Defects, non-abelian t-duality, and the Fourier-Mukai transform of the Ramond-Ramond fields

Eva Gevorgyan^a and Gor Sarkissian^{a,b}

^a*Department of Theoretical Physics, Yerevan State University,
Alex Manoogian 1, 0025 Yerevan*

^b*The Abdus Salam ICTP,
Strada Costiera 11, 34014 Trieste*

E-mail: yevea921228@gmail.com, gor.sarkissian@ysu.am

ABSTRACT: We construct topological defects generating non-abelian T-duality for isometry groups acting without isotropy. We find that these defects are given by line bundles on the correspondence space with curvature which can be considered as a non-abelian generalization of the curvature of the Poincarè bundle. We show that the defect equations of motion encode the non-abelian T-duality transformation. The Fourier-Mukai transform of the Ramond-Ramond fields generated by the gauge invariant flux of these defects is studied. We show that it provides elegant and compact way of computation of the transformation of the Ramond-Ramond fields under the non-abelian T-duality.

KEYWORDS: D-branes, Conformal Field Models in String Theory, String Duality

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1 Introduction

It is well known that dualities play very important role in String theory. Among them T-duality (for review see [1]) is the most well studied one and has enormous significance from the point of view applications as well as from the conceptual point of view. Massless spectrum of String Theory contains NS sector fields $G_{\mu\nu}$, $B_{\mu\nu}$, and the dilaton Φ , and the RR fields $\mathcal{G}_{\mu_1\dots\mu_p}$. Transformations of the NS sector fields were found by Buscher [2]. To derive the transformation of the RR fields several approaches were developed: via dimensional reduction [3, 4], vertex operators for RR fields [5], the gravitino supersymmetry transformation [6], pure spinor formalism [7]. It was found in [5–7], that the transformation rules of the RR fields under T-duality are encoded in the rotation of the bispinor field formed by the RR fields:

$$\hat{P} = P\Omega^{-1} \quad (1.1)$$

where $P = \frac{e^\Phi}{2} \sum_k \frac{1}{k!} \mathcal{G}_{\mu_1\dots\mu_k} \Gamma^{\mu_1\dots\mu_k}$ and k runs the values $k = 1, 3 \dots 9$ in the case of IIB, and the values $k = 0, 2 \dots 10$ in the case of IIA. The curved indices Gamma matrices are defined as usual by contracting with the tetrads e_μ^A . The matrix Ω is spinor representation of the relative twist between the left and right movers. For example for T-duality in the direction of coordinate 1, it is spinor representation of the parity operator in the direction 1: $\Gamma^1\Gamma_{11}$. On the other hand it was suggested in [8] the topological rule of the RR field transformation, which sometimes called Fourier-Mukai transform:

$$\hat{\mathcal{G}} = \int_{T^n} \mathcal{G} \wedge e^{\mathcal{F}} = \int_{T^n} \mathcal{G} \wedge e^{\hat{B}-B+\sum_{i=1}^n d\hat{t}_i \wedge dt^i} \quad (1.2)$$

The integral here is the fiberwise integration, $\mathcal{G}$ is the sum of gauge invariant RR fields, having even and odd ranks in IIA and IIB correspondingly, t^i and $\hat{t}_i$ are coordinates on

T^n and dual $\hat{T}^n$ correspondingly. This formula is explained in detail in section 4.1. The formula (1.2) provides very convenient and elegant way to compute the RR fields transformation. Also as it was noted in [9], the expression (1.2) guarantees that if RR fields $\mathcal{G}$ satisfy supergravity Bianchi identity, the dual fields $\hat{\mathcal{G}}$ satisfy them as well. One can understand the expression (1.2), remembering [10] that T-duality can be elevated to K-theory as Fourier-Mukai transform (which is also related to the Nahm transform for instantons) having as a kernel the Poincarè bundle. The two-form $\sum_{i=1}^n d\hat{t}_i \wedge dt^i$ appearing in exponent is curvature of the Poincarè line bundle. On other hand it was shown in [11] that the two-form $\mathcal{F}$ in the exponent of (1.2) is gauge invariant flux of topological defect performing T-duality.

Recently in papers [12–16], RR fields transformation under non-abelian T-duality [17–23] was studied. To derive dual forms in these works were used the formula (1.1), where now Ω is determined by the relative twist of left and right movers under the non-abelian T-duality, and the dimensional reduction.

In this paper we take the route of the generalization of the Fourier-Mukai transform to the non-abelian T-duality for isometry groups acting without isotropy. To reach this aim we use the following strategy. As we mentioned, the two-form $\mathcal{F}$, appearing in the exponent of the Fourier-Mukai transform, is a gauge invariant flux of the defect performing the corresponding transformations. Hence at the beginning we construct defect performing non-abelian T-duality. Then we use the derived in this way flux to calculate the RR fields transformation. The key result of this paper is that the non-abelian Fourier-Mukai transform of the RR fields reads:

$$\hat{\mathcal{G}} = \int_G \mathcal{G} \wedge e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2} x^a f_{bc}^a L^b \wedge L^c} \quad (1.3)$$

Here L^a and f_{bc}^a are Maurer-Cartan forms and structure constants of the isometry group respectively, x^a are dual coordinates.

