(Q_f) = q_f\,, \tag{2.17}$$

and similarly for $\mathrm{Tr}(R^2Q_f)$ and $\mathrm{Tr}(Q_f^2R)$. This could result in a violation of current conservation in a non-trivial background. Let us define the topological densities

$$\mathcal{P}^{(f)} = \epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}\,, \qquad \mathcal{P}^{(R)} = \epsilon^{\mu\nu\rho\sigma}F^{(R)}_{\mu\nu}F^{(R)}_{\rho\sigma}\,, \qquad \mathcal{P}^{(g)} = \epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\kappa\lambda}R_{\rho\sigma}{}^{\kappa\lambda}\,. \tag{2.18}$$

Here $f_{\mu\nu}$ is the $\mathrm{U}(1)_f$ field strength, $F^{(R)}_{\mu\nu}$ is the field strength of $A^{(R)}_\mu = A_\mu - \frac{3}{2}V_\mu$, which couples to the R-symmetry current, and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor.

In the presence of two supercharges, we have

$$\mathcal{P}^{(f)} = 0\,, \qquad \mathcal{P}^{(R)} = \frac{3}{8}\mathcal{P}^{(g)}\,. \tag{2.19}$$

The first equality follows directly from (2.10) and the second relation was derived in [23] (it is also easily checked from (2.4) and (2.5)). Since $\mathcal{P}^{(f)} = 0$, the $\mathrm{Tr}(Q_f^3)$ and $\mathrm{Tr}(Q_f^2R)$ cubic anomalies do not lead to any violation of current conservation. On the other hand, $\mathcal{P}^{(g)}$ will be non-zero in general. However, one can always write $\mathcal{P}^{(g)} = \nabla_\mu X^\mu$ for X^μ a non-covariant quantity, so that the properly shifted currents are conserved. In general, we can thus preserve $\mathrm{U}(1)_f$ and $\mathrm{U}(1)_R$ at the expense of general covariance [24]. (It was also noted in [23] that the integral of $\mathcal{P}^{(g)}$ vanishes on backgrounds with two supercharges, so that all the integrated anomalies vanish.)

[9] We follow the Riemannian geometry conventions of [1]. In particular, $R < 0$ for a round sphere.

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In any case, for all the backgrounds considered in this paper, one can actually show that $\mathcal{P}^{(g)} = 0$, and therefore the $\mathrm{U}(1)_f$ and $\mathrm{U}(1)_R$ currents are properly conserved.[10] Therefore, we will not need to worry about these global anomalies in the following.

3 Complex structures and supersymmetry on $T^2 \times S^2$

Any complex four-manifold with $T^2 \times S^2$ topology is a ruled surface of genus one [4]. Such surfaces have been classified [4, 25].[11] In this paper, we consider a two-parameter family of complex manifolds obtained as quotients of $\mathbb{C} \times S^2$. Let w, z be the complex coordinates on $\mathbb{C}$ and S^2, respectively. We consider the identifications

$$(w, z) \sim (w + 2\pi, e^{2\pi i\alpha} z)\,, \qquad (w, z) \sim (w + 2\pi\tau, e^{2\pi i\beta} z)\,. \tag{3.1}$$

The quotient space is diffeomorphic to $T^2 \times S^2$. Here $\tau = \tau_1 + i\tau_2$ is the modular parameter of the torus, and α, β are real parameters subject to the identifications $\alpha \sim \alpha+1$, $\beta \sim \beta+1$. We also introduce the complex parameter $\sigma = \tau\alpha - \beta$. Two choices of complex structure moduli τ, σ and τ', σ' are equivalent if they give rise to the same identifications (3.1). These equivalences are generated by:

$$\begin{aligned} S \;:\; & (\tau, \sigma) \mapsto \left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right)\,, & T \;:\; & (\tau, \sigma) \mapsto (\tau + 1, \sigma)\,, \\ U \;:\; & (\tau, \sigma) \mapsto (\tau, \sigma + \tau)\,, & V \;:\; & (\tau, \sigma) \mapsto (\tau, \sigma + 1)\,, \end{aligned} \tag{3.2}$$

These transformations generate a subgroup of $\mathrm{PSL}(3, \mathbb{Z})$, whose corresponding $\mathrm{SL}(3, \mathbb{Z})$ matrices are

$$S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{3.3}$$

It is convenient to introduce real coordinates x, y, θ, φ on $T^2 \times S^2$, where x, y are torus coordinates of period 2π, and $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ are the standard angles on the sphere. The complex structure on the quotient (3.1) can be realized by the complex coordinates

$$w = x + \tau y\,, \qquad z = \tan\frac{\theta}{2} e^{i(\varphi + \alpha x + \beta y)}\,, \tag{3.4}$$

where the identifications (3.1) correspond to $(x, y) \sim (x + 2\pi, y)$ and $(x, y) \sim (x, y + 2\pi)$ on the real torus. The generators (3.2) of non-trivial identifications on the complex structure moduli space correspond to large diffeomorphisms of the underlying real manifold, which are given by the matrices (3.3) acting on the coordinates (x, y, φ) in the obvious way.

[10] One can see that $\mathcal{P}^{(g)} = 0$ if $h = \bar{h} = 0$, with h the metric function appearing in (2.4). One can also check that $\mathcal{P}^{(g)} = 0$ for the $S^3 \times S^1$ background that we will consider in section 5.6.

[11] A ruled surfaces of genus one is a $\mathbb{C}P^1$ fiber bundle over a non-singular elliptic curve Σ_1. The classification of such surfaces follows from the classification of one-dimensional affine fiber bundles over Σ_1, with the ruled surfaces obtained by projectivisation [25]. It was shown in [4] that there are three classes of complex structures on $T^2 \times S^2$, denoted by $\mathcal{S}$, S_{2n} with n any positive integer, and A_0. In this paper we consider the class $\mathcal{S}$, which corresponds to ruled surfaces obtained from degree zero holomorphic line bundles over Σ_1.

3.1 Supergravity background with round metric

One can preserve two supercharges on $T^2 \times S^2$ for any choice of complex structure (3.1). We consider the metric

$$ds^2 = dw d\bar{w} + \frac{4}{(1+z\bar{z})^2} dz d\bar{z}\,, \tag{3.5}$$

which is compatible with the identification (3.1). Note that for generic values of the complex structure moduli, this metric has three real Killing vectors, corresponding to ∂_w, $\partial_{\bar{w}}$ and $z\partial_z - \bar{z}\partial_{\bar{z}}$. (The additional Killing vectors $K_\pm$ in (A.3) are not globally defined unless $\sigma = 0$.) In particular, we have the anti-holomorphic Killing vector $K = \partial_{\bar{w}}$ and we can apply the general formulas (2.4), (2.5) for the supergravity background fields. In terms of the real coordinates (3.4), the metric (3.5) reads

$$ds^2 = (dx + \tau_1 dy)^2 + \tau_2^2 dy^2 + d\theta^2 + \sin^2\theta (d\varphi + \alpha dx + \beta dy)^2\,, \tag{3.6}$$

while the anti-holomorphic Killing vector K is given by

$$K = \partial_{\bar{w}} = \frac{1}{2i\tau_2}\left(\tau\,\partial_x - \partial_y - \sigma\,\partial_\varphi\right)\,. \tag{3.7}$$

Formula (2.5) simplifies greatly because the metric (3.5) is Kähler. We have[12]

$$\begin{aligned} V_\mu dx^\mu &= \kappa K_\mu dx^\mu\,, \\ A_\mu^{(R)} dx^\mu &= \frac{1}{2}(1-\cos\theta)(d\varphi + \alpha dx + \beta dy) - \frac{i}{2}\partial_\mu \log s\, dx^\mu\,, \end{aligned} \tag{3.8}$$

with $A_\mu^{(R)} = A_\mu - \frac{3}{2}V_\mu$. The salient feature of this background is that it involves a non-trivial R-symmetry line bundle L, with first Chern class

$$c_1 = \frac{1}{2\pi}\int_{S^2} dA^{(R)} = 1\,. \tag{3.9}$$

Any field Φ of nonzero R-charge r must be a well-defined section of the line bundle L^r, with transition functions

$$\Phi^{(N)} = e^{ir(\varphi+\alpha x+\beta y)}\Phi^{(S)} \tag{3.10}$$

between the northern (N) and sourthern (S) patches.[13] It follows that the R-charge must be *integer*, $r \in \mathbb{Z}$. In the canonical frame (2.6), $e^1 = dw$, $e^2 = \frac{2}{1+|z|^2}dz$, the Killing spinors are given by (2.7). In order for the holomorphic two-form (2.8) to be well-defined, s must satisfy

$$s \sim e^{-2\pi i\alpha}s\,, \qquad s \sim e^{-2\pi i\beta}s \tag{3.11}$$

under the identifications (3.1). We should offset (3.11) by some R-symmetry transformations, so that s is invariant as we go around the one-cycles of the torus. Therefore, any field Φ of R-charge r has twisted periodicities

$$\Phi \sim e^{i\pi r\alpha}\Phi\,, \qquad \Phi \sim e^{i\pi r\beta}\Phi\,. \tag{3.12}$$

[12] In complex coordinates, $K_\mu dx^\mu = \frac{1}{2}dw$ and $A^{(R)} = -\frac{i}{2}\frac{\bar{z}dz - zd\bar{z}}{(1+z\bar{z})} - \frac{i}{2}d\log s$.

[13] By default, all quantities are written on the northern patch, $\Phi = \Phi^{(N)}$, with complex coordinates w, z. The southern patch has coordinate w, z', with $z' = \frac{1}{z}$ on the overlap.

under (3.1). Finally, note that s is a non-trivial section of the canonical line bundle $\mathcal{K}$ but a trivial section of the total bundle $\mathcal{K}\otimes L^2$, which is therefore trivial [1]. We will take $s=1$ in the following.

Note that the R-symmetry flux through the S^2 is necessary to preserve supersymmetry on $T^2\times S^2$. This is in contrast to lower dimensional cases where supersymmetric backgrounds without flux exist on $S^1\times S^2$ [7] and S^2 [26, 27]. Such backrgounds preserve four supercharges and do not admit an uplift to four dimensions [6].

3.2 Supersymmetric background gauge field

Let us also consider a supersymmetric background gauge multiplet for an Abelian symmetry $\mathrm{U}(1)_f$, which satisfy (2.10). We consider a real gauge field for simplicity. The most general such background gauge field preserving the isometries of (3.5) is given by

$$\begin{aligned} v_\mu dx^\mu &= v_w dw + v_{\bar w} dw - ig\frac{\bar z dz - z d\bar z}{2(1+z\bar z)} \\ &= a_x dx + a_y dy + \frac{g}{2}(1-\cos\theta)(d\varphi + \alpha dx + \beta dy)\,, \end{aligned} \tag{3.13}$$

up to gauge transformations, while the auxiliary field D is given in (2.10). Note that

$$v_w = -\frac{\bar\tau a_x - a_y}{2i\tau_2}\,, \qquad v_{\bar w} = \frac{\tau a_x - a_y}{2i\tau_2}\,. \tag{3.14}$$

The real parameters a_x, a_y are flat connections, which must be identified by $a_x\sim a_x+1$, $a_y\sim a_y+1$ due to large gauge transformations. It is also convenient to define

$$\nu = \tau a_x - a_y\,, \tag{3.15}$$

a line bundle modulus [3] analogous to the complex structure modulus σ. The parameter g is an integer, giving us the quantized flux of (3.13) through the sphere. The discussion of the corresponding $\mathrm{U}(1)_f$ line bundle is analogous to the discussion of the $\mathrm{U}(1)_R$ bundle above. In particular, a field Φ of Q_f-charge q_f has transition function $\Phi^{(N)} = e^{igq_f(\varphi+\alpha x+\beta y)}\Phi^{(S)}$ between the northern and southern patches. It will be useful to define the shifted R-charge

$$\mathbf{r} = r + q_f g\,. \tag{3.16}$$

In the presence of the background gauge field (3.13), the twisted boundary conditions (3.12) for charged fields generalize to[14]

$$\Phi\sim e^{i\pi\mathbf{r}\alpha}\Phi\,, \qquad \Phi\sim e^{i\pi\mathbf{r}\beta}\Phi\,, \tag{3.17}$$

under the identifications (3.1).

[14] It is clear from the transition functions that there must be some twisted periodicities depending on $q_f g$. This choice of boundary conditions is symmetric between the northern and southern patches.

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3.3 More general supersymmetric background

We can consider supersymmetric backgrounds on $T^2 \times S^2$ with more general metrics than (3.5). Indeed, any Hermitian metric of the local form (2.4) is as good as any other. We can still retain the map (3.4) between real and complex coordinates, and thus the explicit form (3.7) for the Killing vector K. Let us note that the real and imaginary parts of the Killing vector K do not close in general, and consequently we have three Killing vectors ∂_x, ∂_y and ∂_φ. Thus we can consider a general background (2.4), (2.5) with the functions $\Omega(z,\bar{z})$, $h(z,\bar{z})$ and $c(z,\bar{z})$ depending on $|z|^2$ only. One can similarly consider more general background gauge fields than (3.13) with the same flux g through the S^2.

For $\sigma = 0$, we need only have two Killing vectors ∂_x and ∂_y. In this special case, the $T^2 \times S^2$ index that we will compute below does not keep track of the J_3 quantum number from the sphere.

