Extremal horizons with reduced symmetry: hyperscaling violation, stripes, and a classification for the homogeneous case

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ABSTRACT: Classifying the zero-temperature ground states of quantum field theories with finite charge density is a very interesting problem. Via holography, this problem is mapped to the classification of extremal charged black brane geometries with anti-de Sitter asymptotics. In a recent paper [1], we proposed a Bianchi classification of the extremal near-horizon geometries in five dimensions, in the case where they are homogeneous but, in general, anisotropic. Here, we extend our study in two directions: we show that Bianchi attractors can lead to new phases, and generalize the classification of homogeneous phases in a way suggested by holography. In the first direction, we show that hyperscaling violation can naturally be incorporated into the Bianchi horizons. We also find analytical examples of "striped" horizons. In the second direction, we propose a more complete classification of homogeneous horizon geometries where the natural mathematics involves real four-algebras with three dimensional sub-algebras. This gives rise to a richer set of possible near-horizon geometries, where the holographic radial direction is non-trivially intertwined with field theory spatial coordinates. We find examples of several of the new types in systems consisting of reasonably simple matter sectors coupled to gravity, while arguing that others are forbidden by the Null Energy Condition. Extremal horizons in four dimensions governed by three-algebras or four-algebras are also discussed.

Contents

1 Introduction

There has been considerable recent interest in possible applications of holography to condensed matter systems [2–5]. Some of the most interesting phenomena arise when one "dopes" an insulator with finite charge density [6]. Related fascinating phenomena are thought to occur also in the phase diagram of QCD as one varies the chemical potential and number of light quark flavors [7]. The problem of exemplifying interesting ground states of doped quantum field theory, and perhaps even classifying them, is then a difficult but well-motivated one.

For the subset of field theories with weakly curved gravitational duals, holography maps these complicated questions about quantum dynamics to simple questions in classical general relativity. Finite temperature states in the field theory are dual to black hole geometries with planar horizons ("black branes"), and states at finite chemical potential (and charge density) map to charged black branes. The low-temperature limit of such a system — dual to the ground state of the doped field theory — is then governed by the extremal charged black brane geometry. Therefore, in holography, the task is to present interesting examples of, or perhaps to classify, extremal black brane geometries in asymptotically anti de Sitter solutions of Einstein gravity coupled to various matter fields (chosen to be typical of the content of low-energy string theory).

Studies of doped matter in AdS/CFT have yielded new near-horizon geometries with novel properties including dynamical scaling [8–11] and hyperscaling violation [12–21]. These are dual to emergent infrared phases which break Lorentz-invariance (as the doping does in the UV), but which respect spatial rotation and translation invariance. However, more typically, one expects to find emergent geometries which break more of the space-time symmetries. In condensed matter physics, for instance, phases with spin or charge density waves, stripe order, nematic order, and other more exotic orders are well known. Similar modulated phases also occur in the phase diagram of finite-density QCD.

There have been interesting holographic studies of some such spatially modulated phases — notably, studies of emergent helical order [22–30], stripe order [31–36], and more elaborate orders [37–40]. One largely open problem is to give simple analytical examples of phases realizing or even combining the various properties mentioned above (e.g. hyperscaling violation with helical order) — many cases discussed in the literature are complicated enough that numerical work is required. But a systematic classification of possible emergent geometries is also something one could hope to achieve. That this is not always hopeless is clear from the success of the Kerr-Newman classification of charged, rotating black holes in asymptotically flat four-dimensional space-time, and perhaps more relevant in this context,

the success of the Bianchi classification of homogeneous cosmological solutions. In fact, in a previous paper [1], we found that the basic strategy of the Bianchi classification in cosmology can also be used to classify homogeneous (but, in general, anisotropic) extremal black brane geometries.

The focus in [1] was the symmetry structure of the 3d spatial slices (the spatial dimensions "where the field theory lives") in asymptotically AdS_5 space. This gives rise to a classification based on real three-dimensional Lie algebras, the original Bianchi classification [41] — very readable expositions appear in [42, 43]. The generators of the Lie algebra correspond to Killing vectors in the geometry, which generate the isometry group of the corresponding spatial slices.[1]

In this paper, our goal is to apply and generalize the results of [1] in several directions. We show that by using standard ideas of dimensional reduction and the solutions of [1], one can give simple analytical examples of Bianchi horizons which exhibit hyperscaling violation. Similarly, we also find analytical examples of "striped" phases, by using techniques of dimensional reduction. And finally, we attempt to give a more general classification of the possible homogeneous, anisotropic near-horizon metrics dual to doped 3+1 dimensional field theories.

The basic generalization of the classification of [1] that remains possible can be easily explained. Even with a focus on static solutions in 5d gravity, there is a more general symmetry structure that could be relevant. The additional "radial" direction of holography could be non-trivially intertwined with the spatial field theory coordinates. This makes it clear that the relevant symmetry structure could more generally involve real four-dimensional Lie algebras with a preferred three sub-algebra (which, roughly, generates the spatial symmetry group in the dual field theory).[2] Happily, the classification of such algebras (with the relevant subalgebras) has also been accomplished, though much more recently — see [44] for a clear exposition and also [45] for the proper history. Here, we use these results to find the more general classification of extremal black brane geometries relevant for holography in 5 bulk dimensions. We also show that a few of the new symmetry structures *cannot* arise in solutions of theories that satisfy simple physical conditions, such as respecting the Null Energy Condition (NEC), and we give simple examples of some interesting new types that do not run afoul of the NEC.

The more general possibilities described above also arise in the four dimensional space-time, implying that one can find homogeneous horizons there governed by real three-dimensional algebras with two-dimensional sub-algebras — i.e., governed by the Bianchi types, but now with the symmetries acting on the "radial" direction in addition to the "field theory" spatial coordinates. We can also potentially realize the real four-algebras in four dimensional space-times. In such cases, the time-direction is nontrivially involved and as a result, we generically obtain either time-dependent metrics or stationary metrics, though static metrics occasionally arise. We describe several such examples in the penultimate section.

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[1]In [1], we also considered some examples where the three-algebra involves the time coordinate, which yield stationary space-times.

[2]The examples in [1] are a degenerate case of this structure where there is a semi-simple Lie algebra with a separate Abelian factor corresponding to the isometry of translations in the radial direction.

The organization of this paper is as follows. In section 2, we describe generic Kaluza-Klein dimensional reduction of Einstein gravity coupled to massive vector fields. In sections 3–5, we discuss how to obtain hyperscaling violating space-time metrics with reduced spatial symmetries. This is done by dimensional reduction of 5d Bianchi metrics in section 3, and by analyzing directly in 5d generalizations of the Bianchi types in section 4 and section 5. In section 6, we give examples of analytical solutions for "striped" phases, by Kaluza-Klein reducing the 5d type VII$_0$ geometries of [1]. In section 7, we change gears and discuss the four-dimensional real Lie algebras and their three-dimensional subalgebras. We also present invariant vector-fields and invariant one-forms which geometrically represent the algebra. One might hope that one could realize all of these symmetry structures in holographic space-times, where the holographic radial direction is nontrivially intertwined with field theory spatial coordinates. However, in section 8, we show that the Null Energy Condition (NEC) gives interesting constraints and rules out several of the potentially relevant algebraic structures. In section 9, we show that prototypical examples of the remaining types enumerated in section 8 do arise as solutions of simple gravity coupled to matter theories. In section 10, we describe 4d static space-times where the radial direction is nontrivially intertwined with "field theory" space-time coordinates by either three-algebras or four-algebras and also show how the NEC constrains or forbids some of these space-times. We conclude in section 11 with a brief discussion of possible future work. Some additional helpful details are relegated to appendices A–C.

2 Kaluza-Klein reduction of gravity coupled to massive vectors

In this section, we review the Kaluza-Klein reduction of gravity coupled to massive vector fields. We will use this result later at section 3 and section 6 in order to obtain hyperscaling violating and striped-phase metrics in lower dimensions.

Consider the $d+1$ dimensional Einstein-Hilbert action coupled to massive vector fields and a cosmological constant,

$$S = \frac{1}{\hat{\kappa}^2} \int d^{d+1}\hat{x} \sqrt{|\hat{g}|} \left[\hat{R} - \frac{1}{4}\hat{F}^{(i)2} - \frac{1}{4}m_i^2 \hat{A}^{(i)2} + \Lambda \right] \tag{2.1}$$

where $\hat{F}^{(i)} = d\hat{A}^{(i)}$ is the field strength corresponding to the i-th massive vector field $A^{(i)}$, and where we note that our convention for Λ requires positive Λ to support AdS space. Quantities with a $\hat{\ }$ correspond to fields in $(d+1)$−dimensions, while hatless objects are $d-$ dimensional. We now follow the calculation of [48].

Let us consider a split of the $d+1$-dimensions into $x^{\hat{\mu}} = (x^\mu, z)$, where $\hat{\mu} = 1, \cdots, d+1$ and $\mu = 1, \cdots, d$. We focus on field configurations where none of the $(d+1)$-dimensional fields depend on the coordinate z, i.e. ∂_z is a Killing vector. Consider the reduction ansatz to be:

$$d\hat{s}^2 = e^{2\alpha_1\phi(x)}ds^2 + e^{2\alpha_2\phi(x)}(dz + B_\mu(x)dx^\mu)^2 , \tag{2.2}$$

$$\hat{A}^{(i)}(x,z) = A_\mu^{(i)}(x)dx^\mu + \chi^{(i)}(x)dz = \left(A^{(i)}(x) - \chi^{(i)}(x)B(x)\right) + \chi^{(i)}(x)(dz + B) , \tag{2.3}$$

where $B = B_\mu dx^\mu$. The determinant of the metric satisfies,

$$\sqrt{|\hat{g}|} = e^{(d\alpha_1 + \alpha_2)\phi}\sqrt{|g|} \ . \tag{2.4}$$

The field strength is given by,

$$\hat{F}^{(i)} = d\hat{A}^{(i)} = \left(dA^{(i)} - d\chi^{(i)} \wedge B\right) + d\chi^{(i)} \wedge (dz + B) \tag{2.5}$$

$$= F^{(i)} + d\chi^{(i)} \wedge (dz + B) \ , \tag{2.6}$$

where $F^{(i)} = dA^{(i)} - d\chi^{(i)} \wedge B$. The vierbein basis is related as,

$$\hat{E}^a = e^{\alpha_1\phi}E^a \ ; \quad \hat{E}^{\bar{z}} = e^{\alpha_2\phi}(dz + B) \ , \tag{2.7}$$

where E^A is the vierbein in $(d+1)$-dimensions, and index $A = (a, \bar{z})$. Now (we will drop the index i for convenience)

$$\hat{F} = \frac{1}{2}\hat{F}_{AB}\hat{E}^A \wedge \hat{E}^B \tag{2.8}$$

$$= \frac{1}{2}e^{2\alpha_1\phi}\hat{F}_{ab}E^a \wedge E^b + e^{(\alpha_1 + \alpha_2)\phi}\hat{F}_{a\bar{z}}E^a \wedge (dz + B) \tag{2.9}$$

$$= \frac{1}{2}F_{ab}E^a \wedge E^b + C_a E^a \wedge (dz + B) \ , \tag{2.10}$$

where $C = d\chi$,

$$\hat{F}_{ab} = e^{-2\alpha_1\phi}F_{ab} \ ; \quad \hat{F}_{a\bar{z}} = e^{-(\alpha_1 + \alpha_2)\phi}C_a \ . \tag{2.11}$$

The components of the potential are related as

$$\hat{A}_a = e^{-\alpha_1\phi}(A - \chi B) \ ; \quad \hat{A}_{\bar{z}} = e^{-\alpha_2\phi}\chi \ . \tag{2.12}$$

So the kinetic term of the gauge field gives

$$-\frac{1}{4}\sqrt{|\hat{g}|}\hat{F}^2 = -\frac{1}{4}\sqrt{|g|}e^{((d-4)\alpha_1 + \alpha_2)\phi}F^2 - \frac{1}{2}\sqrt{|g|}e^{((d-2)\alpha_1 - \alpha_2)\phi}C^2 \ . \tag{2.13}$$

Similarly, the mass term satisfies

$$-\frac{1}{4}m^2\hat{A}^2 = -\frac{1}{4}m^2e^{-2\alpha_1\phi}(A - \chi B)^2 - \frac{1}{4}m^2e^{-2\alpha_2\phi}\chi^2 \ , \tag{2.14}$$

so

$$-\sqrt{|\hat{g}|}\frac{1}{4}m^2\hat{A}^2 = -\frac{1}{4}m^2e^{[(d-2)\alpha_1 + \alpha_2]\phi}(A - \chi B)^2 - \frac{1}{4}m^2e^{(d\alpha_1 - \alpha_2)\phi}\chi^2 \ . \tag{2.15}$$

Now the gravitational part of the action reduces to [48]

$$\sqrt{|\hat{g}|}\hat{R} = \sqrt{|g|}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(d-1)\alpha_1\phi}H^2\right) \ , \tag{2.16}$$

where $H = dB$ and to write the gravitational action in the above form, we have used the identities

$$\alpha_2 = -(d-2)\alpha_1 \ , \qquad \alpha_1^2 = \frac{1}{2(d-1)(d-2)} \ . \tag{2.17}$$

These relations guarantee both that the Weyl rescaling puts the gravitational action in Einstein frame, and that the scalar field has a canonical kinetic term.

In summary, the total dimensionally reduced action becomes:

$$S = \frac{1}{\kappa^2} \int d^d \hat{x} \sqrt{|g|} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(d-1)\alpha_1\phi}H^2 - \frac{1}{4}e^{-2\alpha_1\phi}F^{(i)2} - \frac{1}{2}e^{2(d-2)\alpha_1\phi}(\partial\chi^{(i)})^2 \right.$$
$$\left. - \frac{1}{4}m_i^2(A^{(i)} - \chi^{(i)}B)^2 - \frac{1}{4}m^2 e^{2(d-1)\alpha_1\phi}\chi^2 + \Lambda e^{2\alpha_1\phi} \right] . \tag{2.18}$$

3 Hyperscaling violation via Kaluza-Klein reduction

There has been considerable recent interest in the geometries that include a hyperscaling violation exponent $\theta \neq 0$ [12–21]. This is in part because, in some specific cases, they share many of the properties of theories with a Fermi surface [16]. The cases studied in the literature have all enjoyed standard spatial rotation and translation symmetries. Here, we demonstrate that it is simple to find homogeneous but anisotropic solutions which also enjoy $\theta \neq 0$. These can be duals to ordered phases with hyperscaling violation.

We actually do this in two steps. As a first step, in this section, we show that dimensional reduction allows one to turn some of the 5d Bianchi solutions into 4d Bianchi solutions with hyperscaling violation. (By a 4d Bianchi solution, we mean a solution realizing one of the Bianchi types with the radial direction included as one of the spatial dimensions governed by the algebra). In section 3.2, we demonstrate this for type III solutions, and in section 3.3 we show that the type III solutions of this sort can be obtained from more general actions in 4d which do not arise from dimensional reduction.

As a next step, in section 4, we demonstrate that 5d Bianchi solutions (where the algebra acts only on the field theory "spatial" coordinates) can also enjoy hyperscaling violation. We give several examples there, leaving the special case of type VII$_0$ (which has been especially popular in the literature) to its own section, section 5.

Before proceeding let us note that in the gravity theory hyperscaling violation arises if the metric has a conformal killing vector, i.e. if the metric is left invariant upto a weyl transformation by the corresponding coordinate transformation. In the solutions of this section and in section 4 and section 5, we construct examples where the metric has genuine killing vectors which make it homogenoeus along the field theory directions while possessing an additional conformal killing vector which leads to a scaling transformation in the field theory.

3.1 General idea

In this subsection we discuss the general prescription of obtaining hyperscaling violating solutions from scaling solutions via dimensional reduction, as described in [49]. Let us consider the metric ansatz eq. (2.2), where α_1, α_2 are constants given by eq. (2.17). Let r be the "radial" coordinate in $(d+1)$-dimensions, which will also appear as a coordinate after compactification to d-dimensions. We will consider metric components to be functions of r only, and will allow the higher dimensional metric to be invariant under scale transformations, generated by a shift of r (and appropriate rescaling of the spatial coordinates). Let $\phi(r) = \phi_0 r$:

$$d\hat{s}^2 = e^{2\alpha_1\phi_0 r}ds^2 + e^{2\alpha_2\phi_0 r}(dz + B_\mu(r)dx^\mu)^2 \tag{3.1}$$

Now, requiring that $d\hat{s}^2$ is invariant under $r \to r + \epsilon$ implies that the lower dimensional metric transform as

$$ds^2 = e^{-2\alpha_1\phi_0\epsilon}ds^2 \tag{3.2}$$

So the lower dimensional metric is invariant under the scale transformation up to an overall scaling function in front of the metric, a conformal factor. This corresponds to a metric which exhibits hyperscaling violation.

3.2 Dimensional reduction of 5d type III to 4d space-time

In this subsection, we consider the dimensional reduction of the type III metric as given in eq. (4.38) of [1]. Let us first recall the relevant results of [1].

The action is given by eq. (2.1) with $d = 4$ and only one vector field turned on. For convenience we do not write the quantities with ^ here, and we explicitly mention the space-time dimension of the quantity. The metric in $(4+1)$ dimensions takes the form

$$ds^2 = dr^2 - e^{2\beta_t r}dt^2 + e^{2\beta_2 r}dz^2 + \frac{1}{\rho^2}(d\rho^2 + dx^2) \tag{3.3}$$

with vector field given by

$$A = \sqrt{A_t}e^{\beta_t r}dt \, . \tag{3.4}$$

In order to find a solution, the parameters in the action and in the metric/vector field should be related via:

$$m^2 = 2(1 - \beta_2^2) \, , \qquad\qquad \Lambda = \frac{1}{\beta_2^2} + 2\beta_2^2 \, , \tag{3.5}$$

$$A_t = \frac{2 - 4\beta_2^2}{1 - \beta_2^2} \, , \qquad\qquad \beta_t = \frac{1 - \beta_2^2}{\beta_2} \, . \tag{3.6}$$

Now, consider the dimensional reduction along z by using the techniques of section 3.1. For this subsection we use $\alpha_1 = \frac{1}{2\sqrt{3}}$, $\alpha_2 = -\frac{1}{\sqrt{3}}$ from eq. (2.17) with $d = 4$.

The new 4d action is given by eq. (2.18)

$$S = \int d^4x\sqrt{-g}\left\{R - \frac{1}{2}(\nabla\phi)^2 + e^{2\alpha_1\phi}\Lambda - \frac{1}{4}e^{-2\alpha_1\phi}F^2 - \frac{1}{4}m^2A^2\right\} \, . \tag{3.7}$$

The relevant solution can be read off from the 5d solution above. The metric is

$$ds^2 = e^{\beta_2 r}\left(dr^2 - e^{2\beta_t r}dt^2 + \frac{d\rho^2 + dx^2}{\rho^2}\right) \, , \tag{3.8}$$

and the matter fields are

$$A = \sqrt{A_t}e^{\beta_t r}dt \, , \qquad \phi = \frac{\beta_2}{\alpha_2}r \, . \tag{3.9}$$

Actually eq. (3.8) shows that this is a hyperscaling violating solution of Bianchi type III.