This paper is organized in the following way.

In section 2 we review non-abelian T-duality. In particular we recall the duality relations and demonstrate general formulas for the case of SU(2) principal chiral model. In section 3 we review actions with defects, present defect performing non-abelian T-duality, and show that the defect equations of motion reproduce the duality relations derived in section 1. In section 4.1 we review the Fourier-Mukai transform formula, and recall how it works for abelian T-duality. In section 4.2 using the flux of non-abelian T-duality defect derived in section 3, we derive the Fourier-Mukai transform formula for non-abelian T-duality, and compute the RR fields transformation for SU(2) isometry group. We obtain that our results in agreement with that of [12, 14].

2 Non-abelian T-duality

Here we recall and collect some facts on non-abelian T-duality for isometry groups acting without isotropy [15, 18]. Suppose we have a target space with an isometry group G , with generators T^a , structure constants f_{bc}^a , and coordinates θ^a , and in some coordinates the

metric and the NS two-form take the form

$$ds^2 = G_{\mu\nu}(Y)dY^\mu dY^\nu + 2G_{\mu a}(Y)\Omega_k^a dY^\mu d\theta^k + G_{ab}(Y)\Omega_m^a \Omega_k^b d\theta^m d\theta^k \quad (2.1)$$

$$B = \frac{1}{2}B_{\mu\nu}(Y)dY^\mu \wedge dY^\nu + B_{\mu a}(Y)\Omega_k^a dY^\mu \wedge d\theta^k + \frac{1}{2}B_{ab}(Y)\Omega_m^a \Omega_k^b d\theta^m \wedge d\theta^k \quad (2.2)$$

where Ω_k^a are components of the right-invariant Maurer-Cartan forms L^a :

$$dgg^{-1} = L^a T_a = \Omega_k^a d\theta^k T_a \quad (2.3)$$

The background fields depend on group coordinates θ^a only through the Maurer-Cartan forms. Also as it is clear from the notations they can depend on some spectator coordinates Y . Since $d(dgg^{-1}) = dgg^{-1} \wedge dgg^{-1}$, L^a and Ω_k^a satisfy the Maurer-Cartan relations

$$dL^a = \frac{1}{2}f_{bc}^a L^b L^c \quad (2.4)$$

and

$$\partial_i \Omega_j^c - \partial_j \Omega_i^c = f_{ab}^c \Omega_i^a \Omega_j^b \quad (2.5)$$

The corresponding Lagrangian density is

$$L = Q_{\mu\nu} \partial Y^\mu \bar{\partial} Y^\nu + Q_{\mu a} \Omega_k^a \partial Y^\mu \bar{\partial} \theta^k + Q_{a\mu} \Omega_k^a \partial \theta^k \bar{\partial} Y^\mu + Q_{ab} \Omega_m^a \Omega_k^b \partial \theta^m \bar{\partial} \theta^k \quad (2.6)$$

where

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad Q_{\mu a} = G_{\mu a} + B_{\mu a}, \quad Q_{a\mu} = G_{a\mu} + B_{a\mu}, \quad Q_{ab} = G_{ab} + B_{ab}. \quad (2.7)$$

To find the dual action one can use the Buscher method and write the Lagrangian (2.6) in the form

$$L = Q_{\mu\nu} \partial Y^\mu \bar{\partial} Y^\nu + Q_{\mu a} \partial Y^\mu \bar{A}^a + Q_{a\mu} A^a \bar{\partial} Y^\mu + Q_{ab} A^a \bar{A}^b - x^a (\partial \bar{A}^a - \bar{\partial} A^a - f_{bc}^a A^b \bar{A}^c) \quad (2.8)$$

The equations of motion of the Lagrangian multiplier x^a force the field strength $F_{+-}^a = \partial \bar{A}^a - \bar{\partial} A^a - f_{bc}^a A^b \bar{A}^c$ to vanish. The solution to these constraints is

$$A^a = \Omega_k^a \partial \theta^k \quad \text{and} \quad \bar{A}^a = \Omega_k^a \bar{\partial} \theta^k. \quad (2.9)$$