4 The $T^2 \times S^2$ index and canonical quantization

In this section, we compute the $T^2 \times S^2$ partition function of a chiral multiplet as a supersymmetric index (1.6). The first step is to dimensionally reduce the theory over S^2. Due to the presence of magnetic flux on the sphere, charged fields must be expanded in so-called monopole spherical harmonics [28], which are reviewed in appendix A. The second step is to quantize the resulting theory on the torus. The only contribution to the index will come from a finite number of short multiplets of the $\mathcal{N} = (0,2)$ supersymmetry on the torus, arising from zero modes of the Dirac operator on the S^2 with magnetic flux.

4.1 Sphere reduction and $\mathcal{N} = (0,2)$ multiplets on T^2

Consider the supersymmetric background with round metric (3.5) discussed in the previous section. We consider a free chiral multiplet of R-charge r and Q_f-charge q_f in that background. The bosonic part of the supersymmetric Lagrangian (2.16) can be written

$$\mathscr{L}_{\text{bos}} = 2(D_w + i\gamma)\widetilde{\phi} D_{\bar{w}}\phi + 2D_{\bar{w}}\widetilde{\phi}(D_w - i\gamma)\phi + \widetilde{\phi}\left(\Delta^{\mathbf{r}}_{S^2} + \frac{\mathbf{r}}{2}\right)\phi - \widetilde{F}F\,, \tag{4.1}$$

where we introduced the notation $\gamma = \frac{3}{4}\kappa\left(r - \frac{2}{3}\right)$, with κ the ambiguity in the background fields (3.8), and $\mathbf{r}$ is the shifted R-charge (3.16). Here and in the remainder of this section, the covariant derivatives along $w, \bar{w}$ only contain the $\mathrm{U}(1)_f$ flat connection:

$$D_w = \partial_w - iq_f v_w\,, \quad D_{\bar{w}} = \partial_{\bar{w}} - iq_f v_{\bar{w}}\,. \tag{4.2}$$

The operator $\Delta^{\mathbf{r}}_{S^2}$ in (4.1) is the scalar Laplacian on the sphere with a monopole, which is given by (A.7) appendix A. Note that R-charge r only enters through γ and the shifted R-charge $\mathbf{r}$. Indeed, the scalar field ϕ couples to both $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_f$ through the covariant derivative (2.13), so that it effectively couples with electric charge $\mathbf{r}$ to a monopole of unit flux.[15] Let us also remark that the Laplacian (A.7) must be here interpreted in terms of

[15] We also have couplings of ϕ to R and D in the second line of (2.16), which are crucial for supersymmetry and lead to the shift of the scalar Laplacian by $\frac{\mathbf{r}}{2}$ in (4.1).

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the complex coordinate z in (3.4), effectively shifting the angle φ to $\varphi + \alpha x + \beta y$ in the definition of the monopole harmonics. (The monopole harmonics are then a complete basis of sections of the monopole line bundle on $T^2 \times S^2$ introduced in section 3.1.) We expand the field ϕ in scalar monopole harmonics,

$$\phi(w, \bar{w}, z, \bar{z}) = \sum_{j,m} a_{jm}(w, \bar{w}) Y_{\mathbf{r}\, jm}(z, \bar{z}) , \tag{4.3}$$

where the sum is over $j = \frac{|\mathbf{r}|}{2}, \frac{|\mathbf{r}|}{2} + 1, \ldots$, $m = -j, \ldots, j$. Similarly, we expand the auxiliary field F in monopole harmonics of electric charge $\mathbf{r} - 2$:

$$F(w, \bar{w}, z, \bar{z}) = \sum_{j,m} f_{jm}(w, \bar{w}) Y_{\mathbf{r}-2\, jm}(z, \bar{z}) , \tag{4.4}$$

with $j = \frac{|\mathbf{r}-2|}{2}, \frac{|\mathbf{r}-2|}{2} + 1, \ldots$, $m = -j, \ldots, j$. We similarly expand the bosonic fields of the antichiral multiplet according to $\widetilde{\phi} = \sum \widetilde{a}_{jm} Y^\dagger_{\mathbf{r}\, jm}$ and $\widetilde{F} = \sum \widetilde{f}_{jm} Y^\dagger_{\mathbf{r}-2\, jm}$. Note that there is a mismatch in the SO(3) representations that appear in the expansion of ϕ and F for $\mathbf{r} \neq 1$. When $\mathbf{r} > 1$, all values of $j \geq \frac{\mathbf{r}}{2}$ exist for both fields but one additional representation with $j = \frac{\mathbf{r}}{2} - 1$ is found for F. When $\mathbf{r} < 1$, it is ϕ which has one unmatched representation, with $j = -\frac{\mathbf{r}}{2}$.

The fermionic part of the supersymmetric Lagrangian (2.16) is

$$\mathscr{L}_{\text{fer}} = -2i\tilde{\psi}^{\dot{\beta}} \begin{pmatrix} D_{\bar{w}} & 0 \\ 0 & -(D_w - i\gamma) \end{pmatrix} \psi_\alpha - \tilde{\psi}^{\dot{\beta}} (-i \nabla^{\mathbf{r}}_{S^2})_{\dot{\beta}}{}^{\alpha} \psi_\alpha \, . \tag{4.5}$$

with γ defined above, while the explicit form of the Dirac operator in a monopole background, $-i\nabla^{\mathbf{r}}_{S^2}$, is given in (A.13). The eigenvalues $\lambda_{\mathbf{r}j}$ of $-i\nabla^{\mathbf{r}}_{S^2}$ are given in (A.14). The important thing for us is that there are fermionic zero-modes,

$$\lambda_{\mathbf{r}j} = 0 \qquad \Leftrightarrow \qquad j = \frac{|\mathbf{r} - 1|}{2} - \frac{1}{2} \, . \tag{4.6}$$

We expand the fermions in spinors spherical harmonics of electric charge $\mathbf{r} - 1$. For $\mathbf{r} > 1$, it is convenient to write

$$\psi_\alpha = \sum_{jm} \begin{pmatrix} b_{jm} Y_{\mathbf{r}-2\, jm} \\ -c_{jm} Y_{\mathbf{r}\, jm} \end{pmatrix} + \sum_m \begin{pmatrix} b_{j_0 m} Y_{\mathbf{r}-2\, jm} \\ 0 \end{pmatrix} , \tag{4.7}$$

The second sum comes from the zero modes, with $j_0 = \frac{\mathbf{r}}{2} - 1$. For $\mathbf{r} < 1$, we similarly write

$$\psi_\alpha = \sum_{jm} \begin{pmatrix} b_{jm} Y_{\mathbf{r}-2\, jm} \\ -c_{jm} Y_{\mathbf{r}\, jm} \end{pmatrix} + \sum_m \begin{pmatrix} 0 \\ -c_{j_0 m} Y_{\mathbf{r}\, jm} \end{pmatrix} , \tag{4.8}$$

with $j_0 = \frac{|\mathbf{r}|}{2}$. For the fermion $\widetilde{\psi}$ of opposite chirality, we take

$$\tilde{\psi}^{\dot{\alpha}} = -\sum_{jm} \begin{pmatrix} \tilde{b}_{jm} Y^\dagger_{\mathbf{r}-2\, jm} \\ \tilde{c}_{jm} Y^\dagger_{\mathbf{r}\, jm} \end{pmatrix} - \sum_m \begin{pmatrix} \tilde{b}_{j_0 m} Y^\dagger_{\mathbf{r}-2\, j_0 m} \\ 0 \end{pmatrix} \tag{4.9}$$

in the case $\mathbf{r} > 1$, and similarly for $\mathbf{r} < 1$.

Plugging these mode expansions into the Lagrangian (4.1), (4.5), we obtain a decoupled theory for each angular momentum j, m. For simplicity of notation, we will drop the subscripts j, m on the modes a, b, c, f in the following. The modes organize themselves into representations of an $\mathcal{N} = (0, 2)$ supersymmetry algebra on the torus. Let us consider each case separately:

4.1.1 Long multiplets

Consider first the case when j, m do not correspond to a zero mode, $\lambda \neq 0$. One obtains long multiplets (a, b, c, f) and $(\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{f})$, whose supersymmetry variations follow from (2.14) and (2.15):

$$\begin{aligned}
&\delta a = c\,, && \widetilde{\delta} a = 0\,, && \delta \widetilde{a} = 0\,, && \widetilde{\delta}\widetilde{a} = \widetilde{c}\,,\\
&\delta b = f\,, && \widetilde{\delta} b = i\lambda a\,, && \delta \widetilde{b} = -i\lambda \widetilde{a}\,, && \widetilde{\delta}\widetilde{b} = \widetilde{f}\,,\\
&\delta c = 0\,, && \widetilde{\delta} c = 2iD_{\bar{w}} a\,, && \delta \widetilde{c} = 2iD_{\bar{w}}\widetilde{a}\,, && \widetilde{\delta}\widetilde{c} = 0\,,\\
&\delta f = 0\,, && \widetilde{\delta} f = 2iD_{\bar{w}} b - i\lambda c\,, && \delta \widetilde{f} = 2iD_{\bar{w}}\widetilde{b} + i\lambda\widetilde{c}\,, && \widetilde{\delta}\widetilde{f} = 0\,.
\end{aligned} \tag{4.10}$$

Here we use the notation $\delta = \delta_\zeta$ and $\widetilde{\delta} = \delta_{\widetilde{\zeta}}$. The transformations (4.10) realize a two-dimensional $\mathcal{N} = (0, 2)$ algebra,

$$\delta^2 = 0\,, \qquad \widetilde{\delta}^2 = 0\,, \qquad \{\delta, \widetilde{\delta}\} = 2iD_{\bar{w}}\,. \tag{4.11}$$

The Lagrangian for the (a, b, c, f) multiplet is given by[16]

$$\begin{aligned}
\mathscr{L}_{(a,b,c,f)} = 4D_{\bar{w}}\tilde{a}D_w a + \lambda^2 \tilde{a} a + 2i\gamma(\tilde{a}D_{\bar{w}}a - aD_{\bar{w}}\tilde{a}) - \tilde{f} f\\
+ 2i\tilde{b}D_{\bar{w}}b + 2i\tilde{c}D_w c + 2\gamma\tilde{c}c - i\lambda(\tilde{b}c + \tilde{c}b)\,.
\end{aligned} \tag{4.12}$$

One easily checks that (4.12) is supersymmetric under (4.10).

4.1.2 Short multiplets

The zero modes on the S^2 give rise to short multiplets of $\mathcal{N} = (0, 2)$ supersymmetry on the torus. There are three cases, depending on the shifted R-charge (3.16):

Case $\mathbf{r} = 1$. For $\mathbf{r} = 1$, there are no zero modes.

Case $\mathbf{r} > 1$. The fermionic zero modes for $\mathbf{r} > 1$ were given in (4.7), with $j_0 = \frac{\mathbf{r}}{2} - 1$ and $m = -j_0, \ldots, j_0$. They constitute a spin j_0 representation of SO(3). The corresponding two-dimensional fermions $b_{j_0 m}$ are paired with the corresponding modes $f_{j_0 m}$ of F. The short multiplet (b, f) realizes a fermi multiplet [29] of the $\mathcal{N} = (0, 2)$ supersymmetry,

$$\begin{aligned}
&\delta b = f\,, && \widetilde{\delta} b = 0\,,\\
&\delta f = 0\,, && \widetilde{\delta} f = 2iD_{\bar{w}} b\,.
\end{aligned} \tag{4.13}$$

These variations are simply a truncation of (4.10) for $\lambda = 0$. The supersymmetric Lagrangian reads

$$\mathscr{L}_{(b,f)} = -\tilde{f} f + 2i\tilde{b}D_{\bar{w}} b\,. \tag{4.14}$$

[16] Note the dependence on the ambiguity parameter γ, which is now an arbitrary constant (for simplicity we assumed a constant κ, thus preserving all the isometries of the background). In the following we tacitly assume that γ is real, although the generalization to complex γ is straightforward.

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Case $\mathbf{r} < 1$. For $\mathbf{r} < 1$, the fermionic zero modes were given in (4.8), with $j_0 = \frac{|\mathbf{r}|}{2}$ and $m = -j_0, \dots, j_0$. The corresponding two-dimensional fermions $c_{j_0 m}$ are paired with the $|\mathbf{r}|+1$ bosonic zero modes $a_{j_0 m}$ of the scalar ϕ. The short multiplet (a, c) is a chiral multiplet [29] of $\mathcal{N} = (0, 2)$ supersymmetry,

$$\begin{aligned} \delta a &= c\,, & \widetilde{\delta} a &= 0\,, \\ \delta c &= 0\,, & \widetilde{\delta} c &= 2iD_{\bar{w}} a\,. \end{aligned} \tag{4.15}$$

Its supersymmetric Lagrangian is

$$\mathscr{L}_{(a,c)} = 2D_{\bar{w}}\tilde{a}D_w a + 2D_w\tilde{a}D_{\bar{w}}a + 2i\gamma(\tilde{a}D_{\bar{w}}a - aD_{\bar{w}}\tilde{a}) + 2i\tilde{c}D_w c + 2\gamma\tilde{c}c\,. \tag{4.16}$$

4.1.3 Twisted boundary conditions on the torus

We started with a theory on $T^2 \times S^2$ and reduced it to a theory on a torus with modular parameter τ. At first sight, all dependence on the second complex structure parameter $\sigma = \tau\alpha - \beta$ has disappeared from the effective theories (4.12), (4.14), (4.16). However, this dependence remains through twisted boundary conditions for the fields on the torus. Consider the expansion (4.3) of the scalar ϕ. The monopole harmonics $Y_{\mathbf{r}\,jm}$ satisfy $Y_{\mathbf{r}\,jm} \sim e^{2\pi i(m+\frac{\mathbf{r}}{2})\alpha}Y_{\mathbf{r}\,jm}$ for $x \sim x+2\pi$, and $Y_{\mathbf{r}\,jm} \sim e^{2\pi i(m+\frac{\mathbf{r}}{2})\beta}Y_{\mathbf{r}\,jm}$ for $y \sim y+2\pi$. Since ϕ satisfies the boundary conditions (3.17), each mode a of angular momentum j, m must satisfy

$$\begin{aligned} a(w + 2\pi, \bar{w} + 2\pi) &= e^{-2\pi i m\alpha}a(w, \bar{w})\,, \\ a(w + 2\pi\tau, \bar{w} + 2\pi\bar{\tau}) &= e^{-2\pi i m\beta}a(w, \bar{w})\,. \end{aligned} \tag{4.17}$$

The modes b, c, f also satisfy these boundary conditions.[17] The momentum operators along the one-cycles of the torus on modes a, b, c, f of angular momentum m are

$$P_x = -i\partial_x + m\alpha\,, \qquad P_y = -i\partial_y + m\beta\,. \tag{4.18}$$

In the following, we will consider y to be the "time" coordinate for canonical quantization while x will be the space coordinate. Then P_x corresponds to the momentum denoted by P in (1.7), while $-iP_y$ corresponds to the Hamiltonian.