It is possible to view this solution in different coordinates which make it more transparent that it is a hyperscaling violating solution. We perform the following coordinate transformation

$$r = \frac{1}{\beta_2 + \beta_t}\log\tilde{r} \, . \tag{3.10}$$

In the new coordinates, the solution is given by

$$ds^2 = \left(\frac{1}{\beta_2 + \beta_t}\right)^2 \frac{d\tilde{r}^2}{\tilde{r}^{2\gamma}} - \tilde{r}^{2\gamma}dt^2 + \tilde{r}^{2\beta}\,\frac{d\rho^2 + dx^2}{\rho^2} \tag{3.11}$$

$$A = \sqrt{A_t}\,\tilde{r}^{\frac{\beta_t}{\beta_2+\beta_t}}\,dt\,, \tag{3.12}$$

$$\phi = \frac{\beta_2}{\alpha_2(\beta_2 + \beta_t)}\,\log\tilde{r}\,, \tag{3.13}$$

where

$$2\gamma = \frac{\beta_2 + 2\beta_t}{\beta_2 + \beta_t}\,, \qquad 2\beta = \frac{\beta_2}{\beta_2 + \beta_t}\,. \tag{3.14}$$

For general mass parameter, we have $\beta_2 \neq 0$. Therefore, we generically have $\gamma \neq 1$, which implies that this metric is not scale invariant. Instead, it Weyl rescales under scale transformations, in the way which is characteristic of hyperscaling violating metrics.

3.3 Hyperscaling violating type III metrics in 4d space-time

We saw that the Kaluza-Klein reduced action eq. (3.7) has a hyperscaling violating type III solution of the form eq. (3.11). In this subsection, we define a family of actions which generalizes the 4d action eq. (3.7) of the previous subsection, and we exhibit more generic hyperscaling violating type III solutions of these actions. We note that generic values of the parameters here do not allow an "uplift" to a 5d solution, so these solutions considerably generalize those of section 3.2.

Consider the general Einstein Hilbert action in $(d+1)$-dimensions coupled to a massless scalar field, a massive vector field and a cosmological constant:

$$S = \int d^{d+1}x\sqrt{g}\left\{R - \frac{1}{2}(\nabla\phi)^2 - \frac{e^{2\alpha\phi}}{4}F^2 - \frac{m^2}{4}e^{2\epsilon\phi}A^2 + e^{2\delta\phi}\Lambda\right\}\,. \tag{3.15}$$

In this section we will consider $d = 3$. This action is a natural generalization of the one given by eq. (3.7).

As an ansatz, let us take a hyperscaling violating type III metric of the form:

$$ds^2 = \lambda^2 \frac{dr^2}{r^{2\gamma}} - r^{2\gamma}dt^2 + \frac{r^{2\beta}}{\rho^2}(d\rho^2 + dx^2)\,. \tag{3.16}$$

Also, let the gauge field and the scalar field be

$$A = \sqrt{A_t}r^\theta dt\,, \quad \phi = k\log(r)\,. \tag{3.17}$$

All of the equations become algebraic if we take the following relations among parameters:

$$\gamma = 1 + k\delta\,, \qquad \epsilon = \alpha + \delta\,, \qquad \theta = 1 + k(\delta - \alpha)\,, \qquad \beta = -k\delta\,. \tag{3.18}$$

Plugging in these relations, the Einstein equations along r, t, ρ become

$$-8k\delta(2 + k\delta) + A_t\left(-m^2\lambda^2 + 2(1 + k(-\alpha + \delta))^2\right) - 2\left(-4\lambda^2 + k^2 + 2\Lambda\lambda^2\right) = 0\,, \tag{3.19}$$

$$A_t\left(m^2\lambda^2 + 2(1 - k\alpha + k\delta)^2\right) + 2\left(4\lambda^2 + k^2\left(1 + 4\delta^2\right) - 2\Lambda\lambda^2\right) = 0\,, \tag{3.20}$$

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$$8k\delta(2+k\delta) - A_t\left(m^2\lambda^2 + 2(1+k(-\alpha+\delta))^2\right) + 2\left(4+k^2-2\Lambda\lambda^2\right) = 0\,. \quad (3.21)$$

The equation along x is the same as the equation along ρ because (ρ,x) span an Euclidean AdS_2 factor.

The gauge field equation is

$$m^2\lambda^2 + 2k(-1+k(\alpha-\delta))(\alpha-\delta) = 0\,, \quad (3.22)$$

and the scalar field equation is

$$2k + A_t\left(2\alpha + 2k^2\alpha(\alpha-\delta)^2 + 4k\alpha(-\alpha+\delta) + m^2\lambda^2(\alpha+\delta)\right) + 4\delta\lambda^2\Lambda = 0\,. \quad (3.23)$$

The solutions to the above equations are

$$\epsilon = \alpha+\delta\,, \qquad \gamma = \frac{1+2(\alpha-\delta)\delta\left(1+\lambda^2\right)}{1+4\alpha\delta}\,, \qquad \theta = 1 + \frac{2\left(\alpha^2-\delta^2-(\alpha-\delta)^2\lambda^2\right)}{1+4\alpha\delta}\,,$$

$$\beta = \frac{2\delta\left(\alpha+\delta-\alpha\lambda^2+\delta\lambda^2\right)}{1+4\alpha\delta}\,, \qquad k = \frac{-2(\alpha+\delta)+2(\alpha-\delta)\lambda^2}{1+4\alpha\delta}\,,$$

$$A_t = \frac{2\left(-1+\lambda^2+4\delta^2\left(1+\lambda^2\right)\right)}{-1-2\alpha^2-4\alpha\delta+2\delta^2+2(\alpha-\delta)^2\lambda^2}\,,$$

$$\Lambda = \frac{1}{(\lambda+4\alpha\delta\lambda)^2} \times \left\{\left(1+\lambda^2\right)\left(1+2\delta^2\left(-1+\lambda^2\right)+8\delta^4\left(1+\lambda^2\right)\right)\right.$$

$$\left. +2\alpha^2\left(1-2\lambda^2+\lambda^4+4\delta^2\left(1+\lambda^2\right)^2\right)-4\alpha\delta\left(-2+\lambda^2\left(-3+\lambda^2+4\delta^2\left(1+\lambda^2\right)\right)\right)\right\}\,,$$

$$m^2 = -\frac{4(\alpha-\delta)\left(-1-2\alpha^2-4\alpha\delta+2\delta^2+2(\alpha-\delta)^2\lambda^2\right)\left(\alpha\left(-1+\lambda^2\right)-\delta\left(1+\lambda^2\right)\right)}{(\lambda+4\alpha\delta\lambda)^2}\,. \quad (3.24)$$

Throughout this paper we will take the cosmological constant to be negative, this corresponds to taking $\Lambda > 0$ in our conventions.[3] In addition we will mainly consider geometries where the horizon area vanishes.[4] Choosing the horizon to lie at $r \to 0$ this gives rise to the conditions $\gamma, \beta > 0$. Finally, for a physically acceptable solution $A_t, \lambda > 0$. All these conditions can be met for the solution found above in various open sets of parameter ranges; one such range is $\alpha < \frac{1}{2}\left(-8-3\sqrt{7}\right)$, $-\frac{1}{2} < \delta < 0$, $0 < \lambda < \sqrt{\frac{1-4\delta^2}{1+4\delta^2}}$.

4 Hyperscaling violation in 5d Bianchi VI, III, V attractors

In section 3, we have seen how dimensional reduction of the Bianchi solutions of [1] can give rise to hyperscaling violating solutions. This leads to an expectation that such solutions are fairly easy to find in their own right, also in 5d. We confirm this expectation in this section by finding hyperscaling violating generalizations for many of the solutions found in [1]. In this section, we exhibit hyperscaling violating solutions in 5d for Bianchi types VI, III and V. We leave the discussion of type VII$_0$, which has been especially popular in the literature, to a separate section, section 5.

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[3]Hopefully this will allow the near-horizon geometries we study to smoothly connect with asymptotic AdS space. We leave a study of such interpolating geometries for the future.

[4]Of course, as the extremal RN black brane example shows, this condition can be relaxed in some cases.

We do the calculations in 5d, although we expect similar results to hold in other dimensions too. We begin with the action given by eq. (3.15) for $d = 4$. We can achieve hyperscaling violation by allowing the scalar field to run logarithmically with the radial coordinate. This is similar to the earlier dilatonic realizations of hyperscaling violating geometries studied in [10–14].

Before we move on to individual solutions, it is worth noting that the dimensionless constants in the action are α, δ, ϵ and $\frac{m^2}{\Lambda}$. In the discussion that follows we will find it convenient to take $\epsilon = \alpha + \delta$ since the equations simplify in this case. Furthermore, sometimes the algebra governing the Bianchi type will have some free parameters too — for example, type VI has one parameter h appearing in its real three-algebra.

4.1 Type VI (generic h)

To begin with, let us start with a Type VI metric. We take the ansatz for the metric and matter fields to be of the form

$$ds^2 = \frac{dr^2}{r^{2\gamma}} - r^{2\gamma}dt^2 + \lambda r^{2\beta_x}dx^2 + r^{2\beta_y}e^{-2x}dy^2 + r^{2\beta_z}e^{-2hx}dz^2\,, \tag{4.1}$$

$$A = \sqrt{A_t}\,r^\theta dt\,, \tag{4.2}$$

$$\phi = k\log(r)\,. \tag{4.3}$$

All of the equations become algebraic if we choose

$$\beta_x = -\delta k\,, \qquad \gamma = 1 + \delta k\,, \qquad \epsilon = \alpha + \delta\,, \qquad \theta = 1 + k(\delta - \alpha)\,. \tag{4.4}$$

The Einstein equations, gauge field equation, and scalar field equation then respectively give:

$$A_t(2(1 + k\delta - k\alpha)^2 - m^2) + 8(\beta_y + \beta_z + \beta_y\beta_z) - 2k\left(k + 4\delta + 4k\delta^2\right) + \frac{8\left(1 + h + h^2\right)}{\lambda} - 4\Lambda = 0\,,$$

$$A_t(-2(1 + k\delta - k\alpha)^2 - m^2) - \frac{8\left(1 + h + h^2\right)}{\lambda} - 2\left(k^2 + 4\left(\beta_y^2 + \beta_y\beta_z + \beta_z^2\right) - 2\Lambda\right) = 0\,,$$

$$A_t(-2(1+k\delta-k\alpha)^2 - m^2) + \frac{8h}{\lambda}$$
$$+ 2\left(4\left(1 + \beta_y + \beta_y^2 + \beta_z + \beta_y\beta_z + \beta_z^2\right) + k\left(k + 4(3 + 2\beta_y + 2\beta_z)\delta + 8k\delta^2\right) - 2\Lambda\right) = 0\,,$$

$$A_t(-2(1 + k\delta - k\alpha)^2 - m^2) + \frac{8h^2}{\lambda} + 2\left(4 + 4\beta_z(1 + \beta_z) + k\left(k + 4(2 + \beta_z)\delta + 4k\delta^2\right) - 2\Lambda\right) = 0\,, \tag{4.5}$$

$$A_t(-2(1 + k\delta - k\alpha)^2 - m^2) + \frac{8}{\lambda} + 2\left(4 + 4\beta_y(1 + \beta_y) + k\left(k + 4(2 + \beta_y)\delta + 4k\delta^2\right) - 2\Lambda\right) = 0\,,$$

$$\beta_y + h\beta_z + (1 + h)k\delta = 0\,,$$

$$m^2 - 2(k\alpha + \beta_y + \beta_z)(1 + k\delta - k\alpha) = 0\,,$$

$$2k(1 + \beta_y + \beta_z + k\delta) + A_t\left[m^2(\alpha + \delta) + 2\alpha(1 + k\delta - k\alpha)^2\right] + 4\delta\Lambda = 0\,.$$

Generically, it is difficult to find solutions to these nonlinear algebraic equations. However fortunately, given a choice of h which characterizes the three-algebra, one can show that the following solutions exist:

$$\beta_z = \frac{k(2\delta(-\alpha(h(h+2)-1)+\delta h(2h+1)+\delta)-h+1)-2(h-1)(\alpha+\delta)}{2\left(h^2+1\right)(\alpha-2\delta)},$$

$$\beta_y = \frac{k\left(2\alpha\delta((h-2)h-1)+2\delta^2\left(h^2+h+2\right)+(h-1)h\right)+2(h-1)h(\alpha+\delta)}{2\left(h^2+1\right)(\alpha-2\delta)},$$

$$\lambda = \frac{4\left(h^2+1\right)^2(\alpha-2\delta)^2}{(\alpha(4\delta k+2)-2\delta^2 k+2\delta+k)}$$
$$\times \frac{1}{(k\left(2\alpha\delta((h-4)h+1)+2\delta^2(h+1)^2+(h-1)^2\right)+4\alpha((h-1)h+1)-2\delta(h+1)^2)}, \qquad (4.6)$$

$$A_t = \frac{\left(k+6\delta+6k\delta^2\right)}{(-1+k(\alpha-\delta))(\alpha-2\delta)},$$

$$\Lambda = \frac{1}{4\left(h^2+1\right)(\alpha-2\delta)^2}$$
$$\times \big[4k\{6\alpha^2\delta(h(2h-3)+2)-\alpha\left(3\delta^2(h(h+6)+1)-4h^2+6h-4\right)+\delta\left(12\delta^2\left(h^2+1\right)+(h-6)h+1\right)\}$$
$$+k^2\{2\alpha^2\left(12\delta^2(h-1)^2+h^2+1\right)-6\alpha\delta\left(2\delta^2(h+1)^2-(h-6)h-1\right)$$
$$+12\delta^4(h(3h+2)+3)+16\delta^2\left(h^2+1\right)+3(h-1)^2\}+12\{2\alpha^2((h-1)h+1)-4\alpha\delta h+\delta^2(h-1)^2\}\big],$$

$$m^2 = (1+\delta k-\alpha k)\ \frac{k\left(2\alpha^2\left(h^2+1\right)-4\alpha\delta(h+1)^2+\delta^2\left(6h^2+4h+6\right)+(h-1)^2\right)+2(h-1)^2(\alpha+\delta)}{\left(h^2+1\right)(\alpha-2\delta)}.$$

Note that A_t is h independent, but the rest of the formulae do depend on h.

This messy formula gives, for example, with $h=\frac{1}{2}$:

$$\beta_z = \frac{k+2\alpha+2\delta-k\alpha\delta+8k\delta^2}{5\alpha-10\delta},$$

$$\beta_y = \frac{k+2\alpha+2\delta+14k\alpha\delta-22k\delta^2}{10(-\alpha+2\delta)},$$

$$\lambda = -\frac{25(\alpha-2\delta)^2}{(-12\alpha+18\delta+k(-1+6(\alpha-3\delta)\delta))\left(2(\alpha+\delta)+k\left(1+4\alpha\delta-2\delta^2\right)\right)},$$

$$A_t = \frac{\left(k+6\delta+6k\delta^2\right)}{(-1+k(\alpha-\delta))(\alpha-2\delta)}, \qquad (4.7)$$

$$\Lambda = \frac{1}{20(-\alpha+2\delta)^2}$$
$$\times\{12\left(6\alpha^2-8\alpha\delta+\delta^2\right)+4k\left(8\alpha-7\delta+24\alpha^2\delta-51\alpha\delta^2+60\delta^3\right)$$
$$+k^2\left(3+80\delta^2+228\delta^4+2\alpha^2\left(5+12\delta^2\right)-6\alpha\delta\left(7+18\delta^2\right)\right)\},$$

$$m^2 = (1-k(\alpha-\delta))\ \frac{2(\alpha+\delta)+k\left(1+10\alpha^2-36\alpha\delta+38\delta^2\right)}{5(\alpha-2\delta)}.$$

We take $\Lambda > 0$ and also $\gamma > 0$. An acceptable solution with vanishing horizon area then requires $(\beta_x + \beta_y + \beta_z)$, λ, $A_t > 0$. One possible range where these conditions is met is $0 < \alpha \le \frac{\sqrt{6}}{5}$, $0 < \delta < \frac{\alpha}{2}$, $-\frac{2(\alpha+\delta)}{1+4\alpha\delta-2\delta^2} < k < -\frac{6\delta}{1+6\delta^2}$.

4.2 Type III ($h=0$)

This is a limiting case of type VI, with $h=0$. The solution is obtained from eq. (4.6) by setting $h=0$:

$$\beta_z = \frac{2(\alpha+\delta)+k(1+2\delta(\alpha+\delta))}{2(\alpha-2\delta)},$$

$$\beta_y = -k\delta \,,$$

$$\lambda = \frac{4(\alpha - 2\delta)^2}{\left(2(\alpha + \delta) + k\left(1 + 4\alpha\delta - 2\delta^2\right)\right)\left(4\alpha - 2\delta + k(1 + 2\delta(\alpha + \delta))\right)} \,,$$

$$A_t = \frac{\left(k + 6\delta + 6k\delta^2\right)}{(-1 + k(\alpha - \delta))(\alpha - 2\delta)} \,, \tag{4.8}$$

$$\Lambda = \frac{1}{4(\alpha - 2\delta)^2}$$
$$\times \{12\left(2\alpha^2 + \delta^2\right) + k^2\left(3 + 2\alpha^2 + 6\alpha\delta + 8\left(2 + 3\alpha^2\right)\delta^2 - 12\alpha\delta^3 + 36\delta^4\right)$$
$$+ 4k\left(\delta + 12\alpha^2\delta + 12\delta^3 + \alpha\left(4 - 3\delta^2\right)\right)\} \,,$$

$$m^2 = (1 - k(\alpha - \delta))\,\frac{2(\alpha + \delta) + k\left(1 + 2\alpha^2 - 4\alpha\delta + 6\delta^2\right)}{(\alpha - 2\delta)} \,.$$

An acceptable solution with vanishing horizon area arises if $(\beta_x + \beta_y + \beta_z)$, λ, A_t, Λ, $\gamma > 0$. These conditions can be met, e.g., one possible range of parameters is $0 < \alpha < \sqrt{\frac{2}{3}}$, $0 < \delta < \frac{\alpha}{2}$, $\frac{-2\alpha - 2\delta}{1 + 4\alpha\delta - 2\delta^2} < k < -\frac{6\delta}{1 + 6\delta^2}$.