Putting this solution into (2.8) yields the original action (2.6). On the other hand integrating out gauge fields A^a one obtains:

$$\begin{aligned} M_{ba}^{-1} (Q_{\mu b} \partial Y^\mu + \partial x^b) &= -A^a \\ M_{ab}^{-1} (\bar{\partial} x^b - Q_{b\mu} \bar{\partial} Y^\mu) &= \bar{A}^a \end{aligned} \quad (2.10)$$

where

$$M_{ab} = Q_{ab} + x^c f_{ab}^c \quad (2.11)$$

Inserting expressions (2.10) back in (2.8) we find the dual action:

$$\hat{L} = \hat{E}_{\mu\nu} \partial Y^\mu \bar{\partial} Y^\nu + \hat{E}_{\mu a} \partial Y^\mu \bar{\partial} x^a + \hat{E}_{a\mu} \partial x^a \bar{\partial} Y^\mu + \hat{E}_{ab} \partial x^a \bar{\partial} x^b \quad (2.12)$$

where

$$\begin{aligned}\hat{E}_{\mu\nu} &= Q_{\mu\nu} - Q_{\mu a} M_{ab}^{-1} Q_{b\nu} \\ \hat{E}_{\mu a} &= Q_{\mu b} M_{ba}^{-1} \\ \hat{E}_{a\mu} &= -Q_{b\mu} M_{ab}^{-1} \\ \hat{E}_{ab} &= M_{ab}^{-1}\end{aligned}\tag{2.13}$$

Equating (2.9) and (2.10), one gets the duality relations for non-abelian T-duality [22, 23]

$$M_{ba}^{-1}(Q_{\mu b}\partial Y^\mu + \partial x^b) = -\Omega_k^a \partial \theta^k\tag{2.14}$$

$$M_{ab}^{-1}(\bar{\partial} x^b - Q_{b\mu}\bar{\partial} Y^\mu) = \Omega_k^a \bar{\partial} \theta^k\tag{2.15}$$

Separating in (2.13) symmetric and antisymmetric parts we derive metric and NS form of the dual theory:

$$\hat{G}_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}(Q_{\mu a} M_{ab}^{-1} Q_{b\nu} + Q_{\nu a} M_{ab}^{-1} Q_{b\mu})\tag{2.16}$$

$$\hat{G}_{\mu a} = \frac{1}{2}(Q_{\mu b} M_{ba}^{-1} - Q_{b\mu} M_{ab}^{-1})\tag{2.17}$$

$$\hat{G}_{ab} = \frac{1}{2}(M_{ab}^{-1} + M_{ba}^{-1})\tag{2.18}$$

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{2}(Q_{\mu a} M_{ab}^{-1} Q_{b\nu} - Q_{\nu a} M_{ab}^{-1} Q_{b\mu})\tag{2.19}$$

$$\hat{B}_{\mu a} = \frac{1}{2}(Q_{\mu b} M_{ba}^{-1} + Q_{b\mu} M_{ab}^{-1})\tag{2.20}$$

$$\hat{B}_{ab} = \frac{1}{2}(M_{ab}^{-1} - M_{ba}^{-1})\tag{2.21}$$

Let us recall the SU(2) Principal Chiral Model [12, 24, 25]

$$S(g) = \int k \text{Tr}(g^{-1} \partial g g^{-1} \bar{\partial} g)\tag{2.22}$$

where $g \in \text{SU}(2)$. The metric in the Euler coordinates is

$$ds^2 = k(d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos \theta d\phi d\psi)\tag{2.23}$$

and there is no NS two-form. To obtain the dual background one should compute M_{ab}^{-1} matrix.

Denoting the dual coordinates x^a , $a = 1, 2, 3$, one has here

$$M_{ab} = k\delta_{ab} + \epsilon_{abc}x_c\tag{2.24}$$

and

$$M_{ab}^{-1} = \frac{1}{k^2 + r^2} \left(k\delta_{ab} + \frac{x_a x_b}{k} - \epsilon_{abc}x_c \right)\tag{2.25}$$

Separating symmetric and antisymmetric parts and denoting $r^2 = x^a x^a$ one gets

$$\hat{G}_{ab} = \frac{1}{k^2 + r^2} \left(k\delta_{ab} + \frac{x_a x_b}{k} \right) \quad (2.26)$$

$$\hat{B}_{ab} = -\frac{1}{k^2 + r^2} \epsilon_{abc} x_c \quad (2.27)$$

$$\hat{\Phi} = -\frac{1}{2} \log(k^3 + kr^2) \quad (2.28)$$

and hence one has

$$\hat{ds}^2 = \frac{dr^2}{k} + \frac{kr^2}{k^2 + r^2} ds^2(S^2) \quad (2.29)$$

$$\hat{B} = -\frac{r^3}{k^2 + r^2} \text{Vol}(S^2) \quad (2.30)$$

3 Non-abelian T-duality via defects

3.1 Sigma model with defect

Defects in two-dimensional quantum field theory are lines separating different quantum field theories.

Conformal defects are required to satisfy [26]

$$T^{(1)} - \bar{T}^{(1)} = T^{(2)} - \bar{T}^{(2)} \quad (3.1)$$

Topological defects satisfy [27]

$$T^{(1)} = T^{(2)}, \quad \bar{T}^{(1)} = \bar{T}^{(2)} \quad (3.2)$$

Since the stress-energy tensor is a generator of diffeomorphisms, condition (3.2) implies that the defect is invariant under a deformation of the line to which it is attached. A fusion between a defect and a boundary is defined in the case of topological defects, since the defect can be moved to the boundary without changing the correlator [28].