4.2 Canonical quantization and supersymmetric index

Let us now compute the index (1.6). To do so, we could further reduce the two-dimensional theories of the previous subsection on a circle, to obtain a quantum mechanics for states on $S^1 \times S^2$. However, it will be more illuminating to directly quantize the theory on the torus.

The only states that contribute to the index sit in short multiplets (4.13) or (4.15). We will focus on those in the following. For $\mathbf{r} = 1$ there are no short multiplets, all states are paired by supersymmetry, and the index is simply

$$\mathcal{I}_{\mathbf{r}=1}(q, x, t) = 1\,. \tag{4.19}$$

Let us next consider the non-trivial cases $\mathbf{r} \neq 1$. We choose the direction y on the torus to be the time direction for canonical quantization. The boundary condition along the remaining spatial direction x is given by the first line in (4.17).

[17] Equivalently, we could choose periodic boundary conditions on T^2 and introduce a flat connection which couples to all the modes through their angular momentum $J_3 = m$.

4.2.1 Case $\mathbf{r} > 1$: the Fermi multiplet

The theory (4.14) is particularly simple. The auxiliary field f has no on-shell degrees of freedom and the equation of motion of b is $D_{\bar{w}} b = 0$. It follows from (4.13) that the supercharges annihilate all the states in this sector. Let us also note that the index for $\mathbf{r} > 1$ is trivially independent of the constant γ, since it does not appear in (4.14).

The fermionic wave functions are given by

$$b = e^{iq_f v_{\bar{w}}(\bar{w}-w)} e^{-i\alpha m w} \sum_{k=-\infty}^{\infty} \frac{e^{-ikw}}{\sqrt{2\pi}} b_k\,, \qquad \tilde{b} = e^{-iq_f v_{\bar{w}}(\bar{w}-w)} e^{i\alpha m w} \sum_{k=-\infty}^{\infty} \frac{e^{ikw}}{\sqrt{2\pi}} \tilde{b}_k\,. \tag{4.20}$$

The canonical commutation relations imply

$$\{b_k, \tilde{b}_l\} = \delta_{kl}\,, \tag{4.21}$$

and the Hamiltonian is

$$H_{(b,f)} = -i \sum_k (\tau k + m\sigma + q_f \nu)\, \tilde{b}_k b_k\,. \tag{4.22}$$

Recall that $\sigma = \tau\alpha - \beta$ and $\nu = \tau a_x - a_y$. We assume that the geometric moduli τ, σ and ν are given generic values, such that the Hamiltonian has no zero modes. The operators b_k are either annihilation or creations operators, depending on k. Demanding that the real part of $H_{(b,f)}$ be positive, we find that b_k is an annihilation operator for $k + m\alpha + q_f a_x > 0$, while it is a creation operator for $k + m\alpha + q_f a_x < 0$. Upon reordering b_k and $\tilde{b}_k$ so that all modes have positive excitation energy, we obtain a Casimir energy:

$$\begin{aligned} E_0 &= i \sum_{k > m\alpha + q_f a_x} (k\tau - m\sigma - q_f \nu) \\ &= -i\tau \left(\frac{1}{12} + \frac{l_m(l_m+1)}{2} \right) + i(m\sigma + q_f\nu) \left(\frac{1}{2} + l_m \right)\,. \end{aligned} \tag{4.23}$$

In the second line we used zeta function regularization,[18] and l_m is defined as the integer such that $l_m < m\alpha + q_f a_x < l_m + 1$.

We now compute the index

$$\mathcal{I} = \operatorname{Tr}\left((-1)^F e^{-2\pi H_{(b,f)}} \right)\,. \tag{4.24}$$

Here the trace includes a sum over the $\mathbf{r} - 1$ copies of the (b, f) multiplets (with $m = -\frac{\mathbf{r}}{2} + 1, \ldots, \frac{\mathbf{r}}{2} - 1$). Let us define the fugacities

$$q = e^{2\pi i \tau}\,, \qquad x = e^{2\pi i \sigma}\,, \qquad t = e^{2\pi i \nu}\,. \tag{4.25}$$

Using the explicit form of the Hamiltonian in (4.22), the single particle index is given by

$$\mathcal{I}_{\text{sp}} = - \sum_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1} \left(\sum_{k > m\alpha + q_f \alpha_x} q^k x^{-m} t^{-q_f} + \sum_{k > -m\alpha - q_f a_x} q^k x^m t^{q_f} \right)\,, \tag{4.26}$$

[18] Here we used the Riemann zeta function after splitting the sum in (4.23). Using the Hurwitz zeta function instead, we would find an additional term $-\frac{i}{2\tau}(m\sigma + q_f\nu)^2$. We are assuming that this quadratic term is scheme dependent.

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The first sum over k comes from the single particle states $b_k|0\rangle$ with $k + m\alpha + q_f a_x < 0$. They have quantum numbers $P = -k$, $J_3 = -m$, $Q_f = -q_f$. It is convenient to change $k \to -k$ such that these states are weighted by $q^k x^{-m} t^{-q_f}$. Similarly, the second sum over k in (4.26) corresponds to states $\widetilde{b}_k|0\rangle$ with $k + m\alpha + q_f a_x > 0$. The full index is obtained by plethystic exponentiation [30–33]:

$$\prod_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1} \left[\frac{q^{1/12}}{\sqrt{x^m t^{q_f}}} \frac{q^{l_m(l_m+1)/2}}{(x^m t^{q_f})^{l_m}} \prod_{k > m\alpha + q_f a_x} \left(1 - q^k x^{-m} t^{-q_f}\right) \prod_{k > -m\alpha - q_f a_x} \left(1 - q^k x^m t^{q_f}\right) \right],$$

where we also included the contribution from the Casimir energy (4.23). Up to a constant phase which we discard, the term inside the square brackets can be written as

$$\frac{q^{1/12}}{\sqrt{x^m t^{q_f}}} \prod_{k\geq 0} \left(1 - q^{k+1} x^{-m} t^{-q_f}\right) \left(1 - q^k x^m t^{q_f}\right) = -i \frac{\theta_1(m\sigma + q_f\nu, \tau)}{\eta(\tau)}. \tag{4.27}$$

Here we introduced the theta function [34]

$$\theta_1(\rho, \tau) = 2q^{\frac{1}{8}} \sin \pi\rho \prod_{k=1}^{\infty} (1 - q^k)(1 - q^k y)(1 - q^k y^{-1}), \tag{4.28}$$

with $y = e^{2\pi i \rho}$, and the eta function $\eta(\tau) = q^{1/24} \prod_{k\geq 1}(1 - q^k)$. The index for a four-dimensional $\mathcal{N} = 1$ chiral multiplet of shifted R-charge $\mathbf{r} > 1$ is then given by

$$\begin{aligned} \mathcal{I}_{\mathbf{r}>1} &= \left(\frac{q^{1/12}}{\sqrt{t^{q_f}}}\right)^{\mathbf{r}-1} \prod_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1} \prod_{k\geq 0} \left(1 - q^{k+1} x^{-m} t^{-q_f}\right) \left(1 - q^k x^m t^{q_f}\right) \\ &= \prod_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1} \frac{\theta_1(m\sigma + q_f\nu, \tau)}{i\,\eta(\tau)}. \end{aligned} \tag{4.29}$$

This is the result (1.4) advertised in the introduction. It also agrees with the result of [11, 12] for $\mathcal{N} = (0, 2)$ fermi multiplets.

4.2.2 Case $\mathbf{r} < 1$: the chiral multiplet

Consider the $\mathcal{N} = (0, 2)$ chiral mutiplet with Lagrangian (4.16). Unlike for the fermi multiplet, not every state in this sector contributes to the index. The equations of motion are

$$D_{\bar{w}}(D_w - i\gamma)a = 0, \qquad (D_w - i\gamma)c = 0, \tag{4.30}$$

and their most general solution takes the form

$$a(w, \bar{w}) = e^{iq_f v_{\bar{w}} \bar{w}} a_{\rm H}(w) + e^{i(q_f v_w + \gamma)w} a_{\rm AH}(\bar{w}), \qquad c(w, \bar{w}) = e^{i(q_f v_w + \gamma)w} c_{\rm AH}(\bar{w}). \tag{4.31}$$

It follows from (4.15) that the anti-holomorphic modes $a_{\rm AH}, c_{\rm AH}$ are paired together and therefore do not contribute to the index. We thus focus on the holomorphic modes of a.

We have the mode expansion

$$\begin{aligned}
(D_w - i\gamma)a &= ie^{iq_f v_{\bar w}(\bar w - w)} e^{-i\alpha m w} \sum_{k=-\infty}^{\infty} \frac{e^{-ikw}}{\sqrt{2\pi}} a_k \,, \\
(D_w + i\gamma)\tilde a &= ie^{-iq_f v_{\bar w}(\bar w - w)} e^{i\alpha m w} \sum_{k=-\infty}^{\infty} \frac{e^{ikw}}{\sqrt{2\pi}} \tilde a_k \,.
\end{aligned} \tag{4.32}$$

The canonical commutation relations are equivalent to

$$[a_k, \tilde a_l] = \frac{1}{2}(k + m\alpha + q_f a_x + \gamma)\delta_{kl}\,. \tag{4.33}$$

Depending on the value of k, the operators a_k are either creation or annihilation operators, like for the (b, f) multiplet discussed above. For k such that $k + m\alpha + q_f a_x + \gamma > 0$, $\tilde a_k$ is the creation operators. When $k + m\alpha + q_f a_x + \gamma < 0$, we change notation $k \to -k$ and a_{-k} is the creation operator. As before, we are assuming that the geometric parameters are generic.

The Hamiltonian acting on the "holomorphic" states (created by $\tilde a_k$ or a_k) reads

$$H_{a_{\rm H}} = -i \sum_k (k\tau + m\sigma + q_f \nu) \frac{2\tilde a_k a_k}{(k + m\alpha + q_f a_x + \gamma)}\,. \tag{4.34}$$

The complete Hamiltonian, including the contribution from the anti-holomorphic modes, is rather more complicated and will be omitted here. The operators $a_k, \tilde a_k$ in (4.34) are not normal ordered. As in the $\mathbf{r} > 1$ case, this leads to a zero point energy which we compute by zeta function regularization.

One can obtain the index for the four-dimensional chiral multiplet with $\mathbf{r} < 1$ by summing over the $|\mathbf{r}| + 1$ copies of the (a, c) multiplet and using the same arguments as in the last subsection. One finds

$$\begin{aligned}
\mathcal{I}_{\mathbf{r}<1} &= \left(\frac{\sqrt{t^{q_f}}}{q^{1/12}}\right)^{|\mathbf{r}|+1} \prod_{m=-\frac{|\mathbf{r}|}{2}}^{\frac{|\mathbf{r}|}{2}} \prod_{k\geq 0} \frac{1}{(1 - q^{k+1} x^{-m} t^{-q_f})} \frac{1}{(1 - q^k x^m t^{q_f})} \\
&= \prod_{m=-\frac{|\mathbf{r}|}{2}}^{\frac{|\mathbf{r}|}{2}} i \frac{\eta(\tau)}{\theta_1(m\sigma + q_f \nu, \tau)}\,.
\end{aligned} \tag{4.35}$$

This gives (1.5), and it agrees with the result of [11, 12] for $\mathcal{N} = (0, 2)$ chiral multiplets. Note that the arbitrary parameter γ does not contribute to the final answer.

4.3 Modular properties of $\mathcal{I}^{\Phi}_{T^2\times S^2}$

Let us briefly comment on the modular properties of the $\mathcal{N} = 1$ chiral multiplet index $\mathcal{I}^{\Phi}_{T^2\times S^2}$ obtained above. This partition function depends on three continuous parameters: τ, σ and ν. The first two parameters are the complex structures moduli of $T^2 \times S^2$ while the third is a holomorphic line bundle modulus for the $\mathrm{U}(1)_f$ global symmetry. Those are the only continuous parameters on which the partition function is allowed to depend [3].

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As explained in section 3, two different values of the parameters τ and σ correspond to the same complex structure if they are related by the following transformations:

$$\begin{aligned} S &: (\tau,\sigma)\mapsto\left(-\frac{1}{\tau},\,\frac{\sigma}{\tau}\right), \qquad & T &: (\tau,\sigma)\mapsto(\tau+1,\,\sigma), \\ U &: (\tau,\sigma)\mapsto(\tau,\,\sigma+\tau), \qquad & V &: (\tau,\sigma)\mapsto(\tau,\,\sigma+1), \end{aligned} \tag{4.36}$$

These operations generate a slight generalization of the modular group of the torus. While one may naively expect the partition function to be modular invariant, this is actually not the case. Since these transformations correspond to large diffeomorphisms, the failure of modular invariance is interpreted as a gravitational anomaly. It would be interesting to understand this point better.