4.3 Type V ($h = 1$)

This is a limiting case of type VI, with $h = 1$. The solution is obtained from eq. (4.6) by setting $h = 1$:

$$\beta_y = \beta_z = -k\delta \,,$$

$$\lambda = -\frac{4(\alpha - 2\delta)}{(-1 + k\delta)\left(2(\alpha + \delta) + k\left(1 + 4\alpha\delta - 2\delta^2\right)\right)} \,,$$

$$A_t = \frac{\left(k + 6\delta + 6k\delta^2\right)}{(-1 + k(\alpha - \delta))(\alpha - 2\delta)} \,, \tag{4.9}$$

$$\Lambda = \frac{6\alpha + k\left(2 + 6(\alpha - 2\delta)\delta + k\left(\alpha - 4\left(\delta + 3\delta^3\right)\right)\right)}{2(\alpha - 2\delta)} \,,$$

$$m^2 = 2k(\alpha - 2\delta)(1 + k(-\alpha + \delta)) \,.$$

Again, an acceptable solution with vanishing horizon area arises when $(\beta_x + \beta_y + \beta_z)$, λ, A_t, Λ, $\gamma > 0$. We can find some allowable range of values for α, δ, k where these conditions are met. For example: $0 < \alpha < \sqrt{\frac{2}{3}}$, $0 < \delta < \frac{\alpha}{2}$, $\frac{-2\alpha - 2\delta}{1 + 4\alpha\delta - 2\delta^2} < k < -\frac{6\delta}{1 + 6\delta^2}$.

5 Hyperscaling violation in 5d Bianchi VII$_0$ attractors

In this section we will obtain a hyperscaling violating solutions in 5d with Bianchi type VII$_0$ symmetry using the same theory as before, i.e eq. (3.15) with $d = 4$. Again we will have $\epsilon = \alpha + \delta$, so that the solutions will have $\alpha, \delta, \frac{m^2}{\Lambda}$ as free parameters. The ansatz for the various fields is chosen as

$$ds^2 = C_a^2 \frac{dr^2}{r^{2\gamma}} - r^{2\gamma}dt^2 + r^{2\beta_x}dx^2 + r^{2\beta_{yz}}\left(\left(\omega^2\right)^2 + \lambda\left(\omega^3\right)^2\right) \,. \tag{5.1}$$

$$A = \sqrt{A_2}\, r^\theta \omega^2. \tag{5.2}$$

$$\phi = k \log(r) \,. \tag{5.3}$$

Here, the ω^i are two of the invariant one-forms of Type VII$_0$ geometry:

$$\omega^2 = \cos(x)dy + \sin(x)dz \qquad\qquad \omega^3 = -\sin(x)dy + \cos(x)dz \,. \tag{5.4}$$

All the equations of motion become algebraic if we choose

$$\beta_x = -k\delta \,, \quad \gamma = 1 + k\delta \,, \quad \epsilon = \alpha + \delta \,, \quad \theta = \beta_{yz} - k\alpha \,. \tag{5.5}$$

The Einstein equations, gauge field, and scalar equations respectively give:

$$
\begin{aligned}
-2\left(\beta_{yz}(-8 + (-4 + A_2)\beta_{yz}) + k(-2A_2\alpha\beta_{yz} + 4\delta) + k^2\left(1 + A_2\alpha^2 + 4\delta^2\right)\right)\lambda & \\
+ C_a^2(2 + A_2(2 + m^2\lambda) + 2\lambda(-2 + \lambda - 2\Lambda)) &= 0 \,, \\
2\left(k^2\left(1 + A_2\alpha^2\right) - 2A_2k\alpha\beta_{yz} + (12 + A_2)\beta_{yz}^2\right)\lambda & \\
+ C_a^2(2 + A_2(2 + m^2\lambda) + 2\lambda(-2 + \lambda - 2\Lambda)) &= 0 \,, \\
2\left(4 + \beta_{yz}(8 + (12 + A_2)\beta_{yz}) + 2k(-A_2\alpha\beta_{yz} + 6\delta + 8\beta_{yz}\delta) + k^2\left(1 + A_2\alpha^2 + 8\delta^2\right)\right)\lambda & \\
+ C_a^2(-2 + A_2(-2 + m^2\lambda) - 2\lambda(-2 + \lambda + 2\Lambda)) &= 0 \,, \quad (5.6) \\
2\left(4 + 2A_2k\alpha\beta_{yz} + \beta_{yz}(4 + 4\beta_{yz} - A_2\beta_{yz}) + 4k(2 + \beta_{yz})\delta + k^2\left(1 - A_2\alpha^2 + 4\delta^2\right)\right)\lambda & \\
- C_a^2(A_2(-2 + m^2\lambda) + 2(-3 + \lambda(2 + \lambda + 2\Lambda))) &= 0 \,, \\
2\left(4 - 2A_2k\alpha\beta_{yz} + \beta_{yz}(4 + (4 + A_2)\beta_{yz}) + 4k(2 + \beta_{yz})\delta + k^2\left(1 + A_2\alpha^2 + 4\delta^2\right)\right)\lambda & \\
+ C_a^2\left(-2 + 6\lambda^2 + A_2(-2 + m^2\lambda) - 4\lambda(1 + \Lambda)\right) &= 0 \,, \\
\lambda(2(k\alpha - \beta_{yz})(1 + \beta_{yz} + k(\alpha + \delta))\lambda + C_a^2(2 + m^2\lambda)) &= 0 \,, \\
-A_2\left(2C_a^2\alpha + 2\alpha(-k\alpha + \beta_{yz})^2\lambda + C_a^2m^2(\alpha + \delta)\lambda\right) + 2\lambda\left(k + 2k\beta_{yz} + k^2\delta + 2C_a^2\delta\Lambda\right) &= 0 \,.
\end{aligned}
$$

Note that all of the 7 equations are written in vierbein coordinates. Six of the equations are independent, and can be used to solve for the six parameters C_a^2, Λ, β_{yz}, k, m^2, A_2 in terms of α, δ, λ. Although we were able to solve above equations in terms of α, δ, λ, the solutions are very complicated and not very illuminating. Instead we present below solutions at special values in the (α, δ) parameter space, where the resulting expressions are more compact.

Solution for $\alpha = -\delta$:

$$
\begin{aligned}
C_a^2 &= \frac{2\left(-1 + 6\delta^2\right)\left(11 + 12\delta^2(-1 + \lambda) - 5\lambda\right)(-2 + \lambda)}{(1 + 6\delta^2)^2(-3 + \lambda)^2(-1 + \lambda)} \,, \\
\beta_{yz} &= \frac{2\left(-2 - 3\delta^2(-1 + \lambda) + \lambda\right)}{(1 + 6\delta^2)(-3 + \lambda)} \,, \\
k &= -\frac{6\delta}{1 + 6\delta^2} \,, \quad (5.7) \\
m^2 &= \frac{-22 + 24\delta^2(-1 + \lambda)^2 - 2\lambda(2 + \lambda(-10 + 3\lambda))}{(11 + 12\delta^2(-1 + \lambda) - 5\lambda)\lambda} \,, \\
\Lambda &= \frac{(-1 + \lambda)\left(-22 + 144\delta^4(-1 + \lambda)^3 + \lambda(117 + \lambda(-90 + 19\lambda)) + 12\delta^2(13 + \lambda(-24 + (24 - 7\lambda)\lambda))\right)}{2\left(-1 + 6\delta^2\right)(11 + 12\delta^2(-1 + \lambda) - 5\lambda)(-2 + \lambda)\lambda} \,, \\
A_2 &= -2 - \frac{2}{-2 + \lambda} \,.
\end{aligned}
$$

A solution with vanishing horizon area would arise if $C_a^2 > 0$, $A_2 > 0$, $\beta_x + 2\beta_{yz} > 0$, $\Lambda > 0$, $\lambda > 0$ are satisfied. These conditions are indeed met when $-\frac{1}{\sqrt{6}} < \delta < \frac{1}{\sqrt{6}}$,

$1 < \lambda < 2$. This specific case $\alpha = -\delta$ can be obtained from dimensional reduction, but we can get more generic solutions, which do not allow an uplift to higher dimensions, as follows.

- **Solution for $\alpha = \delta = -1$:**

$$C_a^2 = \frac{-81\left(-12 + \sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right)}{49(-1+\lambda)(27 + \lambda(-14 + 23\lambda))^2}$$
$$+ \frac{\lambda\left(2793 - 938\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right)}{49(-1+\lambda)(27 + \lambda(-14 + 23\lambda))^2}$$
$$+ \frac{\lambda^2\left(-11965 + \lambda(-11109 + 9589\lambda) + 1991\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right)}{49(-1+\lambda)(27 + \lambda(-14 + 23\lambda))^2},$$
$$\beta_{yz} = \frac{30 + \lambda(-21 + 37\lambda) + \sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}}{7(27 + \lambda(-14 + 23\lambda))},$$
$$k = \frac{138 - 28\lambda + 26\lambda^2 - 8\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}}{7(27 + \lambda(-14 + 23\lambda))}, \qquad (5.8)$$
$$m^2 = \frac{1}{5\lambda(-11 + 13\lambda(-6 + 17\lambda))}[26 - 7\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}$$
$$+ \lambda\{-1661 + 104\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}$$
$$+ \lambda\left(2065 + 13\lambda(-79 + 9\lambda) - 13\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right)\}],$$
$$\Lambda = \frac{1}{10\lambda(-11 + 13\lambda(-6 + 17\lambda))}[-463 - 34\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}$$
$$- 6\lambda\left(357 + 52\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right)$$
$$+ \lambda^2\left(10000 + 39\lambda(-166 + 81\lambda) + 754\sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right)],$$
$$A_2 = \frac{1}{4}\left(8 + \lambda(-7 + 9\lambda) + \sqrt{144 + \lambda(-112 + \lambda(113 + 9\lambda(-14 + 9\lambda)))}\right).$$

An acceptable solution with vanishing horizon area arises if $\lambda > 0$ and $\lambda \neq 1$, $\lambda \neq \frac{1}{221}\left(39 + 4\sqrt{247}\right)$.

- **Solution for $\alpha = -3$ and $\delta = -1$:**

$$C_a^2 = \frac{5}{(25 - 39\lambda)^2(-1+\lambda)(11 + 39\lambda)^2}[-12125\left(-30 + \sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)$$
$$+ \lambda\left(220485 - 4978\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)$$
$$+ 39\lambda^2\left(-20395 + 5\lambda(-2075 + 3081\lambda) + 477\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)],$$
$$\beta_{yz} = \frac{-380 + 3\lambda(-83 + 273\lambda) + 9\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}}{(-25 + 39\lambda)(11 + 39\lambda)},$$
$$k = \frac{2\left(185 + \lambda(-82 + 117\lambda) - 8\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)}{(-25 + 39\lambda)(11 + 39\lambda)}, \qquad (5.9)$$
$$m^2 = \frac{1}{5\lambda(-3395 + 109\lambda(-6 + 41\lambda))}[-2425\left(37 + \sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)$$
$$+ \lambda\left(44235 - 654\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)$$
$$+ \lambda^2\left(130567 - 327\lambda(217 + 34\lambda) + 3379\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right)],$$
$$\Lambda = \frac{1}{10\lambda(109\lambda(41\lambda - 6) - 3395)}[\lambda(\lambda(130567 - 327\lambda(34\lambda + 217)) + 44235) - 89725$$
$$+ (109\lambda(31\lambda - 6) - 2425)\sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}],$$
$$A_2 = \frac{1}{40}\left(-10 + \lambda(-19 + 39\lambda) + \sqrt{900 + \lambda(380 + \lambda(-1219 + 39\lambda(-38 + 39\lambda)))}\right).$$

This solution is physically acceptable for all $\lambda > 0$ except $\lambda = \frac{25}{39}, \frac{327 + 4\sqrt{954949}}{4469}, 1$.

These two classes of solutions are quite messy, but are presented (in a somewhat Pyrrhic victory) to illustrate that physically sensible solutions of this sort do exist for several values of α, δ.

6 Striped phases by Kaluza-Klein reduction of 5d type VII$_0$

By using the dimensional reduction we reviewed in section 2, we can also obtain simple analytical examples of striped phases. For that purpose, we start with 5d type VII$_0$ case. See for examples of the other constructions of "striped" phases in the literature [31–36].

Let's consider the 5d type VII$_0$ solution [1], which takes the following ansatz

$$d\hat{s}^2 = dr^2 - e^{2\beta_t r}dt^2 + (dx^1)^2 + e^{2\beta r}\left[(c\,dx^2 + s\,dx^3)^2 + \lambda^2(-s\,dx^2 + c\,dx^3)^2\right], \quad (6.1)$$

$$\hat{A} = \sqrt{\tilde{A}_2}e^{\beta r}(c\,dx^2 + s\,dx^3), \quad (6.2)$$

where $c = \cos x^1$ and $s = \sin x^1$. This is the solution of the 5d action given by eq. (2.1), where

$$m^2 = \frac{2(11 + 2\lambda^2 - 10\lambda^4 + 3\lambda^6)}{\lambda^2(5\lambda^2 - 11)}, \quad (6.3)$$

$$\Lambda = \frac{1}{50}\left(95\lambda^2 + \frac{25}{\lambda^2} - \frac{50}{\lambda^2 - 2} + \frac{144}{5\lambda^2 - 11} - 146\right), \quad (6.4)$$

$$\beta = \sqrt{2}\sqrt{\frac{2 - 3\lambda^2 + \lambda^4}{5\lambda^2 - 11}}, \quad (6.5)$$

$$\beta_t = \frac{\lambda^2 - 3}{\sqrt{2}\,(\lambda^2 - 2)}\sqrt{\frac{2 - 3\lambda^2 + \lambda^4}{5\lambda^2 - 11}}, \quad (6.6)$$

$$\tilde{A}_2 = \frac{2}{2 - \lambda^2} - 2, \quad (6.7)$$

and $1 \leq \lambda < \sqrt{2}$.

In order to dimensionally reduce this 5d solution to 4d along x^2, we re-write the metric and gauge field as

$$d\hat{s}^2 = dr^2 - e^{2\beta_t r}dt^2 + (dx^1)^2 + e^{2\beta r}f(x^1)(dx^3)^2 + e^{2\beta r}(c^2 + \lambda^2 s^2)(dx^2 + B_3 dx^3)^2, \quad (6.8)$$

$$\hat{A} = \left(\sqrt{\tilde{A}_2}e^{\beta r}s - \chi B_3\right)dx^3 + \chi(dx^2 + B_3 dx^3), \quad (6.9)$$

where

$$B_3 = \frac{(1 - \lambda^2)cs}{c^2 + \lambda^2 s^2}, \qquad \chi = \sqrt{\tilde{A}_2}e^{\beta r}c, \qquad f(x^1) = \frac{\lambda^2}{c^2 + \lambda^2 s^2}. \quad (6.10)$$

This metric can be written in the form

$$d\hat{s}^2 = e^{2\alpha_1\phi(x)}ds^2 + e^{2\alpha_2\phi(x)}\left(dz + B_\mu(x)dx^\mu\right)^2 \quad (6.11)$$

where $\alpha_2 = -2\alpha_1 = -\frac{1}{\sqrt{3}}$ using eq. (2.17) and setting $d = 4$. The lower dimensional scalar field depends non-trivially on x^1 through $c = \cos x^1$ and $s = \sin x^1$, so it is clear that this can give geometries dual to striped phases.

The 4d action (after appropriate Weyl rescaling etc) is given by eq. (2.18),

$$S = \int d^4x \sqrt{|g|} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-6\alpha_1\phi}H^2 - \frac{1}{4}e^{-2\alpha_1\phi}F^2 - \frac{1}{2}e^{4\alpha_1\phi}(\partial\chi)^2 \right.$$
$$\left. - \frac{1}{4}m^2(A - \chi B)^2 - \frac{1}{4}m^2 e^{6\alpha_1\phi}\chi^2 + \Lambda e^{2\alpha_1\phi} \right] . \tag{6.12}$$

The 4d solution is given by,

$$ds^2 = e^{\beta r}g(x^1)\left(dr^2 - e^{2\beta_t r}dt^2 + (dx^1)^2 + e^{2\beta r}f(x^1)(dx^3)^2\right) , \tag{6.13}$$

$$g(x^1) = \sqrt{c^2 + \lambda^2 s^2}, \qquad \phi = -\sqrt{3}\left(\beta r + \log g(x^1)\right) , \tag{6.14}$$

$$A = \sqrt{\tilde{A}_2}e^{\beta r}s dx^3 , \qquad B = B_3 dx^3 \tag{6.15}$$

where B_3, χ, $f(x^1)$ are given in (6.10).

If we perform the coordinate change $r = \frac{1}{\beta_t + \beta}\log(\tilde{r})$, the field configuration simplifies slightly to:

$$ds^2 = g(x^1)\left(\frac{1}{(\beta_t + \beta)^2}\frac{d\tilde{r}^2}{\tilde{r}^{2\gamma}} - \tilde{r}^{2\gamma}dt^2 + \tilde{r}^{2\delta}(dx^1)^2 + f(x^1)\tilde{r}^{6\delta}(dx^3)^2\right) , \tag{6.16}$$

$$\phi = -2\sqrt{3}\,\delta\,\log\tilde{r} - \sqrt{3}\log g(x^1) , \qquad \chi = \sqrt{\tilde{A}_2}\tilde{r}^{2\delta}c , \tag{6.17}$$

$$A = \sqrt{\tilde{A}_2}\tilde{r}^{2\delta}s dx^3 , \qquad B = B_3 dx^3 , \tag{6.18}$$

where

$$2\gamma = \frac{2\beta_t + \beta}{\beta_t + \beta} , \qquad 2\delta = \frac{\beta}{\beta_t + \beta} . \tag{6.19}$$

The "striped" structure of the solution is evident and is seen in both the scalar field and in the metric.

7 Classification of four dimensional homogeneous spaces

We have seen so far that by various simple modifications of the Bianchi horizons, it is possible to exhibit analytic striped phases and hyperscaling violation in anisotropic phases. The more ambitious goal of [1] was to classify homogeneous, anisotropic extremal horizons as a tractable starting point for a more general classification.

Here, we try to extend the classification proposed in [1]. The holographic dual of a four-dimensional quantum field theory has four-dimensional spatial slices (including the "radial" direction), and therefore a classification based on four-dimensional real Lie algebras seems more natural for static metrics. This allows for the possibility that the radial direction is more non-trivially intertwined with the "field theory spatial" dimensions.

The classification of [44] yields 12 different classes of four-dimensional real Lie algebras (including some indexed by continuous parameters), with a variety of inequivalent embeddings of fixed three sub-algebras (corresponding to the field theory space) into each. Here, we give the data corresponding to the four-dimensional algebras in section 7.1, and describe the classification of subalgebras in section 7.2.

