Gauge threshold corrections for $\mathcal{N} = 2$ heterotic local models with flux, and mock modular forms

Luca Carlevaro^{a,d} and Dan Israël^{b,c}

^a*Centre de Physique Théorique, Ecole Polytechnique,
91128 Palaiseau, France¹*

^b*Institut d'Astrophysique de Paris,
98bis Bd Arago, 75014 Paris, France²*

^c*Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie,
4 place Jussieu, 75252 Paris CEDEX 05, France³*

^d*LAREMA, Département de Mathématiques, Université d'Angers,
2 Boulevard Lavoisier, 49045 Angers, France⁴*

E-mail: luca.carlevaro@cpht.polytechnique.fr, israel@lpthe.jussieu.fr

ABSTRACT: We determine threshold corrections to the gauge couplings in local models of $\mathcal{N} = 2$ smooth heterotic compactifications with torsion, given by the direct product of a warped Eguchi-Hanson space and a two-torus, together with a line bundle. Using the worldsheet CFT description previously found and by suitably regularising the infinite target space volume divergence, we show that threshold corrections to the various gauge factors are governed by the non-holomorphic completion of the Appell-Lerch sum. While its holomorphic Mock-modular component captures the contribution of states that localise on the blown-up two-cycle, the non-holomorphic correction originates from non-localised bulk states. We infer from this analysis universality properties for $\mathcal{N} = 2$ heterotic local models with flux, based on target space modular invariance and the presence of such non-localised states. We finally determine the explicit dependence of these one-loop gauge threshold corrections on the moduli of the two-torus, and by S-duality we extract the corresponding string-loop and E1-instanton corrections to the Kähler potential and gauge kinetic functions of the dual type I model. In both cases, the presence of non-localised bulk states brings about novel perturbative and non-perturbative corrections, some features of which can be interpreted in the light of analogous corrections to the effective theory in compact models.

KEYWORDS: Flux compactifications, Superstrings and Heterotic Strings, Conformal Field Models in String Theory, Superstring Vacua

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¹Unité mixte de Recherche 7644, CNRS — Ecole Polytechnique

²Unité mixte de Recherche 7095, CNRS — Université Pierre et Marie Curie

³Unité Mixte de Recherche 7589, CNRS — Université Pierre et Marie Curie

⁴Unité mixte de Recherche 6093, CNRS — LAREMA

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1 Introduction

Supersymmetric compactifications of the heterotic string [1] were soon recognised as a very successful approach to string phenomenology. A crucial role is played by the modified Bianchi identity for the field strength of the Kalb-Ramond two-form. It should include a contribution from the Lorentz Chern-Simons three-form coming from the anomaly-cancellation mechanism [2], that cannot be neglected in a consistent low-energy truncation of the heterotic string:

$$d\mathcal{H} = \alpha' (\text{tr } \mathcal{R}(\Omega_-) \wedge \mathcal{R}(\Omega_-) - \text{Tr}_V \mathcal{F} \wedge \mathcal{F}) . \quad (1.1)$$

Consistent torsionless compactifications can be achieved with an embedding of the spin connexion in the gauge connexion. For more general bundles, the Bianchi identity (1.1) is in general not satisfied locally, leading to non-trivial three-form fluxes, i.e. manifolds with non-zero torsion. These compactifications with torsion were explored in the early days of the heterotic string [3, 4]. Their analysis is quite involved, as generically the compactification manifold is not even conformally Kähler. In view of this complexity, it is useful to describe more quantitatively such flux compactifications with non-compact geometries that can be viewed as *local models* thereof. In type IIB flux compactifications [5], an important rôle is devoted to throat-like regions of the compactification manifold, whose flagship is the Klebanov — Strassler background [6].

Heterotic torsional geometries, having only NSNS three-form and gauge fluxes, are expected to allow for a tractable worldsheet description. Recently, it was shown in a series of works [7–14] that worldsheet theories for such flux geometries can be defined as the infrared limit of some classes of $(0, 2)$ gauged linear sigma models. This very interesting approach does not however allow for the moment to perform computations of physical quantities in these torsional backgrounds, as only quantities invariant under RG-flow can be handled.

The most studied examples of supersymmetric heterotic flux compactification are elliptic fibrations $T^2 \hookrightarrow \mathcal{M} \rightarrow K3$, where the $K3$ base is warped. Those backgrounds, that correspond to the most generic $\mathcal{N} = 2$ torsional compactifications [15], can be equipped with a gauge bundle that is the tensor product of a Hermitian-Yang-Mills bundle over the

$K3$ base with a holomorphic line bundle on $\mathcal{M}$. For these geometries, that were found in [16] using string dualities, a proof of the existence of a family of smooth solutions to the Bianchi identity with flux has only appeared recently [17–19].

Considering as a base space a Kummer surface (i.e. the blow-up of a $T^4/\mathbb{Z}_2$ orbifold), an interesting strongly warped regime occurs when the blow-up parameter a of one of the two-cycles is significantly smaller (in string units) than the five-brane charge measured around this cycle, provided small instantons appear in the singular limit. As is shown in [20], one can define a sort of ‘near-bolt’ geometry, that describes the neighbourhood of one of the 16 resolved A_1 singularities, which is decoupled from the bulk. To this end, a *double scaling limit* is defined by sending the asymptotic string coupling g_s to zero, while keeping the ratio g_s/a fixed in string units, which plays the rôle of an effective coupling constant. It consistently defines a local model for this whole class of $\mathcal{N} = 2$ compactifications. More generically, this model can be defined for any value of the five-brane charge.

Remarkably, as we have shown in [20], the corresponding worldsheet non-linear sigma model admits a solvable worldsheet CFT description, as an asymmetrically gauged WZW model. The existence of a worldsheet CFT first implies that these backgrounds are exact heterotic string vacua to all orders in α' , once included the worldsheet quantum corrections to the defining gauged WZW models. Secondly, one can take advantage of the exact CFT description in order, for instance, to determine the full heterotic spectrum as was done in [20]. It involves BPS and non-BPS representations of the $\mathcal{N} = 2$ superconformal algebra, that correspond respectively to states localised in the vicinity of the resolved singularity and to a continuum of delta-function normalisable states that propagate in the bulk.

Having a good knowledge of the worldsheet conformal field theories corresponding to these torsional backgrounds allows to go beyond the large volume limit and tree-level approximation upon which most works on type II flux compactifications are based. In this respect, interesting quantities are gauge threshold corrections, as they both correspond to a one-string-loop effect, which only receives fivebrane instanton corrections, and are sensitive to all order terms in the α' expansion, since the compactification manifold is not necessarily taken in the large-volume limit (which does not exist generically in the heterotic case). In addition, heterotic — type I duality translates one-loop gauge threshold corrections on the heterotic side to perturbative and multi-instanton corrections to the Kähler potential and the gauge kinetic functions on the type I side. In this respect, provided a microscopic theory is available for a given heterotic model, the method of Dixon-Kaplunovsky-Louis (DKL) is instrumental in retrieving (higher) string-loop and Euclidean brane instanton corrections to these type I quantities, from a one-loop calculation on the heterotic side, even when the type I S-dual model is unknown.

This perspective looks particularly enticing from the type I vantage point, since although remarkable advances have been accomplished to understand the perturbative tree-level physics of flux compactifications [21], non-perturbative effects and string-loop corrections continue to often prove fundamental to lift remnant flat directions in the effective potential or ensure a chiral spectrum. Thus, although progresses are still at an early stage, the rôle of Euclidean brane instanton corrections in central issues such as moduli stabilisation [22–24] and supersymmetry breaking [25–28] have been intensively studied. In

addition, non-perturbative effects can also induce new interesting couplings in the superpotential [29–40], while both instanton [41] and string-loop corrections [42] to the Kähler potential of the effective theory prove to be useful to address the problem of the hierarchy of mass scales in large volume scenarii [43, 44].

For all the above reasons, it appears as particularly appealing to be able to explicitly compute one-loop heterotic gauge threshold corrections and determine their moduli dependence for a smooth heterotic background, incorporating back-reacted NSNS flux. To this end, we consider in the present paper a family of *non-compact* models giving a *local* description of the simplest non-Kähler elliptic fibration $T^2 \hookrightarrow \mathcal{M} \rightarrow K3$, where the fibration reduces to a direct product. Locally, the geometry is given by $T^2 \times \widetilde{\text{EH}}$, where $\widetilde{\text{EH}}$ is the warped Eguchi-Hanson space. These $\mathcal{N} = 2$ heterotic backgrounds also accommodate line bundles over the resolved $\mathbb{P}^1$ of the Eguchi-Hanson space, corresponding to Abelian gauge fields which, from the Bianchi identity (1.1) perspective, induce a non-standard embedding of the gauge connection into the Lorentz connection. For the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic theory, the exact CFT description for the warped Eguchi-Hanson base with an Abelian gauge fibration has been constructed in [20] as a gauged WZW model for an asymmetric super-coset of the group $\text{SU}(2)_k \times \text{SL}(2, \mathbb{R})_k$, for which an explicit partition function can be written.

The presence of a line bundle in these non-compact backgrounds breaks the $\text{SO}(32)$ gauge group to $\text{SO}(2m) \times \prod_r \text{SU}(n_r) \times \text{U}(1)^{r-1}$ with $m + \sum_r n_r = 16$, while the r^{th} $\text{U}(1)$ factor is generically lifted by the Green-Schwarz mechanism. One-loop gauge threshold corrections to individual gauge factors can be determined by computing the elliptic index constructed in [45], which we call *modified elliptic genus* as it corresponds to the elliptic genus of the underlying CFT, with the insertion of the regularised Casimir invariant of the gauge factor under consideration. Since the microscopic theory for such heterotic $T^2 \times \widetilde{\text{EH}}$ backgrounds contains as a building block the $\mathcal{N} = 2$ super-Liouville theory, a careful regularisation of the target space volume divergence has to be considered. This concern is also in order for the partition function, for which a holomorphic but non-modular invariant regularisation is usually preferred, as it results in a natural expression in terms of $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ characters. For the elliptic genus in contrast, the seminal work [46] has shown that the correct regularisation scheme based on a path integral formulation is non-holomorphic but preserves modularity. In particular, it has the virtue of taking properly into account not only the contribution to the gauge threshold corrections of states that localise on the resolved $\mathbb{P}^1$ of the warped Eguchi-Hanson space (constructed from discrete $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ representations), but especially the contribution of non-localised bulk states, which compensates for an otherwise present holomorphic anomaly.

Taken separately, the $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ factor in the localised part of the threshold corrections thus transforms as a *Mock modular* form, i.e. a holomorphic form which transforms anomalously under S-transformation, but can be completed into a non-holomorphic modular form, also known as a *Maaß form*, by adding the transform of a what is commonly called a *shadow* function. The concept of Mock modular form¹ goes back to Ramanujan, but a complete classification of such functions and a definite characterisation of their near-

¹Or *Mock theta functions* as he calls them in a letter to Hardy.

modular properties has only been achieved recently by Zwegers [47], despite many insightful papers written since the twenties on Ramanujan's examples (see references in [48]). Recently, Mock modular forms have found their way in string theory. They have in particular been used to address issues central to wall-crossing phenomena for BPS invariants for systems of D-branes [49], and to deriving a reliable index for microstate (quarter-BPS state) counting for single- and multi-centered black holes in $\mathcal{N} = 4$ string theory [50] (see also in the same line more mathematical works [51, 52]). They also appeared in the computation of D-instanton corrections to the hypermultiplet moduli space of type II string theory compactified on a Calabi-Yau threefold [53], and in the investigation of the mysterious decomposition of the elliptic genus of $K3$ in terms of dimensions of irreducible representations of the Mathieu group M_{24} symmetry [54–59]. The theory of Mock modular forms is finally at the core of infinite target space volume regularisation issues in non-compact CFTs [46, 60–63], which directly concerns the calculation of gauge threshold corrections tackled in this paper.

In the present analysis, we will in particular focus on a family of heterotic torsional local models supporting a line bundle $\mathcal{O}(1) \oplus \mathcal{O}(\ell)$ with gauge group $\mathrm{SO}(28) \times \mathrm{U}(1)$ (which is enhanced to $\mathrm{SO}(28) \times \mathrm{SU}(2)$ when $\ell = 1$). The regularised threshold corrections to the these gauge couplings are shown to be given in terms of weak harmonic Maaß forms based on the non-holomorphic completion of Appell-Lerch sums, a major class of Mock modular forms treated by Zwegers. A deeper physical insight into the shadow function featured in the bulk state contribution is achieved by investigating the $\ell = 1$ model, whose interacting part enjoys an enhanced $(4, 4)$ worldsheet superconformal symmetry. We observe in this particular case that localised effects splits on the one hand into $4/\chi(K3)$ of the gauge threshold corrections for a $T^2 \times K3$ model, for which there is a rich literature both in heterotic and type I theories [64–72], and on the other hand into a Mock modular form $F(\tau)$ encoding the presence of warping due to NSNS flux threading the geometry. The non-holomorphic regularisation mentioned above dictates a completion in terms of non-localised bulk states which leads to the harmonic Maaß form $\widehat{F}(\tau) = F(\tau) + g^*(\tau)$, where $g(\tau)$ is the *shadow* function determined from a holomorphic anomaly equation for $\widehat{F}$. Now, some local models such as the $T^2 \times \widetilde{\mathrm{EH}}$ background considered here have a non-trivial boundary at infinity, allowing for non-vanishing five-brane charge, which would globally cancel when patching these models together to obtain a warped $K3$ compactification on $T^2 \times \widetilde{K3}$. The appearance of the Maaß form $\widehat{F}$ thus results from the combination of the non-compactness of the space (with boundary) and the presence of flux with non-vanishing five-brane charge, both things being somehow correlated. This analysis can then be generalised to the $\ell > 1$ models. However, because of reduced worldsheet supersymmetry the interpretation in terms of $K3$ modified elliptic genera is lost for these theories.

We then carry out a careful analysis of the polar structure of the modified elliptic genus determining these gauge threshold corrections, which shows that they share the same features with respect to unphysical tachyons and anomaly cancellation as well-known $\mathcal{N} = 2$ heterotic compactifications. This allows us to identify some universality properties for $\mathcal{N} = 2$ heterotic local models with non-localised bulk states. It also sets the stage to compute explicitly the dependence of these gauge threshold on the T^2 moduli, for the

$\mathcal{O}(1) \oplus \mathcal{O}(1)$ model taken as an example. The modular integrals can be performed by the celebrated *orbit method*, which consists in unfolding the fundamental domain of the modular group against the T^2 lattice sum. From these threshold calculations we recover in particular the β -functions of the effective four-dimensional theory, in perfect agreement with field theory results based on hypermultiplet counting, previously performed by constructing the corresponding massless chiral and anti-chiral primaries in the CFT [20].

We then consider the type I S-dual theory. Contrary to usual orbifold compactifications half D5-branes at the orbifold singularities are absent from these local models as the A_1 singularity is resolved and anomaly cancellation is ensured by U(1) instantons on the blown-up $\mathbb{P}^1$. We proceed to extract the perturbative and non-perturbative corrections to the Kähler potential and the gauge kinetic functions, by the DKL method. The contribution from states that localise on the resolved two-cycle yields corrections similar to those expected for compact models, which separate into string-loop corrections and multi-instanton corrections due to E1 instantons wrapping the T^2 . In addition, as for the original heterotic gauge threshold corrections, non-localised bulk modes bring about novel types of corrective terms, both perturbative and non-perturbative, to the Kähler potential and the gauge kinetic functions. Though recently gauge threshold corrections for local orientifolds in type IIB models have been successfully computed [73, 74], this is to our knowledge the first such calculation carried out for local heterotic models incorporating back-reacted NSNS flux, determining all-inclusively all perturbative and non-perturbative corrections originating from both localised and bulk states.