The modular group also acts on ν in the standard way, $S:\nu\mapsto\nu/\tau$ and $T:\nu\mapsto\nu$. Additionally, since the flat connections a_x and a_y are identified as $a_x\sim a_x+1$, $a_y\sim a_y+1$ by large gauge transformations, we have the additional transformations $U':\nu\mapsto\nu+\tau$ and $V':\nu\mapsto\nu+1$ which are independent of U and V. These large gauge transformations U', V' are also anomalous.

Using the representation of the index in terms of the theta function (4.28), one can find its behaviour under the modular transformations described above. We have the following transformations:

$$\begin{aligned} \frac{\theta_1(\rho,\tau+1)}{\eta(\tau+1)} &= e^{\pi i/6}\,\frac{\theta_1(\rho,\tau)}{\eta(\tau)}\,, \qquad & \frac{\theta_1(\rho+\tau,\tau)}{\eta(\tau)} &= -e^{-i\pi\tau}e^{-2\pi i\rho}\,\frac{\theta_1(\rho,\tau)}{\eta(\tau)}, \\ \frac{\theta_1(\rho+1,\tau)}{\eta(\tau)} &= -\frac{\theta_1(\rho,\tau)}{\eta(\tau)}\,, \qquad & \frac{\theta_1\left(\frac{\rho}{\tau},-\frac{1}{\tau}\right)}{\eta\left(-\frac{1}{\tau}\right)} &= -ie^{\frac{\pi i\rho^2}{\tau}}\,\frac{\theta_1(\rho,\tau)}{\eta(\tau)}\,. \end{aligned} \tag{4.37}$$

In this work, we will not adress the question of whether it is possible to restore some of the modular properties by tuning some local terms. A better understanding of the scheme dependence of our answer would require a systematic understanding of the allowed supersymmetric counterterms, similar to [35, 36] in three dimensions. One can however see by inspection that it is not possible to achieve a fully modular invariant partition function.

5 The chiral multiplet partition function on $\mathcal{M}_4$

In this section, we propose a simple method to compute the $\mathcal{N}=1$ chiral multiplet partition function on any four-dimensional background with two supercharges. We evaluate the path integral explicitly using supersymmetric localization; see for instance [13–15, 37–40] for similar arguments.

After introducing some formalism in subsections 5.1, 5.2 and 5.3, we will explain our path integral result in subsection 5.4. In the remaining subsections we will apply our method to the case of $T^2\times S^2$ and $S^3\times S^1$.

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5.1 Building a frame from the killing spinors

Consider a background with two supercharges. From the Killing spinors ζ and $\widetilde{\zeta}$, we can build the vectors[19]

$$K^\mu = \widetilde{\zeta}\widetilde{\sigma}^\mu\zeta\,, \qquad \bar{K}^\mu = \frac{1}{4}\frac{\widetilde{\zeta}^\dagger\widetilde{\sigma}^\mu\zeta^\dagger}{|\zeta|^2|\widetilde{\zeta}|^2}\,, \qquad Y^\mu = \frac{\widetilde{\zeta}^\dagger\widetilde{\sigma}^\mu\zeta}{2|\widetilde{\zeta}|^2}\,, \qquad \bar{Y}^\mu = -\frac{\widetilde{\zeta}\,\widetilde{\sigma}^\mu\zeta^\dagger}{2|\zeta|^2}\,. \tag{5.1}$$

Note that Y and $\bar{Y}$ have R-charge ± 2, respectively. The vectors K and Y are anti-holomorphic with respect to the complex structure $J^\mu{}_\nu$ defined in (2.2), while $\bar{K}$ and $\bar{Y}$ are holomorphic. Moreover, Y and $\bar{Y}$ are valued in their respective R-symmetry line bundles $L^{\pm 2}$. The vectors (5.1) are normalized such that $K_\mu\bar{K}^\mu = Y_\mu\bar{Y}^\mu = \frac{1}{2}$, and all other contractions of two of the vectors (5.1) vanish. These vectors provide a (R-charged) frame which will be very convenient below.

The vector K is Killing, as already mentioned in section 2, while the three other vectors satisfy

$$\nabla_\mu\bar{K}^\mu = 0\,, \qquad D_\mu Y^\mu = 0\,, \qquad D_\mu\bar{Y}^\mu = 0\,. \tag{5.2}$$

The covariant derivative D_μ was defined in (2.13). These relations follow from the Killing spinor equations (2.1). It will be convenient to define the following operators acting on charged *scalars*,

$$\hat{\mathcal{L}}_K = K^\mu D_\mu\,, \qquad \hat{\mathcal{L}}_{\bar{K}} = \bar{K}^\mu D_\mu\,, \qquad \hat{\mathcal{L}}_Y = Y^\mu D_\mu\,, \qquad \hat{\mathcal{L}}_{\bar{Y}} = \bar{Y}^\mu D_\mu\,. \tag{5.3}$$

Note that $\hat{\mathcal{L}}_Y$ and $\hat{\mathcal{L}}_{\bar{Y}}$ shift the R-charge by ± 2, respectively. The following lemma will be useful:

$$[\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_{\bar{K}}] = 0\,, \qquad [\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_Y] = 0\,, \qquad [\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_{\bar{Y}}] = 0\,. \tag{5.4}$$

This lemma is most easily proven in the adapted coordinates w, z, in terms of which we have

$$\begin{aligned} K &= \partial_{\bar{w}}\,, & Y &= \frac{s}{c\,\Omega^2}\left(\partial_{\bar{z}} - \bar{h}\,\partial_{\bar{w}}\right), \\ \bar{K} &= \frac{1}{\Omega^2}\partial_w\,, & \bar{Y} &= \frac{1}{c\,s}\left(\partial_z - h\,\partial_w\right). \end{aligned} \tag{5.5}$$

Here the functions $\Omega(z,\bar{z})$, $h(z,\bar{z})$, $c(z,\bar{z})$ and $s(z,\bar{z})$ are the ones appearing in (2.4), (2.5). Therefore, we see that (5.4) is equivalent to

$$[D_{\bar{w}}, D_{\bar{z}}] = [D_w, D_{\bar{w}}] = [D_z, D_{\bar{w}}] = 0\,, \tag{5.6}$$

with the covariant derivative acting on scalar fields. For a field of R-charge r and Q_f charge q_f, we have $[D_\mu, D_\nu] = -irF_{\mu\nu} - iq_f f_{\mu\nu}$, where $F_{\mu\nu}$ is the field strength of the R-symmetry gauge field A_μ and $f_{\mu\nu}$ is defined in (2.9). We already stated that $f_{\mu\nu}$ satisfies (2.10). We can similarly show that the Killing spinor equations imply

$$F_{\bar{w}\bar{z}} = F_{w\bar{w}} = F_{z\bar{w}} = 0\,, \tag{5.7}$$

[19] Note that $K, \bar{K}$ and $Y, \bar{Y}$ are not complex conjugates of each other. We hope that this choice of notation will not lead to any confusion.

in the presence of two Killing spinors ζ, $\widetilde{\zeta}$. (The first equation $F_{\bar{w}\bar{z}} = 0$ follows from the existence of a single Killing spinor ζ.) This proves (5.4). One can also compute

$$[\hat{\mathcal{L}}_Y, \hat{\mathcal{L}}_{\bar{Y}}] = i(V^\mu - \kappa K^\mu)D_\mu + \frac{r}{8}(R - 6V^\mu V_\mu) - \frac{q_f}{2}D\,. \tag{5.8}$$

Here R is the Ricci scalar and D the auxiliary background field given by (2.10).

5.2 The chiral multiplet revisited

Consider a chiral multiplet coupled to an external gauge field, as discussed in section 2.3. For a chiral multiplet $\Phi = (\phi, \psi, F)$ of R-charge r and Q_f-charge q_f, we can use the Killing spinor ζ to define

$$B = \frac{1}{\sqrt{2}}\frac{\zeta^\dagger\psi}{|\zeta|^2}\,, \qquad C = \sqrt{2}\zeta\psi\,, \qquad \Leftrightarrow \qquad \psi_\alpha = \sqrt{2}\zeta_\alpha B - \frac{1}{\sqrt{2}}\frac{\zeta^\dagger_\alpha}{|\zeta|^2}C\,. \tag{5.9}$$

The fields B, C are anticommuting scalars of R-charge $r-2$, r, respectively. In terms of the variables (ϕ, B, C, F), the transformation rules (2.14) read

$$\begin{aligned}
\delta\phi &= C\,, & \widetilde{\delta}\phi &= 0\,,\\
\delta B &= F\,, & \widetilde{\delta} B &= -2i\,\hat{\mathcal{L}}_{\bar{Y}}\phi\,,\\
\delta C &= 0\,, & \widetilde{\delta} C &= 2i\,\hat{\mathcal{L}}_K\phi\,,\\
\delta F &= 0\,, & \widetilde{\delta} F &= 2i\left(\hat{\mathcal{L}}_K B + \hat{\mathcal{L}}_{\bar{Y}} C\right)\,.
\end{aligned} \tag{5.10}$$

For an anti-chiral multiplet $\widetilde{\Phi} = (\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ of charges $-r$ and $-q_f$, we use $\widetilde{\zeta}$ to define

$$\widetilde{B} = \frac{1}{\sqrt{2}}\frac{\widetilde{\zeta}^\dagger\widetilde{\psi}}{|\widetilde{\zeta}|^2}\,, \qquad \widetilde{C} = \sqrt{2}\widetilde{\zeta}\widetilde{\psi}\,, \qquad \Leftrightarrow \qquad \widetilde{\psi}_{\dot{\alpha}} = \sqrt{2}\widetilde{\zeta}_{\dot{\alpha}}\widetilde{B} - \frac{1}{\sqrt{2}}\frac{\widetilde{\zeta}^\dagger_{\dot{\alpha}}}{|\widetilde{\zeta}|^2}\widetilde{C}\,. \tag{5.11}$$

The supersymmetry transformations (2.15) become

$$\begin{aligned}
\delta\widetilde{\phi} &= 0\,, & \widetilde{\delta}\widetilde{\phi} &= \widetilde{C}\,,\\
\delta\widetilde{B} &= 2i\,\hat{\mathcal{L}}_Y\widetilde{\phi}\,, & \widetilde{\delta}\widetilde{B} &= \widetilde{F}\,,\\
\delta\widetilde{C} &= 2i\,\hat{\mathcal{L}}_K\widetilde{\phi}\,, & \widetilde{\delta}\widetilde{C} &= 0\,,\\
\delta\widetilde{F} &= 2i\left(\hat{\mathcal{L}}_K\widetilde{B} - \hat{\mathcal{L}}_Y\widetilde{C}\right)\,, & \widetilde{\delta}\widetilde{F} &= 0\,.
\end{aligned} \tag{5.12}$$

Using (5.4), one can see that the transformations (5.10), (5.12) realize the supersymmetry algebra $\{\delta, \widetilde{\delta}\} = 2i\hat{\mathcal{L}}_K$. It should be emphasized that in the new variables the supercharges are scalars and R neutral.

It is illuminating to write the chiral multiplet Lagrangian (2.16) in terms of the new variables. With the help of (5.8), one finds

$$\begin{aligned}
\mathscr{L}_{\Phi\widetilde{\Phi}} = {}& 4\hat{\mathcal{L}}_{\bar{K}}\widetilde{\phi}\hat{\mathcal{L}}_K\phi + 4\hat{\mathcal{L}}_Y\widetilde{\phi}\hat{\mathcal{L}}_{\bar{Y}}\phi + i\kappa(\hat{\mathcal{L}}_K\widetilde{\phi}\phi - \widetilde{\phi}\hat{\mathcal{L}}_K\phi) - \widetilde{F}F\\
&+ 2i\widetilde{B}\hat{\mathcal{L}}_K B + 2i\widetilde{C}\hat{\mathcal{L}}_{\bar{K}}C + 2i\widetilde{B}\hat{\mathcal{L}}_{\bar{Y}}C - 2i\widetilde{C}\hat{\mathcal{L}}_Y B - \kappa\widetilde{C}C.
\end{aligned} \tag{5.13}$$

The results (5.10), (5.12), (5.13) should be compared to (4.10), (4.12). We will see in the following that we can take the analogy to the $T^2 \times S^2$ computation further. For any background $\mathcal{M}_4$ with two supercharges, only modes in shortened multiplets will contribute to the $\mathcal{N}=1$ chiral multiplet partition function $Z^{\Phi}_{\mathcal{M}_4}$ —those short multiplets are of the form (B, F) or (ϕ, C), similarly to the fermi and chiral $\mathcal{N}=(0,2)$ multiplets of section 4.