7.1 Real four-dimensional Lie algebras

A four-dimensional homogeneous space $\mathcal{H}$ has four linearly independent Killing vectors e_i with $i = 1, \cdots 4$, which generate the isometries of $\mathcal{H}$. These satisfy an algebra

$$[e_i, e_j] = C_{ij}^k e_k \tag{7.1}$$

with C_{ij}^k the structure constants of the related four-dimensional real Lie algebra. It also has four invariant one-forms ω^i. The Lie derivatives of the ω^i along all the e_j directions vanish, and the ω^i's can be normalized to satisfy the relation

$$d\omega^i = \frac{1}{2} C_{jk}^i \omega^j \wedge \omega^k . \tag{7.2}$$

Here, we will list the structure constants of the 12 inequivalent four-dimensional algebras, as well as convenient choices for the Killing vectors and one-forms. The one-forms are particularly useful because a metric written in terms of them

$$ds^2 = \cdots + \eta_{ij} \omega^i \otimes \omega^j \tag{7.3}$$

(with $\cdots$ independent of the relevant four dimensions) will be invariant under the isometry group.

In the following, we adopt the notation of [44] in naming the algebras — we call them $A_{4,k}$ with $k = 1, \cdots, 12$. We list only the non-vanishing structure constants (up to obvious permutation of indices).

- $A_{4,1}$: $C_{24}^1 = 1$, $C_{34}^2 = 1$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 \qquad\qquad e_3 = \partial_3 \qquad e_4 = x_2 \partial_1 + x_3 \partial_2 + \partial_4$$
$$\omega^1 = dx^1 - x_4 dx^2 + \frac{1}{2} x_4^2 dx^3 \quad \omega^2 = dx^2 - x_4 dx^3 \quad \omega^3 = dx^3 \quad \omega^4 = dx^4$$

- $A_{4,2}^a$: $C_{14}^1 = a$, $C_{24}^2 = 1$, $C_{34}^2 = 1$, $C_{34}^3 = 1$

$$e_1 = \partial_1 \qquad e_2 = \partial_2 \qquad\qquad e_3 = \partial_3 \qquad\qquad e_4 = ax_1 \partial_1 + (x_2 + x_3) \partial_2 + x_3 \partial_3 + \partial_4$$
$$\omega^1 = e^{-ax_4} dx^1 \quad \omega^2 = e^{-x_4}(dx^2 - x_4 dx^3) \quad \omega^3 = e^{-x_4} dx^3 \quad \omega^4 = dx^4$$

- $A_{4,3}$: $C_{14}^1 = 1$, $C_{34}^2 = 1$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 \qquad\qquad e_3 = \partial_3 \qquad\qquad e_4 = x_1 \partial_1 + x_3 \partial_2 + \partial_4$$
$$\omega^1 = e^{-x_4} dx^1 \qquad \omega^2 = dx^2 - x_4 dx^3 \qquad \omega^3 = dx^3 \qquad \omega^4 = dx^4$$

- $A_{4,4}$: $C_{14}^1 = 1$, $C_{24}^1 = 1$, $C_{24}^2 = 1$, $C_{34}^2 = 1$, $C_{34}^3 = 1$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 \qquad\qquad e_3 = \partial_3$$
$$e_4 = (x_1 + x_2)\partial_1 + (x_2 + x_3)\partial_2 + x_3\partial_3 + \partial_4$$
$$\omega^1 = e^{-x_4}\left(dx^1 - x_4 dx^2 + \frac{1}{2}x_4^2 dx^3\right) \qquad \omega^2 = e^{-x_4}(dx^2 - x_4 dx^3) \quad \omega^3 = e^{-x_4}dx^3 \quad \omega^4 = dx^4$$

- $A_{4,5}^{a,b}$: $C_{14}^1 = 1$, $C_{24}^2 = a$, $C_{34}^3 = b$

$$e_1 = \partial_1 \qquad e_2 = \partial_2 \qquad e_3 = \partial_3 \qquad e_4 = x_1\partial_1 + ax_2\partial_2 + bx_3\partial_3 + \partial_4$$
$$\omega^1 = e^{-x_4}dx^1 \qquad \omega^2 = e^{-ax_4}dx^2 \qquad \omega^3 = e^{-bx_4}dx^3 \qquad \omega^4 = dx^4$$

- $A_{4,6}^{a,b}$: $C_{14}^1 = a$, $C_{24}^2 = b$, $C_{24}^3 = -1$, $C_{34}^2 = 1$, $C_{34}^3 = b$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 \qquad\qquad e_3 = \partial_3$$
$$e_4 = ax_1\partial_1 + (bx_2 + x_3)\partial_2 + (bx_3 - x_2)\partial_3 + \partial_4$$
$$\omega^1 = e^{-ax_4}dx^1 \qquad\qquad \omega^2 = e^{-bx_4}[\cos(x_4)dx^2 - \sin(x_4)dx^3]$$
$$\omega^3 = e^{-bx_4}(\cos(x_4)dx^3 + \sin(x_4)dx^2) \qquad \omega^4 = dx^4$$

- $A_{4,7}$: $C_{14}^1 = 2$, $C_{24}^2 = 1$, $C_{34}^2 = 1$, $C_{34}^3 = 1$, $C_{23}^1 = 1$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \qquad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$$
$$e_4 = 2x_1\partial_1 + (x_2 + x_3)\partial_2 + x_3\partial_3 + \partial_4$$
$$\omega^1 = e^{-2x_4}\left(dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2\right) \qquad \omega^2 = e^{-x_4}(dx^2 - x_4 dx^3)$$
$$\omega^3 = e^{-x_4}dx^3 \qquad\qquad \omega^4 = dx^4$$

- $A_{4,8}$: $C_{23}^1 = 1$, $C_{24}^2 = 1$, $C_{34}^3 = -1$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \qquad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$$
$$e_4 = x_2\partial_2 - x_3\partial_3 + \partial_4$$
$$\omega^1 = dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2 \qquad \omega^2 = e^{-x_4}dx^2 \qquad \omega^3 = e^{x_4}dx^3 \qquad \omega^4 = dx^4$$

- $A_{4,9}^b$: $C_{23}^1 = 1$, $C_{14}^1 = 1 + b$, $C_{24}^2 = 1$, $C_{34}^3 = b$

$$e_1 = \partial_1 \qquad\qquad\qquad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \qquad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$$

$$e_4 = (1+b)x_1\partial_1 + x_2\partial_2 + bx_3\partial_3 + \partial_4$$
$$\omega^1 = e^{-(b+1)x_4}\left(dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2\right) \quad \omega^2 = e^{-x_4}dx^2 \quad \omega^3 = e^{-bx_4}dx^3 \quad \omega^4 = dx^4$$

- $A_{4,10}$: $C_{23}^1 = 1$, $C_{24}^3 = -1$, $C_{34}^2 = 1$

$$e_1 = \partial_1 \qquad\qquad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \qquad\qquad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$$
$$e_4 = -x_2\partial_3 + x_3\partial_2 + \partial_4$$
$$\omega^1 = dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2 \quad \omega^2 = \cos(x_4)dx^2 - \sin(x_4)dx^3 \quad \omega^3 = \cos(x_4)dx^3 + \sin(x_4)dx^2$$
$$\omega^4 = dx^4$$

- $A_{4,11}^a$: $C_{23}^1 = 1$, $C_{14}^1 = 2a$, $C_{24}^2 = a$, $C_{24}^3 = -1$, $C_{34}^2 = 1$, $C_{34}^3 = a$

$$e_1 = \partial_1 \qquad\qquad\qquad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1$$
$$e_3 = \partial_3 + \frac{1}{2}x_2\partial_1 \qquad\qquad e_4 = 2ax_1\partial_1 + (ax_2 + x_3)\partial_2 + (ax_3 - x_2)\partial_3 + \partial_4$$
$$\omega^1 = e^{-2ax_4}\left(dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2\right) \quad \omega^2 = e^{-ax_4}(\cos(x_4)dx^2 - \sin(x_4)dx^3)$$
$$\omega^3 = e^{-ax_4}(\cos(x_4)dx^3 + \sin(x_4)dx^2) \qquad \omega^4 = dx^4$$

- $A_{4,12}$: $C_{13}^1 = 1$, $C_{23}^2 = 1$, $C_{14}^2 = -1$, $C_{24}^1 = 1$

$$e_1 = \partial_1 \qquad\qquad\qquad e_2 = \partial_2$$
$$e_3 = \partial_3 + x_1\partial_1 + x_2\partial_2 \qquad\qquad e_4 = \partial_4 + x_2\partial_1 - x_1\partial_2$$
$$\omega^1 = e^{-x_3}(\cos(x_4)dx^1 - \sin(x_4)dx^2) \qquad \omega^2 = e^{-x_3}(\cos(x_4)dx^2 + \sin(x_4)dx^1)$$
$$\omega^3 = dx^3 \qquad\qquad\qquad \omega^4 = dx^4$$

7.2 Three-dimensional subalgebras

We would like the bulk geometry to reflect homogeneity of the spatial slices in the dual field theory. For this to happen, we wish to embed a three-dimensional real Lie algebra $A_3 \subset A_{4,k}$. The associated Killing vectors will generate isometries of the spatial dimensions in the dual field theory; the non-trivial embedding of A_3 in A_4 reflects the intertwining of the "spatial" and "radial" directions along the flow, which allows us to generalize the solutions of [1], where the full four-dimensional algebra was semi-simple and contained a trivial factor corresponding to scale transformations.

These subalgebras (including inequivalent embeddings of a fixed A_3 into a given $A_{4,k}$) have been classified in [44]. The results are as follows. We name the subalgebras following the convention of [44].[5] We also list the generators after each subalgebra.

- $A_{4,1}$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_4 \\ A_{3,1} & e_4 + xe_3, e_2, e_1 \end{array}$

- $A_{4,2}^a,\ a \neq 0, 1$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,2} & e_4, e_2, e_3 \\ A_{3,4} & e_4, e_1, e_2 \ \ (a = -1) \\ A_{3,5}^z & e_4, e_1, e_2,\ z = a(a < 1),\ z = 1/a(a > 1) \end{array}$

- $A_{4,2}^1$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,2} & e_4, e_2, e_3 + xe_1 \\ A_{3,3} & e_4, e_1, e_2 \end{array}$

- $A_{4,3}$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_2 \oplus A_1 & e_4 + xe_3, e_1; e_2 \\ A_{3,1} & e_3, e_4, e_2 \end{array}$

- $A_{4,4}$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,2} & e_4, e_1, e_2 \end{array}$

- $A_{4,5}^{a,b}\ (-1 \leq a < b < 1, ab \neq 0)$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,5}^a & e_4, e_1, e_2 \\ A_{3,5}^b & e_4, e_1, e_3 \\ A_{3,5}^z & e_4, e_2, e_3 \ \ z = a/b, |a/b| < 1; z = b/a, |a/b| > 1 \end{array}$

- $A_{4,5}^{a,a}\ (-1 \leq a < 1, a \neq 0)$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,3} & e_4, e_2, e_3 \\ A_{3,5}^a & e_4, e_1, e_2 \cos(\phi) + e_3 \sin(\phi) \end{array}$

- $A_{4,5}^{a,1}\ (-1 \leq a < 1, a \neq 0)$: $\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,3} & e_4, e_1, e_3 \\ A_{3,5}^a & e_4, e_1 \cos(\phi) + e_3 \sin(\phi), e_2 \end{array}$

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[5]For a dictionary relating the notation of [44] for three-algebras with the more standard Bianchi nomenclature [43], see appendix B.

- $A_{4,5}^{1,1}$:
$$\begin{array}{ll} 3A_1 & e_1 e_2, e_3 \\ A_{3,3} & e_4, e_1 + xe_3, e_2 + ye_3 \\ & e_4, e_1 + xe_2, e_3 \\ & e_4, e_2, e_3 \end{array}$$

- $A_{4,6}^{a,b}$ $(a \neq 0, b > 0)$:
$$\begin{array}{ll} 3A_1 & e_1, e_2, e_3 \\ A_{3,7}^b & e_4, e_2, e_3 \end{array}$$

- $A_{4,7}$:
$$\begin{array}{ll} A_{3,1} & e_2, e_3, e_1 \\ A_{3,5}^{\frac{1}{2}} & e_4, e_1, e_2 \end{array}$$

- $A_{4,8}$:
$$\begin{array}{ll} A_{3,1} & e_2, e_3, e_1 \\ A_2 \oplus A_1 & e_4, e_2; e_1 \\ & e_4, e_3; e_1 \end{array}$$

- $A_{4,9}^b$ $(0 < |b| < 1)$:
$$\begin{array}{lll} A_{3,1} & e_2, e_3, e_1 \\ A_{3,5}^z & e_4, e_1, e_2 \ z = 1 + b, |1+b| < 1; z = \frac{1}{1+b}, |1+b| > 1 \\ A_{3,4} & e_4, e_1, e_3 \ b = -\frac{1}{2} \\ A_{3,5}^z & e_4, e_1, e_3 \ z = \frac{b}{1+b}, |\frac{b}{1+b}| < 1; z = \frac{1+b}{b}, |\frac{1+b}{b}| > 1 \end{array}$$

- $A_{4,9}^1$:
$$\begin{array}{ll} A_{3,1} & e_2, e_3, e_1 \\ A_{3,5}^{\frac{1}{2}} & e_4, e_1, e_2 \cos(\phi) + e_3 \sin(\phi) \end{array}$$

- $A_{4,9}^0$:
$$\begin{array}{ll} A_{3,1} & e_2, e_3, e_1 \\ A_2 \oplus A_1 & e_1, e_4; e_3 \\ A_{3,3} & e_4, e_1, e_2 \\ A_{3,2} & e_4 + xe_3, e_1, e_2 \ x \neq 0 \end{array}$$

- $A_{4,10}$: $A_{3,1}$ e_2, e_3, e_1

- $A_{4,11}$ $(0 < a)$: $A_{3,1}$ e_2, e_3, e_1

- $A_{4,12}$:
$$\begin{array}{ll} A_{3,3} & e_3, e_1, e_2 \\ A_{3,6} & e_4, e_1, e_2 \\ A_{3,7}^{|x|} & e_4 + xe_3, e_1, e_2 \ x \neq 0 \end{array}$$

8 Null energy condition for 5d space-times with four-algebras

We have proposed a classification of extremal near-horizon metrics for 5d black branes using four-algebras. The first check of validity for space-times which realize these symmetries is whether they can be supported by reasonable matter content, i.e. whether the stress energy tensor that supports the space-time satisfies the Null Energy Condition (NEC):

$$T_{\mu\nu}N^\mu N^\nu \geq 0 \tag{8.1}$$

for all future directed null vectors N^μ. Via Einstein's equations this translates into a constraint on the geometry, namely that the Einstein tensor has to satisfy

$$G_{\mu\nu}N^\mu N^\nu \geq 0. \tag{8.2}$$

We will work in an abstract orthonormal basis, in which the metric takes the form

$$ds^2 = \sum_{i=1}^4 (\sigma^i)^2 - (\sigma^t)^2, \tag{8.3}$$

for $\sigma^i \equiv \lambda_{ij}\omega^j$, where ω^j are the invariant one-forms listed before. For simplicity we shall only take the matrix λ_{ij} to be diagonal: $\lambda_{ij} = \lambda_i\delta_{ij}$ and $\sigma^i = \lambda_i\omega^i$ (no sum for i). The time-like one-form is given by $\sigma^t = \sqrt{g_{tt}(r)}dt$, where we will only consider a simple scaling red-shift factor: $g_{tt}(r) = e^{2\beta_t r}$. The radial coordinate r is identified with one of the exact one-forms $\lambda_i dr = \sigma^i$ such that the other 3 one-forms form a sub-algebra. In this basis an arbitrary future-directed null vector can easily be written by using 4 real parameters s_i ($i = 1, 2, 3, 4$) in the form:

$$\vec{N} = \left(\sum_{i=1}^4 s_i^2\right)^{\frac{1}{2}} X_t + \sum_{i=1}^4 s_i X_i \tag{8.4}$$

where X_t, X_i are dual vectors to the invariant one-forms σ^t, σ^i. Equation (8.2) then becomes a bilinear function

$$G_{\mu\nu}N^\mu N^\nu \equiv M_{ij}s^i s^j \geq 0, \tag{8.5}$$

for arbitrary choices of the four real parameters $s^i, i = 1, 2, 3, 4$. Since the NEC amounts to imposing positive definiteness of this bilinear function, it then requires that the eigenvalues of the matrix M_{ij} must be non-negative.

We will study natural multiparameter families of metrics realizing a given symmetry structure. We conclude that if a given set of parameters violates the NEC, it cannot be supported by physically reasonable matter fields. In this case, we view the metric with those parameters as capturing a physically unrealizable geometry. In some cases the entire "natural" set of possibilities for realizing a given four-algebra can be eliminated — in this case, we conclude that the related type is not realized physically.

In the following we will study the NEC for each of the 12 types of space-times (in a natural metric parametrization realizing the associated four-algebra), by writing out specifically the eigenvalues of the M_{ij}, and analyzing the constraints obtained by requiring that all of these eigenvalues be non-negative.

Before listing the results of applying the NEC to each type, let's discuss the constraint on β_t, β_i. In general, a structure coefficient of the form $C_{ir}^i = k_i$ signifies a scaling of the form $e^{-k_i r}$ for the one-form σ^i. The red-shift factor $g_{tt}(r)$ is $e^{2\beta_t r}$, and the limit where the red-shift factor g_{tt} approaches zero corresponds to the IR in the dual theory. A field theory spatial volume $\sim e^{-\sum_{i=1}^{3} k_i r}$ in the IR limit corresponds to a ground state entropy density of the field theory. In order to satisfy the laws of thermodynamics, we require that the field theory entropy density goes to zero in the IR. On the gravity side, this means that the spatial volume $\sim e^{-\sum_{i=1}^{3} k_i r}$ should have the same qualitative scaling behavior as the red-shift factor, in the sense that

$$\mathbf{Nernst's\ Law}: \beta_t \sum_{i=1}^{3} C_{ir}^i < 0\,. \tag{8.6}$$

We will impose this, as well as the NEC, as a criterion of reasonableness, which we'll call "Nernst's law".[6]

We will now study the NEC separately for each of the $A_{4,k}$ cases, where the natural metric ansatz we will consider is

$$ds^2 = -e^{2\beta_t r}dt^2 + \sum_{i=1}^{4}\lambda_i^2(\omega^i)^2\,. \tag{8.7}$$

- $A_{4,1}$: We identify x^4 as a radial coordinate; $\lambda_4 dr = \sigma^4 = \lambda_4 \omega^4$ gives $x^4 = r$.

  It is straightforward to calculate the matrix M defined by eq. (8.5). One of the eigenvalues of the matrix M becomes

  $$M_1 = -\frac{\lambda_1^2\lambda_3^2 + \lambda_2^4}{2\lambda_2^2\lambda_3^2\lambda_4^2} < 0\,, \tag{8.8}$$

  which is negative. Therefore this geometry is ruled out.