In order to be able to make sensible phenomenological predictions, one should of course properly engineer the gluing of sixteen of these heterotic local models into a $T^2 \times \widetilde{K3}$ compactification, which would give us a proper effective field theory understanding of bulk state contributions. This could be of particular interest, on the dual type I side, to clarify the rôle of these novel bulk state contributions we find in E1-instanton corrections, which include an infinite sum over descendants of the modified elliptic genus, as functions of the induced T^2 moduli. These could then be put into perspective with supergravity [75, 76] or field theory [36] calculations of Euclidean brane instanton corrections for compact models.

This work is organized as follows. In section 2 we define the heterotic supersymmetric solutions of interest, and recall their worldsheet description. In section 3 we set the stage for the threshold corrections and provide general aspects of the latter. In section 4 we compute the modified elliptic genus that enters into the modular integral. Finally in section 5 we compute the integral over the fundamental domain in order to recover the moduli dependence, and discuss in section 6 the type I dual interpretation in terms of perturbative and non-perturbative corrections. Some material about superconformal characters, modular form, and some lengthy computations are given in the various appendices.

2 Heterotic flux backgrounds on Eguchi-Hanson space

In this section we briefly describe the heterotic solution of interest, for which the threshold corrections computations will be done, both from the point of view of supergravity and worldsheet conformal field theory.

2.1 The geometry

We consider a family of heterotic backgrounds whose transverse geometry is described by the six-dimensional space $\mathcal{M}_6 = T^2 \times \widetilde{\text{EH}}$, where the four-dimensional non-compact factor $\widetilde{\text{EH}}$ is the warped Eguchi-Hanson space, the Eguchi-Hanson space (EH) being the resolution by blowup of a $\mathbb{C}^2/\mathbb{Z}_2$, or A_1 , singularity. It provides a workable example of a smooth background with intrinsic torsion induced by the presence of NSNS three-form flux. In the following, we will be concerned with the heterotic $\text{Spin}(32)/\mathbb{Z}_2$ theory, but our results can be straightforwardly extended to the $E_8 \times E_8$ gauge group.

The two-torus is characterised by two complex moduli, the Kähler class and the complex structure, which we denote respectively by T and U , related to the string frame metric and B-field as:

$$T = T_1 + iT_2 = \frac{B_{12} + i\sqrt{\det G}}{\alpha'}, \quad U = U_1 + iU_2 = \frac{G_{12} + i\sqrt{\det G}}{G_{11}}. \quad (2.1)$$

Accordingly, the full six-dimensional torsional geometry takes the form:

$$ds_6^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\alpha' T_2}{U_2} |dx^1 + U dx^2|^2 + H(r) ds_{\text{EH}}^2. \quad (2.2)$$

where the torus coordinates have periodicity $(x^1, x^2) \sim (x^1 + 2\pi, x^2 + 2\pi)$ and the A_1 space is locally described by the Eguchi-Hanson (EH) metric:

$$ds_{\text{EH}}^2 = \frac{dr^2}{1 - \frac{a^4}{r^4}} + \frac{r^2}{4} \left((\sigma_1^L)^2 + (\sigma_2^L)^2 + \left(1 - \frac{a^4}{r^4}\right) (\sigma_3^L)^2 \right), \quad (2.3)$$

here given in terms of the $\text{SU}(2)$ left-invariant one-forms:

$$\sigma_1^L = \sin \psi d\theta - \cos \psi \sin \theta d\phi, \quad \sigma_2^L = -(\cos \psi d\theta + \sin \psi \sin \theta d\phi), \quad \sigma_3^L = d\psi + \cos \theta d\phi, \quad (2.4)$$

with $\theta \in [0, \pi]$ and $\phi, \psi \in [0, 2\pi]$. Note in particular that the ψ coordinate runs over half of its original span, since for the EH space to be smooth, an extra $\mathbb{Z}_2$ orbifold is necessary to eliminate the bolt singularity at $r = a$.

The EH manifold is homotopic to the blown-up $\mathbb{P}^1$ resolving the original $\mathbb{C}^2/\mathbb{Z}_2$ singularity. This two-cycle is given geometrically by the non-vanishing two-sphere $ds_{\mathbb{P}^1}^2 = \frac{a^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2)$ and is Poincaré dual to a closed two-form which has the following *local* description:

$$\omega = -\frac{a^2}{4\pi} d\left(\frac{\sigma_3^L}{r^2}\right), \quad \text{with} \quad \int_{\mathbb{P}^1} \omega = 1, \quad \text{and} \quad \int_{\text{EH}} \omega \wedge \omega = -\frac{1}{2}. \quad (2.5)$$

In particular, the last integral yields minus the inverse Cartan matrix of A_1 , as expected for a resolved ADE singularity. The second cohomology thus reduces to $H^{1,1}(\mathcal{M}_{\text{EH}})$, as it is spanned by a single generator $[\omega]$, given by the harmonic and anti-selfdual two-form (2.5). Globally EH can hence be shown to have the topology of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$.

2.2 The heterotic solutions

The six dimensional space (2.2) can be embedded in heterotic supergravity, with a background including an NSNS three-form² $\mathcal{H}$ and a varying dilaton:

$$e^{2\Phi(r)} = g_s^2 H(r) = g_s^2 \left(1 + \frac{2\alpha' \mathcal{Q}_5}{r^2} \right), \quad (2.6a)$$

$$\mathcal{H} = -H *_\text{EH} dH = 4\alpha' \mathcal{Q}_5 \left(1 - \frac{a^4}{r^4} \right) \text{Vol}(S^3), \quad (2.6b)$$

where $\mathcal{Q}_5$ is the charge of the stack of back-reacted NS five-branes wrapped around the T^2 which are recovered in the blowdown limit, opening a throat at $r = 0$. When the A_1 singularity is resolved the NS five-branes are no longer present and we obtain a smooth non-Kähler geometry threaded by three-form flux, with non-vanishing five-brane charge $4\pi^2\alpha'\mathcal{Q}_5 = -\int_{\text{EH}} \mathcal{H}$ due to the boundary $\partial\mathcal{M}_{\text{EH}} = \mathbb{R}P^3$.

This background preserves $\mathcal{N}_{\text{ST}} = (0, 2_4)$, resulting from the existence of a pair of $\text{Spin}(6)$ spinors ϵ^i , $i = 1, 2$ constant with respect to only one of the two generalised spin connections $\Omega_{\pm}^a{}_b = \omega_{\text{EH}}^a{}_b \pm \frac{1}{2}\mathcal{H}^a{}_b$:

$$\left(\partial_\mu + \frac{1}{4} \Omega_+^{ab}{}_\mu \Gamma_{ab} \right) \epsilon^i = 0, \quad i = 1, 2, \quad (2.7)$$

where μ and a, b are six-dimensional space and frame indices respectively.

Bianchi identity and line bundle. In addition to satisfying the supersymmetry equations, anomaly cancellation requires a heterotic background to solve the Bianchi identity:

$$d\mathcal{H} = -\alpha' \left(\text{Tr}_V \mathcal{F} \wedge \mathcal{F} - \text{tr} \mathcal{R}(\Omega_-) \wedge \mathcal{R}(\Omega_-) \right). \quad (2.8)$$

For non-zero fivebrane charge $\mathcal{Q}_5$ the NSNS three-form (2.6b) is not closed. A non-standard embedding of the Lorentz connection into the gauge connection has therefore to be used to satisfy the Bianchi identity. This can be achieved by considering a multi-line bundle

$$\mathcal{L} = \bigoplus_{a=1}^{16} \mathcal{O}_{\mathbb{P}^1}(\ell_a) \quad (2.9)$$

where the individual line bundles, labelled by a , are embedded in an Abelian principal bundle valued in the Cartan subalgebra of $\text{SO}(32)$. The resulting heterotic gauge field, characterised by a vector of magnetic charges (or 'shift vector') $\vec{\ell}$, reads:

$$\mathcal{F} = -2\pi\omega \sum_{a=1}^{16} \ell_a \mathbf{H}^a, \quad \mathbf{H}^a \in \mathfrak{h}(\text{SO}(32)) \quad \text{with} \quad \text{Tr} \mathbf{H}^a \mathbf{H}^b = -2\delta^{ab}. \quad (2.10)$$

Since the above gauge field is along the anti-selfdual and harmonic two-form of EH, it satisfies the Hermitian Yang-Mills (or Uhlenbeck-Donaldson-Yau) equations: $J \lrcorner \mathcal{F} = 0$ and

²The volume of the three-sphere is given in terms of the Euler angles as follows: $\text{Vol}(S^3) = \frac{1}{8} \sigma_1^L \wedge \sigma_2^L \wedge \sigma_3^L = \frac{1}{8} d(\cos \theta) \wedge d\phi \wedge d\psi$.

$\mathcal{F}^{(0,2)} = \mathcal{F}^{(2,0)} = 0$. Hence it does not further break the existing spacetime supersymmetry of the background.

Furthermore, it solves the Bianchi identity (2.8) in the regime where the gravitational contribution is negligible, i.e. in the large five-brane charge limit:

$$\mathcal{Q}_5 = -\frac{1}{4\pi^2\alpha'} \int_{\mathbb{R}P_3, \infty} \mathcal{H} = \vec{\ell}^2 \gg 1. \quad (2.11)$$

As we will see later on, in a specific double-scaling limit of the metric (2.2) the background (2.6) admits an exact worldsheet CFT description, even beyond this large-charge limit.

Beyond the large-charge approximation, one can consider corrections resulting from the integrated Bianchi identity, which are captured by the tadpole equation:

$$\frac{1}{4\pi^2\alpha'} \int_{\text{EH}} \left[(d\mathcal{H} + \alpha' (\text{Tr}_V \mathcal{F} \wedge \mathcal{F} - \text{tr} \mathcal{R}(\Omega_-) \wedge \mathcal{R}(\Omega_-))) \right] = 0 \implies \mathcal{Q}_5 = \vec{\ell}^2 - 6. \quad (2.12)$$

This in particular determines the allowed shift vectors for a given five-brane charge, and the resulting breaking of the gauge group.

In addition to the tadpole equation, dictated by anomaly cancellation, two more constraints restrict the value of the shift vector $\vec{\ell}$, namely:

- i) a Dirac quantisation condition for the adjoint representation of $\text{SO}(32)$, requiring the integrated first Chern class of the line bundle $\mathcal{L}$ to have only integer or half-integer entries corresponding to bundles with or without vector structure respectively:

$$\begin{cases} \vec{\ell} \in \mathbb{Z}^{16}, & \Rightarrow \text{bundle with vector structure} \\ \vec{\ell} \in (\mathbb{Z} + \frac{1}{2})^{16} & \Rightarrow \text{bundle without vector structure} \end{cases} \quad (2.13)$$

- ii) a so-called 'K-theory' condition which must be further imposed on the first Chern class of $\mathcal{L}$ to ensure that the gauge bundle admits spinors:

$$c_1(\mathcal{L}) \in H^2(\text{EH}, 2\mathbb{Z}) \implies \sum_a \ell_a \equiv 0 \pmod{2}. \quad (2.14)$$

2.3 The double-scaling limit

We will now introduce a consistent *double-scaling limit* of the torsional background (2.2)–(2.6), which decouples the bulk physics from the physics in the vicinity of the resolved A_1 singularity:

$$g_s \rightarrow 0 \quad , \quad \lambda = \frac{g_s \sqrt{\alpha'}}{a} \text{ fixed}. \quad (2.15)$$

This specific regime isolates the dynamics near the blownup two-cycle, but still keeps the singularity resolved. In particular if we wrap five-branes around the two-cycle, their tension will be proportional to $\text{Vol}(\mathbb{P}^1)/g_s^2$ and thus held fixed, so that no extra massless degrees of freedom appear in the double scaling limit. This procedure results in an interacting theory whose effective coupling constant is set by the double-scaling parameter. Interestingly

enough, it has been shown in [20] that in this limit the heterotic fluxed background admits a solvable CFT, which we will introduce shortly.

The resulting near-horizon geometry arising in this regime can best be expressed in the new radial coordinate $\cosh \rho = (r/a)^2$:

$$ds_6^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\alpha' T_2}{U_2} |dx^1 + U dx^2|^2 + \frac{\alpha' Q_5}{2} \left[d\rho^2 + (\sigma_1^\perp)^2 + (\sigma_2^\perp)^2 + \tanh^2 \rho (\sigma_3^\perp)^2 \right]. \quad (2.16)$$

Furthermore, while the dilaton is affected by the near-horizon limit, the gauge field and the three-form, which are localised respectively on the blown-up two-cycle and on the $\mathbb{R}P^3$ boundary of EH, remain untouched. Their formulation in the new coordinate are:

$$\mathcal{H} = -4\alpha' Q_5 \tanh^2 \rho \text{Vol}(S^3), \quad e^{2\Phi(\rho)} = \frac{2\lambda^2 Q_5}{\cosh \rho} \quad (2.17a)$$

$$\mathcal{F} = -\frac{1}{2 \cosh \rho} \left(\tanh \rho d\rho \wedge \sigma_3^\perp - \sigma_1^\perp \wedge \sigma_2^\perp \right) \sum_{a=1}^{16} \ell_a H^a. \quad (2.17b)$$

Finally, the tadpole equation correcting the five-brane charge is also modified:

$$Q_5|_{\text{n.h.}} = \vec{\ell}^2 - 4. \quad (2.18)$$

The change with respect to expr. (2.12), namely the jump of -2 units in the integrated first Pontryagin class of the six-dimensional manifold, is due to the decoupling of the boundary of the space, because of the now asymptotically vanishing conformal factor $H(\rho)$.

2.4 The worldsheet cft description

The exact CFT description for the double-scaling limit of the heterotic background (2.16)–(2.17) for a line bundle $\bigoplus_{a=1}^{16} \mathcal{O}_{\mathbb{P}^1}(\ell_a)$ satisfying the tadpole equation (2.18) has been derived in [20]. The interacting part is given by an asymmetrically gauged $\text{SU}(2)_k \times \text{SL}(2, \mathbb{R})_{k'}$ super-WZW model with $\mathcal{N}_{\text{ws}} = (0, 1)$ worldsheet supersymmetry:

$$\frac{(\text{SU}(2)_k / \mathbb{Z}_2) \times \text{SL}(2, \mathbb{R})_{k'}}{\text{U}(1)_L \times \text{U}(1)_R}. \quad (2.19)$$

The gauging in this theory is asymmetric and results from acting on the group elements $(g_1, g_2) \in \text{SU}(2) \times \text{SL}(2, \mathbb{R})$ as follows:

$$(g_1, g_2) \rightarrow (g_1 e^{i\sigma_3 \alpha}, e^{i\sigma_3 \beta} g_2 e^{i\sigma_3 \alpha}) \quad (2.20)$$

The $\text{SU}(2)_k$ factor is also modded out by the $\mathbb{Z}_2$ action $\mathcal{I} : g_1 \mapsto -g_1$, which leaves the current algebra invariant. This orbifold is at the CFT level the algebraic equivalent of the geometric $\mathbb{Z}_2$ orbifold reducing the periodicity of the angular coordinate ψ to $[0, 2\pi]$ (see section 2.1). The 16 left-moving Weyl fermions are also minimally coupled to the worldsheet gauge fields with charge $\{\ell_i, i = 1, \dots, 16\}$.