5.3 Reality condition and inner product

We would like to evaluate the path integral of a chiral multiplet with Lagrangian (5.13) explicitly. In order to define the Euclidean path integral, we need to choose a contour of integration. We will assign the following reality conditions to the fields ϕ, B, C, F in a chiral multiplet of R-charge r:

$$\widetilde{\phi} = \frac{\Omega^r}{|s|^r}\phi^\dagger\,, \qquad \widetilde{C} = \frac{\Omega^r}{|s|^r}C^\dagger\,, \qquad \widetilde{B} = \frac{\Omega^{r-2}}{|s|^{r-2}}B^\dagger\,, \qquad \widetilde{F} = -\frac{\Omega^{r-2}}{|s|^{r-2}}\phi^\dagger\,, \tag{5.14}$$

with Ω and s the quantities defined in section 2. This prescription is consistent with the complexified R-symmetry transformations and dimensional analysis (recall that s has R-charge 2). It will also be useful to consider appropriate inner products on field space. For the fields ϕ and C of R-charge r, let us define the inner product

$$\langle \phi_1, \phi_2\rangle_r = \int d^4x\sqrt{g}\,\frac{\Omega^r}{|s|^r}\frac{\phi_1^\dagger\,\phi_2}{\Omega^2}\,. \tag{5.15}$$

Similarly, for the fields B and F of R-charge $r-2$, we define

$$\langle B_1, B_2\rangle_{r-2} = \int d^4x\sqrt{g}\,\frac{\Omega^{r-2}}{|s|^{r-2}}B_1^\dagger\, B_2\,. \tag{5.16}$$

We will denote by $\mathcal{H}_r$ and $\mathcal{H}_{r-2}$ the corresponding Hilbert spaces. Note that the operator $i\hat{\mathcal{L}}_{\bar{Y}}$ acting on ϕ or C and the operator $i\Omega^2\hat{\mathcal{L}}_Y$ acting on B or F are maps between different Hilbert spaces,

$$i\hat{\mathcal{L}}_{\bar{Y}} \;: \mathcal{H}_r \to \mathcal{H}_{r-2}\,, \qquad\qquad i\Omega^2\hat{\mathcal{L}}_Y \;: \mathcal{H}_{r-2} \to \mathcal{H}_r\,, \tag{5.17}$$

while $i\mathcal{L}_K$ and $i\Omega^2\hat{\mathcal{L}}_{\bar{K}}$ act inside $\mathcal{H}_r$ or $\mathcal{H}_{r-2}$ (depending on the R-charge). Moreover, the two operators (5.17) are mutually adjoint (see appendix B):

$$\begin{aligned} \langle \phi, i\Omega^2\hat{\mathcal{L}}_Y B\rangle_r &= \langle i\hat{\mathcal{L}}_{\bar{Y}}\phi, B\rangle_{r-2}\,, \\ \langle B, i\hat{\mathcal{L}}_{\bar{Y}}\phi\rangle_{r-2} &= \langle i\Omega^2\hat{\mathcal{L}}_Y B, \phi\rangle_r\,. \end{aligned} \tag{5.18}$$

5.4 Localization and unpaired eigenvalues

Let us consider the following δ-exact deformation of the $\mathcal{N}=1$ chiral multiplet theory (5.13):

$$\mathscr{L}_{\rm loc} = \delta\left(-2i\widetilde{B}\hat{\mathcal{L}}_{\bar{Y}}\phi - 2i\widetilde{C}\hat{\mathcal{L}}_{\bar{K}}\phi - \widetilde{F}B\right)\,. \tag{5.19}$$

This term is equal to (5.13) itself with $\kappa = 0$. (Note that the full (5.13) is also δ-exact.) It can be written

$$\mathscr{L}_{\rm loc} = 4\,\widetilde{\phi}\big(-\hat{\mathcal{L}}_{\bar{K}}\hat{\mathcal{L}}_K - \hat{\mathcal{L}}_Y\hat{\mathcal{L}}_{\bar{Y}}\big)\phi + 2i\,\widetilde{\Psi}\begin{pmatrix}\hat{\mathcal{L}}_K & \hat{\mathcal{L}}_{\bar{Y}} \\ -\hat{\mathcal{L}}_Y & \hat{\mathcal{L}}_{\bar{K}}\end{pmatrix}\Psi\;-\widetilde{F}F\,, \tag{5.20}$$

where we introduced $\Psi = (B, C)^T$ and $\widetilde{\Psi} = (\widetilde{B}, \widetilde{C})$. Let us define the kinetic operators

$$\Delta_{\text{bos}} = \Omega^2\big(- \hat{\mathcal{L}}_{\bar{K}} \hat{\mathcal{L}}_K - \hat{\mathcal{L}}_Y \hat{\mathcal{L}}_{\bar{Y}} \big) \,, \qquad \Delta_{\text{fer}} = i \begin{pmatrix} \hat{\mathcal{L}}_K & \hat{\mathcal{L}}_{\bar{Y}} \\ -\Omega^2 \hat{\mathcal{L}}_Y & \Omega^2 \hat{\mathcal{L}}_{\bar{K}} \end{pmatrix} . \tag{5.21}$$

Note the appearance of factors of Ω^2 in this definition, necessary to make these operators dimensionless. Using (5.14) and (5.21), we can write the localizing action in term of the inner products (5.15) and (5.16) as

$$S_{\text{loc}} = \int d^4x \sqrt{g}\, \mathscr{L}_{\text{loc}} = 4 \langle \phi, \Delta_{\text{bos}} \phi \rangle_r + 2 \langle \Psi, \Delta_{\text{fer}} \Psi \rangle_{r,r-2} + \langle F, F \rangle_{r-2} \,, \tag{5.22}$$

where we defined

$$\langle \Psi_1, \Psi_2 \rangle_{r,r-2} = \langle B_1, B_2 \rangle_{r-2} + \langle C_1, C_2 \rangle_r \,. \tag{5.23}$$

By a standard supersymmetric localization argument, we can deform the original theory by (5.22) with an arbitrarily large coefficient without affecting the path integral, which therefore reduces to a ratio of functional determinants,

$$Z^{\Phi}_{\mathcal{M}_4} = \frac{\det \Delta_{\text{fer}}}{\det \Delta_{\text{bos}}} \,. \tag{5.24}$$

In deriving (5.24), we are assuming that $\phi = 0$ is the only relevant saddle point. (This is true for generic values of the complex structure and line bundle moduli.) This result can be simplified further. Let us first note that

$$i \begin{pmatrix} \hat{\mathcal{L}}_K & \hat{\mathcal{L}}_{\bar{Y}} \\ -\Omega^2 \hat{\mathcal{L}}_Y & \Omega^2 \hat{\mathcal{L}}_{\bar{K}} \end{pmatrix} \begin{pmatrix} 1 & -i\hat{\mathcal{L}}_{\bar{Y}} \\ 0 & i\hat{\mathcal{L}}_K \end{pmatrix} = \begin{pmatrix} i\hat{\mathcal{L}}_K & 0 \\ -i\Omega^2 \hat{\mathcal{L}}_Y & \Delta_{\text{bos}} \end{pmatrix} , \tag{5.25}$$

where we used $[\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_{\bar{Y}}] = 0$. Taking the determinant on both sides of (5.25), we find

$$\frac{\det \Delta_{\text{fer}}}{\det \Delta_{\text{bos}}} = \frac{\det(i\hat{\mathcal{L}}_K^{(r-2)})}{\det(i\hat{\mathcal{L}}_K^{(r)})} \,. \tag{5.26}$$

Here we denoted by $i\hat{\mathcal{L}}_K^{(r-2)}$ and $i\hat{\mathcal{L}}_K^{(r)}$ the operator $i\hat{\mathcal{L}}_K$ acting on $\mathcal{H}_{r-2}$ and $\mathcal{H}_r$, respectively. Consider next the adjoint operators (5.17). A standard argument shows that

$$\text{Ker}(i\Omega^2 \hat{\mathcal{L}}_Y) = \text{Ker}(\hat{\mathcal{L}}_{\bar{Y}} \Omega^2 \hat{\mathcal{L}}_Y) \,, \qquad \text{Ker}(i\hat{\mathcal{L}}_{\bar{Y}}) = \text{Ker}(\Omega^2 \hat{\mathcal{L}}_Y \hat{\mathcal{L}}_{\bar{Y}}) \,, \tag{5.27}$$

and that the non-zero eigenvalues of $\hat{\mathcal{L}}_{\bar{Y}} \Omega^2 \hat{\mathcal{L}}_Y$ and $\Omega^2 \hat{\mathcal{L}}_Y \hat{\mathcal{L}}_{\bar{Y}}$ are in one-to-one correspondence. In other words, $i\Omega^2 \hat{\mathcal{L}}_Y$ provides an isomorphism between the space of eigenfunctions of $\hat{\mathcal{L}}_{\bar{Y}} \Omega^2 \hat{\mathcal{L}}_Y$ in $\mathcal{H}_{r-2}$ with non-vanishing eigenvalues and the space of eigenfunctions of $\Omega^2 \hat{\mathcal{L}}_Y \hat{\mathcal{L}}_{\bar{Y}}$ in $\mathcal{H}_r$ with non-vanishing eigenvalues. Additionally, by (5.4) this isomorphism commutes with $i\hat{\mathcal{L}}_K$. It follows that all the eigenvalues of $i\hat{\mathcal{L}}_K$ which lie outside of $\text{Ker}(i\Omega^2 \hat{\mathcal{L}}_Y)$ or $\text{Ker}(i\hat{\mathcal{L}}_{\bar{Y}})$ cancel from (5.26). Therefore, we find that

$$Z^{\Phi}_{\mathcal{M}_4} = \frac{\det_{\text{Ker}\hat{\mathcal{L}}_Y}(i\hat{\mathcal{L}}_K^{(r-2)})}{\det_{\text{Ker}\hat{\mathcal{L}}_{\bar{Y}}}(i\hat{\mathcal{L}}_K^{(r)})} \,. \tag{5.28}$$

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Similar arguments appeared, for instance, in [13, 15, 40]. Practically speaking, the partition function reduces to

$$Z^{\Phi}_{\mathcal{M}_4} = \frac{\prod \lambda_B}{\prod \lambda_\phi}\,, \tag{5.29}$$

with the eigenvalues λ_B, λ_ϕ determined by the following BPS-like linear equations:

$$\begin{aligned} i\hat{\mathcal{L}}_K B &= \lambda_B\, B\,, \qquad & \hat{\mathcal{L}}_Y B &= 0\,, \\ i\hat{\mathcal{L}}_K \phi &= \lambda_\phi\, \phi\,, \qquad & \hat{\mathcal{L}}_{\bar{Y}} \phi &= 0\,. \end{aligned} \tag{5.30}$$

It is also useful to rewrite these equations in terms of the complex coordinates w, z using (5.5):

$$\begin{aligned} iD_{\bar w} B &= \lambda_B\, B\,, \qquad & (D_{\bar z} - \bar h\, D_{\bar w})B &= 0\,, \\ iD_{\bar w} \phi &= \lambda_\phi\, \phi\,, \qquad & (D_z - h D_w)\phi &= 0\,. \end{aligned} \tag{5.31}$$

Let us note that the equation $\hat{\mathcal{L}}_Y B = 0$ in (5.30) is a condition for the shortening of the $\mathcal{N} = 1$ chiral multiplet. For a solution of this equation, it is consistent to set $C = 0$ (and also $\phi = 0$) because B disappears from its equation of motion. We are then left with a short multiplet (B, F) akin to the (b, f) multiplet we found for $T^2 \times S^2$ in section 4. Similar remarks apply to (ϕ, C) and the (a, c) multiplet of our $T^2 \times S^2$ index computation.

The equations (5.31) are also consistent with known constraints on the parameter dependence of $\mathcal{N} = 1$ supersymmetric partition functions [3]. We will see in the examples that the eigenvalues λ_B, λ_ϕ depend holomorphically on complex structure moduli and line bundle moduli (up to an arbitrary overall rescaling). Moreover, for any background with two supercharges the equations (5.31) are independent of the ambiguity κ in (2.5). It is indeed expected that the partition function of a chiral multiplet be independent of κ, because this free theory possesses an FZ multiplet [3].

In the rest of this section, we will compute (5.29) in two interesting examples. For $\mathcal{M}_4 = T^2 \times S^2$, we will reproduce the canonical quantization result of section 4. For $\mathcal{M}_4 = S^3 \times S^1$, we will reproduce the previously known result [16, 17] in a new and elegant manner.

5.5 The case of $T^2 \times S^2$

Let us compute the one-loop determinant (5.29) for the $T^2 \times S^2$ background of section 3. Consider the B modes first. They are solutions of the first line in (5.31), which becomes

$$i\,(\partial_{\bar w} - iq_f v_{\bar w})\, B = \lambda_B\, B\,, \qquad \left(\partial_{\bar z} + \frac{\mathbf{r}-2}{2}\,\frac{z}{1+|z|^2}\right) B = 0\,. \tag{5.32}$$

Here $\mathbf{r}$ is the shifted R-charge and $v_{\bar w}$ the $\mathrm{U}(1)_f$ flat connection, as defined in section 3.2. The second equation in (5.32) implies $B = f_1(z)(1+|z|^2)^{-\frac{\mathbf{r}-2}{2}} f_2(w, \bar w)$, with $f_1(z)$ holomorphic. It is convenient to consider eigenmodes of the angular momentum operator $J_3 = z\partial_z - \bar z \partial_{\bar z} - \frac{\mathbf{r}-2}{2}$ on the sphere (see appendix A). $J_3 B = mB$ implies $f_1 \propto z^{m+\frac{\mathbf{r}-2}{2}}$. We also consider definite momentum $n_x, n_y \in \mathbb{Z}$ on the torus. Taking into account the boundary conditions (3.17), we find

$$B = e^{\frac{1}{2\tau_2}(n_x\tau - n_y - m\sigma)\bar w - \frac{1}{2\tau_2}(n_x\bar\tau - n_y - m\bar\sigma) w}\, \frac{z^{m+\frac{\mathbf{r}-2}{2}}}{(1+z\bar z)^{\frac{\mathbf{r}-2}{2}}}\,. \tag{5.33}$$

Moreover, B is normalizable if and only if $\frac{2-\mathbf{r}}{2} \leq m \leq \frac{\mathbf{r}-2}{2}$. In particular, solutions exist only if $\mathbf{r} \geq 2$. We easily see that (5.33) solves (5.32) with eigenvalue

$$\lambda_B = \frac{i}{2\tau_2}(n_x\tau - n_y - m\sigma - q_f\nu)\,. \tag{5.34}$$

Note that the B modes correspond to the first kind of spinor zero modes in (A.17). A similar analysis for the modes ϕ solving (5.31) leads to

$$\phi = e^{\frac{1}{2\tau_2}(n_x\tau - n_y - m\sigma)\bar{w} - \frac{1}{2\tau_2}(n_x\bar{\tau} - n_y - m\bar{\sigma})w}\,\frac{(1+z\bar{z})^{\frac{\mathbf{r}}{2}}}{z^{m+\frac{\mathbf{r}}{2}}}\,, \tag{5.35}$$

with $\frac{\mathbf{r}}{2} \leq m \leq -\frac{\mathbf{r}}{2}$. In particular such solutions exist only for $\mathbf{r} \leq 0$. The eigenvalues λ_ϕ are given by

$$\lambda_\phi = \frac{i}{2\tau_2}(n_x\tau - n_y - m\sigma - q_f\nu)\,. \tag{5.36}$$

These modes correspond to the second set of spinor zero modes in (A.17).