- $A_{4,2}^a$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

  $$M_1 = -\frac{(a+2-\beta_t)(a+\beta_t)}{\lambda_4^2}\,, \quad M_2 = -\frac{2\lambda_3^2\left(a^2+(a+2)\beta_t+2\right)+\lambda_2^2}{2\lambda_3^2\lambda_4^2}\,, \tag{8.9}$$

  $$M_{3,4} = \frac{(\beta_t+1)(-a-2+\beta_t)}{\lambda_4^2} \pm \frac{\sqrt{\lambda_2^2\lambda_3^4\lambda_4^4\left(\lambda_3^2(a+2-\beta_t)^2+\lambda_2^2\right)}}{2\lambda_3^4\lambda_4^4}\,. \tag{8.10}$$

  Nernst's law forces

  $$\beta_t(a+2) < 0\,. \tag{8.11}$$

  It can be shown that the NEC and Nernst's law are satisfied in several open sets of parameter space; for example: with all $\lambda_i = O(1)$ for $i = 1, 2, 3, 4$, it is easy to see that at large negative values of β_t with $-2 < a$, all conditions are satisfied.

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[6]As already mentioned in some of the discussion above this may in fact be too strong. For instance, $AdS_2 \times R^d$ near-horizon geometries, which violate this criterion, are ubiquitous in simple approximations to string theory, and show fascinating physical properties. However, these have $k_i = 0$, and one could contemplate softening the condition above to $\leq$, at least in some cases.

- $A_{4,3}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = \frac{\beta_t^2 - 1}{\lambda_4^2}, \quad M_2 = -\frac{2\beta_t + \frac{\lambda_2^2}{\lambda_3^2} + 2}{2\lambda_4^2}, \tag{8.12}$$

$$M_{3,4} = \frac{2(\beta_t - 1)\beta_t \lambda_3^4 \lambda_4^2 \pm \sqrt{\lambda_2^2 \lambda_3^4 \lambda_4^4 \left((\beta_t - 1)^2 \lambda_3^2 + \lambda_2^2\right)}}{2\lambda_3^4 \lambda_4^4}. \tag{8.13}$$

Nernst's law forces $\beta_t < 0$. The NEC can be easily satisfied at large negative values of β_t.

- $A_{4,4}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = -\frac{\frac{\lambda_1^2}{\lambda_2^2} + \frac{\lambda_2^2}{\lambda_3^2} + 6 + 6\beta_t}{2\lambda_4^2} \tag{8.14}$$

The other three eigenvalues $M_{2,3,4}$ are roots of a higher order polynomial, which is too cumbersome to write down. However in the large negative limit $\beta_t \to -\infty$, we can easily check that the rest M eigenvalues behave as

$$M_{2,3,4} \to \frac{\beta_t^2}{\lambda_4^2} + O(\beta_t). \tag{8.15}$$

On the other hand, Nernst's law forces $\beta_t < 0$. Therefore the NEC can be easily satisfied in the regime with large negative values of β_t.

- $A_{4,5}^{a,b}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = -\frac{(1 + \beta_t)(a + b - \beta_t + 1)}{\lambda_4^2}, \quad M_2 = -\frac{(a + \beta_t)(a + b - \beta_t + 1)}{\lambda_4^2}, \tag{8.16}$$

$$M_3 = -\frac{(b + \beta_t)(a + b - \beta_t + 1)}{\lambda_4^2}, \quad M_4 = -\frac{1 + a^2 + b^2 + \beta_t(1 + a + b)}{\lambda_4^2}. \tag{8.17}$$

Nernst's law forces

$$\beta_t(1 + a + b) < 0. \tag{8.18}$$

We see that there is an open parameter set satisfying both the NEC and Nernst's law for large negative β_t with $1 + a + b > 0$.

- $A_{4,6}^{a,b}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = -\frac{(a + \beta_t)(a + 2b - \beta_t)}{\lambda_4^2}, \tag{8.19}$$

$$M_2 = -\frac{2a^2 + 2a\beta_t + 4b(b + \beta_t) + \frac{\lambda_3^2}{\lambda_2^2} + \frac{\lambda_2^2}{\lambda_3^2} - 2}{2\lambda_4^2}, \tag{8.20}$$

$$M_{3,4} = -\frac{(b + \beta_t)(a + 2b - \beta_t)}{\lambda_4^2}$$

$$\pm \frac{\sqrt{\lambda_2^4 \lambda_4^4 \left(\lambda_3^4 - \lambda_2^2 \lambda_3^2\right)^2 \left(\lambda_2^2 \lambda_3^2 ((a + 2b - \beta_t)^2 + 2) + \lambda_2^4 + \lambda_3^4\right)}}{2\lambda_2^4 \lambda_3^4 \lambda_4^4}. \tag{8.21}$$

Nernst's law forces

$$\beta_t(a + 2b) < 0 \,. \tag{8.22}$$

We see that there is an open parameter set satisfying both the NEC and Nernst's law for large negative β_t with $a + 2b > 0$.

- $A_{4,7}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = \frac{\beta_t^2 - 2\beta_t - 8}{\lambda_4^2} + \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} \,, \quad M_2 = -\frac{8\beta_t + \frac{\lambda_2^2}{\lambda_3^2} + 12}{2\lambda_4^2} \,, \tag{8.23}$$

$$M_{3,4} = \frac{(\beta_t - 4)(\beta_t + 1)}{\lambda_4^2} - \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} \pm \frac{\sqrt{\lambda_2^{10}\lambda_3^4\lambda_4^4\left((\beta_t - 4)^2\lambda_3^2 + \lambda_2^2\right)}}{2\lambda_2^4\lambda_3^4\lambda_4^4} \,. \tag{8.24}$$

Nernst's law gives

$$\beta_t < 0 \,. \tag{8.25}$$

It is obvious that there is an open parameter range satisfying the NEC and Nernst's law at large negative β_t.

- $A_{4,8}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = -\frac{2}{\lambda_4^2} \,, \quad M_2 = \frac{\beta_t^2}{\lambda_4^2} + \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} \,, \tag{8.26}$$

$$M_{3,4} = -\frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} + \frac{\beta_t(\beta_t \pm 1)}{\lambda_4^2} \,. \tag{8.27}$$

We find that the matrix M has one negative eigenvalue, $M_1 = -\frac{2}{\lambda_4^2}$, hence this type of metric is ruled out by the NEC.

- $A_{4,9}^b$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = -\frac{(2b - \beta_t + 2)(1 + \beta_t)}{\lambda_4^2} - \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} \,, \tag{8.28}$$

$$M_2 = -\frac{(2b - \beta_t + 2)(b + \beta_t)}{\lambda_4^2} - \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} \,, \tag{8.29}$$

$$M_3 = -\frac{(2b - \beta_t + 2)(b + \beta_t + 1)}{\lambda_4^2} + \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2} \,, \tag{8.30}$$

$$M_4 = -\frac{2(b(b + \beta_t + 1) + \beta_t + 1)}{\lambda_4^2} \,. \tag{8.31}$$

Nernst's law forces

$$\beta_t(b + 1) < 0 \,. \tag{8.32}$$

Therefore, the NEC and Nernst's law are satisfied on an open set where $b > -1$ and large negative β_t.

- $A_{4,10}$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_2 = -\frac{\left(\lambda_2^2 - \lambda_3^2\right)^2}{2\lambda_2^2\lambda_3^2\lambda_4^2}, \quad M_1 = \frac{\beta_t^2}{\lambda_4^2} + \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2}, \tag{8.33}$$

$$M_{3,4} = \frac{2\beta_t^2\lambda_2^4\lambda_3^4\lambda_4^2 - \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^4 \pm \sqrt{\lambda_2^4\lambda_4^4\left(\lambda_3^4 - \lambda_2^2\lambda_3^2\right)^2\left(\left(\beta_t^2+2\right)\lambda_2^2\lambda_3^2+\lambda_2^4+\lambda_3^4\right)}}{2\lambda_2^4\lambda_3^4\lambda_4^4}. \tag{8.34}$$

Notice that M_1 is negative and hence violates the NEC unless $\lambda_2 = \lambda_3$. However, in the case where $\lambda_2 = \lambda_3$, we have the metric

$$ds^2 = -e^{2\beta_t r}dt^2 + \lambda_4^2 dr^2 + (\lambda_1)^2(\omega^1)^2 + \lambda_2^2((\omega^2)^2 + (\omega^3)^2) \tag{8.35}$$

where

$$\omega^1 = dx^1 + \frac{1}{2}x^2 dx^3 - \frac{1}{2}x^3 dx^2, \quad \omega^2 = dx^2, \quad \omega^3 = dx^3. \tag{8.36}$$

This is a type II Bianchi geometry[7] of the sort studied in section 4.1 of [1]. Therefore, the generic case of $A_{4,10}$ with $\lambda_2 \neq \lambda_3$ is ruled out by the NEC. In the case $\lambda_2 = \lambda_3$, the NEC imposes the constraint $\beta_t^2 > \lambda_1^2\lambda_4^2/2\lambda_2^4$.

- $A_{4,11}^a$: We identify x^4 as a radial coordinate ($x^4 = r$). The eigenvalues of M are

$$M_1 = \frac{-8a^2 - 2a\beta_t + \beta_t^2}{\lambda_4^2} + \frac{\lambda_1^2}{2\lambda_2^2\lambda_3^2}, \quad M_2 = -\frac{12a^2+8a\beta_t+\frac{\lambda_3^2}{\lambda_2^2}+\frac{\lambda_2^2}{\lambda_3^2}-2}{2\lambda_4^2}, \tag{8.37}$$

$$M_{3,4} = -\frac{2\lambda_2^2\lambda_3^2(4a - \beta_t)(a + \beta_t) + \lambda_1^2}{2\lambda_2^2\lambda_3^2\lambda_4^2}$$
$$\pm\frac{\sqrt{\lambda_2^4\lambda_4^4\left(\lambda_3^4 - \lambda_2^2\lambda_3^2\right)^2\left(\lambda_2^2\lambda_3^2\left((\beta_t - 4a)^2 + 2\right) + \lambda_2^4 + \lambda_3^4\right)}}{2\lambda_2^4\lambda_3^4\lambda_4^4}. \tag{8.38}$$

Nernst's law forces

$$a\beta_t < 0. \tag{8.39}$$

Therefore, there is an open parameter range for $a > 0$ and large negative β_t where both the NEC and Nernst's law are satisfied.

- $A_{4,12}$: in this class both x^3 and x^4 are obvious candidates for the radial coordinate. *Case 1: $r = x^4$*

If we pick $r = x^4$, the eigenvalues of M are

$$M_1 = -\frac{\left(\lambda_1^2 - \lambda_2^2\right)^2}{2\lambda_1^2\lambda_2^2\lambda_4^2}, \quad M_2 = \frac{\beta_t^2}{\lambda_4^2} - \frac{2}{\lambda_3^2}, \tag{8.40}$$

$$M_{3,4} = \frac{\beta_t^2}{\lambda_4^2} - \frac{2}{\lambda_3^2} \pm \frac{\sqrt{\lambda_3^4\left(\lambda_1^2 - \lambda_2^2\right)^2\left(\left(\beta_t^2 + 2\right)\lambda_1^2\lambda_2^2 + \lambda_1^4 + \lambda_2^4\right)}}{2\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2}. \tag{8.41}$$

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[7]One can easily check that basis (8.36) gives $d\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k$ with $C^1_{23} = -C^1_{32} = 1$ and the rest $C^i_{jk} = 0$.

Similar to the $A_{4,10}$ case, we see that in this case M_1 is negative and hence the NEC is violated, unless $\lambda_1 = \lambda_2$.

In the case $\lambda_1 = \lambda_2$, the geometry becomes a product of the form

$$ds^2 = -e^{2\beta_t r}dt^2 + \lambda_4^2 dr^2 + \lambda_3^2 d\vec{x}^2_{EAdS3}\,, \tag{8.42}$$

where $d\vec{x}^2_{EAdS3}$ is the Euclidean AdS_3 metric with unit length-scale. In this case the NEC then imposes just that $\beta_t^2 > 2\lambda_4^2/\lambda_3^2$.

Case 2: $r = x^3$

If we pick $r = x^3$ instead, the eigenvalues of M are then

$$M_1 = -\frac{2(\beta_t + 1)}{\lambda_3^2}\,, \quad M_2 = \frac{(\beta_t - 2)\beta_t}{\lambda_3^2} - \frac{\left(\lambda_1^2 - \lambda_2^2\right)^2}{2\lambda_1^2\lambda_2^2\lambda_4^2}\,, \tag{8.43}$$

$$M_{3,4} = \frac{(\beta_t - 2)(\beta_t + 1)}{\lambda_3^2} \pm \frac{\lambda_1^4 - \lambda_2^4}{2\lambda_1^2\lambda_2^2\lambda_4^2}\,. \tag{8.44}$$

Nernst's law forces

$$\beta_t < 0\,. \tag{8.45}$$

Therefore, both the NEC and Nernst's law are easily satisfied at large negative β_t.

In summary, we have seen that the types of space-times that are eliminated by the NEC are types $A_{4,1}$, $A_{4,8}$, $A_{4,10}$, and $A_{4,12}$ with the $x^4 = r$ choice. The rest of the classes all contain at least one open set of parameters where both the NEC and "Nernst's law" are satisfied, around large negative values of β_t.

The reader should note that we have made the "obvious" choice of the holographic radial coordinate in each case in the analysis above.[8] Generically, the radial direction r can be a more complicated function of x^i's. While this freedom does not enter in any of the data involving the spatial 4-slices, it is relevant in the construction of our 5d space-time. This is because we have selected the redshift factor $g_{tt}(r)$ "by hand" to have the form $e^{2\beta_t r}$ and its contributions in the NEC calculations clearly depend upon, among other things, which coordinate we choose to consider as r. It is quite possible that more elaborate choices than those considered here could resurrect some of the algebraic structures we have left for dead.

9   5d space-time avatars of four-algebras

In this section we will demonstrate that some of the afore-mentioned geometries are indeed realizable from reasonable matter content. In particular, we will work out a few examples explicitly, in a system with a similar effective action to one we've discussed before

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[8]"Obvious" since we have taken r simply identified with one of the x^i, but generically we can take more complicated combinations of them as radial coordinate. For example, in the type $A_{4,12}$, one can also take any linear combinations of x^3 and x^4 as radial coordinates, r.

(eq. (3.15)):[9]

$$S = \int dx^5 \sqrt{-g} \left\{ R - 2(\nabla\phi)^2 + e^{2\delta\phi}\Lambda - \frac{1}{4}e^{2\alpha\phi}(F_A^2 + F_B^2) - \frac{1}{4}e^{2\beta\phi}(M_A^2 A^2 + M_B^2 B^2) \right\} \quad (9.1)$$

with A, B two abelian gauge fields, and F_A, F_B their field strengths.

However, we are going to relax the requirement that the spacetime be scale-invariant, and look for a possibly hyperscaling violating generalization of the space times based on four-algebras. In the orthonormal basis notation, this means that we are going to generalize the radial one-form to be

$$\sigma^r = \frac{dr}{f(r)}$$

where

$$f(r) = e^{\theta r}$$

for a hyperscaling violating metric (with hyperscaling-violation exponent θ). The structure constants involving the radial index will need to change via: $C_{rj}^i \to C_{rj}^i f(r)$. The equations of motion will be modified in the orthonormal basis to be:

- Electric field:

$$\left\{ \left[A_t'(r)f(r) + A_t(r)C_{rt}^t(r) \right] e^{2\alpha\phi} \right\}' f(r) + \left[A_t'(r)f(r) + A_t(r)C_{rt}^t(r) \right] C_{ri}^i e^{2\alpha\phi}$$

$$= \frac{M_A^2}{2}e^{2\beta\phi}A_t(r) \quad (9.2)$$

with a similar equation for B.

- Magnetic field:

$$\left[A_i'(r)f(r) + A_j(r)C_{ri}^j(r) \right] C_{pq}^k \epsilon^{ikl}\epsilon^{pql}e^{2\alpha\phi} = 0 \quad (9.3)$$

$$\frac{M_A^2}{2}e^{2\beta\phi}A_m(r) = \left\{ \left[A_m'(r)f(r) + A_j(r)C_{rm}^j(r) \right] e^{2\alpha\phi} \right\}' f(r)$$

$$+ \left[A_m'(r)f(r) + A_j(r)C_{rm}^j(r) \right] C_{rt}^t(r)e^{2\alpha\phi}$$

$$- \frac{1}{4}A_i(r)C_{pq}^i C_{kl}^j \epsilon^{jpq}\epsilon^{mkl}e^{2\alpha\phi} + [A_i'(r)f(r) + A_j(r)C_{ri}^j(r)]C_{rp}^k(r)\epsilon^{ikl}\epsilon^{mpl}e^{2\alpha\phi} \quad (9.4)$$

with a similar equation for B.

- Scalar field:

$$2\phi'(r)f(r)\epsilon^{mjk}C_{ri}^m(r)\epsilon^{ijk} + 4\phi'(r)f(r)C_{rt}^t(r) + 4(\phi'(r)f(r))'f(r)$$

$$- \frac{\alpha}{2}e^{2\alpha\phi}(F_A^2 + F_B^2) - \frac{\beta}{2}e^{2\beta\phi}(M_A^2 A^2 + M_B^2 B^2) + 2\delta e^{2\delta\phi}\Lambda = 0 \quad (9.5)$$

The gauge curvature is given by

$$F_A(r) = \left[A_i'(r)f(r) + A_j(r)C_{ri}^j(r) \right]\sigma^r \wedge \sigma^i + [A_t'(r)f(r) + A_t(r)C_{rt}^t(r)]\sigma^r \wedge \sigma^t$$

$$+ \frac{1}{2}A_i(r)C_{jk}^i\sigma^j \wedge \sigma^k \quad (9.6)$$

with a similar equation for B.

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[9]In fact the above action is the same as eq. (3.15) generalized to include two gauge fields, up to the following simple field redefinitions: $\phi_{here} = \frac{1}{2}\phi_{there}$, $\alpha_{here} = 2\alpha_{there}$, $\delta_{here} = 2\delta_{there}$, $\beta = 2\epsilon$.