In order to obtain a gauge-invariant worldsheet action the following conditions on the levels of the affine superconformal algebras are obtained:

$$k' = k, \quad k = 2(\vec{\ell}^2 - 1). \quad (2.21)$$

In particular, we recognise in the second constraint the CFT equivalent of the tadpole equation (2.18). To simplify the notations and the computations we will restrict to $U(1)^2$ bundles with shift vector $\vec{\ell} = (1, \ell, 0^{14})$. In this subclass of models the left superconformal symmetry of the $SL(2, \mathbb{R})/U(1)$ factor is enhanced to $\mathcal{N}_{\text{ws}} = 2$. For this specific choice of shift vector, the condition (2.21) fixes $k = 2\ell^2$. The K-theory condition (2.13), in this case, restricts ℓ to be an odd-integer (as we shall see below, this condition is also needed in the CFT).

Integrating out the worldsheet gauge fields classically, one finds a non-linear sigma model [20] whose background metric, B-field and dilaton exactly reproduce the double-scaling limit of the torsional background of interest, given in eq. (2.17).

One-loop partition function. To write down the partition function for $\text{Spin}(32)/\mathbb{Z}_2$ heterotic strings in the torsional background (2.17) we combine the partition function for the four-dimensional coset CFT with the flat space-time part (in the light-cone gauge), the remaining 28 free left-moving Majorana-Weyl fermions and a toroidal lattice, written in the Lagrangian formulation:

$$\Gamma_{2,2}(T, U) = \frac{T_2}{\tau_2} \sum_{n_1, n_2, m_1, m_2} \exp \left[2\pi i T \det A - \frac{\pi T_2}{\tau_2 U_2} \left| (1, U) A \begin{pmatrix} \tau \\ -1 \end{pmatrix} \right|^2 \right], \quad (2.22)$$

where the matrix A encodes the topologically non-trivial mapping of the string worldsheet onto the target-space torus:

$$A = \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}, \quad n_i, m_i \in \mathbb{Z}, \quad i = 1, 2. \quad (2.23)$$

The representations that appear in the spectrum of the coset theory (2.19) are labelled in particular by the spin of $SL(2, \mathbb{R})$ irreducible representations, that fit into two classes:

- a discrete spectrum of normalisable states localised on the blown-up two-cycle at the resolved A_1 singularity. These are labelled by a real spin J , which runs over the range: $\frac{1}{2} < J < \frac{k+1}{2}$. The corresponding coset representations are BPS and have massless ground states. We will denote their contribution to the partition function by T_d .
- a continuous spectrum of δ -function normalisable states, which live in the weakly coupled asymptotic region $\varrho \rightarrow \infty$. They are labelled by a continuous $SL(2, \mathbb{R})$ spin $J = \frac{1}{2} + iP$, with $P \in \mathbb{R}^+$ and correspond to non-BPS massive representations in the coset. We denote their contribution to the partition function by T_c .

Combining all together, we obtain the total partition function for all models with line-

bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell)$:

$$\begin{aligned}
 T &= T_d + T_c \\
 &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} \frac{\Gamma_{2,2}(T, U)}{|\eta|^8} \frac{1}{2} \sum_{a,b=0}^1 (-)^{a+b} \frac{\bar{\vartheta} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^2}{\bar{\eta}^2} \frac{1}{2} \sum_{\gamma, \delta=0}^1 \sum_{2j=0}^{k-2} (-)^{\delta(2j+(\frac{k}{2}-1)\gamma)} \sum_{m \in \mathbb{Z}_{2k}} \bar{C}_m^j \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \\
 &\quad \times \chi_{k-2}^{j+\gamma(\frac{k}{2}-2j-1)} \frac{1}{2} \sum_{u,v=0}^1 \left(\Gamma_m^d \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] + \Gamma_m^c \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] \right) \frac{\vartheta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]^{14}}{\eta^{14}}. \tag{2.24}
 \end{aligned}$$

The contribution to the partition function (2.24) of the compact part of the coset CFT decomposes, on the left-moving side, into the affine characters χ_{k-2}^j of the bosonic $SU(2)_{k-2}$ (A.3) affine algebra, and on the right-moving side, into the super-parafermion characters $C_m^j \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]$ of the supersymmetric $SU(2)_k/U(1)$ (A.13). The contributions from $SL(2, \mathbb{R})_k/U(1)$ characters with $\mathcal{N}_{\text{ws}} = (2, 2)$ superconformal symmetry are repackaged in expression $\Gamma_m^{\text{d,c}} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]$. Localised states, in particular, are captured by the following partition function for discrete $SL(2, \mathbb{R})_k/U(1)$ representations:

$$\begin{aligned}
 \Gamma_m^d \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] &= \sum_{2J=1}^k \overline{\text{Ch}}_d \left(J, \frac{m}{2} - J - \frac{a}{2} \right) \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \sum_{n \in \mathbb{Z}_{2\ell}} e^{-i\pi v(n+\frac{u}{2})} \text{Ch}_d \left(J, \ell \left(n + \frac{u}{2} \right) - J - \frac{u}{2} \right) \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] \\
 &\quad \times \delta_{2J-m+a,0}^{[2]} \delta_{2J-(\ell-1)u,0}^{[2]}, \tag{2.25}
 \end{aligned}$$

with $\delta^{[2]}$ the mod-two Kronecker symbol.³ We refer the reader to appendix A for the definition of extended characters for discrete (A.20) and continuous (A.21) $SL(2, \mathbb{R})_k/U(1)$ representations. It should also be pointed out that when the $SU(2)/\mathbb{Z}_2$ orbifold is combined with the projection by the mod-two Kronecker symbol $\delta_{2J-(\ell-1)u,0}^{[2]}$ and the K-theory condition (2.13), representations with half-integer spin are projected out.

The contribution of δ -function normalisable bulk states is encoded in the partition function for continuous $SL(2, \mathbb{R})_k/U(1)$ representations:

$$\Gamma_m^c \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] = \int_0^\infty dp \, \overline{\text{Ch}}_c \left(\frac{1}{2} + ip, \frac{m}{2} \right) \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \sum_{n \in \mathbb{Z}_{2\ell}} e^{-i\pi v(n+\frac{u}{2})} \text{Ch}_c \left(\frac{1}{2} + ip, \ell \left(n + \frac{u}{2} \right) \right) \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]. \tag{2.26}$$

Regularisation of the infinite volume divergence: the decomposition of the partition function (2.24) in terms of characters of discrete and continuous representations of the chiral $\mathcal{N}_{\text{ws}} = 2$ superconformal algebra results from adopting a particular regularisation scheme of the infinite volume divergence in target-space.⁴ This regularisation

³Note that we have included in the above partition function contributions from the ‘boundary’ representation $J = 1/2$. It will be in practice projected out in the partition function with all other half-integer spin states of $SL(2, \mathbb{R})_k/U(1)$ but we nevertheless include it to make the connection with the elliptic genus of the orbifolded super-Liouville theory more palpable.

⁴To be more precise, this regularisation leads in principle to a non-trivial regularised density of continuous representations, see [77]. However this is not necessary for our present purpose, which is to summarise the full string spectrum in a compact way.

preserves holomorphicity of the characters; however, as the infinite volume divergence cannot be factored out as the volume of a symmetry group, it breaks modular invariance. Although characters for discrete and continuous representations separately close under a T-transformation, they mix under an S-transformation. Schematically we have:

$$\begin{aligned} (\text{discrete rep.}) &\xrightarrow{S} (\text{discrete rep.}) + (\text{continuous rep.}) \\ (\text{continuous rep.}) &\xrightarrow{S} (\text{continuous rep.}) \\ (\text{identity rep.}) &\xrightarrow{S} (\text{discrete rep.}) + (\text{continuous rep.}). \end{aligned} \quad (2.27)$$

Therefore, the full partition function (2.24) is not modular invariant, but the continuous representation term T_c is on its own.

From now on, the one-loop gauge threshold corrections (3.1) that we will tackle shortly can be formulated in terms of a modified supersymmetric index, similar in spirit to the elliptic genus of the microscopic theory, for which a different kind of regularisation should be prescribed, which is modular invariant but not holomorphic [46].

Blowdown limit. From the perspective of a correspondence between geometrical (supergravity) and algebraic (CFT) data, we observe that the contribution T_d from discrete representations localises at the bolt of the manifold and is thus related, on the geometric side, to the resolution of the A_1 singularity. Consequently, the blowdown limit of the space (2.16) will be described at the microscopic level only by continuous representations in T_c . This is actually in keep with the fact that T_c is by itself modular invariant, while extended characters for discrete representations do not close under the action of the modular group, and in particular transform into *discrete + continuous* extended characters under S-transformation.

Correspondingly, in the $a \rightarrow 0$ limit of the supergravity solution (2.6), we see genuine coincident heterotic fivebranes transverse to the A_1 singularity emerging, for which the T_c partition function gives a microscopic description of the near-horizon geometry. The corresponding worldsheet theory is actually a $\mathbb{Z}_2$ orbifold of the Callan-Harvey-Strominger (CHS) solution [78], together with a linear dilaton of charge $Q = \sqrt{2/\alpha'k}$:

$$\mathbb{R}^{3,1} \times T^2 \times \mathbb{R}_Q \times \text{SU}(2)_k/\mathbb{Z}_2. \quad (2.28)$$

2.5 The massless spectrum

The partition function (2.24) gives the full spectrum of heterotic string on warped Eguchi-Hanson space endowed with the line bundle consisting of two U(1) instantons with magnetic charges one and ℓ . The unbroken gauge group G is the commutant of $\text{U}(1)_1 \times \text{U}(1)_\ell$ in $\text{SO}(32)$. It contains two Abelian factors, but only one of them, corresponding to the left $\text{U}(1)_R$ of the $\text{SL}(2, \mathbb{R})/\text{U}(1)$ super-coset, remains massless. The orthogonal combination, whose embedding in $\text{SO}(32)$ is given by $\vec{\ell} \cdot \vec{H}$, is lifted by the Green-Schwarz mechanism. Thus for $\ell \neq 1$ the actual massless gauge group is $G = \text{SO}(28) \times \text{U}(1)_R$. When $\ell = 1$, it is enhanced to $G = \text{SO}(28) \times \text{SU}(2)$.

In table 1 the complete list of massless hypermultiplets charged under G is given for the $\vec{\ell} = (1, \ell, 0^{14})$ theories. Generically, these states are constructed by tensoring the

$\vec{\ell}$	Untwisted sector	Twisted sector	Gauge bosons
$(1, 1, 0^{14})$	$(\mathbf{28}, \mathbf{2})$ + $2(\mathbf{1}, \mathbf{1})$		$\text{SO}(28) \times \text{SU}(2)$ (non-normalisable) . $\text{U}(1)$ with mass $\mathbf{m} = \frac{2}{\sqrt{\alpha'}}$.
$(1, \ell, 0^{14})$ $\ell \in 2\mathbb{N}^* + 1$	$\mathbf{28}_{-1} + (2\ell - 1)\mathbf{28}_{\frac{1}{\ell}}$ + $\mathbf{1}_0 + (2\ell - 1)\mathbf{1}_{\frac{1}{\ell}-1}$	$(2\ell^2 - 1)\mathbf{28}_1 + (2\ell^2 - 2\ell + 1)\mathbf{28}_{-\frac{1}{\ell}}$ + $(2\ell^2 - 1)\mathbf{1}_0 + (2\ell^2 - 2\ell + 1)\mathbf{1}_{1-\frac{1}{\ell}}$	$\text{SO}(28) \times \text{U}(1)_R$ (non-normalisable) . $\text{U}(1)$ with mass $\mathbf{m} = \frac{2}{\sqrt{\alpha' \ell }}$.

Table 1. Spectra of hypermultiplets and gauge bosons for models with integer shift vectors $\vec{\ell} = (1, \ell, 0^{16})$.

combination $[(\bar{c}, \bar{c}) + (\bar{a}, \bar{a})]$ of right-moving chiral and anti-chiral primaries of the supercosets $\frac{\text{SU}(2)_k}{\text{U}(1)} \otimes \frac{\text{SL}(2, \mathbb{R})_k}{\text{U}(1)}$ with either a chiral c_u/t or anti-chiral a_u/t left-moving primary of $\frac{\text{SL}(2, \mathbb{R})_k}{\text{U}(1)} \otimes \text{SU}(2)_{k-2}$, in the untwisted (u label) or twisted (t label) sector of the $\mathbb{Z}_2$ orbifold (2.19) acting on the compact $\text{SU}(2)_L$. The detailed CFT construction of these states can be found in [20].

These hypermultiplets of $d = 6$, $\mathcal{N} = 1$ supersymmetry obtained by ‘compactification’ of heterotic strings on the warped Eguchi-Hanson space are supplemented by the extra multiplets coming from the compactification on T^2 to $d = 4$. The latter, being neutral, do not contribute to the threshold corrections discussed below.

In the particular case of ‘minimal’ magnetic charge $\ell = 1$, the left superconformal symmetry is enhanced to $\mathcal{N} = 4$, hence the $\text{U}(1)_R$ worldsheet R-symmetry is enhanced to $\text{SU}(2)_2$. Since in this case, the action of the $\mathbb{Z}_2$ orbifold is trivialised, hypermultiplets coming from its twisted sectors are altogether absent, while the ‘untwisted’ hypermultiplets organise into a doublet and two singlets of $\text{SU}(2)_2$. In the other cases, i.e. for $\ell \in 2\mathbb{N}^* + 1$, the emergence of twisted sectors of the $\mathbb{Z}_2$ orbifold enhances the spectrum of hypermultiplets.

Hypermultiplet multiplicities and accidental $\text{SU}(2)$ symmetry: the hypermultiplet multiplicity factors in table 1 take into account the $(2j+1)$ state degeneracy characterising operators with internal left-moving $\text{SU}(2)_{k-2}$ spin j . This $\text{SU}(2)_L$ symmetry should indeed be regarded as an accidental global symmetry of the local model for which one computes the gauge threshold corrections, that can be understood in supergravity as counting KK modes originating from the $\mathbb{P}^1$ reduction; in a genuine $T^2 \times \widetilde{K3}$ compactification, this symmetry is absent. Another way of phrasing things is to say that modifications to the worldsheet theory necessary to glue the local model onto a full-fledged compactification will inevitably break this $\text{SU}(2)_L$ symmetry.

Worldsheet non-perturbative effects. The ‘K-theory’ constraint (2.14) is actually a necessary condition for the CFT (2.19) to make sense, as was shown in [20]. The supercoset $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ worldsheet action receives non-perturbative corrections in the form of a dynamically generated $\mathcal{N}_{\text{ws}} = (2, 0)$ Liouville potential. In the present case, the corresponding vertex operator is given by the hypermultiplet which is an uncharged singlet of $\text{SO}(32)$ and belongs to the twisted sector of the $\mathbb{Z}_2$ orbifold (cf. table 1), making the latter

mandatory.⁵ Requiring this particular operator to be both orbifold and GSO-invariant further imposes respectively that $k \equiv 2 \bmod 4$ and $\sum_i \ell_i = 1 + \ell \equiv 0 \bmod 2$, hence the $\ell \in 2\mathbb{N}^* + 1$ condition in table 1, the latter being nothing else than the K-theory constraint (2.14).

3 Threshold corrections and the elliptic genus: general aspects

We consider a generic compactification of the heterotic string theory to four dimensions, with $\mathcal{N}_{\text{ST}} = 2$ space-time supersymmetry and an unbroken gauge group $G = \prod_{a \leq 16} G_a \subset \text{SO}(32)$.

The one-loop correction to the gauge coupling constants takes the generic form:

$$\left. \frac{4\pi^2}{g_a^2(\mu^2)} \right|_{1\text{-loop}} = \frac{k_a}{L} + \frac{b_a}{4} \log \left(\frac{M_s^2}{\mu^2} \right) + \frac{\Delta_a(M, \overline{M})}{4}, \quad (3.1)$$

where L is the linear multiplet associated to the dilaton, M_s is the string scale, μ an infrared cutoff that will be discussed below later, M the compactification moduli and k_a the Kac-Moody levels determining the normalisation of the gauge group generators. One can alternatively express (3.1) in terms the complexified axio-dilaton S multiplet by using the relation:

$$L^{-1} = \text{Im } S - \frac{1}{4} \Delta_{\text{univ}}(M, \overline{M}), \quad (3.2)$$

with Δ_{univ} a *universal* (group independent) function of the moduli.