Let us briefly review the construction of an action with defects [11, 29]. Let us locate the defect at the vertical line Z defined by the condition $\sigma = 0$. Denote by Σ_1 the left half-plane ($\sigma \leq 0$), and by Σ_2 the right half-plane ($\sigma \geq 0$), and a pair of maps $X : \Sigma_1 \rightarrow M_1$ and $\tilde{X} : \Sigma_2 \rightarrow M_2$, where M_1 and M_2 are the target spaces for the two theories. Assume we have a submanifold Q of the product of target spaces: $Q \subset M_1 \times M_2$, with a connection one-form A , and a combined map:

$$\begin{aligned} \Phi : Z &\rightarrow M_1 \times M_2 \\ z &\mapsto (X(z), \tilde{X}(z)) \end{aligned} \quad (3.3)$$

which takes values in the submanifold Q .

In this setup one can write the action:

$$S = \int_{\Sigma_1} L_1 dx^+ dx^- + \int_{\Sigma_2} L_2 dx^+ dx^- + \int_Z \Phi^* A \quad (3.4)$$

where

$$L_i = E_{mn}^{(i)} \partial X^m \bar{\partial} X^n, \quad E^{(i)} = G^{(i)} + B^{(i)}, \quad i = 1, 2, \quad (3.5)$$

and

$$x^\pm = \tau \pm \sigma. \quad (3.6)$$

3.2 Defects implementing non-abelian T-duality

Consider the action (3.4) with a defect as in the situation above, where M_1 is the target space with the coordinates (Y^μ, θ^k) and has metric and NS 2-form given by (2.1) and (2.2), M_2 is the space with the coordinates (Y^μ, x^a) and with metric and 2-form given by (2.16)–(2.21), and Q is the correspondence space, with the coordinates (Y^μ, θ^k, x^a) , the connection

$$A = -x^a L^a = -x^a \Omega_k^a d\theta^k \quad (3.7)$$

and the curvature

$$F = dA = -\left(dx^a L^a + \frac{1}{2} x^a f_{bc}^a L^b L^c\right) \quad (3.8)$$

To derive (3.8) we used the Maurer-Cartan relation (2.4). By other words we take as L_1 in (3.4) the L given by (2.6), and as L_2 the $\hat{L}$ given by (2.12).

The conditions (3.7) and (3.8) define a line bundle $\mathcal{P}^{\text{NA}}$ over Q , with the curvature (3.8), which can be called non-abelian Poincaré line bundle. In this case the action (3.4) yields the following equations of motion on the defect line:

$$Q_{\mu a} \partial Y^\mu + Q_{ba} \Omega_m^b \partial \theta^m - Q_{a\mu} \bar{\partial} Y^\mu - Q_{ab} \Omega_m^b \bar{\partial} \theta^m = -x^c \Omega_m^b f_{ba}^c \partial_\tau \theta^m - \partial_\tau x^a \quad (3.9)$$

$$\hat{E}_{\mu a} \partial Y^\mu + \hat{E}_{ba} \partial x^b - \hat{E}_{a\mu} \bar{\partial} Y^\mu - \hat{E}_{ab} \bar{\partial} x^b = -\Omega_k^a \partial_\tau \theta^k. \quad (3.10)$$

$$\begin{aligned} Q_{\mu\alpha} \partial Y^\mu + Q_{a\alpha} \Omega_k^a \partial \theta^k - Q_{\alpha\mu} \bar{\partial} Y^\mu - Q_{\alpha a} \Omega_k^a \bar{\partial} \theta^k \\ - \hat{E}_{\mu\alpha} \partial Y^\mu - \hat{E}_{a\alpha} \partial x^a + \hat{E}_{\alpha\mu} \bar{\partial} Y^\mu + \hat{E}_{\alpha a} \bar{\partial} x^a = 0 \end{aligned} \quad (3.11)$$

In the first line we used the second of the Maurer-Cartan relations (2.5).

Solving equations (3.9)–(3.11) we obtain the duality relations of non-abelian T-duality (2.14) and (2.15)

$$Q_{\mu a} \partial Y^\mu + M_{ba} \Omega_m^b \partial \theta^m = -\partial x^a \quad (3.12)$$

$$Q_{a\mu} \bar{\partial} Y^\mu + M_{ab} \Omega_m^b \bar{\partial} \theta^m = \bar{\partial} x^a \quad (3.13)$$

$$M_{ba}^{-1} (Q_{\mu b} \partial Y^\mu + \partial x^b) = -\Omega_m^a \partial \theta^m \quad (3.14)$$

$$M_{ab}^{-1} (\bar{\partial} x^b - Q_{b\mu} \bar{\partial} Y^\mu) = \Omega_m^a \bar{\partial} \theta^m \quad (3.15)$$