9.1 $A_{4,2}^a$

Many of the space-times in the classification before are simply scaling geometries with a spatial 3-fold belonging to one of the 3d Bianchi types. They have already been dealt with in previous sections. We will therefore focus on the geometries with non-trivial radial actions on the spatial geometries, and $A_{4,2}^a$ is the first such case. A hyperscaling violating version of the $A_{4,2}^a$ geometry takes the following metric:

$$ds^2 = -e^{2\beta_t r}dt^2 + \frac{dr^2}{f(r)^2\beta_1^2} + (\omega^1)^2 + \beta_2^2(\omega^2)^2 + (\omega^3)^2 \tag{9.7}$$

The structure coefficients in the orthonomal basis becomes:

$$C_{34}^2(r) = \beta_2\beta_1 f(r), \quad C_{14}^1(r) = a\beta_1 f(r), \quad C_{24}^2(r) = C_{34}^3(r) = \beta_1 f(r), \tag{9.8}$$

where again $f(r) = e^{\theta r}$. We will turn on the two massive vector fields and the dilaton as:

$$A(r) = A_t e^{-\omega\phi(r)}\sigma^t, \quad B(r) = B_2 e^{-\omega\phi(r)}\sigma^2 + B_3 e^{-\omega\phi(r)}\sigma^3, \quad \phi(r) = kr.$$

If we take

$$\alpha = \beta - \frac{\theta}{k}, \quad \delta = \frac{\theta}{k}, \quad \theta = k(\beta - \omega),$$

then the equations of motion becomes algebraic:

- EOM for A:
$$M_A^2 + 2(2 + a - k\beta)\beta_1^2(\beta_t - k\omega) = 0 \tag{9.9}$$

- EOM for B:

$$2B_3\beta_1^2\beta_2(1 + k\omega) + B_2(M_B^2 + 2\beta_1^2(-1 - a + k\beta + \beta_2^2$$
$$+\beta_t + k(-1 - a + k\beta + \beta_t)\omega)) = 0 \tag{9.10}$$
$$2B_2\beta_1^2\beta_2(-1 - a + k\beta + \beta_t) + B_3(M_B^2 - 2\beta_1^2(1 + a - k\beta - \beta_t)(1 + k\omega)) = 0 \tag{9.11}$$

- EOM for ϕ:

$$- B_2^2 M_B^2\beta - B_3^2 M_B^2\beta - 8(2 + a)k\beta_1^2 + 8k^2\beta\beta_1^2 + 8k\beta_1^2\beta_t + 4\Lambda(\beta - \omega)$$
$$-2\beta_1^2(B_3^2 + 4k^2 + B_2^2(1 + \beta_2^2) + 2\beta_2 B_2 B_3)\omega - 4k\beta_1^2(B_2^2 + B_3^2 + B_2 B_3\beta_2)\omega^2$$
$$-2(B_2^2 + B_3^2)k^2\beta_1^2\omega^3 + A_t^2(M_A^2\beta + 2\beta_1^2\omega(\beta_t - k\omega)^2) = 0 \tag{9.12}$$

- Einstein's Equations:

$$- 4\Lambda + B_3^2(M_B^2 + 2\beta_1^2) + 4B_2 B_3\beta_1^2\beta_2 + B_2^2(M_B^2 + 2\beta_1^2(1 + \beta_2^2))$$
$$+2\beta_1^2(12 + \beta_2^2 - 8\beta_t + 4(k^2 + k\beta(-2 + \beta_t) + \beta_t^2)) - A_t^2(M_A^2 + 2\beta_1^2(\beta_t - k\omega)^2)$$
$$+2k\beta_1^2\omega(8 + 2B_2 B_3\beta_2 - 4\beta_t + B_2^2(2 + k\omega) + B_3^2(2 + k\omega)) = 0 \tag{9.13}$$

$$-4\Lambda + B_3^2(M_B^2 + 2\beta_1^2) + 4B_2 B_3\beta_1^2\beta_2 - B_2^2(M_B^2 - 2\beta_1^2(-1 + \beta_2^2))$$

$$+2\beta_1^2(4+4a^2+3\beta_2^2-4\beta_t+4(k^2+k\beta(-1+\beta_t)+\beta_t^2)-4a(-1+k\beta+\beta_t))$$
$$-A_t^2(M_A^2+2\beta_1^2(\beta_t-k\omega)^2)$$
$$+2k\beta_1^2\omega(4+4a+2B_2B_3\beta_2-4\beta_t-B_2^2(2+k\omega)+B_3^2(2+k\omega))=0 \quad (9.14)$$

$$-2B_2^2\beta_1^2\beta_2(1+k\omega)-2\beta_1^2\beta_2(2+a-k\beta-\beta_t+k\omega)-B_2B_3(M_B^2+2(\beta_1+k\beta_1\omega)^2)=0 \quad (9.15)$$

$$-4\Lambda-B_3^2(M_B^2+2\beta_1^2)-4B_2B_3\beta_1^2\beta_2+B_2^2(M_B^2-2\beta_1^2(-1+\beta_2^2))$$
$$+2\beta_1^2(4+4a^2-\beta_2^2-4\beta_t+4(k^2+k\beta(-1+\beta_t)+\beta_t^2)-4a(-1+k\beta+\beta_t))$$
$$-A_t^2(M_A^2+2\beta_1^2(\beta_t-k\omega)^2)$$
$$+2k\beta_1^2\omega(4+4a-2B_2B_3\beta_2-4\beta_t+B_2^2(2+k\omega)-B_3^2(2+k\omega))=0 \quad (9.16)$$

$$-4\Lambda+B_3^2(M_B^2-2\beta_1^2)-4B_2B_3\beta_1^2\beta_2+B_2^2(M_B^2-2\beta_1^2(1+\beta_2^2))$$
$$-2\beta_1^2(-4+4k^2+\beta_2^2+4a(\beta_t-2)+8\beta_t)-A_t^2(M_A^2-2\beta_1^2(\beta_t-k\omega)^2)$$
$$-2k\beta_1^2\omega(2B_2B_3\beta_2+B_2^2(2+k\omega)+B_3^2(2+k\omega))=0 \quad (9.17)$$

$$4\Lambda-B_3^2(M_B^2+2\beta_1^2)-4B_2B_3\beta_1^2\beta_2-B_2^2(M_B^2+2\beta_1^2(1+\beta_2^2))$$
$$-2\beta_1^2(4(3+2a+a^2+k^2-(a+2)k\beta)+\beta_2^2)-A_t^2(M_A^2+2\beta_1^2(\beta_t-k\omega)^2)$$
$$-2k\beta_1^2\omega(8+4a+2B_2B_3\beta_2+B_2^2(2+k\omega)+B_3^2(2+k\omega))=0 \quad (9.18)$$

A general solution is impractical for such a system of algebraic equations. Instead, we will show that a solution exists for some particular values of the parameters, which gives a thermodynamically reasonable space-time. In particular, we pick $a=2, \beta_1=\frac{3}{2}, \beta_t=-2, \beta_2=2$. The following solution is obtained:

$$k=-\frac{6}{\epsilon}, \quad \Lambda=-54+\frac{162}{\epsilon^2}, \quad B_2=\epsilon\frac{\sqrt{2(1+\sqrt{26+\frac{12\beta(3\beta+5\epsilon)}{\epsilon^2}})}}{6\beta+5\epsilon},$$

$$A_t=\frac{\sqrt{36+\epsilon^2(-12+\sqrt{26+\frac{12\beta(3\beta+5\epsilon)}{\epsilon^2}})}}{3\beta+2\epsilon},$$

$$B_3=B_2\frac{\epsilon(1+\sqrt{26+\frac{12\beta(3\beta+5\epsilon)}{\epsilon^2}})}{6\beta+5\epsilon}, \quad M_A^2=-18\frac{(3\beta+2\epsilon)^2}{\epsilon^2},$$

$$M_B^2=-\frac{9}{2\epsilon^2}\left(36\beta^2+60\beta\epsilon+\epsilon^2\left(27-2\sqrt{26+\frac{12\beta(3\beta+5\epsilon)}{\epsilon^2}}\right)\right), \quad (9.19)$$

where $\epsilon=\omega-\beta$. We see that a large open set of β,ϵ will yield real solutions with a positive value of Λ (so that it is, quite plausibly, ultimately gluable to AdS_5). For example, taking $\beta=-\frac{1}{2}, \epsilon=1$, we have $k=-6, A_t=2\sqrt{24+\sqrt{5}}, B_2=\sqrt{(1+\sqrt{5})/2}, B_3=(1+\sqrt{5})^{3/2}/2\sqrt{2}, \Lambda=108, M_A^2=-\frac{9}{2}, M_B^2=-\frac{9}{2}(6-2\sqrt{5})$. Also notice that the solution is not valid when $\epsilon=0$, which corresponds to the conformal case $\theta=0$. This fact remains true for other values of the parameters, indicating that in this system $A_{4,2}^a$ is not realizable unless it has non-zero hyperscaling violation exponent θ.

9.2 $A_{4,3}$

This is a simplified version of $A_{4,2}^a$ with the structure coefficients

$$C_{34}^2(r) = \beta_2\beta_1 f(r), \quad C_{14}^1(r) = \beta_1 f(r). \tag{9.20}$$

It can be shown that with the same matter content as the previous example, this class of geometries is also realizable. Again, if we fix that $\beta_1 = \frac{3}{2}, \beta_t = -2, \beta_2 = 2$, the solutions take the form:

$$k = -\frac{3}{\epsilon}, \quad \Lambda = -\frac{9}{2} + \frac{81}{2\epsilon^2}, \quad B_2 = \epsilon\frac{\sqrt{2(1 + \sqrt{10 + \frac{9\beta(\beta+2\epsilon)}{\epsilon^2}})}}{3(\beta + \epsilon)},$$

$$A_t = \frac{2\sqrt{9 + \epsilon^2(-1 + \sqrt{10 + \frac{9\beta(\beta+2\epsilon)}{\epsilon^2}})}}{3\beta + \epsilon}, \quad M_A^2 = -\frac{9(3\beta + \epsilon)^2}{2\epsilon^2},$$

$$B_3 = B_2\frac{\epsilon(1 + \sqrt{10 + \frac{9\beta(\beta+2\epsilon)}{\epsilon^2}})}{3(\beta + \epsilon)},$$

$$M_B^2 = -\frac{9}{2\epsilon^2}\left(9\beta^2 + 18\beta\epsilon + \epsilon^2\left(11 - 2\sqrt{10 + \frac{9\beta(\beta + 2\epsilon)}{\epsilon^2}}\right)\right). \tag{9.21}$$

Similarly to the previous case, a large open set of β and ϵ can yield real solutions with positive Λ; and this system can only support geometries with non-zero hyperscaling violation.

9.3 $A_{4,6}^{a,b}$

This geometry has the interesting feature that the radial action involves both scaling and rotation on the spatial 3-manifold. The hyperscaling violating version of the geometry has the following metric:

$$ds^2 = -e^{2\beta_t r}dt^2 + \frac{dr^2}{f(r)^2\beta_1^2} + (\omega^1)^2 + (\omega^2)^2 + \lambda^2(\omega^3)^2$$

with structure coefficients:

$$C_{14}^1(r) = a\beta_1 f(r), \quad C_{24}^2(r) = C_{34}^3(r) = b\beta_1 f(r), \quad C_{34}^2(r) = \frac{\beta_1}{\lambda}f(r), \quad C_{24}^3(r) = -\lambda\beta_1 f(r)$$

Since the radial action rotates the 2-3 plane of the geometry, choosing one of the vector fields to be aligned with either of σ^2, σ^3 will suffice to be general. Therefore we will turn on the following fields:

$$A(r) = A_t e^{-\omega\phi(r)}\sigma^t, \quad B(r) = B_2 e^{-\omega\phi(r)}\sigma^2, \quad \phi(r) = kr$$

As before, taking $\alpha = \beta - \frac{\theta}{k}, \delta = \frac{\theta}{k}, \theta = k(\beta - \omega)$ will reduce the equations to algebraic equations:

- A_t:

$$M_A^2 + 2(a + 2b - k\beta)\beta_1^2(\beta_t - k\omega) = 0 \tag{9.22}$$

- B_2 :

$$M_B^2\lambda^2 + 2\beta_1^2(1 - (a + b - k\beta - \beta_t)\lambda^2(b + k\omega)) = 0 \qquad (9.23)$$
$$a + b - k\beta - \beta_t + b\lambda^2 + k\lambda^2\omega = 0 \qquad (9.24)$$

- ϕ :

$$(B_2^2 M_B^2\beta - 8k\beta_1^2(-a - 2b + k\beta + \beta_t))\lambda^2 + 2\beta_1^2(B_2^2 + (b^2 B_2^2 + 4k^2)\lambda^2)\omega$$
$$+ 4b B_2^2 k\beta_1^2\lambda^2\omega^2 + 2B_2^2 k^2\beta_1^2\lambda^2\omega^3 + 4\Lambda\lambda^2(-\beta + \omega)$$
$$- A_t^2\lambda^2(M_A^2\beta + 2\beta_1^2\omega(\beta_t - k\omega)^2) = 0 \quad (9.25)$$

- Einstein's Equations:

$$(-4\Lambda - A_t^2 M_A^2 + B_2^2 M_B^2)\lambda^2 + 2\beta_1^2\Big(1 + B_2^2(1 + \lambda^2(b + k\omega)^2)$$
$$+ \lambda^2(-2 + 12b^2 - (-4 + A_t^2)\beta_t^2 + \lambda^2 - 8b(\beta_t + k(\beta - \omega))$$
$$+ 2k\beta_t(2\beta + (A_t^2 - 2)\omega) + k^2(4 - A_t^2\omega^2))\Big) = 0 \quad (9.26)$$

$$- (4\Lambda + A_t^2 M_A^2 + B_2^2 M_B^2)\lambda^2 + 2\beta_1^2\Big(3 + (-2 + 4(a^2 + ab + b^2))\lambda^2$$
$$- B_2^2((b\lambda + k\lambda\omega)^2 - 1) - \lambda^2(4k\beta(a + b - \beta_t) + \beta_t(4(a + b) + (A_t^2 - 4)\beta_t)$$
$$+ \lambda^2 - 2k(2(a + b) + (A_t^2 - 2)\beta_t)\omega + k^2(A_t^2\omega^2 - 4))\Big) = 0 \quad (9.27)$$

$$(-4\Lambda - A_t^2 M_A^2 + B_2^2 M_B^2)\lambda^2 + 2\beta_1^2\Big(-1 + (-2 + 4(a^2 + ab + b^2))\lambda^2$$
$$+ B_2^2((b\lambda + k\lambda\omega)^2 - 1) + \lambda^2(-4k\beta(a + b - \beta_t) - \beta_t(4(a + b)$$
$$+ (A_t^2 - 4)\beta_t) + 3\lambda^2 + 2k(2(a + b) + (A_t^2 - 2)\beta_t)\omega + k^2(4 - A_t^2\omega^2))\Big) = 0 \quad (9.28)$$

$$(-4\Lambda - A_t^2 M_A^2 + B_2^2 M_B^2)\lambda^2 + 2\beta_1^2\Big(-1 + (2 + 4b^2 - 4k^2 + a(8b - 4\beta_t)$$
$$- 8b\beta_t - \lambda^2 + A_t^2(\beta_t - k\omega)^2)\lambda^2 - B_2^2(1 + \lambda^2(b + k\omega)^2)\Big) = 0 \quad (9.29)$$

$$(4\Lambda - A_t^2 M_A^2 - B_2^2 M_B^2)\lambda^2 + 2\beta_1^2\Big(-1 - B_2^2((b\lambda + k\lambda\omega)^2 + 1)$$
$$- \lambda^2(-2 + 4a^2 + 12b^2 + 4k^2 + A_t^2\beta_t^2 + \lambda^2 + 8bk(\omega - \beta)$$
$$+ A_t^2 k\omega(k\omega - 2\beta_t) + 4a(2b + k(\omega - \beta)))\Big) = 0 \quad (9.30)$$

$$- b(2 + B_2^2 - 2\lambda^2) + a(\lambda^2 - 1) - (k\beta + \beta_t)(\lambda^2 - 1) + k(\lambda^2 - B_2^2 - 1)\omega = 0 \quad (9.31)$$

Again, we are not going to give a general solution, but instead simply observe that at $b = 1, \beta_t = -2, \lambda = 2$, the following solution is obtained:

$$k = -\frac{1}{\omega}, \qquad\qquad B_2 = \sqrt{15}, \qquad A_t = \frac{\sqrt{4 + 16\beta\omega + 34\omega^2}}{\omega}$$

$$\Lambda = \frac{\beta_1^2(16 + 64\beta\omega + 121\omega^2)}{8\omega^2}, \qquad a = -3 - \frac{\beta}{\omega},$$

$$M_A^2 = -2\beta_1^2, \qquad\qquad M_B^2 = -\frac{\beta_1^2}{2}. \tag{9.32}$$

It is easy to see that by making for example $\beta = -4\omega \sim \varepsilon$ for some $\varepsilon \ll 1$, we can obtain real solutions with $a > 0$, which appear thermodynamically stable; and $\Lambda > 0$, which allows it to be glued to AdS_5. The conformal case $\theta = 0 \rightarrow \beta = \omega$ is apparently valid, except that it will make $a + 2b < 0$, hence is not thermodynamically allowed (by "Nernst's law" of section 8).

9.4 Comment

In summary, in this section, we have found (hyperscaling-violating analogues of) extremal scaling metrics governed by several of the four-algebras consistent with the NEC and "Nernst's law." It is interesting that we have been unable, as yet, to find such metrics without hyperscaling violationi, i.e. with $\theta = 0$. We leave this to further work. All the examples we have considered have a three-dimensional sub-algebra which acts on the spatial directions in which the field theory lives. The gravity description suggests there could be even more novel possibilities where the bulk geometry is homogeneous with a symmetry group which does not have such a three-dimensional sub-algebra that acts on the field theory directions alone. An exploration of such geometries and their field theory duals is also left for the future.

10 4d metrics governed by three and four-algebras

In section 8 and section 9, we have discussed 5d space-time. In this section, we discuss 4d space-time where the radial direction is nontrivially involved in realizations of the real three and four-algebras we've been discussing. We have seen that the fairly general static homogeneous horizons in 5d can be governed by a four-algebra involving the three "field theory" spatial coordinates and the radial direction. Quite analogously, the Bianchi three-algebras can arise as symmetry algebras of general horizons in 4d. In such cases, the Bianchi three-algebra mixes the two field theory spatial coordinates and the radial direction. In the first part of this section, we investigate examples where the radial and field theory spatial coordinates are intertwined in a non-trivial way by the Bianchi three-algebras. Then, in the second part of this section, we investigate the examples where four-algebras can arise as symmetry algebras of horizons in 4d, where the radial and field theory spatial and time coordinates can all be intertwined. We restrict our attention to only the static metrics, and discuss conditions that the NEC places on them. We will see that some of the types are excluded.

10.1 Bianchi three-algebras for radial and two-spatial directions

We first discuss 4d space-times where the radial direction is nontrivially involved in the three-algebras. Here, we give some examples of this sort, which should be easily generalizable.

We will illustrate the type III examples. The Killing vectors e_i of type III satisfy $[e_i, e_j] = C^k_{ij} e_k$ where $C^1_{13} = -C^1_{31} = 1$ and the rest of the $C^i_{j,k} = 0$. The Killing vectors and invariant one forms are given as

$$
\begin{aligned}
e_1 &= \partial_2 & \omega^1 &= e^{-x^1} dx^2 \\
e_2 &= \partial_3 & \omega^2 &= dx^3 \\
e_3 &= \partial_1 + x^2 \partial_2 & \omega^3 &= dx^1 \, .
\end{aligned}
$$

This three-algebra has three two-dimensional sub-algebras given by $\{e_1, e_2\}$, $\{e_2, e_3\}$ and $\{e_1, e_3\}$. We consider two possible embeddings of the two-algebras in the three-algebra.