The β -function coefficients b_a are given by a fixed linear combination of the quadratic Casimir invariants of the gauge group. For $\mathcal{N}_{\text{ST}} = 2$ theories, when G_a is non-Abelian, these coefficients are determined by

$$b_a = 2 \sum_{\mathbf{R}} n_{\mathbf{R}} T_a(\mathbf{R}) - 2 T_a(\text{Adj}_a), \quad (3.3)$$

where $n_{\mathbf{R}}$ counts the number of matter multiplets in the representation $\mathbf{R}$ of G_a .

When one of the gauge factors G_a is Abelian, its β -function is given by

$$b_{\text{U}(1)} = 2 \sum_{\mathbf{R}} n_{\mathbf{R}} \eta_{\mathbf{R}} \dim(\mathbf{R}) Q_{\mathbf{R}}^2, \quad (3.4)$$

in terms of the $\text{U}(1)$ charges $Q_{\mathbf{R}}$ of the representations of the non-abelian factors G_a which appear in the hypermultiplet spectrum and the respective normalisation $\eta_{\mathbf{R}}$ of their generators. Typically, hypermultiplets which are singlets of G_a will not contribute to (3.3) but will appear in (3.4).

⁵Note that for the $\vec{\ell} = (1^2, 0^{14})$ model, the Liouville potential is still present, despite the trivialisation of the $\mathbb{Z}_2$ orbifold. In this case, the corresponding operator sits in the same $(\mathbf{1}, \mathbf{1})$ hypermultiplet as the dynamical current-current deformation triggering the blowup.

3.1 The modified elliptic genus

Heterotic $\mathcal{N}_{\text{st}} = 2$ gauge threshold corrections are determined at one-loop by a properly regularised three-point function in the worldsheet CFT on the torus, integrated over the fundamental domain of the modular group $\text{PSL}(2, \mathbb{Z})$:

$$\mathcal{F} = \left\{ \tau \in \mathcal{H} \mid -\frac{1}{2} \leq \tau_1 < \frac{1}{2}, |\tau| \geq 1 \right\}, \quad (3.5)$$

where $\mathcal{H}$ is the upper half complex plane.

The non-universal part of the threshold (3.1) is the given by the integral over $\mathcal{F}$ of a modification of the supersymmetric index introduced in [45, 79, 80]⁶

$$\Lambda_a \equiv \frac{b_a}{4} \log \frac{M_s^2}{\mu^2} + \frac{\Delta_a}{4} = \frac{1}{4} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \hat{\mathcal{B}}_a(\tau), \quad (3.7)$$

given by a descendant of an elliptic index modified by the insertion of the (regularised) Casimir operator of the corresponding gauge group factor:

$$\hat{\mathcal{B}}_a(\tau) = \frac{i}{\eta^2} \text{Tr}_{\mathcal{H}_{\text{R}}^{(22,9)}} \left(\left[Q_a^2 - \frac{k_a}{4\pi\tau_2} \right] e^{i\pi \bar{J}_0^R} \bar{J}_0^R q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} e^{2\pi i \vec{\nu} \cdot \vec{J}_0} \right) \Big|_{\vec{\nu}=\vec{0}}, \quad (3.8)$$

where $\{J^i, i = 1, \dots, 16\}$ denote the Cartan currents of $\text{SO}(32)_1$. The trace in $\hat{\mathcal{B}}_a$ projects onto the ground states of the right-moving twisted Ramond sector of the internal six-dimensional $(c, \bar{c}) = (22, 9)$ CFT. Also, the insertion of the total right-moving $\text{U}(1)_{\bar{R}}$ current zero-mode $\bar{J}_0^R$ is there to remove the extra zero-modes coming from the two-torus CFT which would otherwise make the index (3.8) vanish altogether.

The quantity $\hat{\mathcal{B}}_a$ only depends on the topology of the manifold and of the gauge bundle. In particular, if we remove the regularised Casimir operator in expression (3.8), $-i\eta^2 \hat{\mathcal{B}}_a$ reduces to an elliptic generalisation of the Dirac-Witten index [79, 80], counting the difference between vector- and hypermultiplets (and including non-physical states violating the level matching condition, which are required by modular invariance of the index). This elliptic genus is thus stable under an arbitrary chiral marginal deformation and is as such invariant under deformations of the hypermultiplet moduli.

We remind that the $\frac{k_a}{\tau_2}$ term in (3.8), which results from a modular invariant regularisation of worldsheet short distance singularities appearing when two vertex operators

⁶Another procedure for computing the non-holomorphic modular form $\hat{\mathcal{B}}_a$ is the so-called background field method [64, 81, 82], where a magnetic B-field is turned on along two of the spatial directions in four-dimensional Minkowski space. Expanding in the weak field limit the one-loop vacuum energy in powers of B , we recover the gauge threshold corrections (3.7) as the quadratic term in the expansion, the zero order term vanishing because of supersymmetry. We then obtain in the $\overline{\text{DR}}$ renormalisation scheme:

$$\hat{\mathcal{B}}_a(\tau) = -\frac{i}{\pi} \frac{1}{|\eta|^4} \sum_{(b,c) \neq (1,1)} \partial_{\bar{\tau}} \left(\frac{\bar{\vartheta} \left[\frac{b}{c} \right]}{\bar{\eta}} \right) \left[Q_a^2 - \frac{k_a}{4\pi\tau_2} \right] Z \left[\frac{b}{c} \right](\tau), \quad (3.6)$$

where $Z \left[\frac{a}{b} \right]$ is the partition function of the internal six-dimensional theory. This procedure is however not very handy in our case, where ν -derivatives of $\text{SL}(2, \mathbb{R})_k / \text{U}(1)$ characters lack most of the useful identities enjoyed by characters of the CFTs associated to heterotic toroidal orbifold compactifications.

collide, has no analogue in QFT.⁷ In string theory this term, which is in fact universal, contains in particular the gravitational corrections to the gauge couplings.

In the class of models coming from a toroidal reduction of a six-dimensional compactification, one can further simplify the expression (3.8) by using the decomposition of the right $U(1)_R$ current as

$$\bar{J}^R = \bar{J}^t + 2\bar{J}^3, \quad (3.9)$$

the former being the R-current of the free T^2 $\mathcal{N}_{\text{ws}} = (0, 2)$ CFT⁸ and the latter being the Cartan of the $SU(2)$ R-symmetry of the remaining interacting CFT with $(c, \bar{c}) = (22, 6)$ which has an $\mathcal{N}_{\text{ws}} = (0, 4)$ extended superconformal symmetry.

It can be shown that for any representation (h, I) of the right-moving $\mathcal{N}_{\text{ws}} = 4$ and $\bar{c} = 6$ superconformal algebra (see appendix B.2), the following trace vanishes:

$$\text{Tr}_{\mathcal{H}_{(h,I)}^{(22,6)}} e^{2\pi i \bar{J}_0^3} \bar{J}_0^3 q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} = 0. \quad (3.10)$$

for both the continuous and discrete spectrum of states, since discrete representations come in opposite pairs of eigenvalues under $\bar{J}_0^3$ and since for continuous representations (B.12) expression (3.10) contains factors $\theta_i(\tau|0)\theta'_i(\tau|0)$ with $i = 1, \dots, 4$, which vanish.

Hence, using the decomposition of the left R-current (3.9) in the index (3.8) one obtains that the one-loop gauge threshold corrections factorise as follows:

$$\Lambda_a = \frac{1}{8} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \hat{\mathcal{A}}_a(\tau). \quad (3.11)$$

The contribution of the four-dimensional warped EH space and of the the gauge bundle is now encoded in a non-holomorphic Jacobi form obtained by projecting the trace over the Hilbert space of the theory onto the right-moving twisted Ramond ground state of the $(c, \bar{c}) = (20, 6)$ CFT:⁹

$$\hat{\mathcal{A}}_a(\tau) = \frac{1}{\eta^4} \text{Tr}_{\mathcal{H}_{\bar{R}}^{(20,6)}} \left(\left[Q_a^2 - \frac{k_a}{4\pi\tau_2} \right] e^{2i\pi \bar{J}_0^3} q^{\bar{L}_0 - \frac{1}{4}} \bar{q}^{L_0 - \frac{5}{6}} \right). \quad (3.12)$$

Universality of $\mathcal{N} = 2$ threshold corrections. It has been often emphasised how universal features of $\mathcal{N}_{\text{ST}} = 2$ heterotic gauge threshold corrections can be completely determined on the one hand by requiring the absence of tachyons and cancellation of tadpoles, and on the other hand from the global symmetries dictated by the background geometry [85, 86].

Thus, by considering its $T^2 \times (T^4/\mathbb{G})$ orbifold limit, with $\mathbb{G}$ inducing a breaking of the $SO(32)$ gauge group to $G = \prod_{a \leq 16} G_a$, one can show that the one-loop threshold

⁷Such a term originates from a loop of charged or uncharged string states coupling universal to two external gauge bosons via the dilaton, and corresponds to one-particle reducible diagram [83].

⁸The $\bar{J}_0^t$ insertion in the trace absorbs the zero-modes of the free Weyl fermion in the two-torus CFT, ensuring that the index does not vanish.

⁹Note that the normalisation used here for (3.12) differs from some conventions in the literature by a sign, for instance from that of ref. [84], with conversion $\hat{\mathcal{A}}^{\text{ours}} = -\hat{\mathcal{A}}^{\text{theirs}}$.

corrections to the gauge couplings g_a^{-2} for the corresponding resolved heterotic $T^2 \times K3$ compactification are fixed uniquely by the following linear combination [83]:

$$\Lambda_a = \frac{1}{8} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) (k_a \hat{\mathcal{C}} + 2b_a) \quad (3.13)$$

in terms of the following quasi-holomorphic genus:

$$\hat{\mathcal{C}} = \frac{1}{12} \left(-\frac{\hat{E}_2 E_{10}}{\eta^{24}} + j - 1008 \right) \equiv \frac{D_{10} E_{10} - 528 \eta^{24}}{20 \eta^{24}} \quad (3.14)$$

with the Klein invariant $j = E_4^3/\eta^{24}$ and $D_{10} E_{10}$ the modular covariant derivative (C.18).

In particular, in the first expression of $\hat{\mathcal{C}}$, the combination $-\hat{E}_2 E_{10} + j \eta^{24}$ is fixed by requiring no q^{-1} pole to be present in (3.14), which would signal the presence of a tachyon. Such a would-be tachyon being uncharged under the gauge group, the potential single pole coming from η^{-24} should not appear in the gauge threshold correction. Nevertheless, gauge threshold corrections for $\mathcal{N}_{\text{ST}} = 2$ heterotic compactifications allow for a $(\tau_2 q)^{-1}$ behaviour of $\hat{\mathcal{A}}_a$ (3.11), as $q \rightarrow 0$, stemming from the IR regulator in $\hat{E}_2$. This pole, associated to an unphysical tachyon, will be referred to as 'dressed' pole in the following, in contrast to the 'bare' q^{-1} pole, which should be absent from a gauge threshold correction. In consequence, $\hat{\mathcal{C}}$ is fixed by the linear combination of two modular forms of weight 12: the quasi-holomorphic modular form $D_{10} E_{10}$ and the cusp form η^{24} , a feature which we will also observe for non-compact models.

In addition, gauge and gravitational anomaly cancellation in six-dimensional vacua fixes the constant term in $\hat{\mathcal{A}}_a$ and fixes the coefficients of the linear combination (3.13) to be the β -functions b_a and the levels k_a of the corresponding Kac-Moody algebras. Another way to look at the decomposition (3.13) is to observe that the $\hat{\mathcal{C}}$ dependent piece is IR-finite when integrated over $\mathcal{F}$ thanks to the regulator τ_2^{-1} , while the constant b_a contribution exhibits an IR divergence, signaling the presence of massless states. These are precisely the massless hypermultiplets and the vector multiplet in the four-dimensional effective field theory which contribute to the β -functions (3.3).

As a consequence of these universality properties, the two-by-two difference of threshold corrections for different gauge factors satisfy, for such heterotic $\mathcal{N}_{\text{ST}} = 2$ vacua, the relation:

$$\frac{\Lambda_{a_1}}{k_{a_1}} - \frac{\Lambda_{a_2}}{k_{a_2}} = \frac{1}{4} \left(\frac{b_{a_1}}{k_{a_1}} - \frac{b_{a_2}}{k_{a_2}} \right) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U). \quad (3.15)$$

A reformulation of the the threshold correction Λ_a associated to a $T^2 \times T^4/\mathbb{G}$ heterotic vacuum is particularly useful to understand the topology of the gauge bundle supported by the string compactification. Merging the combination (3.13) into a single contribution yields [87]:

$$\Lambda_a = \frac{k_a}{8} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \frac{1}{12 \eta^{24}} \left(-\hat{E}_2 E_{10} + \frac{n_a}{24} E_6^2 + \frac{m_a}{24} E_4^3 \right), \quad (3.16)$$

with the identification:

$$n_a = 14 - \frac{b_a}{3k_a}, \quad m_a = 10 + \frac{b_a}{3k_a}. \quad (3.17)$$

Then, the tadpole equation is reproduced by the constraint:

$$n_a + m_a = \chi(K3) = 24. \quad (3.18)$$

One can achieve some insight into the topology of the gauge bundle after resolution in the smooth $K3$ limit of the $T^4/\mathbb{G}$ orbifold by rewriting $n_a = 12 + t_a$ and $m_a = 12 - t_a$. In particular, the various β -functions depend on t_a as follows:

$$b_a = 3k_a(2 - t_a), \quad (3.19)$$

where t_a is the number of $SU(2)$ instantons now present in the resolved $T^2 \times K3$ geometry.¹⁰ Depending on the value of t_a , partial or total Higgsing of the gauge group G is possible.

3.2 Threshold corrections for local models

Before embarking, in the next section, on discussing the intricacies of how to evaluate gauge threshold corrections for $T^2 \times \widetilde{EH}$ models, it is worthwhile to put them in a wider perspective. The non-compact nature of these backgrounds will have drastic consequences, both at the physical and mathematical levels, as we will discuss below.

Vector and hyper multiplets. In order to build vertex operators corresponding to gauge bosons in space-time, one needs, as far as right-movers are concerned, to tensor a standard vector operator of the free $\mathbb{R}^{3,1}$ theory with an operator of dimension zero in the internal CFT. The latter is necessarily built on the *identity representation* (A.19) of the $SL(2, \mathbb{R})/U(1)$ coset (with spin $J = 0$), since the conformal weights for the $SU(2)/U(1)$ coset theory are non-negative.

As the identity representation of $SL(2, \mathbb{R})/U(1)$ is non-normalisable, we readily see that vector multiplets do not appear in the spectrum obtained from the partition function (2.24). In means that, assuming that these local models can be glued to a full-fledged compactification with flux, the wave-functions corresponding to the gauge bosons are not localised in the throat regions that are decoupled from the bulk by the double-scaling limit (2.15). Hence, they cannot be considered as fluctuating fields in the path-integral.

We can then interpret the result of the computation that we perform here as a one-loop correction to the gauge couplings in the effective four-dimensional theory from hypermultiplets whose higher-dimensional wave-function is localized in a particular region of the compactification manifold with strong warping, near a resolved A_1 singularity — provided the gauge group G is not further broken by global effects in the full theory.