Using expressions (2.13) and the duality relations (3.12) and (3.13) we obtain

$$T = \hat{T} \quad \text{and} \quad \bar{T} = \hat{\bar{T}} \quad (3.16)$$

where

$$T = G_{\mu\nu} \partial Y^\mu \partial Y^\nu + 2G_{\mu a} \Omega_k^a \partial Y^\mu \partial \theta^k + G_{ab} \Omega_m^a \Omega_k^b \partial \theta^m \partial \theta^k \quad (3.17)$$

$$\bar{T} = G_{\mu\nu} \bar{\partial} Y^\mu \bar{\partial} Y^\nu + 2G_{\mu a} \Omega_k^a \bar{\partial} Y^\mu \bar{\partial} \theta^k + G_{ab} \Omega_m^a \Omega_k^b \bar{\partial} \theta^m \bar{\partial} \theta^k \quad (3.18)$$

$$\hat{T} = \hat{G}_{\mu\nu} \partial Y^\mu \partial Y^\nu + 2\hat{G}_{\mu a} \partial Y^\mu \partial x^a + \hat{G}_{ab} \partial x^a \partial x^b \quad (3.19)$$

$$\hat{\bar{T}} = \hat{G}_{\mu\nu} \bar{\partial} Y^\mu \bar{\partial} Y^\nu + 2\hat{G}_{\mu a} \bar{\partial} Y^\mu \bar{\partial} x^a + \hat{G}_{ab} \bar{\partial} x^a \bar{\partial} x^b \quad (3.20)$$

what means that the defect is topological.

4 Transformation of the Ramond-Ramond fields

4.1 Defects and Fourier-Mukai transform

As we mentioned, a topological defect can be fused with a boundary, producing new boundary condition from the old one. From the other side boundary conditions correspond to D-branes, which can be characterized by their RR charges or by elements of the K-theory. Therefore an action of the defect on the Ramond-Ramond charges and K-theory elements can be defined. It is expected [11, 29–33] that the action should be “Fourier-Mukai” type with a kernel given by the exponent of the gauge invariant flux $\mathcal{F} = \hat{B} - B + F$ on defect, or by the defect bundle $\mathcal{P}$ correspondingly. Saying Fourier-Mukai type transform we mean the following construction.¹ Suppose we can associate to a target space X a ring $D(X)$ (e.g. cohomology groups, K-theory groups, etc.), in a way that for a map $p : X_1 \rightarrow X_2$ exist pullback $p^* : D(X_2) \rightarrow D(X_1)$ and pushforward $p_* : D(X_1) \rightarrow D(X_2)$ maps. Assume one has an element $K \in D(X \times Y)$. Now we can define the Fourier-Mukai transform, $FM(F) : D(X) \rightarrow D(Y)$ with the kernel K by the formula:

$$FM(F) = p_*^Y (K \cdot p^{X*} F) \quad (4.1)$$

where $F \in D(X)$, and $p^X : X \times Y \rightarrow X$, $p^Y : X \times Y \rightarrow Y$ are projections. One can see that usual Fourier transform has this form with the Riemann integral as pushforward map.

Consider for example the T-duality transformation of the RR fields.

It is found in [8] that the Ramond-Ramond fields of the theory on $T^n \times M$ and those of the T-dual theory on $\hat{T}^n \times M$ are related by a Fourier-Mukai transform:

$$\hat{\mathcal{G}} = \int_{T^n} \mathcal{G} \wedge e^{\mathcal{F}} = \int_{T^n} \mathcal{G} \wedge e^{\hat{B} - B + \sum_{i=1}^n dt_i \wedge dt^i} \quad (4.2)$$

Here B is the Neveu-Schwarz B -field and $\mathcal{G} = \sum_p \mathcal{G}_p$ is the sum of gauge invariant RR field strength where the sum is over $p = 0, 2, 4, \dots$ for Type IIA and $p = 1, 3, \dots$ for Type IIB. The integrand in (4.2) is considered as a differential form on the space $M \times T^n \times \hat{T}^n$ and pushforward map is fiberwise integration $\int_{T^n}$, mapping differential forms on $M \times T^n \times \hat{T}^n$ to differential forms on $M \times \hat{T}^n$. The integral acts on the differential forms of the highest degree n in dt_i and sets to zero differential forms of lower degree in dt_i [36]:

$$f(x, \hat{t}_i, t^i) p^* \omega \wedge dt_{i_1} \wedge \dots \wedge dt_{i_r} \mapsto 0, \quad r < n \quad (4.3)$$

$$f(x, \hat{t}_i, t^i) p^* \omega \wedge dt_1 \wedge \dots \wedge dt_n \mapsto \omega \int_{T^n} f(x, \hat{t}_i, t^i) dt_1 \wedge \dots \wedge dt_n$$

Here p is the projection $M \times T^n \times \hat{T}^n \rightarrow M \times \hat{T}^n$, ω is a differential form on $M \times \hat{T}^n$, $f(x, \hat{t}_i, t^i)$ is an arbitrary function and x denotes a point in M .