Therefore, we have three distinct cases, depending on the shifted R-charge $\mathbf{r} \in \mathbb{Z}$. If $\mathbf{r} = 1$, all the bosonic and fermionic modes are paired together by supersymmetry and therefore the partition function is

$$Z^{\Phi}_{T^2\times S^2;\,\mathbf{r}=1} = 1\,. \tag{5.37}$$

If $\mathbf{r} > 1$, only the B modes contribute to the partition function, with eigenvalues (5.34):

$$Z^{\Phi}_{T^2\times S^2;\,\mathbf{r}>1} = \prod_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1}\ \prod_{n_x,n_y=-\infty}^{\infty} \left(\tau n_x - n_y - m\sigma - q_f\nu\right). \tag{5.38}$$

Here we rescaled away the overall factor of $\frac{i}{2\tau_2}$ in (5.34).[20] If $\mathbf{r} < 1$, the ϕ mode eigenvalues (5.36) are the ones which contribute:

$$Z^{\Phi}_{T^2\times S^2;\,\mathbf{r}<1} = \prod_{m=-\frac{|\mathbf{r}|}{2}}^{\frac{|\mathbf{r}|}{2}}\ \prod_{n_x,n_y=-\infty}^{\infty} \frac{1}{\tau n_x - n_y - m\sigma - q_f\nu}\,. \tag{5.39}$$

The infinite products (5.38), (5.39) need to be properly regularized, which can be done using zeta function regularizaton —see in particular Example 13 of [41]. This leads to (1.4), (1.5).

5.6 The case of $S^3 \times S^1$: the elliptic gamma function

The supersymmetric partition function of $\mathcal{N} = 1$ theories on $S^3 \times S^1$, $Z_{S^3\times S^1}$, has been well-studied in the literature. It computes a supersymmetric index for the theory quantized on S^3 [30, 42], which has been computed exactly for rather general $\mathcal{N} = 1$ gauge theories [16, 17]. In particular, the chiral multiplet partition function is given by a certain elliptic

[20] Such an overall rescaling is arbitrary and does not affect the final answer in terms of ζ-function regularized products.

gamma function [17]. In the following, we will rederive that last result using the method of section 5.4.

Let us consider a complex manifold $\mathcal{M}_4^{p,q}$ diffeomorphic to $S^3 \times S^1$ called a primary Hopf surface of the first type. It is obtained as a quotient of $\mathbb{C}^2 - (0,0)$:

$$(z_1, z_2) \sim (p\, z_1, q\, z_2)\,, \qquad 0 < |p| \leq |q| < 1\,. \tag{5.40}$$

Here z_1, z_2 are the coordinates on $\mathbb{C}^2 - (0,0)$, and p, q are complex structure parameters. It was realized recently that the partition function $Z_{S^3\times S^1}$ is a locally holomorphic function on the complex structure moduli space of $\mathcal{M}_4^{p,q}$ [3]. We refer to [3] for more details and references.[21]

It will be convenient to introduce two complex parameters $\sigma = \sigma_1 + i\sigma_2$ and $\tau = \tau_1 + i\tau_2$ (with $\sigma_{1,2}, \tau_{1,2}$ real) such that

$$p = e^{2\pi i \sigma}\,, \qquad q = e^{2\pi i \tau}\,, \qquad 0 < \tau_2 \leq \sigma_2\,, \tau_1 \sim \tau_1 + 1\,, \sigma_1 \sim \sigma_1 + 1\,. \tag{5.41}$$

For generic values of p, q, $\mathcal{M}_4^{p,q}$ admits two supercharges [3]. In the rest of this subsection, we will consider the subcase $\sigma_2 = \tau_2$ for simplicity. We consider the following Hermitian metric on $\mathcal{M}_4^{p,q}$ (with $|p| = |q|$):

$$ds^2 = \frac{1}{|z_1|^2 + |z_2|^2} \left(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2\right)\,. \tag{5.42}$$

To take advantage of the general discussion of section 2, we introduce new coordinates

$$w = -i \log z_1\,, \qquad z = \frac{z_2}{z_1}\,. \tag{5.43}$$

These coordinates cover $S^3 \times S^1$ except for the locus $z_1 = 0$, which can be covered with coordinates $w' = w - i\log z$, $z' = \frac{1}{z}$. The w, z coordinates (5.43) are subject to the following identifications:

$$(w, z) \sim \left(w + 2\pi\sigma,\, e^{2\pi i(\tau_1 - \sigma_1)} z\right)\,, \qquad (w, z) \sim \left(w + 2\pi,\, z\right)\,, \tag{5.44}$$

It is convenient to consider real coordinates x, θ, φ, χ on $S^3 \times S^1$, with $x \in [0, 2\pi)$ the coordinate on the S^1 and $\theta \in [0, \pi]$, $\varphi, \chi \in [0, 2\pi)$ the coordinates on the S^3. In terms of these angular coordinates,

$$w = \sigma x + \varphi - i\log\left(\cos\frac{\theta}{2}\right)\,, \qquad z = e^{i(\tau_1 - \sigma_1)x} \tan\frac{\theta}{2} e^{i(\chi - \varphi)}\,. \tag{5.45}$$

The periodicities (5.44) correspond to $x \sim x + 2\pi$ and $\varphi \sim \varphi + 2\pi$, respectively. In terms of the w, z coordinates, the metric (5.42) takes the form

$$ds^2 = \left(dw - \frac{i\bar{z}}{1+|z|^2} dz\right)\left(d\bar{w} + \frac{iz}{1+|z|^2} d\bar{z}\right) + \frac{1}{(1+|z|^2)^2} dz d\bar{z}\,. \tag{5.46}$$

[21] Note that our coordinates z_1, z_2 are denoted by w, z in [3]. In the present paper we reserve the notation w, z for the special coordinates adapted to two supercharges, such that $K = \partial_{\bar{w}}$ is the Killing vector built from the two Killing spinors.

The resulting supersymmetric background is studied in more detail in appendix C. In addition to the supergravity background fields, we also consider a background gauge field for a $\mathrm{U}(1)_f$ internal symmetry,

$$a_\mu dx^\mu = \frac{1}{2i\sigma_2}(a_r + ia_i)\left(dw - \frac{i\bar{z}}{1+|z|^2}dz\right) - \frac{1}{2i\sigma_2}(a_r - ia_i)\left(d\bar{w} + \frac{iz}{1+|z|^2}d\bar{z}\right), \quad (5.47)$$

which preserves the same supercharges as (5.46). Let us also define the fugacity

$$u = e^{i(a_r - ia_i)}. \quad (5.48)$$

The complex parameter $a_r - ia_i$ is a holomorphic line bundle modulus [3].

For generic values of τ_1, σ_1 and $\sigma_2 = \tau_2$, the background (5.46) has a $\mathrm{U}(1)^3$ isometry, corresponding to rotations along the real angles x, φ, χ mentioned above. Supersymmetry dictates twisted periodicities for R-charged fields going around these angles. The corresponding momentum operators acting on fields of R-charge r are given by

$$\begin{aligned} P_x &= (\tau_1 - \sigma_1)(z\partial_z - \bar{z}\partial_{\bar{z}}) - i(\sigma\partial_w + \bar{\sigma}\partial_{\bar{w}}) - \frac{r}{2}(\tau_1 - \sigma_1), \\ P_\varphi &= -(z\partial_z - \bar{z}\partial_{\bar{z}}) - i(\partial_w + \partial_{\bar{w}}) + r, \\ P_\chi &= (z\partial_z - \bar{z}\partial_{\bar{z}}). \end{aligned} \quad (5.49)$$

These operators have integer eigenvalues. Note that

$$K = \partial_{\bar{w}} = \frac{1}{2\sigma_2}\left(-P_x + \sigma P_\varphi + \tau P_\chi - \frac{r}{2}(\sigma + \tau)\right). \quad (5.50)$$

Let us compute the partition function for a chiral multiplet of R-charge r and Q_f-charge q_f, according to (5.29), (5.31). Using the results of appendix C, the eigenvalue equations for the unpaired modes of type B read

$$i(\partial_{\bar{w}} - iq_f v_{\bar{w}})B = \lambda_B B, \qquad \left(\partial_{\bar{z}} + \frac{r-2}{2}\frac{z}{1+|z|^2} - \frac{iz}{1+|z|^2}\partial_{\bar{w}}\right)B = 0, \quad (5.51)$$

where $v_{\bar{w}} = -\frac{1}{2i\sigma_2}(a_r - ia_i)$. We also require B to be an eigenmode of the operators (5.49), with

$$P_x B = n_0 B, \qquad P_\varphi B = n_1 B, \qquad P_\chi B = n_2 B, \qquad n_0, n_1, n_2 \in \mathbb{Z}. \quad (5.52)$$

Therefore, the first equation in (5.51) together with (5.50) imply

$$\lambda_B = \frac{i}{2\sigma_2}\left(-n_0 + n_1\sigma + n_2\tau - \frac{r-2}{2}(\sigma+\tau) + q_f(a_r - ia_i)\right). \quad (5.53)$$

The B modes are given explicitly by

$$B = z^{n_2}(1+|z|^2)^{\lambda_B - q_f v_{\bar{w}} - \frac{r-2}{2}}\, e^{-i(\lambda_B - q_f v_{\bar{w}})\bar{w}} e^{-i(\bar{\lambda}_B - qv_w)w}. \quad (5.54)$$

A similar expression holds in the w', z' patch, with $|B| \sim |z'|^{n_1}$ near $z' = 0$. Normalizabiltiy of the modes restricts the allowed values of n_1, n_2 to $n_1, n_2 \geq 0$. Similarly, the ϕ modes solving (5.31) are given by

$$\phi = \frac{1}{\bar{z}^{n_2}}(1+|z|^2)^{-(\bar{\lambda}_\phi - q_f v_w - \frac{r}{2})}\, e^{-i(\lambda_\phi - q_f v_{\bar{w}})\bar{w}} e^{-i(\bar{\lambda}_\phi - qv_w)w}. \quad (5.55)$$

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with eigenvalues

$$\lambda_\phi = \frac{i}{2\sigma_2}\left(-n_0 + n_1\sigma + n_2\tau - \frac{r}{2}(\sigma+\tau) + q_f(a_r - ia_i)\right) . \tag{5.56}$$

The integers n_0, n_1, n_2 are again the eigenvalues of (5.49). Nomalizability imposes $n_1, n_2 \le 0$. Plugging the eigenvalues (5.53), (5.56) into (5.29) and renaming some of the integers, we find the partition function

$$Z^{\Phi}_{S^3\times S^1} = \prod_{n_0=-\infty}^{\infty}\prod_{n_1,n_2=0}^{\infty} \frac{n_0 + \sigma n_1 + \tau n_2 - \frac{r-2}{2}(\sigma+\tau) + q_f(a_r - ia_i)}{n_0 + \sigma n_1 + \tau n_2 + \frac{r}{2}(\sigma+\tau) - q_f(a_r - ia_i)} . \tag{5.57}$$

Note that this formula is a natural generalization of the three-dimensional localization result for the squashed S^3 [15, 39].

The result (5.57) can be regularized using Barnes' multiple zeta function [43, 44].[22] In term of the parameters p, q, u defined in (5.41) and (5.48), we obtain

$$\begin{aligned} Z^{\Phi}_{S^3\times S^1}(p,q,u) &= e^{i\pi\mathcal{A}} \prod_{j,k=0}^{\infty} \frac{1 - u^{-q_f} p^{j+1-\frac{r}{2}} q^{k+1-\frac{r}{2}}}{1 - u^{q_f} p^{j+\frac{r}{2}} q^{k+\frac{r}{2}}} \\ &= e^{i\pi\mathcal{A}}\, \Gamma_e(u^{q_f}(pq)^{\frac{r}{2}}; p, q) . \end{aligned} \tag{5.58}$$

The function $\Gamma_e(t;p,q)$ is the elliptic gamma function.[23] We thus reproduced the known result for the $\mathcal{N}=1$ chiral multiplet $S^3\times S^1$ partition function, without relying on the supersymmetric index point of view of [16–18].[24] The answer is locally holomorphic in the geometric moduli τ, σ and $a_r - ia_i$, as expected [3]. In addition to the expected gamma function, there is an interesting prefactor $e^{i\pi\mathcal{A}}$ in (5.58), with $\mathcal{A}$ given by

$$\mathcal{A}(\mathbf{w}) = \frac{1}{\sigma\tau}\left(\mathbf{w}^3 - \frac{\tau^2+\sigma^2-2}{12}\mathbf{w}\right) , \qquad \mathbf{w} = (r-1)\frac{\tau+\sigma}{2} - q_f(a_r - ia_i) . \tag{5.59}$$

Note that $\mathcal{A}$ is a cubic polynomial in $r-1$ and q_f, which are the R- and Q_f-charges of the fermion in the $\mathcal{N}=1$ chiral multiplet. It would be interesting to understand whether this $\mathcal{A}$ is physical or whether it can be removed by a local counterterm.