Since vector multiplets, being 'frozen', are expected not to contribute to the β -functions, the factor (3.3) in the one-loop correction (3.1) will thus be modified as:

$$b_a^{\text{loc}} = 2 \sum_{\mathbf{R}} n_{\mathbf{R}} T_a(\mathbf{R}). \quad (3.20)$$

¹⁰In these models tadpole cancellation usually requires the presence of a certain number of small instantons hidden at orbifold singularities. Performing a slight resolution of the singularities brings out these instantons in the open, in the guise of $SO(2)$ instantons embedded in $SO(32)$. But when realizing a full blow-up to a smooth $K3$ geometry, these $U(1)$ instantons cannot be defined anymore on the blown-up $\mathbb{P}^1$'s and are replaced by $SU(2)$ instantons with instanton number t_a .

In the class of models studied here, with shift vectors of the form $\vec{\ell} = (1, \ell, 0^{14})$, whose spectrum is given in table 1, the β -functions for the gauge group factors are given accordingly by:

$$\begin{aligned} \ell = 1 : & \quad b_{\text{SO}(28)}^{\text{loc}} = 4, & b_{\text{SU}(2)}^{\text{loc}} = 56, \\ \ell = 2\mathbb{N}^* + 1 : & \quad b_{\text{SO}(28)}^{\text{loc}} = 8\ell^2, & b_{\text{U}(1)_R} = 4(29\ell^2 - 2\ell + 29). \end{aligned} \quad (3.21)$$

The useful Casimir invariants are $T(\square) = 1$ for $\text{SO}(2N)_1$, and $T(\square) = \frac{k_{\text{SU}(2)}}{2} = 1$ for the $\text{SU}(2)_2$ factor. The level of the latter is fixed by its embedding into the $\text{SO}(32)_1$ gauge algebra and is determined by identifying its Cartan with the $\text{U}(1)_R$ charge generator, which generically has level $k_{\text{U}(1)_R} = \frac{k+2}{k}$.

Perspectives on non-holomorphicity. For the heterotic local models considered here, one observes some deviations from the standard computation of threshold corrections for $T^2 \times K3$ compactifications. These are not peculiar to one-loop gauge threshold corrections, but can already be found at the level of the elliptic genus. They are due both to the non-compactness of target space and to the presence of non-zero five-brane charge at infinity.

The modified elliptic genus (3.12) for the four-dimensional warped Eguchi-Hanson theory will schematically take the form:

$$\hat{\mathcal{A}}_a(\tau) = \hat{\mathcal{A}}_a^{\text{d}}(\tau) + k_a \mathcal{R}^c(\tau) = \sum_{g=0}^{g_{\text{max}}} \frac{1}{\tau_2^g} \left(\sum_{n=-1}^{\infty} c_{gn}^{\text{d}} q^n + \sum_{m \in \mathbb{Z}} c_{gm}^c(\tau_2) q^m \right). \quad (3.22)$$

We can already give an overview of some prominent features of (3.22) which will be made more precise in the following:

- the $\hat{\mathcal{A}}_a^{\text{d}}$ contribution in (3.22) arises from states which are obtained from discrete $\text{SL}(2, \mathbb{R})_k / \text{U}(1)$ representations, i.e. from states which localise on the blown-up $\mathbb{P}^1$. As such, it retains some characteristics of its compact $K3$ analogues (3.16): it is quasi-holomorphic and, as we require no charged tachyon to appear in the spectrum, a 'bare' q^{-1} pole at infinity is absent from its Fourier expansion ($c_{0,-1}^{\text{d}} = 0$). However as for $K3$ compactifications, $\hat{\mathcal{A}}_a^{\text{d}}$ generically has poles dressed by IR regulators, namely $(\tau_2^g q)^{-1}$, which are the only source of non-holomorphicity. The maximal power g_{max} for such non-holomorphic factors is fixed by supersymmetry, as it relates to the regularisation of worldsheet divergences caused by g pairs of vertex operators colliding at the corners of the moduli space and giving rise to a massless state. Mathematically this translates as the presence of $\hat{E}_2^g$ factors in $\hat{\mathcal{A}}_a^{\text{d}}$. For a background preserving $\mathcal{N}_{\text{ST}} = 2$ in four dimensions, the effective action starts with two legs, entailing $g_{\text{max}} = 1$.¹¹ The term $\hat{\mathcal{A}}_a^{\text{d}}$ however differs from its $K3$ counterpart in that it transforms anomalously under S-transformation. It actually transforms as a *Mock modular form*, which will be discussed below.¹²

¹¹In contrast, for an $\mathcal{N}_{\text{ST}} = 1$ background, threshold corrections would derive from an effective action with four legs, inducing $g_{\text{max}} = 2$ and $\hat{E}_2^2$ factors in (3.22).

¹²It is not *strictly speaking* a Mock modular form since it contains a finite number of non-holomorphic terms, but can be recast as a sum of Mock modular forms multiplied by almost-holomorphic Jacobi forms, as we will shortly see.

- This anomalous behaviour of $\hat{\mathcal{A}}_a^d$ comes from considering only the contribution of BPS representations to the index, as we are instructed to do in the compact case. The usual argument fails here, as the fermionic zero-modes are compensated by the infinite-volume divergence. Indeed, by resorting to a modular invariant regularisation of this divergence – that adds extra non-holomorphic contributions to the index – one obtains the additional term $\mathcal{R}^c$, which decomposes on a continuous spectrum of states and will be shown to be independent of the gauge group, and universal for a fixed value of the five-brane charge $\mathcal{Q}_5$.¹³ This non-holomorphic completion seems at first sight to exhibit an infinite number of poles in q , with arbitrarily large order. However, in the theory of non-holomorphic Jacobi forms reviewed below, $\mathcal{R}^c$ contains the transform of the *shadow* of a Mock modular form. This dictates a specific form for the functions $c_{gm}(\tau_2)$ (see [47]). In particular, as a sum $\mathcal{R}^c$ can be shown to be absolutely and uniformly convergent for $\tau \in \mathcal{H}$ (upper half complex plane) in such a way as only to possess a single ‘dressed’ pole at $\tau_2 \rightarrow \infty$. In this case however the non-holomorphic regulator comprises additional exponential terms which are a distinguishable feature of non-localised states, the real part of this polar term being generically bounded by $\frac{1}{\tau_2} \sum_{n \in I} c_n e^{-\frac{\pi n}{2k(k-2)} \tau_2} e^{2\pi \tau_2}$, where $I \subset \mathbb{N}$ is a finite set. Thus $\mathcal{R}^c$ has a polar structure even less divergent at $\tau_2 \rightarrow \infty$ than the $(\tau_2 q)^{-1}$ cusp of $\hat{\mathcal{A}}_a^d$. Hence, to determine explicitly the moduli dependence of the threshold corrections (3.11), one can proceed as for toroidal orbifolds.

3.3 A brief review on Mock modular forms

In the previous section, we mentioned that the gauge threshold correction (4.5) incorporates contributions from non-localised states, which enter into the function $\mathcal{R}^c$ and recombine into the transform of the shadow function of some Mock modular form. We find it useful to recall here some facts about Mock modular forms and their isomorphism to weak harmonic Maaß forms, and in particular to clarify the notion of shadow. In this perspective, we synthesise among other things the illuminating presentation of [48].

Disregarding possible dependence on elliptic variables, a Mock modular form h of weight r is a function of the upper half-plane $\mathcal{H} = \{\tau \in \mathbb{C} | \tau_2 \geq 0\}$, which *almost* transforms as a modular form of corresponding weight. The space of all such forms, which we call $\mathbb{M}_r$, contains as subspace the space $M_r^!$ of weak holomorphic modular forms of weight r , which are allowed to have exponential growth, that is q^{-N} singularities, at cusps. Then, associated to a Mock modular form $h \in \mathbb{M}_r$ there exists a *shadow* $g = \mathcal{S}[h]$, which is an ordinary holomorphic modular form of weight $2 - r$. As such it has expansion

$$g(\tau) = \sum_{\nu \geq 0} b_\nu q^\nu, \quad (3.23)$$

where ν runs over some arithmetic progression in $\mathbb{Q}$.

¹³Discussions on non-holomorphicity of the elliptic genus are also central to the question of deriving a reliable index for micro-states counting for systems of multi-centered black-holes [50].

The shadow map $\mathcal{S}$ is $\mathbb{R}$ -linear in h and can be given by defining an associated function g^* , which is the following transform of g :

$$\begin{aligned} g^*(\tau) &= \left(\frac{i}{2}\right)^{r-1} \int_{-\bar{\tau}}^{i\infty} \frac{\overline{g(-\bar{\tau})}}{(z+\tau)^r} dz \\ &= \frac{\bar{b}_0}{(r-1)(4\tau_2)^{r-1}} + \pi^{r-1} \sum_{\nu>0} \nu^{r-1} \bar{b}_\nu \Gamma(1-r, 4\pi\nu\tau_2) q^{-\nu}, \end{aligned} \quad (3.24)$$

where ν belongs to an arithmetic progression in $\mathbb{Q}$, and $\Gamma(x, s)$ is the upper incomplete gamma function:

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt, \quad x > 0. \quad (3.25)$$

The function g^* is such that the combination

$$\hat{h}(\tau) = h(\tau) + g^*(\tau) \quad (3.26)$$

transforms, for all $\gamma \in \begin{pmatrix} \bullet & \bullet \\ c & d \end{pmatrix} \in \Gamma$, a suitable subgroup of $\mathrm{SL}(2, \mathbb{Z})$, as a modular form of weight r :

$$\hat{h}(\gamma\tau) = \rho(\gamma)(c\tau + d)^r \hat{h}(\tau),$$

where ρ is a character of Γ . As $\mathcal{S}$ is surjective and in addition vanishes when h is (a weakly holomorphic) modular form, we have the following exact sequence over $\mathbb{R}$:

$$0 \longrightarrow M_r^! \longrightarrow \mathbb{M}_r \xrightarrow{\mathcal{S}} M_{2-r} \longrightarrow 0 \quad (3.27)$$

and $\mathbb{M}_r$ can be regarded as an extension of a space of classical modular forms.

As the non-holomorphicity of $\hat{h}$ is integrally encoded in the shadow function g^* , we can reverse the perspective and obtain h by acting with Cauchy-Riemann operator $\partial/\partial\bar{\tau}$ on $\hat{h}$, which by combining (3.26) and (3.24) gives:

$$\frac{\partial \hat{h}}{\partial \bar{\tau}} = -\frac{2i}{(4\tau_2)^r} \overline{g(\tau)}, \quad (3.28)$$

by which we recover $h = \hat{h} - g^*$. Through this procedure we can establish a canonical isomorphism $\mathbb{M}_r \cong \widehat{\mathbb{M}}_r$ between the space $\widehat{\mathbb{M}}_r$ of non-holomorphic weak modular forms of weight r , to which $\hat{h}$ belongs, and the space of Mock modular forms of corresponding weight.

We can now push further and show that the space $\widehat{\mathbb{M}}_r$ is actually the space of weak harmonic Maaßforms. To this end, we define $\mathfrak{M}_{r,l}$ the space of modular forms of weight (r, l) , i.e. which transform as $F(\gamma\tau) = \rho(\gamma)(c\tau + d)^r (c\bar{\tau} + d)^l F(\tau)$ for $\gamma \in \Gamma \subset \mathrm{SL}(2, \mathbb{R})$, such that $\mathfrak{M}_r = \mathfrak{M}_{r,0}$ reduces to the space of real-analytic modular forms in $\tau \in \mathcal{H}$ of weight r . In addition we introduce an operator $\tau_2^s \partial_{\bar{\tau}}$ which sends $\mathfrak{M}_{r,l} \xrightarrow{\cong} \mathfrak{M}_{r,l+2} \longrightarrow \mathfrak{M}_{r-s,l-s+2}$, where the first map is an isomorphism for $s \in \mathbb{Z}$.

Applying this operator to $\mathfrak{M}_r = \mathfrak{M}_{r,0}$ and further acting with the holomorphic derivative we obtain the commutative diagram:

$$\begin{array}{ccccc}
 \mathfrak{M}_r = \mathfrak{M}_{r,0} & \xrightarrow{\partial/\partial\bar{\tau}} & \mathfrak{M}_{r,2} & \xrightarrow[\cong]{\tau_2^r} & \mathfrak{M}_{0,2-r} & \xrightarrow{\partial/\partial\tau} & \mathfrak{M}_{2,2-r} \\
 & & & & \uparrow & \nearrow 0 \\
 & & & & \overline{M_{2-r}} & &
 \end{array}$$

It follows from this diagram that $\widehat{\mathbb{M}}_r$ is defined as the space of real-analytic modular forms $F \in \mathfrak{M}_r$ such that $\tau_2^r \partial_{\bar{\tau}} F$ belongs to $\overline{M_{2-r}}$, in other words for which it is antiholomorphic:

$$\widehat{\mathbb{M}}_r = \left\{ F \in \mathfrak{M}_r \mid \frac{\partial}{\partial\tau} \left(\tau_2^r \frac{\partial}{\partial\bar{\tau}} F(\tau) \right) = 0 \right\}. \quad (3.29)$$

Now since $\partial_{\tau}(\tau_2^r \partial_{\bar{\tau}}(\bullet))$ is up to an additive constant proportional to the weight r Laplace operator Δ_r , namely:

$$\Delta_r F = \frac{1}{4} \frac{\partial}{\partial\tau} \left(\tau_2^r \frac{\partial}{\partial\bar{\tau}} F \right) + \frac{r}{2} \left(1 - \frac{r}{2} \right) F \quad (3.30)$$

$\widehat{\mathbb{M}}_r$ is thus the space of real-analytic modular forms which are allowed exponential growth at cusps and that are harmonic with $\frac{r}{2}(1 - \frac{r}{2})$ eigenvalue under the weight K Laplacian. This is precisely the definition of *weak harmonic Maaß forms* according to Bruiner and Funke, which completes the identification.

Appell-Lerch sums. The simplest and most familiar example of a Mock modular form is the almost modular Eisenstein series E_2 , whose shadow $g(\tau) = -\frac{12}{\pi}$ is a constant. Using formula (3.26) for a weight 2 Mock modular form, we get the well known non-holomorphic completion $\widehat{E}_2 = E_2 - \frac{3}{\pi\tau}$.

In this work, we will be particularly interested in a more involved class of Mock modular forms, the *Appell-Lerch sums*. The Appell-Lerch sums of level K are functions of the upper half plane $\tau \in \mathcal{H}$ and depend on two elliptic variables $u \in \mathbb{C}$ and $v \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$:

$$A_K(u, v|\tau) = a^{\frac{K}{2}} \sum_{n \in \mathbb{Z}} (-)^{Kn} \frac{q^{\frac{K}{2}n(n+1)} b^n}{1 - aq^n}, \quad \text{with } a = e^{2\pi i u}, \quad b = e^{2\pi i v}. \quad (3.31)$$

The investigation of the near modular behaviour of these functions can be deduced from the transformation properties of the level one A_1 sum, since for an arbitrary level K we can reexpress:

$$\begin{aligned}
 A_K(u, v|\tau) &= \sum_{m=0}^{K-1} a^m A_1 \left(Ku, v + m\tau + \frac{K-1}{2} | K\tau \right) \\
 &\equiv \frac{a^{\frac{K-1}{2}}}{K} \sum_{m \in \mathbb{Z}_K} a^m A_1 \left(u, \frac{v+m}{K} + \frac{(K-1)\tau}{2K} \mid \frac{\tau}{K} \right).
 \end{aligned} \quad (3.32)$$

We can thus concentrate on the level one case. In particular, the almost modularity of A_1 consists in its failure to transform as modular form under S-transformation:

$$A_1 \left(\frac{u}{\tau}, \frac{v}{\tau} \mid -\frac{1}{\tau} \right) = \tau e^{\pi i \frac{(2v-u)u}{\tau}} \left[A_1(u, v|\tau) - \frac{1}{2} M(u-v|\tau) i \vartheta_1(v|\tau) \right] \quad (3.33)$$

where the second term on the r.h.s. contains the function M of $\tau \in \mathcal{H}$ and $\nu \in \mathbb{C}$ first studied by Mordell, which is defined in terms of the integral:

$$M(\nu|\tau) = \int_{\mathbb{R}} dx \frac{q^{\frac{x^2}{2}} e^{-2\pi x \nu}}{\cosh(\pi x)}. \quad (3.34)$$

There is a clear reminiscence of this behaviour in the S-transformation of discrete $\mathrm{SL}(2, \mathbb{R})_k/\mathrm{U}(1)$ characters (2.27), which will be made explicit in a moment.