Since the gauge invariant flux $\mathcal{F}$ satisfies the condition

$$d\mathcal{F} = \hat{H} - H \quad (4.4)$$

¹The paragraph below is neither a rigorous nor a precise definition of the Fourier-Mukai transform, and only has a goal to outline basic ideas. For the rigorous definitions see [34, 35] and references therein.

and the exterior differentiation d commutes with the fiberwise integration [36], one can show that the dual forms satisfy the equation [9]:

$$(d - \hat{H}) \wedge \hat{\mathcal{G}} = \int_{T^n} e^{\mathcal{F}} \wedge (d - H) \wedge \mathcal{G} \quad (4.5)$$

This implies that $d_H = d - H$ closed forms mapped to $d_{\hat{H}} = d - \hat{H}$ closed form. This means that if the RR fields $\mathcal{G}$ satisfy the supergravity Bianchi identity, so do the dual RR fields $\hat{\mathcal{G}}$.

The kernel of the Fourier-Mukai transform (4.2) is indeed the exponent of the gauge invariant combination of the B fields and the flux $\sum_{i=1}^n d\hat{t}_i \wedge dt^i$ of the T-duality defect given by the Poincaré bundle [11] (many detailed explanations on the T-duality defects can be also found in [33])

$$e^{\mathcal{F}} = e^{\hat{B} - B + \sum_{i=1}^n d\hat{t}_i \wedge dt^i} \quad (4.6)$$

As preparation to the calculations for the case of non-abelian T-duality in the next section, now we show how the formula (4.2) produces the known transformation rules of the Ramond-Ramond fields for the case of the abelian T-dualization in the direction of one coordinate, which we choose to be the first one. Remember that in this case the Buscher transformation rules of the metric G and NS two-form B are:

$$\begin{aligned} \hat{G}_{11} &= \frac{1}{G_{11}} \\ \hat{G}_{1M} &= \frac{B_{1M}}{G_{11}} \\ \hat{B}_{1M} &= \frac{G_{1M}}{G_{11}} \\ \hat{G}_{MN} &= G_{MN} - \frac{1}{G_{11}}(G_{M1}G_{1N} + B_{1N}B_{M1}) \\ \hat{B}_{MN} &= B_{MN} - \frac{1}{G_{11}}(G_{M1}B_{1N} + G_{1N}B_{M1}) \end{aligned} \quad (4.7)$$

Here capital latin letters run from 2 to the dimension of the target spaces. With (4.7) at hand (4.2) takes the form:

$$\hat{\mathcal{G}} = \int_{S^1} \mathcal{G} \wedge e^{(A_1 + d\hat{t}^1) \wedge (A_2 + dt^1)} = \int_{S^1} \mathcal{G} \wedge (1 + (A_1 + d\hat{t}^1) \wedge (A_2 + dt^1)) \quad (4.8)$$

where

$$A_1 = B_{1N}dX^N \quad \text{and} \quad A_2 = \frac{G_{1N}}{G_{11}}dX^N \quad (4.9)$$

Taking $\mathcal{G}$ in the form

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(1)} \wedge dt^1 \quad (4.10)$$

and using the rules (4.3), one obtains

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}^{(0)} + \hat{\mathcal{G}}^{(1)} \wedge d\hat{t}^1 \quad (4.11)$$

where

$$\hat{\mathcal{G}}^{(0)} = \mathcal{G}^{(1)} + \mathcal{G}^{(0)} \wedge A_1 + \mathcal{G}^{(1)} \wedge A_1 \wedge A_2 \quad (4.12)$$

and

$$\widehat{\mathcal{G}}^{(1)} = \mathcal{G}^{(0)} - \mathcal{G}^{(1)} \wedge A_2 \quad (4.13)$$

These are indeed the RR fields transformation rules under the abelian T-duality [4].

4.2 Non-abelian T-duality Fourier-Mukai transform of the Ramond-Ramond fields

Taking into account that the curvature of the defect generating the non-abelian T-duality is given by the formula (3.8), the Fourier-Mukai transform of the RR fields takes the form:

$$\widehat{\mathcal{G}} = \int_G \mathcal{G} \wedge e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2} x^a f_{bc}^a L^b \wedge L^c} \quad (4.14)$$

Here we apply this formula to the case of background considered in [12, 14], namely:

$$ds^2 = ds^2(M_7) + k(Y) ds^2(S^3) \quad (4.15)$$

Here M_7 is a seven-dimensional manifold, Y are coordinates on M_7 , $k(Y)$ is a function of Y . One can have also B field on M_7 . The second term is actually the $SU(2)$ principal chiral model, considered in section 2. Therefore, using formulae (2.29) and (2.30) the dual model takes the form:

$$\widehat{ds}^2 = ds_{M_7}^2(Y) + \frac{dr^2}{k} + \frac{kr^2}{k^2 + r^2} ds^2(S^2) \quad (4.16)$$

and

$$\hat{B} = B - \frac{r^3}{k^2 + r^2} \text{Vol}(S^2) \quad (4.17)$$

Consider the following RR forms:

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}_a^{(1)} \wedge L^a + \frac{1}{2} \mathcal{G}_{ab}^{(2)} \wedge L^a \wedge L^b + \mathcal{G}^{(3)} \wedge L^1 \wedge L^2 \wedge L^3 \quad (4.18)$$

Here $\mathcal{G}^{(0)}$, $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ are forms on M_7 .