5.7 Computation with arbitrary hermitian metrics

In the two examples above, we chose rather simple Hermitian metrics for ease of presentation. However, it is easy to generalize the computation of section 5.4 to arbitrary backgrounds with two supercharges. Consider for instance the case of $T^2\times S^2$ with an arbitrary Hermitian metric compatible with a given complex structure (with parameters

[22] One can rewrite (5.57) in terms of triple gamma functions and use Corollary 6.2 of [44].

[23] It can be defined by

$$\Gamma_e(t;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - t^{-1}p^{j+1}q^{k+1}}{1 - tp^j q^k} .$$

[24] See [45] for a related computation of the $\mathcal{N}=4$ index using localization.

τ, σ). Using the w, z coordinates, we can solve the eigenvalue equations (5.30) with the ansatz

$$B = e^{i(n_x + \frac{\mathrm{r}-2}{2}\alpha)x} e^{i(n_y + \frac{\mathrm{r}-2}{2}\beta)y} e^{i(m + \frac{\mathrm{r}-2}{2})\varphi} B_0(\theta)\,,$$
$$\phi = e^{i(n_x + \frac{\mathrm{r}}{2}\alpha)x} e^{i(n_y + \frac{\mathrm{r}}{2}\beta)y} e^{i(m + \frac{\mathrm{r}}{2})\varphi} \phi_0(\theta)\,, \tag{5.60}$$

in the northern patch. Here x, y, φ, θ are the real coordinates introduced in (3.4). This ansatz is dictated by the periodicities (3.17) (and by the general properties of the spin operator J_3 on S^2 with magnetic flux), which are independent of the details of the metric (although they do depend on the choice of complex coordinates). Using the expression (3.7) for $\partial_{\bar{w}}$, we directly find the eigenvalues λ_B (5.34) and λ_ϕ (5.36). Possible restrictions on the values of the integers in $\lambda_{B,\phi}$ can be obtained from a careful analysis of the profiles $B_0(\theta), \phi_0(\theta)$ near $\theta = 0, \pi$, asking that the modes be normalizable. (Understanding the limit $\theta \sim 0, \pi$ is much simpler than solving for the full profiles, which cannot be done in general.)

A similar analysis can be done for $S^3 \times S^1$. One thus confirms explicitly, in our simple examples, that the partition function on $\mathcal{M}_4$ is independent of the Hermitian metric [3].

Acknowledgments

It is a pleasure to thank Ofer Aharony, Francesco Benini, Lorenzo Di Pietro, Thomas Dumitrescu, Guido Festuccia, Itamar Yaakov and especially Zohar Komargodski for stimulating discussions and for comments on the manuscript. We also thank Joel Fine for pointing out reference [25]. I.S. would like to extend his gratitude to Efrat, Guy, Nizan, Ran and Amir for countless discussions from which he has benefited. C.C. would like to thank the PMIF group at ULB for its warm hospitality during the completion of this work. The work of I.S. is supported by the ERC STG grant number 335182, by the Israel Science Foundation under grant number 884/11. I.S. would also like to thank the United States-Israel Binational Science Foundation (BSF) for support under grant number 2010/629. In addition, the research of I.S. is supported by the I-CORE Program of the Planning and Budgeting Committee and by the Israel Science Foundation under grant number 1937/12. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the funding agencies.

A Monopole harmonics on the sphere

Consider S^2 with angular coordinates $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. We introduce the complex coordinate

$$z = \tan\frac{\theta}{2} e^{i\varphi}\,, \tag{A.1}$$

in the northern patch, which covers the sphere except for the south pole at $\theta = \pi$. The southern patch is similarly covered by a coordinate z', with $z' = \frac{1}{z}$ on the overlap $\theta \in (0, \pi)$. In this paper we always work on the northern patch, unless otherwise stated.

Consider the sphere with the round metric and one unit of magnetic flux:

$$ds^2 = \frac{4}{(1+|z|^2)^2} dz d\bar{z}\,, \qquad A = -\frac{i}{2(1+|z|^2)}(\bar{z}dz - zd\bar{z})\,. \tag{A.2}$$

Here A is the gauge field for the monopole. This background is invariant under SO(3) rotations. The metric in (A.2) had three Killing vectors,

$$K_+ = z^2\partial_z + \partial_{\bar z}\,, \qquad K_- = -\partial_z - \bar z^2\partial_{\bar z}\,, \qquad K_3 = z\partial_z - \bar z\partial_{\bar z}\,, \tag{A.3}$$

while the monopole gauge field is invariant along (A.3) up to gauge transformations,

$$\mathcal{L}_{K_+}A + \frac{i}{2}dz = 0\,, \qquad \mathcal{L}_{K_-}A + \frac{i}{2}d\bar z = 0\,, \qquad \mathcal{L}_{K_3}A = 0\,. \tag{A.4}$$

Here $\mathcal{L}_K$ is the Lie derivative along K. Consequently, the SO(3) transformations of fields coupling to A must be accompanied by gauge transformations, which are determined from (A.4) up to integration constants. These integrations constants are fixed by the SO(3) algebra

$$[J_+, J_-] = 2J_3\,, \qquad [J_3, J_\pm] = \pm J_\pm\,. \tag{A.5}$$

Therefore, the infinitesimal SO(3) transformations on any field coupling to the monopole with electric charge $\mathbf{r}$ are generated by

$$J_- = \mathcal{L}_{K_+} - \frac{\mathbf{r}}{2}z\,, \qquad J_+ = \mathcal{L}_{K_-} - \frac{\mathbf{r}}{2}\bar z\,, \qquad J_3 = \mathcal{L}_{K_3} - \frac{\mathbf{r}}{2}\,. \tag{A.6}$$

Let us also define $J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$.

A.1 Scalar monopole harmonics

The scalar Laplacian on the monopole background (A.2) is given by

$$\Delta^{\mathbf{r}}_{S^2} = -(1+z\bar z)^2\partial_z\partial_{\bar z} - \frac{\mathbf{r}}{2}(1+z\bar z)\left(z\partial_z - \bar z\partial_{\bar z} - \frac{\mathbf{r}}{2}\right) - \frac{\mathbf{r}^2}{4}\,. \tag{A.7}$$

It acts on scalar fields of electric charge $\mathbf{r}$. Due to the relation $\Delta^{\mathbf{r}}_{S^2} + \frac{\mathbf{r}^2}{4} = J^2$, we can diagonalize (A.7) together with J^2, J_3:

$$\Delta^{\mathbf{r}}_{S^2}Y_{\mathbf{r}\,jm} = \left(j(j+1) - \frac{\mathbf{r}^2}{4}\right)Y_{\mathbf{r}\,jm}\,,\ J^2Y_{\mathbf{r}\,jm} = j(j+1)Y_{\mathbf{r}\,jm}\,,\ J_3Y_{\mathbf{r}\,jm} = mY_{\mathbf{r}\,jm}\,. \tag{A.8}$$

Not all possible SO(3) representations appear. The allowed values of j, m are

$$j = \frac{|\mathbf{r}|}{2}, \frac{|\mathbf{r}|}{2} + 1, \ldots\,, \qquad m = -j, \ldots, j\,. \tag{A.9}$$

Note that for $\mathbf{r}$ odd the allowed values for the angular momentum are half-integer, therefore a scalar can behave like a fermion in a monopole background [46].

The eigenfunctions $Y_{\mathbf{r}\,jm}$ are known as monopole harmonics [28]. We will not need their explicit form, just the fact that they are orthonormal for fixed $\mathbf{r}$,

$$\int_{S^2} d^2x\sqrt{g}\,Y^\dagger_{rjm}Y_{rj'm'} = \delta_{jj'}\delta_{mm'}\,. \tag{A.10}$$

Let us emphasize that the monopole harmonics are really sections of a non-trivial line bundle, with transitions functions

$$Y^{(N)}_{\mathbf{r}\,jm} = \left(\frac{z}{\bar z}\right)^{\frac{\mathbf{r}}{2}} Y^{(S)}_{\mathbf{r}\,jm} \tag{A.11}$$

between the northern (N) and southern (S) patches.

A.2 Spinor monopole harmonics

Consider spinor fields on the sphere with a monopole. (See [47, 48] for related discussions.) We choose the complex veilbein and gamma matrices on S^2:

$$e = \frac{2}{1+|z|^2}dz\,, \quad \bar{e} = \frac{2}{1+|z|^2}d\bar{z}\,, \quad \gamma^{(e)} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\,, \quad \gamma^{(\bar{e})} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\,. \tag{A.12}$$

In this frame, the Dirac operator on (A.2), acting on a spinor of electric charge $\mathbf{r}-1$, is given by

$$-i\nabla\!\!\!\!/\,^{\mathbf{r}}_{S^2} = -i\begin{pmatrix} 0 & (1+z\bar{z})\partial_z - \frac{1}{2}\mathbf{r}\bar{z} \\ (1+z\bar{z})\partial_{\bar{z}} + \frac{1}{2}(\mathbf{r}-2)z & 0 \end{pmatrix}. \tag{A.13}$$

One can find its eigenvalues,

$$-i\nabla\!\!\!\!/\,^{\mathbf{r}}_{S^2}\psi^{\pm}_{\mathbf{r}-1\,jm} = \pm\lambda_{\mathbf{r}j}\,\psi^{\pm}_{\mathbf{r}-1\,jm}\,, \qquad \lambda_{\mathbf{r}j} = \sqrt{\left(j+\frac{1}{2}\right)^2 - \frac{(\mathbf{r}-1)^2}{4}}\,, \tag{A.14}$$

where j, m are the eigenvalues of J^2, J_3, respectively. The allowed values are

$$j = \frac{|\mathbf{r}-1|}{2} - \frac{1}{2}, \frac{|\mathbf{r}-1|}{2} + \frac{1}{2}, \ldots\,, \qquad m = -j, \ldots, j\,, \tag{A.15}$$

with the lowest level for j appearing only when $\mathbf{r} \neq 1$. The lowest level $j = \frac{|\mathbf{r}-1|}{2} - \frac{1}{2}$ corresponds to zero modes. The eigenspinors $\psi^{\pm}_{\mathbf{r}-1\,jm}$ can be expressed in terms of the scalar monopole harmonics. We have

$$(\psi^{\pm}_{\mathbf{r}-1\,jm})_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} iY_{\mathbf{r}-2\,jm} \\ \pm Y_{\mathbf{r}\,jm} \end{pmatrix}\,, \tag{A.16}$$

for the non-zero modes, $j > \frac{|\mathbf{r}-1|}{2} - \frac{1}{2}$, and

$$(\psi_{\mathbf{r}-1\,jm})_\alpha = \begin{pmatrix} Y_{\mathbf{r}-2\,jm} \\ 0 \end{pmatrix}\,, \qquad\qquad (\psi_{\mathbf{r}-1\,jm})_\alpha = \begin{pmatrix} 0 \\ Y_{\mathbf{r}\,jm} \end{pmatrix}\,, \tag{A.17}$$

for the zero modes, where on the left we have the case $\mathbf{r} > 1$ (with $j = \frac{\mathbf{r}}{2} - 1$), and on the right the case $\mathbf{r} < 1$ (with $j = -\frac{\mathbf{r}}{2}$). The eigenspinors are orthonormal for fixed $\mathbf{r}$,

$$\int_{S^2} d^2x\sqrt{g}\,(\psi^{\epsilon}_{\mathbf{r}-1\,jm})^\dagger\psi^{\epsilon'}_{\mathbf{r}-1\,j'm'} = \delta^{\epsilon\epsilon'}\delta_{jj'}\delta_{mm'}, \tag{A.18}$$

where $\epsilon, \epsilon' = \pm 1$.

B Comments on the supersymmetric pairing of eigenvalues

In this appendix, we show that the operators (5.17) are adjoint. We also give an alternative argument for the supersymmetric pairing of eigenvalues discussed in section 5.4.

B.1 Adjoint operators

Let us prove that the operators $i\hat{\mathcal{L}}_{\bar{Y}}$ and $i\Omega^2\hat{\mathcal{L}}_Y$ in (5.17) are mutually adjoint. We have

$$Y^\mu A_\mu = -\frac{is}{2\Omega^2 c}\partial_{\bar{z}}\log(\Omega^2 cs)\,, \qquad \bar{Y}^\mu A_\mu = \frac{i}{2sc}\partial_z \log\left(\frac{\Omega^4 c}{s}\right)\,, \tag{B.1}$$

in the notation of section 2. It follows directly from the definition (5.3) that

$$\begin{aligned} i\hat{\mathcal{L}}_Y B &= i s^{\frac{r}{2}}(\Omega^2 c)^{\frac{r}{2}-2}(\partial_{\bar{z}} - \bar{h}\partial_{\bar{w}})\left(\frac{B}{(\Omega^2 c\, s)^{\frac{r}{2}-1}}\right)\,, \\ i\hat{\mathcal{L}}_{\bar{Y}}\phi &= i s^{\frac{r}{2}-1}\frac{\Omega^4}{(\Omega^4 c)^{\frac{r}{2}+1}}(\partial_z - h\partial_w)\left(\left(\frac{\Omega^4 c}{s}\right)^{\frac{r}{2}}\phi\right)\,, \end{aligned} \tag{B.2}$$

on fields B and ϕ of R-charge $r-2$ and r, respectively. Recalling that Ω, c and s are functions of $z, \bar{z}$ only, with Ω and c real, and that $\sqrt{g} = \frac{1}{4}\Omega^4 c^2$, one can use (B.2) to prove (5.18) by direct computation.