To construct the non-holomorphic completion of the Appell-Lerch sums (3.31), it then suffices to consider the level $K = 1$ example, in which case it actually proves more convenient to normalise this sum by a ϑ -function:

$$\mu(u, v|\tau) = -\frac{i}{\vartheta_1(v|\tau)} A_1(u, v|\tau), \quad (3.35)$$

also called the *Appell function*. Then, by studying the modular transformation properties of the function M (3.34) and by noticing that the near modularity of the Appell function μ only depends on the difference $u - v$, Zwegers was able to construct its function g^* :

$$R(\nu|\tau) = \sum_{n \in \mathbb{Z}} (-)^n \left(\operatorname{sgn} \left(n + \frac{1}{2} \right) - E \left(\left[n + \frac{1}{2} + \frac{\nu_2}{\tau_2} \right] \sqrt{2\tau_2} \right) \right) z^{-(n+\frac{1}{2})} q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \quad (3.36)$$

where $\nu_2 = \operatorname{Im} \nu$, and $E(z)$ is the error function, defined as follows:

$$E(z) = 2 \int_0^z e^{-\pi w^2} dw, \quad z \in \mathbb{C}, \quad (3.37)$$

which is an odd and entire function of z . Since the argument of E in (3.36) is real, we can alternatively express $R(\nu|\tau)$ in terms of the incomplete gamma function (3.25)

$$E(x) = \operatorname{sign}(x) \left[1 - \frac{1}{\sqrt{\pi}} \Gamma \left(\frac{1}{2}, \pi x^2 \right) \right], \quad x \in \mathbb{R}, \quad (3.38)$$

by means of the following identity:

$$\operatorname{erfc}(\sqrt{\pi}|x|) = 2 \int_{|x|}^{\infty} e^{-\pi u^2} du = \int_{x^2}^{\infty} v^{-\frac{1}{2}} e^{-\pi v} dv = \frac{1}{\sqrt{\pi}} \Gamma \left(\frac{1}{2}, \pi x^2 \right), \quad (3.39)$$

where $\operatorname{erfc}(\sqrt{\pi}x) = 1 - E(x)$ is the complementary error function. One sees that $R(\nu|\tau)$ is indeed of the form propounded in (3.24).

The Appell function can thus be completed into a non-holomorphic Jacobi form of two elliptic variables:

$$\widehat{\mu}(u, v|\tau) = \mu(u, v|\tau) - \frac{1}{2} R(u - v|\tau), \quad (3.40)$$

which is furthermore a harmonic Maaß form for the Laplace operator $\Delta_{1/2}$ (3.30), and thus transforms as a Jacobi form of weight $1/2$.

In particular for $u = v = \nu$, the non-holomorphic completion of the Appell function of one elliptic variable, which we denote by $\widehat{\mu}(\nu|\tau) \equiv \widehat{\mu}(\nu, \nu|\tau)$ in the following, reads

$$\widehat{\mu}(\nu|\tau) = -\frac{i}{\vartheta_1(\nu|\tau)} \sum_{n=0}^{\infty} (-)^n \frac{q^{\frac{1}{2}n(n+1)} z^{n+\frac{1}{2}}}{1 - zq^n} - \sum_{n=0}^{\infty} (-)^n \operatorname{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{2}(n+\frac{1}{2})^2} \quad (3.41)$$

and is characterised by a shadow function which can be extracted from the relation (3.28):

$$\frac{\partial \hat{\mu}(\nu|\tau)}{\partial \bar{\tau}} = \frac{i}{2\sqrt{2}} \frac{\overline{\eta(\tau)}^3}{\sqrt{\tau_2}}. \quad (3.42)$$

The shadow of $\mu(\nu|\tau)$ is thus the holomorphic modular form $g(\tau) = -\frac{1}{2\sqrt{2}}\eta(\tau)^3$ with weight $3/2$, as expected for a Mock Jacobi form of weight $1/2$.

The full modular transformation properties of $\hat{\mu}$ are neatly given by:

$$\hat{\mu}(u, v|\tau + 1) = e^{-\frac{\pi i}{4}} \hat{\mu}(u, v|\tau), \quad \hat{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau} \middle| -\frac{1}{\tau}\right) = -e^{-\frac{\pi i}{4}} \sqrt{\tau} e^{-\pi i \frac{(u-v)^2}{\tau}} \hat{\mu}(u, v|\tau) \quad (3.43)$$

from which we deduce its index to be $\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$. Transformations of $\hat{\mu}$ under shifts in the elliptic variables can also be worked out (note that $\hat{\mu}$ is symmetric in u and v):

$$\hat{\mu}(u + 1, v|\tau) = a^{-1} b q^{-\frac{1}{2}} \hat{\mu}(u + \tau, v|\tau) = -\hat{\mu}(u, v|\tau). \quad (3.44)$$

In addition, $\hat{\mu}$ satisfies:

$$\begin{aligned} \hat{\mu}(u + \lambda, v + \lambda|\tau) - \hat{\mu}(u, v|\tau) &= \mu(u + \lambda, v + \lambda|\tau) - \mu(u, v|\tau) \\ &= -\frac{\eta(\tau)^3 \vartheta_1(u + v + \lambda|\tau) \vartheta_1(\lambda|\tau)}{\vartheta_1(u|\tau) \vartheta_1(v|\tau) \vartheta_1(u + \lambda|\tau) \vartheta_1(v + \lambda|\tau)}, \\ &\quad u, v, u + \lambda, v + \lambda \notin \mathbb{Z}\tau + \mathbb{Z}. \end{aligned} \quad (3.45)$$

By using the reformulation of the Appell-Lerch sums at arbitrary level K in terms of A_1 , as in (3.32), and the non-holomorphic completion (3.36), we can generalise the construction of similar corrective terms for all sums A_K :

$$\begin{aligned} \hat{A}_K(u, v|\tau) &= A_K(u, v|\tau) \\ &\quad - \frac{1}{2} \sum_{m=0}^{K-1} a^m R\left(Ku - v - m\tau - \frac{K-1}{2} \middle| K\tau\right) i\vartheta_1\left(v + m\tau + \frac{K-1}{2} \middle| K\tau\right) \\ &= A_K(u, v|\tau) \\ &\quad - \frac{a^{\frac{K-1}{2}}}{2K} \sum_{m \in \mathbb{Z}_K} R\left(u - \frac{v+m}{K} - \frac{(K-1)\tau}{2K} \middle| \frac{\tau}{K}\right) i\vartheta_1\left(\frac{v+m}{K} + \frac{(K-1)\tau}{2K} \middle| \frac{\tau}{K}\right). \end{aligned} \quad (3.46)$$

One can then show that these non-holomorphic Appell-Lerch sums indeed transform under the modular group as a Jacobi form of two elliptic variables:

$$\hat{A}_K(u, v|\tau + 1) = \hat{A}_K(u, v|\tau), \quad \hat{A}_K\left(\frac{u}{\tau}, \frac{v}{\tau} \middle| -\frac{1}{\tau}\right) = \tau e^{\pi i \frac{(2v-Ku)u}{\tau}} \hat{A}_K(u, v|\tau), \quad (3.47)$$

and display the following elliptic transformations:

$$\begin{aligned} \hat{A}_K(u + 1, v|\tau) &= (-)^K \hat{A}_K(u, v|\tau), & \hat{A}_K(u, v + 1|\tau) &= \hat{A}_K(u, v|\tau), \\ \hat{A}_K(u + \tau, v|\tau) &= (-)^K a^K b^{-1} q^{\frac{K}{2}} \hat{A}_K(u, v|\tau), & \hat{A}_K(u, v + \tau|\tau) &= a^{-1} \hat{A}_K(u, v|\tau), \end{aligned} \quad (3.48)$$

which makes them into non-holomorphic Jacobi form of weight 1 and index $\begin{pmatrix} -K/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

4 Computations of the gauge threshold corrections

After the preliminary discussions of section 3 we are now ready to get to the heart of the matter, namely the actual computation of the threshold corrections. We need to consider each gauge factor separately, namely the $U(1)$ (enhanced to $SU(2)$ for $\ell = 1$) and the $SO(28)$ factor, since the former comes from the R-symmetry of the interacting CFT and the latter from the remaining free left-moving fermions. We will start with the $SO(28)$ case, which is simpler, and consider in more detail the special case $\ell = 1$ for which the superconformal symmetry is enhanced.

4.1 The $SO(28)$ gauge threshold corrections: discrete representations

In this section, we consider the one-loop corrections to the $SO(28)$ gauge coupling (3.22). For the sake of clarity, we start by computing the contribution $\hat{\mathcal{A}}_{SO(28)}^d$ from discrete (BPS) representations that localises on the resolved A_1 singularity, which can be determined algebraically from the partition function (2.24). As stressed before, this contribution is not modular-invariant by itself, and needs a non-holomorphic completion, namely $\mathcal{R}^c$ in (3.22) coming from non-BPS non-localised states to be free of modular anomalies.

Keeping in mind that the Kac-Moody level of this orthogonal factors is $k_{SO(28)} = 1$, the contribution to the modified elliptic genus (3.12) which localises on the resolved singularity is obtained by projecting the right-moving sector of the internal four-dimensional theory (2.19) onto its twisted Ramond ground state, while summing over all states in the right-moving sector. To facilitate the calculation we split the genus into left- and right-moving contributions:

$$\hat{\mathcal{A}}_{SO(28)}^d(\tau) = \sum_{2j=0}^{k-2} \sum_{2J=2}^k A_L^{(j,J)} A_R^{(j,J)}. \quad (4.1)$$

The right-movers part yields a Witten type index identifying the $SL(2, \mathbb{R})_k/U(1)$ discrete spin and the $SU(2)_k/U(1)$ one:

$$\begin{aligned} A_R^{(j,J)} &= \sum_{m \in \mathbb{Z}_{2k}} \overline{C}_m^j \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau, 0) \overline{\text{Ch}}_d \left(J, \frac{m}{2} - J - \frac{1}{2}; \tau, 0 \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta_{2J-m,1}^{[2]} \\ &= \sum_{m \in \mathbb{Z}_{2k}} (\delta_{m,2j+1} - \delta_{m,-(2j+1)}) \delta_{\frac{m}{2}-J+\frac{1}{2},0}^{[k]} \delta_{2J-m,1}^{[2]} \\ &= (\delta_{j,J-1}^{[k]} - \delta_{j,-J}^{[k]}) \delta_{2j,2J}^{[2]} \\ &= \delta_{j,J-1}. \end{aligned} \quad (4.2)$$

In particular, we observe from the second line of the above expression that this index counts representations built on right-moving anti-chiral primaries of $SL(2, \mathbb{R})/U(1)$, see appendix A. As the extended discrete $SL(2, \mathbb{R})/U(1)$ character in expression (4.2) takes into account all winding sectors of the model by incorporating all $\mathbb{Z}_{2k}$ orbits of spectral flow, the latter condition selects all states with:

$$m - 2J = -1 \bmod 2k. \quad (4.3)$$

To determine the contributions $A_L^{(j,J)}$ from the left-moving sector, we observe that the quadratic Casimir operator acts on $\text{SO}(28)$ characters as $Q^2 \chi_{\text{SO}(28)}(\nu_1, \dots, \nu_{14}|\tau) = -\frac{1}{4\pi^2} \partial_{\nu_1}^2 \chi_{\text{SO}(28)}(\nu_1, \dots, \nu_{14}|\tau)$. Hence we obtain

$$\begin{aligned} A_L^{(j,J)} &= \frac{1}{2\eta^4} \sum_{\gamma, \delta=0}^1 (-)^{\delta(2j + [\frac{k}{2}-1]\gamma)} \chi_{k-2}^{j+\gamma(\frac{k}{2}-2j-1)} \\ &\times \sum_{u,v=0}^1 \sum_{n \in \mathbb{Z}_{2\ell}} e^{-i\pi v(n + \frac{u}{2})} \text{Ch}_d \left(J, \ell \left(n + \frac{u}{2} \right) - J - \frac{u}{2} \right) \begin{bmatrix} u \\ v \end{bmatrix} \\ &\times \left(-\frac{1}{4\pi^2} \right) \left[\frac{\vartheta'' \begin{bmatrix} u \\ v \end{bmatrix}}{\vartheta \begin{bmatrix} u \\ v \end{bmatrix}} + \frac{\pi}{\tau_2} \right] \frac{\vartheta \begin{bmatrix} u \\ v \end{bmatrix}^{14}}{\eta^{14}} \delta_{2J, (\ell-1)u}^{[2]}, \end{aligned} \quad (4.4)$$

We note that the $\mathbb{Z}_2$ orbifold (2.19) and the K-theory condition (2.14) combine to project out half-integer $\text{SU}(2)_{k-2}$ and $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ spins j and J , which are identified through (4.2).

If we tried to use this algebraic method to determined the contribution of continuous $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ representations to the modified index on the basis of how they enter into the partition function, for which a non-modular invariant regularisation has been adopted, we would obtain zero. The reason is that non-localised states behave like the 'untwisted' sector of an orbifold compactification, hence do not contribute to the index because of their fermionic zero-modes. As we shall see shortly, continuous $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ representations nonetheless enter into the modified elliptic genus, if we adopt a non-holomorphic regularisation of the path integral.

Collecting both left- and right-moving contributions from localised states, and leaving for the moment the term $\mathcal{R}^c$ unspecified, the one-loop threshold correction to the $\text{SO}(28)$ gauge coupling for arbitrary five-brane charge $\mathcal{Q}_5 = k/2 = \ell^2$ reads:

$$\begin{aligned} \Lambda_{\text{SO}(28)}[\mathcal{Q}_5] &= \frac{1}{96} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \\ &\times \left[\sum_{(u,v) \neq (0,0)} \Phi_k \begin{bmatrix} u \\ v \end{bmatrix} \frac{\vartheta \begin{bmatrix} u \\ v \end{bmatrix}^{14}}{\eta^{14}} \frac{\widehat{E}_2 + (-1)^v \vartheta \begin{bmatrix} u+v+1 \\ u \end{bmatrix}^4 - (-1)^u \vartheta \begin{bmatrix} v \\ u+v+1 \end{bmatrix}^4}{\eta^4} + 12 \mathcal{R}^c[\mathcal{Q}_5] \right]. \end{aligned} \quad (4.5)$$

The contributions from the $(c, \bar{c}) = (6, 6)$ interacting CFT with is encoded in the localised elliptic indices with mixed left / right boundary conditions:

$$\begin{aligned} \Phi_k \begin{bmatrix} u \\ v \end{bmatrix}(\nu|\tau) &= \text{Tr}_{\mathcal{H}_{* \otimes \mathbb{R}}^{(6,6)} \text{discrete}} \left(e^{\pi i (2\bar{J}_0^3 + v J_0^R)} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} z^{J_R} \right) \\ &= \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) \sum_{n \in \mathbb{Z}_{2\ell}} e^{-i\pi v(n + \frac{u}{2})} \text{Ch}_d(J, \ell n - J; \nu|\tau) \begin{bmatrix} u \\ v \end{bmatrix}, \end{aligned} \quad (4.6)$$

where $*$ stands for NS when $u = 0$ and for R when $u = 1$.