- **Case 1:**
  Consider $\{e_1, e_2\}$ to correspond to symmetries along the field theory spatial directions and $x^1 = r$ to be the radial direction. Then the metric can be written as

  $$
  ds^2 = L^2 \left[-e^{2\beta_t r} dt^2 + dr^2 + e^{-2r}(dx^2)^2 + (dx^3)^2 \right] \, . \tag{10.1}
  $$

  Time is added as an extra direction, modifying e_3 to $e_3 = \partial_r + x^2 \partial_2 + \beta_t t \partial_t$ and adding $e_4 = \partial_t$ to the set of Killing vectors.

- **Case 2:**
  Consider $\{e_1, e_3\}$ to correspond to symmetries along the field theory spatial directions and $x^3 = r$ to be the radial direction. Then the metric can be written as

  $$
  ds^2 = L_1^2 \left[-e^{2\beta_t r} dt^2 + dr^2 \right] + L_2^2 \left[(dx^1)^2 + e^{-2x^1}(dx^2)^2 \right] \, . \tag{10.2}
  $$

  Time is added as an extra direction, modifying e_2 to $e_2 = \partial_r + \beta_t t \partial_t$ and adding $e_4 = \partial_t$ to the set of Killing vectors.

We will illustrate that both case 1 and case 2 can be obtained as solutions of Einstein gravity coupled to a (either massive or massless) gauge field with/without scalar fields.

- **Case 1**

  Case 1 can be obtained from the Einstein action coupled to a massive vector and a massless scalar field,

  $$
  S = \int dx^4 \sqrt{-g} \left[R + \Lambda - \frac{1}{4} F^2 - \frac{1}{4} m^2 A^2 - 2(\partial\phi)^2 \right] \, . \tag{10.3}
  $$

  Along with the metric ansatz given by eq. (10.1), we consider the ansatz for the gauge field and scalar field:

  $$
  A = \sqrt{A_t} e^{\beta_t r} dt \, , \tag{10.4}
  $$
  $$
  \phi = \phi_1 x^3 \, . \tag{10.5}
  $$

  The scalar field equation is identically satisfied. The vector field equation gives

  $$
  \sqrt{A_t}(m^2 L^2 + 2\beta_t) = 0 \, . \tag{10.6}
  $$

The Einstein equations are given by

$$\frac{A_t}{L^2}\left(m^2L^2 + 2\beta_t^2\right) + 8(1+\phi_1^2) - 4\Lambda L^2 = 0$$

$$\frac{A_t}{L^2}\left(m^2L^2 - 2\beta_t^2\right) + 8(\beta_t - \phi_1^2) + 4\Lambda L^2 = 0$$

$$\frac{A_t}{L^2}\left(m^2L^2 + 2\beta_t^2\right) - 8(\beta_t^2 + \phi_1^2) + 4\Lambda L^2 = 0$$

$$\frac{A_t}{L^2}\left(m^2L^2 + 2\beta_t^2\right) + 8(\beta_t - \beta_t^2 - 1 + \phi_1^2) + 4\Lambda L^2 = 0\,.$$

The equations can be solved to get

$$m^2L^2 = -2\beta_t\,, \tag{10.7}$$

$$\Lambda L^2 = 2 - \beta_t + \beta_t^2\,, \tag{10.8}$$

$$A_t = 2L^2(1 + \frac{1}{\beta_t})\,, \tag{10.9}$$

$$\phi_1^2 = \frac{1}{2}(1 - \beta_t)\,. \tag{10.10}$$

So we have $\Lambda > 0$. In order to satisfy Nernst's law (i.e. that the horizon area vanishes at the horizon), we need $\beta_t < 0$, and therefore $m^2 > 0$ and $\phi_1^2 > 0$. With this, $A_t > 0$ implies $\beta_t < -1$.

- **Case 2**

  Case 2 can be obtained as a solution of the Einstein-Maxwell action,

  $$S = \int dx^4 \sqrt{-g}\, \left[R + \Lambda - \frac{1}{4}F^2\right]\,. \tag{10.11}$$

  Along with the metric ansatz given by eq. (10.2), we consider the gauge field ansatz to be

  $$A = \sqrt{A_t}\, e^{\beta_t r} dt\,. \tag{10.12}$$

  The Maxwell equation is identically satisfied. Then the Einstein equations give only two independent equations:

  $$\frac{A_t \beta_t^2}{L_1^2} + 4\frac{L_1^2}{L_2^2} - 2\Lambda L_1^2 = 0 \tag{10.13}$$

  $$\frac{A_t \beta_t^2}{L_1^2} - 4\beta_t^2 + 2\Lambda L_1^2 = 0\,. \tag{10.14}$$

  The solution is given by

  $$A_t = 2L_1^2\left(1 + \frac{1}{1 - L_2^2\Lambda}\right)\,, \tag{10.15}$$

  $$\beta_t^2 = L_1^2\left(-\frac{1}{L_2^2} + \Lambda\right)\,. \tag{10.16}$$

$\beta_t^2 > 0$ implies $\Lambda L_2^2 > 1$.

10.2 Bianchi four-algebras $A_{4,k}$ in 4d Bianchi attractors

So far we investigated examples of the three-algebras realized in 4d space-time. We now investigate 4d realizations of the four-algebras $A_{4,k}$. Since this involves the field theory time coordinates as well, generically this induces a time-dependent metric. In this paper, we are rather interested in time-independent metrics, which may be dual to (time-independent) ground states of doped field theories. Therefore we seek metrics which do not involve the time-coordinate explicitly. This leaves us with only two possibilities; either the metric should be static, or it should be stationary. Here we restrict our attention to static metrics in 4d, but it is straightforward to generalize the analysis to include the stationary cases.[10]

A suitable static metric is as follows. For simplicity, we consider the diagonal metric ansatz and takes the "obvious" choice for the coordinate identification. Then, the metric ansatz is

$$ds^2 = \sum_{i=1}^{4} \eta_i (\omega^i)^2 \tag{10.17}$$

where ω^i are the invariant one-forms of $A_{4,k}$ and we require all $\eta_i > 0$. Since the static property restricts the metric to the form

$$ds^2 = -e^{2\beta_t r} dt^2 + ds^2_{3D\,\text{Bianchi}} , \tag{10.18}$$

we will see later that this form can be obtained only from the four-algebras $A_{4,2}^a$, $A_{4,3}$, $A_{4,5}^{a,b}$, $A_{4,6}^{a,b}$. The three dimensional subgroup G, which acts on the field theory spatial coordinates and radial direction, turns out to be type IV, II, VI, VII$_b$ in the 3d Bianchi's classification (or, $A_{3,2}$, $A_{3,1}$, $A_{3,5}^a$, $A_{3,7}^b$ in the notation of [44]) for the $A_{4,k}$ with $k = 2, 3, 5, 6$ respectively.

We will investigate the metric ansatz for each $A_{4,k}$ type given by eq. (10.17) and also the corresponding null energy condition eq. (8.2) and "Nernst law" constraint as in section 8 for each class of static metrics. We choose our null vectors in investigating the NEC constraints as in eq. (8.4), but here i only runs over three values.

- $A_{4,1}$:

  The invariant one-forms are manifestly dependent on x^4. So we should not choose x^4 as time if we wish to obtain a static metric. Furthermore, if we choose time as one of the x^i $(i = 1, 2, 3)$, then the diagonal metric ansatz yields a stationary metric. For example, if we choose x^1 as the time and x^4 as the radial directions, the metric ansatz becomes like;

  $$ds^2 = -\left(dt - r dx + \frac{1}{2} r^2 dy\right)^2 + \eta_r dr^2 + \eta_x (dx - r dy)^2 + \eta_y dy^2 , \tag{10.19}$$

  and this generically induces closed time-like curves (CTC). So we discard this case without further exploration.

---

[10]Once we allow the stationary cases, we have to worry about the possible presence of closed time-like curves, as seen in [1].

- $A^a_{4,2}$:

Again in this case, we should not choose x^4 as time since then the metric will be manifestly time-dependent. Then, the best choice is to select x^1 as time and x^4 as the radial direction, yielding a metric ansatz

$$ds^2 = -e^{-2ar}dt^2 + \eta_r dr^2 + e^{-2r}\left(\eta_x (dx - rdy)^2 + \eta_y dy^2\right), \qquad (10.20)$$

which is static. We have $\eta_r > 0$, $\eta_x > 0$, $\eta_y > 0$. In this case, the spatial part of the metric coordinatized by (x, y, r) forms type IV of Bianchi's 3d classification, or $A_{3,2}$ in the notation of [44]. (See appendix B).

The non-zero components of the Einstein tensor are given by

$$G_{11} = -\frac{\eta_x + 12\eta_y}{4\eta_r\eta_y}, \qquad G_{22} = 2a + 1 - \frac{\eta_x}{4\eta_y}, \qquad (10.21)$$

$$G_{33} = \frac{\eta_x\left(3\eta_x + 4(a^2 + a + 1)\eta_y\right)}{4\eta_r\eta_y}, \qquad G_{34} = -\frac{(a+2)\eta_x}{2\eta_r}, \qquad (10.22)$$

$$G_{44} = -\frac{\eta_x - 4(a^2 + a + 1)\eta_y}{4\eta_r}. \qquad (10.23)$$

Then, for the arbitrary null vector $N^\mu = (\sqrt{\sum_{i=1}^{3}(s^i)^2}, \frac{s^1}{\sqrt{\eta_r}}, \frac{s^2}{\sqrt{\eta_x}}, \frac{s^3}{\sqrt{\eta_y}})$, the null energy condition gives:

$$N^\mu N^\nu T_{\mu\nu} = N^\mu N^\nu G_{\mu\nu} = \frac{f_1(s^1)^2 + f_2(s^2)^2 + f_3 s^2 s^3 + f_4(s^3)^2}{2\eta_r\eta_y} \geq 0 \quad (10.24)$$

where

$$f_1 = -\eta_x + 4(a-1)\eta_y, \qquad f_2 = \eta_x + 2(a-1)(a+2)\eta_y, \qquad (10.25)$$

$$f_3 = -2(a+2)\sqrt{\eta_x\eta_y}, \qquad f_4 = -\eta_x + 2(a-1)(a+2)\eta_y, \qquad (10.26)$$

for arbitrary s^1, s^2, and s^3. Therefore we need

$$f_i \geq 0 \quad (i = 1, 2, 4), \qquad 4f_2 f_4 \geq (f_3)^2. \qquad (10.27)$$

Also, to satisfy Nernst's law, we need $a > 0$.

Let us set $\eta_x = 1$ by a coordinate change. We can then show for large positive a and η_y,

$$f_1 \to 4a\eta_y > 0, \qquad f_2 \to 2a^2\eta_y > 0, \qquad f_4 \to 2a^2\eta_y > 0,$$
$$4f_2 f_4 \to 16a^2\eta_y^2 \gg (f_3)^2 \to 4a^2\eta_y. \qquad (10.28)$$

Therefore, eq. (10.27) allows at least solutions at large (a, η_y).

If we choose x^2 or x^3 as the radial direction instead, but keep using x^1 as time in order to obtain a static metric, the warp factor is not a function of the radial direction alone. And if we do not choose time as x^1, then the metric becomes stationary at most, but not static. We postpone further investigation of these geometries.

- $A_{4,3}$:

  This has a very similar structure to the $A_{4,2}^a$ case above. So again to obtain a static metric, the simplest approach is to choose x^1 as time. If we choose x^4 as the radial direction, then the metric ends up with following form:

  $$ds^2 = -e^{-2r}dt^2 + \eta_r dr^2 + \eta_x \left(dx - rdy\right)^2 + \eta_y dy^2 \,. \tag{10.29}$$

  The only difference between this case and $A_{4,2}^a$ is the way in which radial warping appears in various components. In these cases, the spatial part of the metric coordinated by (x, y, r) form Bianchi's 3d classification type II, or $A_{3,1}$ (appendix B).

  Note that in this case the horizon volume is independent of the radial coordinate. So special care must be taken for analysis of Nernst's law or "physical reasonableness". Let us first check whether the Null Energy Condition is satisfied.

  The non-zero components of the Einstein tensor are given by

  $$G_{11} = -\frac{\eta_x}{4\eta_r\eta_y} \,, \qquad G_{22} = -\frac{\eta_x}{4\eta_y} < 0 \,, \tag{10.30}$$

  $$G_{33} = \frac{\eta_x \left(3\eta_x + 4\eta_y\right)}{4\eta_r\eta_y} \,, \qquad G_{34} = -\frac{\eta_x}{2\eta_r} \,, \tag{10.31}$$

  $$G_{44} = -\frac{\eta_x - 4\eta_y}{4\eta_r} \,. \tag{10.32}$$

  It follows that for a null vector $N^\mu = (1\,, \frac{1}{\sqrt{\eta_r}}\,, 0\,, 0)$, the Null Energy Condition is violated;

  $$N^\mu N^\nu G_{\mu\nu} = -\frac{\eta_x}{2\eta_r\eta_y} < 0 \,. \tag{10.33}$$

  Therefore we cannot obtain this space-time from physical matter systems.

- $A_{4,4}$:

  This has a very similar structure to the $A_{4,1}$ case. Choosing x^4 as the radial direction, the warp factors as a function of r are the only difference. Again, there are three choices for time from x^1, x^2, x^3, but any choice always admits at most a stationary metric, but not a static one.

- $A_{4,5}^{a,b}$:

  In this case, the simplest choice for a radial direction is x^4, since it admits warp factors in the metric which are functions of r only, consistent with the algebraic structure. (If we do not choose x^4 as the radial direction, then the warp factors are not functions of radius alone). There are three equally good choice for time: x^1, x^2, or x^3. All give equivalently good static metric ansatzes. For example, setting x^3 to be time, we have a static metric

  $$ds^2 = -e^{-2br}dt^2 + \eta_r dr^2 + e^{-2r}dx^2 + e^{-2ar}dy^2 \,. \tag{10.34}$$

Similarly setting either x^1 or x^2 as time, we obtain respectively,

$$ds^2 = -e^{-2r}dt^2 + \eta_r dr^2 + e^{-2ar}dx^2 + e^{-2br}dy^2 \,, \qquad (10.35)$$

and

$$ds^2 = -e^{-2ar}dt^2 + \eta_r dr^2 + e^{-2r}dx^2 + e^{-2br}dy^2 \,. \qquad (10.36)$$

Note that these are just generalized Lifshitz geometries. In all cases, the three spatial coordinates $(x,\, y,\, r)$ are in type VI of Bianchi's classification, or $A_{3,5}^a$ (appendix B) for generic $a \neq 0, 1$. For the special case $a = 1$, the metric reduces to Bianchi's type V, or $A_{3,3}$ (appendix B).

Let's consider the metric ansatz eq. (10.35). It is pretty straightforward to calculate the Einstein tensor, and we obtain it in diagonal form as

$$G_{11} = -\frac{a^2 + ba + b^2}{\eta_r} \,, \qquad G_{22} = ba + a + b \,, \qquad (10.37)$$

$$G_{33} = \frac{b^2 + b + 1}{\eta_r} \,, \qquad G_{44} = \frac{a^2 + a + 1}{\eta_r} \,. \qquad (10.38)$$

Therefore, for the choice of null vectors $N^\mu = (1, \frac{1}{\sqrt{\eta_r}}, 0, 0)$, $N^\mu = (1, 0, 1, 0)$, $N^\mu = (1, 0, 0, 1)$, respectively, we obtain

$$\frac{a - a^2 + b - b^2}{\eta_r} \geq 0 \,, \quad -\frac{(a-1)(a+b+1)}{\eta_r} \geq 0 \,, \quad -\frac{(b-1)(a+b+1)}{\eta_r} \geq 0 \,. \, (10.39)$$

The black brane horizon is at $r \to \infty$ where $g_{tt} \to 0$. In order to satisfy the Nernst's law, we need

$$e^{-(a+b)r} \to 0 \Rightarrow a + b > 0 \,. \qquad (10.40)$$

Then, with $\eta_r > 0$, (10.39) gives

$$a \leq 1 \,, \qquad b \leq 1 \,. \qquad (10.41)$$

Assuming that $g_{xx} \to 0$ and $g_{yy} \to 0$ at the horizon, we have a parameter regime where the Null Energy Condition is satisfied,

$$0 \leq a \leq 1 \,, \qquad 0 \leq b \leq 1 \,. \qquad (10.42)$$

Interestingly, on one boundary $a = b = 0$, we have $AdS_2 \times R^2$, and on the other boundary $a = b = 1$, we have AdS_4.

- $A_{4,6}^{a,b}$:

  Again, the best choice is to select x^1 as time and x^4 as the radial direction. Then, we obtain the static metric ansatz

$$ds^2 = e^{-2ar}dt^2 + \eta_r dr^2 + e^{-2br}\left(\eta_x \left(\cos r\, dx - \sin r\, dy\right)^2 + \eta_y \left(\cos r\, dy + \sin r\, dx\right)^2\right) \,. \quad (10.43)$$

The three spatial coordinate (x, y, r) form Bianchi's 3d classification type VII_b, or $A_{3,7}^b$ in the notation of [44][11] (appendix B).

It is pretty straightforward to calculate the Einstein tensor. Then, for the arbitrary null vector, $N^\mu = (\sqrt{\sum_{i=1}^3 (s^i)^2}, \frac{s^1}{\sqrt{\eta_r}}, \frac{s^2}{\sqrt{\eta_x}}, \frac{s^3}{\sqrt{\eta_y}})$, we have the NEC

$$N^\mu N^\nu T_{\mu\nu} = N^\mu N^\nu G_{\mu\nu} = \frac{f_1(s^1)^2 + f_2(s^2)^2 + f_3 s^2 s^3 + f_4(s^3)^2}{2\eta_x \eta_y \eta_r} \geq 0 \quad (10.45)$$

for arbitrary s^1, s^2, and s^3 with $\eta_r > 0$, $\eta_x > 0$, $\eta_y > 0$. Here, the f_i are given by:

$$f_1 = 2\eta_x \eta_y (2b(a-b) + 1) - \eta_x^2 - \eta_y^2, \quad (10.46)$$

$$f_2 = 2\eta_x \eta_y (a-b)(a+2b) + \eta_x^2 - \eta_y^2, \quad (10.47)$$

$$f_3 = 2(a+2b)(-\eta_x + \eta_y)\sqrt{\eta_x \eta_y}, \quad (10.48)$$

$$f_4 = 2\eta_x \eta_y (a-b)(a+2b) - \eta_x^2 + \eta_y^2. \quad (10.49)$$

Therefore we need,

$$f_i \geq 0 \quad (i = 1, 2, 4), \qquad 4f_2 f_4 \geq (f_3)^2. \quad (10.50)$$

In appendix C we discuss the explicit parameter ranges where these conditions are satisfied.

Other choices of coordinates yield either non-static metrics, or field-theory position-dependent warping factors in the metric.

- $A_{4,7}$ - $A_{4,12}$: By completely analogous reasoning to that appearing above, if we assume a diagonal metric ansatz and make the "obvious" choice(s) for the time and radial coordinates, we can obtain at most stationary metrics, but not static metrics.