B.2 Another derivation of the eigenvalue pairing

Let us give another, complementary explanation of the claims of section 5.4 by exhibiting explicitly the supersymmetric pairing between bosonic and fermionic eigenmodes. Consider the eigenvalue equations

$$\Delta_{\rm bos}\phi = \Lambda_{\rm b}\phi\,, \qquad \Delta_{\rm fer}\Psi = \Lambda_{\rm f}\Psi\,, \tag{B.3}$$

with the kinetic operators (5.21). We want to compute the quantity

$$Z^{\Phi}_{\mathcal{M}_4} = \frac{\det\Delta_{\rm fer}}{\det\Delta_{\rm bos}} = \frac{\prod\Lambda_f}{\prod\Lambda_b}\,. \tag{B.4}$$

Let us consider ϕ a bosonic solution of (B.3), which we can also take to be an eigenstate of $i\hat{\mathcal{L}}_K$ since $i\hat{\mathcal{L}}_K$ commutes with $\Delta_{\rm bos}$. Let us moreover *assume* that the operator $i\Omega^2\hat{\mathcal{L}}_{\bar{K}}$ also commutes with $\Delta_{\rm bos}$, so that we can diagonalize $\Delta_{\rm bos}, i\hat{\mathcal{L}}_K$ and $i\Omega^2\hat{\mathcal{L}}_{\bar{K}}$ simultaneously.[25] We thus have

$$\Delta_{\rm bos}\phi = \Lambda_{\rm b}\phi, \qquad i\hat{\mathcal{L}}_K\phi = l_K\phi, \qquad i\Omega^2\hat{\mathcal{L}}_{\bar{K}}\phi = l_{\bar{K}}\phi\,. \tag{B.5}$$

We can use ϕ to construct its matching fermionic eigenstates. Define

$$\Psi_1 = \begin{pmatrix} i\hat{\mathcal{L}}_{\bar{Y}}\phi \\ 0 \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 0 \\ \phi \end{pmatrix}. \tag{B.6}$$

[25] We easily see that $[\Delta_{\rm bos}, \Omega^2\hat{\mathcal{L}}_{\bar{K}}] = 0$ if and only if $F_{zw} = F_{\bar{z}w} = 0$, in addition to (5.7). We have $F_{zw} = \partial_z A_w$, $F_{\bar{z}w} = \partial_{\bar{z}}A_w$, and $A_w = \frac{3}{4}\kappa\Omega^2 - \frac{i}{2c^2}\partial_z\bar{h}$. Recall that κ is a function satisfying $K^\mu\partial_\mu\kappa = 0$ but otherwise arbitrary. The partition function of a chiral multiplet is independent of κ [3] and we are free to make a convenient choice. Even though such a choice is possible locally, there could be subtleties with setting A_w to a constant globally.

This mapping is precisely the one we would naively expect from the supersymmetry transformations (5.10). The fermions Ψ_1, Ψ_2 generate an invariant subspace of Δ_{fer}, which reads

$$\Big[\Delta_{\text{fer}}\Big]_{\{\Psi_1,\Psi_2\}} = \begin{bmatrix} l_K & 1 \\ l_K l_{\bar{K}} - \Lambda_{\text{b}} & l_{\bar{K}} \end{bmatrix} . \tag{B.7}$$

One can diagonalize this matrix to find two eigenvalues $\Lambda_{\text{f},\pm}$ whose product equals Λ_b. Therefore, a bosonic eigenstate ϕ with eigenvalue Λ_{b} is generally paired with two fermionic eigenstates with eigenvalues $\Lambda_{\text{f},\pm}$, such that $\Lambda_{\text{f},+}\Lambda_{\text{f},-} = \Lambda_{\text{b}}$. One can easily invert this map. Therefore, the only contributions to (B.4) come from unpaired modes, which are of two types.

Firstly, we can have a bosonic mode ϕ which is not completely cancelled by its fermionic partners. This happens if and only if $\hat{\mathcal{L}}_{\bar{Y}}\phi = 0$, and one can show that the net contribution to the partition function is the eigenvalue of the operator $i\hat{\mathcal{L}}_K$:

$$i\hat{\mathcal{L}}_K \phi = \lambda_\phi\, \phi\,, \qquad \hat{\mathcal{L}}_{\bar{Y}}\phi = 0\,. \tag{B.8}$$

The eigenvalue λ_ϕ effectively contributes to the denominator of (B.4). Secondly, we can have a fermionic eigenmode with no paired boson, which must be of the type $\Psi_0 = (B, 0)^T$. Such a mode is a solution of the eigenvalue equations:

$$i\hat{\mathcal{L}}_K B = \lambda_B\, B\,, \qquad \hat{\mathcal{L}}_Y B = 0\,. \tag{B.9}$$

The eigenvalue λ_B contributes to the numerator of (5.24). This proves (5.29), (5.30).

While this argument is very concrete, it is however less general than the discussion above (5.28) since we had to assume that $[\Delta_{\text{bos}}, \Omega^2\hat{\mathcal{L}}_{\bar{K}}] = 0$.

C $\;S^3 \times S^1$ supersymmetric backgrounds

In order to preserve at least two supercharges on $S^3 \times S^1$, we need to consider the following quotient of $\mathbb{C}^2 - (0,0)$ [3]:

$$(z_1, z_2) \sim (p\, z_1, q\, z_2)\,, \qquad 0 < |p| \leq |q| < 1\,. \tag{C.1}$$

This quotient is a complex manifold $\mathcal{M}_4^{p,q}$ diffeomorphic to $S^3 \times S^1$. Let us introduce the complex parameters $\sigma = \sigma_1 + i\sigma_2$ and $\tau = \tau_1 + i\tau_2$, defined by $p = e^{2\pi i\sigma}, q = e^{2\pi i\tau}$. It is also convenient to introduce the real coordinates x, θ, φ, χ, with $x \in [0, 2\pi)$ the S^1 coordinate and $\theta \in [0, \pi]$, $\varphi, \chi \in [0, 2\pi)$ the S^3 coordinates, with

$$z_1 = e^{i\sigma x} \cos\frac{\theta}{2}\, e^{i\varphi}\,, \qquad z_2 = e^{i\tau x} \sin\frac{\theta}{2}\, e^{i\chi}\,. \tag{C.2}$$

The identification (C.1) corresponds to $x \sim x + 2\pi$. We consider the following Hermitian metric on $\mathcal{M}_4^{p,q}$:

$$ds^2 = e^{2\sigma_2 x} dz_1 d\bar{z}_1 + e^{2\tau_2 x} dz_2 d\bar{z}_2\,. \tag{C.3}$$

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The background fields V_μ, A_μ can be obtained from (2.5) with $\sqrt{g} = \frac{1}{4} e^{2(\sigma_2+\tau_2)x}$, in the z_1, z_2 coordinates. In the complex frame $(e^1, e^2) = (e^{\sigma_2 x} dz_1, e^{\tau_2 x} dz_2)$, the Killing spinors are given by

$$\zeta_\alpha = \sqrt{\frac{s}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \widetilde{\zeta}^{\dot\alpha} = \frac{1}{\sqrt{2s}} \frac{1}{\sqrt{\sigma_2 \tau_2}} \begin{pmatrix} i\sigma_2\, e^{2\sigma_2 x} \bar z_1 \\ -i\tau_2\, e^{\tau_2 x} \bar z_2 \end{pmatrix} , \tag{C.4}$$

while the anti-homolorphic Killing vector (2.3) is given by $K = -\frac{i}{\sqrt{\sigma_2\tau_2}} (\sigma_2 \bar z_1 \partial_{\bar z_1} + \tau_2 \bar z_2 \partial_{\bar z_2})$. Note that s must transform as

$$s \sim e^{-2\pi i(\sigma_1+\tau_1)} s \tag{C.5}$$

under the identification (C.1), so that the holomorphic two-form (2.8) is well-defined. We compensate (C.5) by an R-symmetry transformation, and s is then a scalar in the sense of [1], which we can set to 1. Consequently, any field Φ of R-charge r satisfies the twisted boundary condition

$$\Phi \sim e^{i\pi r(\sigma_1+\tau_1)} \Phi \tag{C.6}$$

as $x \sim x + 2\pi$. Note that this condition depends on the choice of holomorphic coordinates.[26] We will discuss a different choice of coordinates in the following, with different resulting boundary conditions for the fields.

In section 5.6, we found convenient to use coordinates w, z adapted to two supercharges in the sense of section 2, such that $K = \partial_{\bar w}$. A simple choice is

$$w = -i \sqrt{\frac{\tau_2}{\sigma_2}} \log z_1 , \qquad z = \frac{(z_2)^{\sqrt{\frac{\sigma_2}{\tau_2}}}}{(z_1)^{\sqrt{\frac{\tau_2}{\sigma_2}}}} \tag{C.7}$$

for the "northern" patch with $z_1 \neq 0$, and $w' = w - i \log z$, $z' = \frac{1}{z}$ to cover the "southern" patch with $z_2 \neq 0$. These coordinates satisfy various identifications that follow from their definition and from (C.1). In the following we will focus on the case $\tau_2 = \sigma_2$ for simplicity. In the general case, it is actually more convenient to use the z_1, z_2 coordinates and the metric (C.3).[27]

C.1 Background fields in w, z coordinates, for $|p| = |q|$

In the special case $\tau_2 = \sigma_2$, the w, z coordinates (C.7) simplify to (5.43). The metric then takes the canonical form (2.4) with

$$\Omega = 1 , \qquad h = -\frac{i \bar z}{1+|z|^2} , \qquad \bar h = \frac{i z}{1+|z|^2} , \qquad c = \frac{1}{1+|z|^2} . \tag{C.8}$$

[26] Upon a change of holomorphic coordinates, s shifts by a phase (it transforms as $pg^{-\frac{1}{4}}$, with p a section of the canonical line bundle $\mathcal{K}$). That phase is removed by a $U(1)_R$ gauge transformation, which affects the boundary conditions of R-charged fields.

[27] One can of course apply the formalism of section 5 in any coordinate system, solving the eingenvalue equations (5.30). Doing that in the z_1, z_2 coordinates, one recovers (5.57). We will not present the details here, as it does not give anything new with respect to the $|p| = |q|$ subcase.

This gives the Hermitian metric (5.46) considered in the main text. The background fields V_μ, A_μ take the simple form

$$\begin{aligned} V_\mu dx^\mu &= \frac{1}{2}(1+\kappa)(dw+hdz) + \frac{1}{2}(d\bar{w}+\bar{h}d\bar{z})\,, \\ A_\mu dx^\mu &= \frac{1}{2}hdz + \frac{1}{2}\bar{h}d\bar{z} + \frac{1}{2}\left(1+\frac{3}{2}\kappa\right)(dw+hdz)\,. \end{aligned} \tag{C.9}$$

The two Killing spinors take the form (2.7). In the special case $\tau_1 = -\sigma_1$, we can have four Killing spinors upon setting $\kappa = -2$ [1, 6].[28] The present background has a $\mathrm{U}(1)^3$ isometry in general, corresponding to the three Killing vectors $\partial_x, \partial_\varphi, \partial_\chi$. In these w, z coordinates, R-charged fields satisfy the twisted boundary conditions

$$\Phi \sim e^{\pi i r(\tau_1-\sigma_1)}\Phi\,,\ (x\sim x+2\pi)\,, \qquad \Phi\sim e^{-2\pi i r}\Phi\,,\ (\varphi\sim\varphi+2\pi)\,, \tag{C.10}$$

as we circle the x and φ coordinates, respectively. (Note that the S^1 spanned by φ does not shrink on the w, z patch. A similar boundary condition for $\chi \sim \chi + 2\pi$ holds on the w', z' patch.) Correspondingly, the momentum operators along the $\mathrm{U}(1)^3$ isometries are given by

$$P_x = -i\partial_x - \frac{r}{2}(\tau_1-\sigma_1)\,, \qquad P_\varphi = -i\partial_\varphi + r\,, \qquad P_\chi = -i\partial_\chi\,, \tag{C.11}$$

for R-charged fields. This gives (5.49) in the w, z coordinates.

We also consider a background gauge field for a $\mathrm{U}(1)_f$ symmetry as discussed in section 2.2. On $S^3\times S^1$, we can have background gauge fields corresponding to a one-dimensional family of holomorphic line bundles [3],

$$a_\mu dx^\mu = -\frac{1}{2}(a_r+ia_i)\omega^{1,0} - \frac{1}{2}(a_r-ia_i)\omega^{0,1}\,. \tag{C.12}$$

Here $\omega^{1,0}$ is an element of $H^{1,0}(\mathcal{M}_4^{p,q})$, which can be taken as $\omega^{1,0} = \partial(-2x)$. This gives (5.47) in the main text. The parameter a_r is a flat connection on the S^1, while the combination $a_r - ia_i$ is the holomorphic line bundle modulus entering the supersymmetric partition function [3].

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