The elliptic indices (4.6) can be obtained by spectral flows of what is commonly known as the *elliptic genus* of the $(c, \bar{c}) = (6, 6)$ CFT underlying the solution (2.16). This topological invariant is obtained by projecting the trace on the (discrete representation) Hilbert

space onto the $\tilde{R} \otimes \tilde{\tilde{R}}$ ground state of the CFT. It is nothing but

$$\Phi_k(\nu|\tau) \equiv \Phi_k \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\nu|\tau). \quad (4.7)$$

It will be convenient for later us to package the contribution of $SL(2, \mathbb{R})_k/U(1)$ characters with spin J in the single function:

$$\mathcal{Y}_k^J(\nu|\tau) = \sum_{n \in \mathbb{Z}_{2\ell}} (-)^{n+1} i\text{Ch}_d \left(J, \ell \left(n + \frac{1}{2} \right) - J - \frac{1}{2}; \nu|\tau \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (4.8)$$

In terms of Φ_k , the remaining genera (4.6) are easily recovered by spectral flow:

$$\begin{aligned} & \Phi_k \begin{bmatrix} 1-a \\ 1-b \end{bmatrix} (\nu|\tau) \\ &= (-)^b e^{\frac{\pi i(\ell+1)}{2}(a+b+ab)} q^{\frac{k+2}{16}a^2} z^{-\frac{k+2}{4\ell}a} \sum_{J=1}^{k/2} \left(\chi^{J-1} + \chi^{k/2-J} \right) \mathcal{Y}_k^J \left(\nu - \frac{\ell(a\tau+b)}{2} | \tau \right), \end{aligned}$$

with $\ell \in 2\mathbb{N}^* + 1$.

As these elliptic genera are restricted to localised states, they can be given the same interpretation as for compact models. More specifically, $\Phi_k \begin{bmatrix} 0 \\ v \end{bmatrix}$ are elliptic generalisations of the Dirac index, which keep track of how antisymmetric tensor representations of the $SO(2)^2 \subset SO(4)$ embedding of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell)$ are counted, whether with a plus ($v=0$) or minus ($v=1$) sign [45, 88]. In contrast, the index $\Phi_k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ captures the coupling of the elliptic generalisation of the Dirac index to the spinor bundles associated to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell)$.

Starting from the discrete representations contribution to the elliptic genus (4.5), using equations (A.5) and (A.16), one reproduces the inverse cusp form η^{-24} characteristic of the polar behaviour of $\mathcal{N}_{\text{ST}} = 2$ heterotic gauge threshold corrections (3.16), related to the would-be tachyon (see section 3.2 for discussion). This confirms that the contribution of localised states to the gauge threshold correction (4.5) is similar in nature to what is expected for a genuine heterotic compactification.

4.2 Infinite volume regularisation and non-holomorphic completion of the Appel-Lerch sum

We have shown above how to express the contribution to the gauge threshold correction of states localised on the resolved singularity, see eq. (4.5), in terms of a combination of holomorphic $SL(2, \mathbb{R})/U(1)$ characters, given by eq. (4.8). As we discussed above, the modular properties of these characters, given in appendix A, imply that the result is not a modular form as it should.

This problem can be traced back to the partition function (2.24), from which the elliptic genus has been extracted, which displays a holomorphic anomaly, since the infinite-volume divergence has been removed in a rather cavalier way, which preserves the splitting of the theory into holomorphic and anti-holomorphic characters of the chiral algebra but spoils modular invariance.

Completing the elliptic genus. A modular-covariant regularization of the $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ elliptic genus has been developed first in [46] and subsequent [61, 63]. The idea behind this work, which is summarised in appendix D, was to reformulate the elliptic genus directly in terms of a path integral. The poles in the zero-mode integral, corresponding to the infinite target-space volume divergence, were regularised in a way that preserves modular invariance, thus giving an unambiguous prescription to evaluate the elliptic genus. As explained in more details in appendix D, the result of this evaluation splits into a holomorphic contribution coming from the discrete representations, which can be resummed into the Appell-Lerch sum A_{2k} , and a non-holomorphic contribution coming from continuous representations:

$$\begin{aligned}\widehat{\mathcal{Z}}_k(\nu|\tau) &= \sum_{2J=1}^k \mathrm{Ch}_d(J, -1; \tau, \nu) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{i}{\pi} \sum_n \int_{\mathbb{R}-i\varepsilon} \frac{dp}{2ip+n} \mathrm{Ch}_c\left(\frac{1}{2} + ip, \frac{n}{2}; \nu|\tau\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{q}^{\frac{p^2}{k} + \frac{n^2}{4k}} \\ &= -\hat{A}_{2k}\left(\frac{\nu}{k}, 2\nu|\tau\right) \frac{i\theta_1(\nu|\tau)}{\eta(\tau)^3}.\end{aligned}\tag{4.9}$$

In this non-holomorphic regularisation of the infinite target-space volume divergence, continuous representations supply the precise counter-term needed to cancel the holomorphic anomaly, which is none other than the (transformed) shadow function $R(u|\tau)$ (3.36), summed as in expression (3.46).

In the cases considered here a similar procedure can be carried out. However, since we have already computed the discrete representations contribution, i.e. the Mock Jacobi form of interest, it will suffice to use the *shadow map* $\mathcal{S}$ dictated by the regularisation scheme (4.9) in order to get a genuine modular form. To this end we rewrite the contribution of discrete $\mathrm{SL}(2, \mathbb{R})_k/\mathrm{U}(1)$ representations (4.8) as:

$$\begin{aligned}\mathcal{Y}_k^J(\nu|\tau) &= \sum_{n \in \mathbb{Z}_{2\ell}} (-)^{n+1} i \mathrm{Ch}_d\left(J, \ell\left(n + \frac{1}{2}\right) - J - \frac{1}{2}; \tau, \nu\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \sum_{n \in \mathbb{Z}_{2\ell}} (-)^{n+1} \sum_{m \in \mathbb{Z}} \frac{q^{\frac{1}{2}(2\ell m + n + \frac{1}{2})^2 - \frac{1}{k}(J - \frac{1}{2})^2} z^{\frac{1}{\ell}(2\ell m + n + \frac{1}{2})}}{1 - zq^{\ell(2\ell m + n) + \frac{\ell+1}{2} - J}} \frac{i\vartheta_1(\tau, \nu)}{\eta(\tau)^3} \\ &= q^{\frac{1}{8} - \frac{1}{k}(J - \frac{1}{2})^2} \sum_{s \in \mathbb{Z}} (-)^{s+1} \frac{q^{\frac{1}{2}s(s+1)} z^{\frac{1}{\ell}(s + \frac{1}{2})}}{1 - zq^{\ell s + \frac{\ell+1}{2} - J}} \frac{i\vartheta_1(\tau, \nu)}{\eta(\tau)^3} \\ &= -q^{-\frac{1}{k}(J - \frac{\ell+1}{2})^2} \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} A_1\left(\frac{1}{\ell} \left(\nu + \left(\frac{\ell+1}{2} - J\right)\tau + m\right), \frac{\nu}{\ell}|\tau\right) \frac{i\vartheta_1(\tau, \nu)}{\eta(\tau)^3}.\end{aligned}\tag{4.10}$$

To express the result in terms of a level 1 Appell sum we used the following identity:¹⁴

$$\frac{1}{1 - aq^{\ell n}} = \frac{1}{\ell} \sum_{m=0}^{\ell-1} \frac{1}{1 - e^{2\pi i \frac{m}{\ell}} a^{\frac{1}{\ell}} q^n}, \quad a = e^{2\pi i u}.\tag{4.11}$$

¹⁴We are particularly grateful to S. Zwegers for suggesting this formula.

Then, the regularisation of the infinite target-space volume divergence goes through¹⁵ like in eq. (4.9). Using (3.35) and its completion into a Maß form (3.40), the full expression for the $\mathrm{SL}(2, \mathbb{R})_k/\mathrm{U}(1)$ factor (4.10) can be nicely repackaged in a sum of non-holomorphic Appell functions:

$$\widehat{\mathcal{Y}}_k^J(\nu|\tau) = q^{-\frac{1}{k}(J-\frac{\ell+1}{2})^2} \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} \widehat{\mu} \left(\frac{1}{\ell} \left(\nu + \left(\frac{\ell+1}{2} - J \right) \tau + m \right), \frac{\nu}{\ell} \middle| \tau \right) \frac{\vartheta_1(\frac{\nu}{\ell}|\tau) \vartheta_1(\nu|\tau)}{\eta(\tau)^3}. \quad (4.12)$$

From the above expression, we obtain the regularised expression of the elliptic genus (4.7), now also including continuous representation resummed in the (transform) shadow function $R(u|\tau)$ (3.36), as the following weight 0 Maß form:

$$\widehat{\Phi}_k(\nu|\tau) = \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) \widehat{\mathcal{Y}}_k^J(\nu|\tau). \quad (4.13)$$

Spectral flow and gauge threshold corrections. Using this result one can recover the full set of regularised genera (4.9) for the $(c, \bar{c}) = (6, 6)$ theory by spectral flowing the elliptic genus (4.13):

$$\begin{aligned} \widehat{\Phi}_k \left[\begin{matrix} u \\ v \end{matrix} \right] (\nu|\tau) &= \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) q^{-\frac{1}{k}(J-\frac{\ell+1}{2})^2} \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} \times \\ &\times \widehat{\mu} \left(\frac{1}{\ell} \left(\nu + \left(\frac{\ell u + 1}{2} - J \right) \tau + \frac{\ell(v-1)}{2} + m \right), \frac{\nu}{\ell} + \frac{(u-1)\tau + v - 1}{2} \middle| \tau \right) \times \\ &\times \frac{\vartheta \left[\frac{u}{v} \right] (\frac{\nu}{\ell}|\tau) \vartheta \left[\frac{u}{v} \right] (\nu|\tau)}{\eta(\tau)^3}. \end{aligned} \quad (4.14)$$

The threshold corrections to the $\mathrm{SO}(28)$ gauge coupling for arbitrary $\mathcal{Q}_5 = k/2$ units of five-brane flux then reads:

$$\begin{aligned} \Lambda_{\mathrm{SO}(28)}[\mathcal{Q}_5] &= \frac{1}{96} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \Gamma_{2,2}(T, U) \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) q^{-\frac{1}{k}(J-\frac{\ell+1}{2})^2} \times \\ &\times \sum_{(u,v) \neq (1,1)} \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} \frac{\widehat{\mu} \left(\left(\frac{\ell u + 1}{2} - J \right) \frac{\tau}{\ell} + \frac{v-1}{2} + \frac{m}{\ell}, \frac{(u-1)\tau + (v-1)}{2} \middle| \tau \right)}{\eta} \times \\ &\times \frac{\left(\widehat{E}_2 + (-)^v \vartheta \left[\frac{u+v+1}{u} \right]^4 - (-)^u \vartheta \left[\frac{v}{u+v+1} \right]^4 \right) \vartheta \left[\frac{u}{v} \right]^{16}}{\eta^{20}}. \end{aligned} \quad (4.15)$$

¹⁵It is interesting to note that initial $\mathbb{Z}_{2\ell}$ orbifold of the $\mathrm{SL}(2, \mathbb{R})_k/\mathrm{U}(1)$ theory in (4.10) is rewritten in terms of a $\mathbb{Z}_\ell$ orbifold of the Appell sum A_1 . As can be seen by combining (4.23) and (4.25), A_1 encodes the discrete representation (i.e. holomorphic) contribution to the elliptic genus of the $(\mathrm{SL}(2, \mathbb{R})_2/\mathrm{U}(1))/\mathbb{Z}_2$ orbifold:

$$\frac{1}{2} \sum_{\gamma, \delta \in \mathbb{Z}_2} z^{2\gamma} q^{\frac{\gamma^2}{2}} \mathcal{Z}_2^d(\nu + \gamma\tau + \delta|\tau) = -A_1(\nu|\tau) \frac{i\vartheta_1(\nu|\tau)}{\eta(\tau)^3},$$

$\mathcal{Z}_2^d$ being the localised part of the elliptic genus (D.1). By virtue of a relation similar to (3.32), the holomorphic piece in the elliptic genus of the $(\mathrm{SL}(2, \mathbb{R})_k/\mathrm{U}(1))/\mathbb{Z}_{2\ell}$ theory can thus be rewritten, for $k = 2\ell^2$, in terms of a $\mathbb{Z}_\ell$ orbifold of the $(\mathrm{SL}(2, \mathbb{R})_2/\mathrm{U}(1))/\mathbb{Z}_2$ theory, and eventually of the Appell sum A_1 .

We can in particular extract the contribution of non-localised bulk states from expression (4.15):

$$\begin{aligned}
 \mathcal{R}^c[\mathcal{Q}_5] &= -\frac{1}{12} \sum_{J=1}^{k/2} \left(\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J} \right) \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} q^{-\frac{1}{k} (J - \frac{\ell+1}{2})^2} R\left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} | \tau \right) \times \\
 &\quad \times \frac{\widehat{E}_2 E_8 - E_{10}}{\eta^{21}} \\
 &= \frac{1}{16} \sum_{J=1}^{k/2} \left(\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J} \right) \sum_{n \in \mathbb{Z}} (-)^n \operatorname{sgn}\left(n + \frac{1}{2}\right) \operatorname{erfc} \left(|(n+1)k - 2J + 1| \sqrt{\frac{\pi \tau_2}{k}} \right) \times \\
 &\quad \times q^{-\frac{1}{4k} ((n+1)k - 2J + 1)^2} \frac{D_8 E_8}{\eta^{21}} \\
 &= \frac{1}{8} \sum_{J=1}^{k/2} \left(\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J} \right) \sum_{n=0}^{\infty} (-)^n \operatorname{erfc} \left(|nk + 2J - 1| \sqrt{\frac{\pi \tau_2}{k}} \right) q^{-\frac{1}{4k} (nk + 2J - 1)^2} \frac{D_8 E_8}{\eta^{21}}.
 \end{aligned} \tag{4.16}$$

The intermediate steps that bring us from the first to the second line are explicitly given in appendix E.1. To go from the second to the third line, we have exploited the $\mathbb{Z}_2$ symmetry of the $\mathrm{SU}(2)_{k-2}$ factor.

As anticipated in expression (3.22), the bulk state contribution (4.16) is (up to a factor k_a) universal, i.e. gauge group independent for both $\mathrm{SO}(28)$ and $\mathrm{U}(1)$ factors. This will become clearer in section 4.4.

Polar structure. It is worth spending some time discussing the polar behaviour of the modified elliptic genus appearing in the gauge threshold (4.15), thereby clarifying its physical signification. Firstly, the polar structure of the contribution of localised states $\hat{\mathcal{A}}_{\mathrm{SO}(28)}^d$ has already been addressed in section 4.1. It has been shown that it reproduces the inverse cusp form η^{-24} in the denominator of $\hat{\mathcal{A}}_{\mathrm{SO}(28)}^d$. Further analysing the Fourier expansion of the localised part of expression (4.5), we can show that this expression has no more than a dressed single pole $(\tau_2 q)^{-1}$, also characteristic of heterotic $K3$ compactifications.