In summary, for the diagonal and static metric ansatz (10.17), the allowed cases for which the NEC and Nernst law constraints can be met are, $A_{4,2}^a$, $A_{4,5}^{a,b}$, and $A_{4,6}^{a,b}$.

11 Discussion

In this paper, we have extended the program initiated in [1], of trying to classify a wide variety of less symmetric extremal near-horizon geometries for black branes, in several directions. We have demonstrated that one can simply modify these geometries to incorporate hyperscaling violation in the dual field theory. We have given examples of such hyperscaling

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[11]The fact that the three coordinates (x, y, r) span a manifold of type VII_b in Bianchi's classification, can be seen as follows (see [43]). We have chosen e_1 as the generator of time-translations, and the spatial subset e_2, e_3, e_4 forms a real three-algebra where antisymmetric structure constants are given by

$$C_{24}^2 = b, \; C_{24}^3 = -1, \; C_{34}^2 = 1, \; C_{34}^3 = b. \quad (10.44)$$

Defining the two-index constants C^{dc} by $C_{ab}^c \equiv \epsilon_{abd} C^{dc}$, and furthermore decomposing them into a symmetric and anti-symmetric part as $C^{ab} \equiv n^{ab} + \epsilon^{abc} a_c$, we can see that n^{ab} has two unit eigenvalues and one zero eigenvalue, and $a_c = (b, 0, 0)$. These are precisely the characteristics of Bianchi type VII_b.

violation both in 4d and 5d bulk space-time metrics which realize the Bianchi types, with the radial direction included in the algebra. We have also shown that one can easily obtain analytical "striped" metrics starting from the horizons of [1]. It should be clear in each case that in providing examples, we have barely scratched the surface of what are likely very rich sets of solutions to the Einstein equations or appropriate low-energy limits of string theory.

In the direction of finding a more complete classification, we discussed the possible application of the larger algebraic structures uncovered in [44, 45] to classify 5d extremal near-horizon geometries in terms of real four-algebras with preferred 3d subgroups. While we found that several of these possibilities will remain unrealized in sensible gravity coupled to matter theories (which satisfy the Null Energy Condition), others can be realized with simple matter sectors coupled to gravity, and likely arise as duals to suitable infrared phases in strongly-coupled quantum field theory. Similarly, we have classified the 4d extremal static near-horizon space-times with symmetries governed by the four-algebras, and we have seen that some of the types are forbidden due to the NEC, but others are likely attainable.

A number of issues remain to be clarified. In many cases, these near-horizon geometries manifest infrared singularities similar to that of the Lifshitz space-time [8, 50, 51]. In the Lifshitz case, various physical smoothings or more subtle "resolutions" of the metric singularity have been discussed in [52, 53]. The issue has also been addressed in isotropic space-times with hyperscaling violation in [54, 55]. It would be interesting to see if similar physics arises for the anisotropic space-times described here.

In addition, while we have given evidence in [1] that some of these horizons can be glued into asymptotically AdS space-time by suitable RG flows, this has not been discussed in anything close to a comprehensive way for the full classes of anisotropic metrics we've described. A careful study of which classes of geometries are truly infrared phases of doped CFTs (perhaps with additional currents activated on the boundary) would be worthwhile.

Another important question is to study the stability of the solutions found both in this paper and in [1]. Such a study could include analysing both whether the solutions are perturbatively stable, i.e., whether they have modes which grow with time, and also whether they are stable with respect to changes in boundary conditions, i.e., the presence of relevant deformations with respect to RG flow.

As always, one can wonder to what extent the rich set of possible phases found here manifest themselves in UV complete models derived from string theory. It would be useful to explore embeddings of these solutions into gauged supergravity theories[12] that can be derived from consistent truncation of IIA and IIB supergravity, for instance. It would also be interesting to find proposals for phases of matter in real systems which could give rise to some of the more exotic symmetry groups discussed here.

A simple, possibly interesting, extension of this work is to relax the condition of the geometry being static. In section 10, we have seen that the real four-algebras in 4d space-time naturally induce metrics which are not static, but can be stationary. However, one must be careful with such metrics, since they can easily contain closed time-like curves, as illustrated in [1] (see also [58]). Classifying 4d stationary space-times governed by the

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[12]See also [56, 57] for the study of interesting black brane solutions in gauged supergravity.

four-algebras which do not contain any such pathological features, could lead to interesting duals. In field theory systems, if external sources perturb the system, they can induce currents to flow. This would naturally correspond to a stationary metric in the putative gravity dual of the system. A concrete example is that of a fluid subjected to a temperature or an electric potential which is time independent and varying slowly in the spatial directions. Gravity duals of extremal geometries subject to such potential gradients would be worth studying further.[13]

The most ambitious possible extension would be to try and classify all extremal inhomogeneous, anisotropic black brane horizons. In light of the interesting scaling features shown in holographic transport in the simplest inhomogeneous geometries [59–61], this problem could be interesting for "applied holography" in addition to its intrinsic interest as a question in general relativity and string theory. Needless to say, finding all such inhomogeneous phases is a challenging question since the analysis cannot be reduced to merely solving algebraic equations now and instead requires us to confront coupled partial differential equations in their full glory. The striped phase discussed in section 6 is an example of an inhomogeneous phase and our discussion in that section can be viewed as a small step in this direction. Clearly though, much more effort is needed to make progress on this issue.

Acknowledgments

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[13]We thank Shiraz Minwalla, Mukund Rangamani and especially Sayantani Bhattacharyya for a discussion of this issue.

A Einstein gravity coupled to massive vector fields

In many sections of the paper, we will want to exhibit explicit solutions of Einstein gravity coupled to a reasonable matter sector, to show that the near-horizon geometries we propose can arise in physically sensible systems. We will usually choose our matter sector to consist of a set of massive Abelian gauge fields A_a, described by the Lagrangian:

$$S = \int d^5x \sqrt{-g} \{R + \Lambda - \sum_a (\frac{1}{4}F_a^2 + \frac{1}{4}m_a^2 A_a^2)\} \tag{A.1}$$

In this section, we discuss some basic facts about the Einstein equations for the theory with matter (A.1).

A.1 Ricci curvature

We will be interested in static solutions of Einstein gravity. The spatial slices spanned by $\{r, x_1, x_2, x_3\}$ will be homogeneous spaces as in [1], though we will allow for slightly more general possibilities intertwining the radial and "field theory" spatial coordinates in the later sections of the paper.

As in [1], the basic objects of interest are invariant one-forms ω^i, annihilated by the Killing vectors e_i which generate the 4-dimensional isometry group $\mathcal{G}$. Metrics which are written purely in terms of the ω^i with constant coefficients, and with trivial t dependence:

$$ds^2 = -g_{tt}(r)dt^2 + \lambda_{ij}\omega^i \otimes \omega^j \tag{A.2}$$

will automatically be $\mathcal{G}$-invariant. Here we slightly modify the notation of [1]; the factors of $e^{-\beta_i r}$ there, which incorporate the scaling of the coordinates under radial translations, are built into the forms now, and the radial scaling symmetry is treated as part of $\mathcal{G}$.

In order to exploit the underlying symmetries of the homogeneous geometries, we will write down the Einstein-Maxwell equations in an orthnomal (vielbein) basis, in which the metric tensor takes the form:

$$ds^2 = \sum_{i=1}^3 (\sigma^i)^2 + (\sigma^r)^2 - (\sigma^t)^2 \tag{A.3}$$

The vielbein elements are simple linear combinations of the ωs; in the most trivial case, $\sigma^i = \omega^i$. In this formalism, we will in general lose the advantage that the one forms σ^μ are exact, and hence could be integrated into coordinates; instead they form an orthonormal non-commuting basis.

We are interested in scale-invariant (or conformally scale-invariant, in the case of hyperscaling-violating metrics) near-horizon geometries. In order for the generalised scaling to make sense, we require that the radial coordinate r to be identified with one of the exact one-forms $\sigma^r = \lambda_4 dr = dx^4$, such that the other 3 one-forms form a sub-algebra. In this appendix A, we set $\lambda_4 = 1$. We further demand that $(\sigma^t)^2 = g_{tt}(r)dt^2$, in keeping with (A.2). Therefore, the orthnomal non-commuting basis satisfies the following commutation relation:

$$d\sigma^\mu = \frac{1}{2}C^\mu_{\nu\alpha}\sigma^\nu \wedge \sigma^\alpha . \tag{A.4}$$

The constants $C^i_{\alpha\beta}$ for $i \in \{1, 2, 3\}$ and $\alpha, \beta \in \{1, 2, 3, r\}$ are given by the data of the four-algebra; $C^r_{\mu\nu} = 0$; $C^t_{tr} = -\frac{1}{2}g'_{tt}(r)/g_{tt}(r)$. With this set-up the connection form $\Gamma^\mu_{\nu\beta}$ are no longer given by the Christoffel symbols, but instead are given by the structure constants via:

$$\Gamma^\mu_{\alpha\beta} = -\frac{1}{2}(g_{\tau\alpha}g^{\mu\sigma}C^\tau_{\sigma\beta} + g_{\tau\beta}g^{\mu\sigma}C^\tau_{\sigma\alpha} - C^\mu_{\alpha\beta}) \ . \tag{A.5}$$

$g_{\mu\nu}$ is equal to the Minkowski $\eta_{\mu\nu}$ in this basis. See [43] for detail. The Riemann curvature tensor is given in this basis by the connection forms via:

$$R^\sigma_{\mu\alpha\beta} = \Gamma^\sigma_{\mu\beta,\alpha} - \Gamma^\sigma_{\mu\alpha,\beta} + \Gamma^\tau_{\mu\beta}\Gamma^\sigma_{\tau\alpha} - \Gamma^\tau_{\mu\alpha}\Gamma^\sigma_{\tau\beta} - (\Gamma^\tau_{\beta\alpha} - \Gamma^\tau_{\alpha\beta})\Gamma^\sigma_{\mu\tau} \tag{A.6}$$

Hence we can compute the Ricci curvature $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ completely in terms of the structure constants, which only depend on r. In the case where we also have scaling symmetry along the time direction, the entire space-time is homogeneous and the Ricci curvature is therefore algebraic. We will see that in this formalism, the Einstein equations reduce to algebraic equations, as in the story of generalized attractors discussed in e.g. [46, 47].

A.2 Maxwell's equations

Here, we present the Maxwell equations for a single massive Abelian gauge field. Since there are no cross-couplings between the vector fields A_a, the generalization to multiple vectors is trivial.

Assume that the vector potential takes the form $A(r) = A_t(r)\sigma^t + \sum_i A_i(r)\sigma^i$. Then the curvature is given by

$$F = [A'_i(r) + A_j(r)C^j_{ri}]\sigma^r \wedge \sigma^i + [A'_t(r) + A_t(r)C^t_{rt}]\sigma^r \wedge \sigma^t + \frac{1}{2}A_iC^i_{jk}\sigma^j \wedge \sigma^k \tag{A.7}$$

with components given by $F = \frac{1}{2}F_{\mu\nu}\sigma^\mu \wedge \sigma^\nu$. The Maxwell equations for a massive vector field are given in differential form as:

$$d \star_5 F = -\frac{1}{2}m^2 \star_5 A \tag{A.8}$$

Since the metric tensor is now Minkowskian, the Levi-Civita tensor reduces to the usual flat case. It is therefore straight-forward to obtain the following Maxwell's equations.

A.2.1 Magnetic field

In this case, $A_t(r) = 0$. The Maxwell's equations are:

$$(A'_i(r) + A_j(r)C^j_{ri})C^k_{mn}\epsilon^{ikl}\epsilon^{mnl} = 0 \tag{A.9}$$

$$\frac{1}{2}m^2 A_m(r) = (A'_m(r) + A_j(r)C^j_{rm})' + (A'_m(r) + A_j(r)C^j_{rm})C^t_{rt}$$
$$-\frac{1}{4}A_i(r)C^i_{pq}C^j_{kl}\epsilon^{jpq}\epsilon^{mkl} + (A'_i(r) + A(r)_jC^j_{ri})C^k_{rn}\epsilon^{ikl}\epsilon^{mnl} \tag{A.10}$$

Algebra in the notation of [44]	Algebra in the notation of [42, 43]
$3A_1$	Bianchi I
$A_{3,1}$	Bianchi II
$A_2 \oplus A_1$	Bianchi III
$A_{3,2}$	Bianchi IV
$A_{3,3}$	Bianchi V
$A_{3,5}^a \ (0 < |a| < 1)$	Bianchi VI
$A_{3,7}^a \ (a > 0)$	Bianchi VII
$A_{3,8}$	Bianchi VIII
$A_{3,9}$	Bianchi IX

Table 1. Classification of Real 3 dimensional Lie algebras.

A.2.2 Electric field

In this case, $A_i(r) = 0$. There is only one component of Maxwell's equation:

$$(A_t'(r) + A_t(r)C_{rt}^t)' + (A_t'(r) + A_t(r)C_{rt}^t)C_{ri}^i = \frac{m^2}{2}A_t(r) \tag{A.11}$$

Similar to before, in a scaling solution, we expect that the components of the vector potential are constants, reducing Maxwell's equations to a set of algebraic equations.

A.3 Einstein's equations

The stress energy tensor of a massive gauge field is given by:

$$T_{\mu\nu} = \frac{1}{2}F_{\mu\lambda}F_\nu^\lambda + \frac{1}{4}m^2 A_\mu A_\nu - \frac{1}{2}\eta_{\mu\nu}\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{m^2}{4}A_\rho A^\rho\right) \tag{A.12}$$

where index contraction is done using $\eta_{\mu\nu}$.

Therefore we see that the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(R + \Lambda) = \sum_a T_{\mu\nu}^a \tag{A.13}$$

are reduced to a set of ordinary differential equations in r; or, in the case of scaling solutions, a set of algebraic equations.

B Rosetta stone relating different nomenclatures for the Bianchi classification

In the table 1 above, we provide a dictionary relating two different common nomenclatures for the classification of 3d real Lie algebras, as given in [44] and [42, 43].

- Bianchi VII$_0$, which we use many times in this paper, is the special limit of Bianchi VII. More precisely, Bianchi VII has nonzero structure constant $C_{23}^1 = -C_{32}^1 = -1$, $C_{13}^2 = -C_{31}^2 = 1$ and $C_{23}^2 = -C_{32}^2 = a$. By setting $a = 0$, Bianchi VII reduces to Bianchi VII$_0$. The $a = 0$ limit is called $A_{3,6}$ in [44].

- $A_{3,4}$ in [44] is the special limit of the $A_{3,5}^a$ in [44] with $a = -1$.
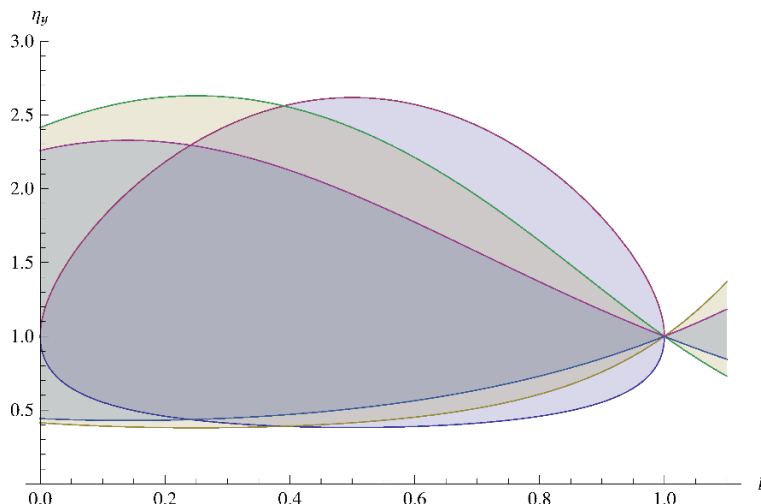
Figure 1. The darkest parameter region is the allowed region in the (η_y, b) parameter space. We have set $\eta_x = 1$ and $a = 1$.

C Null energy condition for 4d $A_{4,6}$

We need

$$f_i \geq 0 \quad (i = 1, 2, 4), \qquad 4f_2 f_4 \geq (f_3)^2. \tag{C.1}$$

for f_i given by eq. (10.46)–(10.49). We set $\eta_x = 1$ by performing a coordinate re-parameterization. Then, with $\eta_r > 0$, $f_i > 0$ $(i = 1, 2, 4)$ gives

$$2\eta_y(2b(a - b) + 1) - 1 - \eta_y^2 \geq 0, \tag{C.2}$$

$$2\eta_y(a - b)(a + 2b) + 1 - \eta_y^2 \geq 0, \tag{C.3}$$

$$2\eta_y(a - b)(a + 2b) - 1 + \eta_y^2 \geq 0. \tag{C.4}$$

Furthermore, by flipping $r \leftrightarrow -r$, we can always make $a > 0$. So the horizon is at $r \to \infty$. In order to satisfy the Nernst's law, we require $b > 0$.

Let's set $a = 1$. Then the above 3 conditions become

$$2\eta_y(2b(1 - b) + 1) - 1 - \eta_y^2 \geq 0, \tag{C.5}$$

$$2\eta_y(1 - b)(1 + 2b) + 1 - \eta_y^2 \geq 0, \tag{C.6}$$

$$2\eta_y(1 - b)(1 + 2b) - 1 + \eta_y^2 \geq 0. \tag{C.7}$$

In the regime where

$$b > 0, \qquad \eta_y > 0, \tag{C.8}$$

the first condition (C.5) gives

$$1 + 2b - 2b^2 - 2\sqrt{b - 2b^3 + b^4} \leq \eta_y \leq 1 + 2b - 2b^2 + 2\sqrt{b - 2b^3 + b^4}. \tag{C.9}$$

The second the third conditions (C.6) and (C.7) give respectively

$$\eta_y \leq \frac{1}{2}\left(2 + 2b - 4b^2 + \sqrt{4 + (2 + 2b - 4b^2)^2}\right), \tag{C.10}$$

$$\eta_y \geq \frac{1}{2}\left(-2 - 2b + 4b^2 + \sqrt{4 + (2 + 2b - 4b^2)^2}\right). \tag{C.11}$$

Finally the condition $4f_2 f_4 \geq (f_3)^2$ gives additionally

$$\frac{1}{4}\left(g_1 - g_2\right) \leq \eta_y \leq \frac{1}{4}\left(g_1 + g_2\right) \tag{C.12}$$

where

$$g_1 = \sqrt{8b\left(10b^3 - 4b^2 + b + 9\right) + 41} - 4b(b+1) - 1, \tag{C.13}$$

$$g_2 = \sqrt{2}\sqrt{-(2b+1)^2\left(\sqrt{8b\left(10b^3 - 4b^2 + b + 9\right) + 41} - 12(b-1)b - 13\right)}. \tag{C.14}$$

We plot the allowed parameter ranges satisfying (C.9)–(C.12) in figure 1. The limit $\eta_y = b = 1$ corresponds to AdS_4.

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