Denote the forms in the exponent of (4.14) as

$$A^{(2,0)} = \hat{B} - B \quad (4.19)$$

$$A^{(1,1)} = -dx^a \wedge L^a \quad (4.20)$$

$$A^{(0,2)} = -\frac{1}{2} x^a f_{bc}^a L^b \wedge L^c \quad (4.21)$$

In this notations we indicate by the first number the degree of the form in dx^a , and by the second in L^a . Expanding the exponent and remembering that one can have at most third degree terms in the both kinds of 1-forms we get:

$$\begin{aligned} e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2} x^a f_{bc}^a L^b \wedge L^c} &= 1 + A^{(2,0)} + A^{(1,1)} + A^{(0,2)} \\ &+ \frac{1}{2} A^{(1,1)} \wedge A^{(1,1)} + A^{(2,0)} \wedge A^{(1,1)} + A^{(1,1)} \wedge A^{(0,2)} + A^{(2,0)} \wedge A^{(0,2)} \\ &+ \frac{1}{6} A^{(1,1)} \wedge A^{(1,1)} \wedge A^{(1,1)} + A^{(2,0)} \wedge A^{(1,1)} \wedge A^{(0,2)} \end{aligned} \quad (4.22)$$

Using the rules of the fiberwise integration we obtain that the dual of the first term comes from the all third order terms in L^a appearing in the expansion of the exponent:

$$\widehat{\mathcal{G}^{(0)}} = \mathcal{G}^{(0)} \wedge \omega^{(3)} \quad (4.23)$$

where

$$\omega^{(3)} = \int_G \frac{1}{6} A^{(1,1)} \wedge A^{(1,1)} \wedge A^{(1,1)} + A^{(2,0)} \wedge A^{(1,1)} \wedge A^{(0,2)} + A^{(0,2)} \wedge A^{(1,1)} \quad (4.24)$$

One can explicitly compute that

$$\frac{1}{6} A^{(1,1)} \wedge A^{(1,1)} \wedge A^{(1,1)} = dx^1 \wedge dx^1 \wedge dx^2 \wedge \text{vol}(\text{SU}(2)) \quad (4.25)$$

where we introduced $\text{vol}(\text{SU}(2)) = L^1 \wedge L^2 \wedge L^3$,

$$A^{(0,2)} \wedge A^{(1,1)} = x^a dx^a \wedge \text{vol}(\text{SU}(2)) = r dr \wedge \text{vol}(\text{SU}(2)) \quad (4.26)$$

$$A^{(2,0)} \wedge A^{(1,1)} \wedge A^{(0,2)} = -\frac{r^4 dr}{k^2 + r^2} \wedge \text{Vol}(S^2) \wedge \text{vol}(\text{SU}(2)) \quad (4.27)$$

To derive (4.27) we used the expressions (4.17) and (4.19) for $A^{(2,0)}$. Collecting all and using that $dx^1 \wedge dx^2 \wedge dx^3 = r^2 dr \wedge \text{vol}(S^2)$ we obtain

$$\widehat{\mathcal{G}^{(0)}} = \mathcal{G}^{(0)} \wedge \left(\frac{r^2 k^2 dr}{k^2 + r^2} \wedge \text{vol}(S^2) + r dr \right) \quad (4.28)$$

Similarly collecting all the second order terms in L^a in the expansion of the exponent one obtains the dual of the second term:

$$\begin{aligned} \widehat{\mathcal{G}_a^{(1)}} \wedge L^a &= \int_G \frac{1}{2} \mathcal{G}_a^{(1)} \wedge L^a \wedge A^{(1,1)} \wedge A^{(1,1)} + \mathcal{G}_a^{(1)} \wedge L^a \wedge A^{(0,2)} \\ &\quad + \int_G \mathcal{G}_a^{(1)} \wedge L^a \wedge A^{(2,0)} \wedge A^{(0,2)} \\ &= -\frac{1}{2} \epsilon_{abc} \mathcal{G}_a^{(1)} \wedge dx^b \wedge dx^c - \mathcal{G}_a^{(1)} x^a - A^{(2,0)} \wedge \mathcal{G}_a^{(1)} x^a \end{aligned} \quad (4.29)$$

Picking up the first order terms in L^a gives us the dual of the third term:

$$\begin{aligned} \widehat{\mathcal{G}_{ab}^{(2)}} \wedge L^a \wedge L^b &= \int_G \mathcal{G}_{ab}^{(2)} \wedge L^a \wedge L^b \wedge A^{(1,1)} + \mathcal{G}_{ab}^{(2)} \wedge L^a \wedge L^b \wedge A^{(1,1)} \wedge A^{(2,0)} \\ &= -\epsilon_{abc} \mathcal{G}_{ab}^{(2)} \wedge dx^c + \epsilon_{abc} \mathcal{G}_{ab}^{(2)} x^c \wedge \frac{r^2 dr}{k^2 + r^2} \wedge \text{vol}(S^2) \end{aligned} \quad (4.30)$$

And finally the dual of the last term is given by the terms not containing L^a at all:

$$\int_G \mathcal{G}^{(3)} \wedge L^1 \wedge L^2 \wedge L^3 \wedge e^{\hat{B}-B-dx^a \wedge L^a - \frac{1}{2} x^a f_{bc}^a L^b \wedge L^c} = \mathcal{G}^{(3)} + \mathcal{G}^{(3)} \wedge (\hat{B} - B) \quad (4.31)$$

Rearranging the terms in order of dx^a we can write for the non-abelian T-dual of $\mathcal{G}$:

$$\widehat{\mathcal{G}} = \widehat{\mathcal{G}^{(0)}} + \widehat{\mathcal{G}^{(1)}} + \widehat{\mathcal{G}^{(2)}} + \widehat{\mathcal{G}^{(3)}} \quad (4.32)$$

where

$$\widehat{\mathcal{G}}^{(0)} = -\mathcal{G}_a^{(1)} x^a + \mathcal{G}^{(3)} \quad (4.33)$$

$$\widehat{\mathcal{G}}^{(1)} = \mathcal{G}^{(0)} \wedge r dr - \frac{1}{2} \epsilon_{abc} \mathcal{G}_{ab}^{(2)} \wedge dx^c \quad (4.34)$$

$$\widehat{\mathcal{G}}^{(2)} = -\frac{1}{2} \epsilon_{abc} \mathcal{G}_a^{(1)} \wedge dx^b \wedge dx^c - (\hat{B} - B) \wedge \mathcal{G}_a^{(1)} x^a + \mathcal{G}^{(3)} \wedge (\hat{B} - B) \quad (4.35)$$

$$\widehat{\mathcal{G}}^{(3)} = \mathcal{G}^{(0)} \wedge \frac{r^2 k^2 dr}{k^2 + r^2} \wedge \text{Vol}(S^2) + \frac{1}{2} \epsilon_{abc} \mathcal{G}_{ab}^{(2)} x^c \wedge \frac{r^2 dr}{k^2 + r^2} \wedge \text{vol}(S^2) \quad (4.36)$$

As we have explained before, since the gauge invariant flux on the defect, which appears in the exponent of (4.14), satisfies the relation (4.4), and the exterior differentiation commutes with the fiberwise integration, the dual fields satisfy the relation:

$$(d - \hat{H}) \wedge \hat{\mathcal{G}} = \int_G e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2} x^a f_{bc}^a L^b \wedge L^c} \wedge (d - H) \wedge \mathcal{G} \quad (4.37)$$

The relation (4.37) guarantees that the hatted forms satisfy the supergravity Bianchi identity given that so do the original forms $\mathcal{G}$. In [12, 14], the non-abelian T-duality transformation of the RR fields was performed for backgrounds (4.15), using the approaches based on equation (1.1) and the dimensional reduction, with the RR fields having the form:

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(3)} \wedge L^1 \wedge L^2 \wedge L^3 \quad (4.38)$$

The results obtained in these works are in agreement with the formulae (4.33)–(4.36) for this case.

5 Discussion

One of the exciting direction of the further work is study of the elevation of the non-abelian T-duality Fourier-Mukai transform to K-theory. As we know the D-branes are elements of K-theory. The arguments of section 4.1 imply that if a brane given by an element D of the K-theory of the space M_1 , under non-abelian T-duality it is mapped to the following element of the K-theory of the dual space M_2 :

$$FM(D) = p_{2!} \left(\mathcal{P}^{NA} \otimes p_1^! D \right) \quad (5.1)$$

Here p_1 and p_2 are projections of the correspondence space Q to M_1 and M_2 correspondingly, upper and lower shrieks denote pullback and pushforward maps in K-theory and $\mathcal{P}^{NA}$ is line bundle on Q with the curvature $F = dx^a L_a + \frac{1}{2} x^a f_{bc}^a L^b L^c$ constructed in section 3.2.

Next it is interesting to use the technique developed in this paper to study Ramond-Ramond fields transformation under non-abelian T-duality for other groups than $SU(2)$.

Another important direction is to generalize these results to non-abelian T-duality with isometry group acting with isotropy.

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