Turning to the contribution from bulk states to the threshold corrections, given by eq. (4.16), we observe, following [47] that, for $\tau_2 > 0$, $n \geq 0$ and $J \in [1, \dots, \frac{k}{2}]$:

$$\begin{aligned}
 \left| (-)^n \operatorname{erfc} \left(\frac{1}{2\ell} (nk + 2J - 1) \sqrt{2\pi \tau_2} \right) q^{-\frac{1}{4k} (nk + 2J - 1)^2} \right| &\leq e^{-\frac{\pi (nk + 2J - 1)^2}{k} \tau_2} \left| q^{-\frac{1}{4k} (nk + 2J - 1)^2} \right| \\
 &= e^{-\frac{\pi (nk + 2J - 1)^2}{2k} \tau_2}.
 \end{aligned} \tag{4.17}$$

Hence all terms in the sum over n in (4.16) are exponentially suppressed as $\tau_2 \rightarrow \infty$. Thus, including the η^{-3} factor coming from $\chi_{k-2}^J(\tau)$, the only pole at $\tau_2 \rightarrow \infty$ in expression (4.16) comes from:

$$\frac{D_8 E_8}{\eta^{24}} = \frac{4}{\pi q \tau_2} - 960 + \frac{2016}{\pi \tau_2} + \mathcal{O}(q). \tag{4.18}$$

Taking into account the Fourier expansion of the characters $\chi_{k-2}^J(\tau)$, cf. eq. (A.5), we see that $\mathcal{R}^c$ only has a q^{-1} cusp at $\tau_2 \rightarrow \infty$, whose real part is bounded by:

$$\frac{1}{\tau_2} \sum_{n_1 \in [1, \dots, 2\sqrt{2}\ell[} \sum_{|n_2| \in [1, \dots, 2\sqrt{k-2}[} c_{n_1 n_2} e^{-\frac{\pi}{2} \left(\frac{n_1^2}{k} + \frac{n_2^2}{k-2} \right) \tau_2} \frac{1}{|q|}, \quad (4.19)$$

with n_1 and n_2 following some progression in $\mathbb{Z}$. More specifically we have $n_1 = nk + 2J - 1 \geq 0$ and $n_2 = 2(k-2)m + 2J - 1$ for $m \in \mathbb{Z}$. The contribution from bulk states thus has a similar polar behaviour as the localised part $\mathcal{A}_{\text{SO}(28)}^d$, with a simple pole 'dressed' by a regulator; the difference being that the regulator is now exponentially suppressed for $\tau_2 \rightarrow \infty$, which we interpret as the signature of an unphysical tachyon appearing in the spectrum of non-localised states.

Thus we conclude that by considering a regime where $T_2 > 1$ we can compute the integral (4.15) by unfolding the fundamental domain $\mathcal{F}$ against the lattice sum $\Gamma_{2,2}(T, U)$, similarly in every respect to calculations of heterotic gauge thresholds for toroidal orbifold compactifications (3.16).

4.3 Threshold corrections for $\mathcal{Q}_5 = 1$ and $\mathcal{N} = 4$ characters

After having discussed the SO(28) threshold for a generic value of the fivebrane charge, we would like to discuss here in detail the particular case $\mathcal{Q}_5 = 1$, which is somehow degenerate, but displays interesting features. In this case, the worldsheet supersymmetry of the (6,6) CFT is further enhanced to $\mathcal{N}_{\text{ws}} = (4,4)$, so that the result can be nicely repackaged, as we shall see, into $\mathcal{N}_{\text{ws}} = 4$ superconformal characters at level $\kappa = 1$. This will help making contact with the known threshold corrections for $T^2 \times K3$.

For $k = 2$, the contributions to (4.5) from discrete representations greatly simplifies. In particular, as the $\text{SL}(2, \mathbb{R})_k / \text{U}(1)$ spin can now only take the value $J = 1$, the $\mathbb{Z}_2$ orbifold which selects integer spins in (4.5) becomes trivial. In addition, the $\text{SU}(2)_{k-2}$ theory reduces now to the identity. Then:

$$\Lambda_{\text{SO}(28)}[1] = \frac{1}{96} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \left[\sum_{(u,v) \neq (1,1)} \sum_{n \in \mathbb{Z}_2} e^{-i\pi v(n + \frac{u}{2})} \text{Ch}_d(1, n-1) \begin{bmatrix} u \\ v \end{bmatrix} \times \right. \\ \left. \times \frac{\left(\widehat{E}_2 + (-1)^v \vartheta \left[\begin{smallmatrix} u+v+1 \\ u \end{smallmatrix} \right]^4 - (-1)^u \vartheta \left[\begin{smallmatrix} v \\ u+v+1 \end{smallmatrix} \right]^4 \right) \vartheta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]^{14}}{\eta^{18}} + 12 \mathcal{R}^c[1] \right]. \quad (4.20)$$

We will discuss now how to rephrase this result in terms of $\mathcal{N} = 4$ characters. We refer the reader to the appendix B, in particular to subsection B.2, for details on the subject.

Representations of the $\mathcal{N}_{\text{ws}} = 4$ superconformal algebra at level κ are distinguished by two quantum numbers (h, I) , namely their conformal weight and their spin. Unitary representations are:

- BPS representations labeled by discrete quantum numbers (h, I) and with massless ground states, which are obtained by saturating the unitary bounds, i.e. by setting:

$$h = I \text{ in the NS sector,} \quad h = \frac{\kappa}{4} \text{ in the R sector,} \quad \text{for spin range } 0 \leq I \leq \frac{\kappa}{2}$$

$$\begin{array}{ccc}
 \text{ch}_{1, \frac{1}{4}, 0}^{\widetilde{\text{R}}} & \xrightarrow{\nu \mapsto \nu - \frac{\tau}{2}} & \text{ch}_{1, \frac{1}{2}, \frac{1}{2}}^{\widetilde{\text{NS}}} \\
 \downarrow \scriptstyle \begin{smallmatrix} \sim \\ \sim \\ \sim \\ \sim \end{smallmatrix} & & \downarrow \\
 \text{ch}_{1, \frac{1}{4}, 0}^{\text{R}} & \longrightarrow & \text{ch}_{1, \frac{1}{2}, \frac{1}{2}}^{\text{NS}}
 \end{array}$$

Figure 1. Spectral flow of $\mathcal{N}_{\text{ws}} = 4$ characters at level $\kappa = 1$: $(h, I) = (1/2, 1/2)$ orbit

- non-BPS massive representations with discrete spin values I but continuous conformal weight h bounded from below:

$$\begin{aligned}
 h > I \quad \text{with} \quad 0 \leq I \leq \frac{\kappa - 1}{2} \quad \text{in the NS sector,} \\
 h > \frac{\kappa}{4} \quad \text{with} \quad \frac{1}{2} \leq I \leq \frac{\kappa}{2} \quad \text{in the R sector}
 \end{aligned}$$

Focusing on discrete representations, we can exploit the branching relations of the $\mathcal{N}_{\text{ws}} = 4$ characters at level $\kappa = 1$ into $\mathcal{N}_{\text{ws}} = 2$ representations with $c = 6$ in order to rewrite the localised elliptic genus Φ_2 in terms of the $\mathcal{N}_{\text{ws}} = 4$ character for the only normalisable BPS (discrete) representation in the twisted Ramond sector, defined in eq. (B.13):

$$\Phi_2(\nu|\tau) = \sum_{n \in \mathbb{Z}_2} e^{-i\pi(n + \frac{1}{2})} \text{Ch}_d(1, n - 1; \nu|\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{ch}_{1, \frac{1}{4}, 0}^{\widetilde{\text{R}}}(\nu|\tau) \quad (4.21)$$

The other elliptic indices with mixed boundary conditions (4.6) can then be obtained by spectral flow, as previously explained, namely

$$\Phi_2 \begin{bmatrix} 1 - a \\ 1 - b \end{bmatrix}(\nu|\tau) = (-)^{a(1+b)} q^{\frac{a^2}{4}} z^{-a} \text{ch}_{1, \frac{1}{4}, 0}^{\widetilde{\text{R}}} \left(\nu - \frac{a\tau + b}{2} | \tau \right), \quad a, b \in [0, 1], \quad (4.22)$$

with different values of the ‘spin structure’ reproducing all the characters for normalisable BPS representations listed in (B.13), for instance $\Phi_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\nu|\tau) = \text{ch}_{1, \frac{1}{2}, \frac{1}{2}}^{\text{NS}}(\nu|\tau)$. This corresponds pictorially to circumnavigating the orbit under spectral flow of the $(h, I) = (1/2, 1/2)$ representation in the NS sector, as illustrated by the diagram in figure 1.¹⁶ Identities (4.22) belong to the more general case of branching relations of $\mathcal{N}_{\text{ws}} = 4$ super-conformal representations into $\mathcal{N}_{\text{ws}} = 2$ ones [89]. We shall see shortly how these relations can be exploited to rephrase the gauge threshold corrections for $\mathcal{Q}_5 = 1$ in a more suggestive way.

Since we are dealing with a degenerate case where the $\text{SU}(2)_{k-2}$ factor in the $\widetilde{\text{EH}}$ CFT is the identity, the localised part of the elliptic genus is directly given by expression (4.10) for $k = 2$ and takes the simple form:

$$\Phi_2(\nu|\tau) = -A_1(\nu|\tau) \frac{i\vartheta_1(\nu|\tau)}{\eta(\tau)^3} \equiv \mu(\nu|\tau) \frac{\vartheta_1(\nu|\tau)^2}{\eta(\tau)^3}, \quad (4.23)$$

¹⁶We should emphasise that the orbit for the other existing discrete representation with $(h, I) = (0, 0)$ in the NS sector does not contribute to the localised part of the threshold correction, since these representations are non-normalisable, as is the corresponding identity representation of $\text{SL}(2, \mathbb{R})_2/\text{U}(1)$.

$\mathcal{N} = 4$ characters and $\mathcal{N} = 2$ Liouville theory. We have seen above that the genera $\Phi_2 \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]$ organise into an orbit under spectral flow, comprising the $\mathcal{N}_{\text{ws}} = 4$ character $\text{ch}_{1, \frac{1}{4}, 0}^{\tilde{\text{R}}}$. Now, from (4.21), we observe that this character is precisely the holomorphic part of the elliptic genus of a $\mathbb{Z}_2$ orbifold of the $\mathcal{N} = 2$ Liouville theory at level $k = 2$ (see appendix D), as has already been pointed out in [90]. From eq. (D.3) one finds indeed:

$$\text{ch}_{1, \frac{1}{4}, 0}^{\tilde{\text{R}}}(\nu|\tau) = \frac{1}{2} \sum_{\gamma, \delta \in \mathbb{Z}_2} z^{2\gamma} q^{\frac{\gamma^2}{2}} \mathcal{Z}_2^{\text{d}}(\nu + \gamma\tau + \delta|\tau) \quad (4.24)$$

As already explained, by using the regularisation scheme (4.9) we can compute the non-holomorphic completion of the localised part of the elliptic genus, by which we determine the complete elliptic genus of the orbifolded $\mathcal{N} = 2$ Liouville theory at level $k = 2$:

$$\widehat{\Phi}_2(\nu|\tau) = \frac{1}{2} \sum_{\gamma, \delta \in \mathbb{Z}_2} z^{2\gamma} q^{\frac{\gamma^2}{2}} \widehat{\mathcal{Z}}_2(\nu + \gamma\tau + \delta|\tau) = \widehat{\mu}(\nu|\tau) \frac{\vartheta_1(\nu|\tau)^2}{\eta(\tau)^3}. \quad (4.25)$$

in keep with the general formula (4.10). By making use of the modular and elliptic properties of the Appell function (3.43), we find that $\widehat{\Phi}_2$ transforms as Jacobi form weight 0 and index 1, in accordance with the corresponding transformations of the $(\text{SL}(2, \mathbb{R})_k / \text{U}(1)) / \mathbb{Z}_2$ theory, see (D.12) and (D.13).¹⁷

By spectral flow one obtains the remaining regularised genera:

$$\widehat{\Phi}_2 \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] (\nu|\tau) = \widehat{\mu} \left(\nu + \frac{(u-1)\tau + (v-1)}{2} \middle| \tau \right) \frac{\vartheta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]^2 (\nu|\tau)}{\eta(\tau)^3}, \quad (4.26)$$

leading to the $\text{SO}(28)$ threshold corrections for five-brane charge $\mathcal{Q}_5 = 1$:

$$\begin{aligned} \Lambda_{\text{SO}(28)}[1] &= \frac{1}{96} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \times \\ &\times \sum_{(u,v) \neq (1,1)} \frac{\widehat{\mu} \left(\frac{(u-1)\tau + (v-1)}{2} \middle| \tau \right) \left(\widehat{E}_2 + (-1)^v \vartheta \left[\begin{smallmatrix} u+v+1 \\ u \end{smallmatrix} \right]^4 - (-1)^u \vartheta \left[\begin{smallmatrix} v \\ u+v+1 \end{smallmatrix} \right]^4 \right) \vartheta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]^{16}}{\eta^{21}}. \end{aligned} \quad (4.27)$$

As for the general $\mathcal{Q}_5 > 1$ case (4.16), from the decomposition of the elliptic genus (4.25) into discrete and continuous $\text{SL}(2, \mathbb{R})_2 / \text{U}(1)$ representations, we may single out the contribution of non-localised bulk modes:

$$\mathcal{R}^{\text{c}}[1] = - \frac{R(\tau) (\widehat{E}_2 E_8 - E_{10})}{12 \eta^{21}} = \frac{R(\tau) D_8 E_8}{16 \eta^{21}} \quad (4.28)$$

which factorises in terms of the (transform) shadow function $R(\tau) \equiv R(0|\tau)$ and the covariant derivative $D_8 E_8$ (C.18). This contribution is universal (up to a k_a factor) for both $\text{SO}(28)$ and $\text{SU}(2)$ thresholds.

¹⁷The only trace of the $\mathbb{Z}_2$ orbifold is found in the extension of the allowed range for the shift parameters in the elliptic transformations (D.13), which is now $\mu, \lambda \in \mathbb{Z}$ instead of $2\mathbb{Z}$.

Local thresholds vs. $K3$ thresholds. We shall now exploit the $\mathcal{N}_{\text{ws}} = 4$ superconformal algebra at level $\kappa = 1$ that appears for $\mathcal{Q}_5 = 1$ in order to make contact with the well-known threshold corrections for $T^2 \times K3$ compactifications. In the same vein, we will show in the next section that the $SU(2)$ threshold for $\mathcal{Q}_5 = 1$ can be cast in the same universal form.

The relation between $\mathcal{N}_{\text{ws}} = 4$ characters at level $\kappa = 1$ and $K3$ characters can be illustrated by considering the S-transformation of say the twisted Ramond character for (normalisable) discrete representations (B.13):

$$\begin{aligned} \text{ch}_{1, \frac{1}{4}, 0}^{\tilde{R}} \left(\frac{\nu}{\tau} \middle| - \frac{1}{\tau} \right) &= e^{2\pi i \frac{\nu^2}{\tau}} \left[\text{ch}_{1, \frac{1}{4}, 0}^{\tilde{R}}(\nu|\tau) - \frac{1}{2} \int_{\mathbb{R}} \frac{dx}{\cosh \pi x} \text{ch}_{1, \frac{x^2}{2} + \frac{3}{8}, \frac{1}{2}}^{\tilde{R}}(\nu|\tau) \right] \\ &= e^{2\pi i \frac{\nu^2}{\tau}} \left[\text{ch}_{1, \frac{1}{4}, 0}^{\tilde{R}}(\nu|\tau) - \frac{1}{2} M(0|\tau) \frac{\vartheta_1(\nu|\tau)^2}{\eta(\tau)^3} \right]. \end{aligned} \quad (4.29)$$

As can alternatively be inferred from combining identities (4.21) and (4.23), the transformation law (4.29) indicates that $\text{ch}_{1, \frac{1}{4}, 0}^{\tilde{R}}(\nu|\tau)$ is a Mock Jacobi form. In particular, the extra piece appearing in the r.h.s. of (4.29) and breaking modular covariance can be reexpressed in terms of continuous twisted Ramond $\mathcal{N}_{\text{ws}} = 4$ characters (B.12) or alternatively repackaged into the Mordell integral (3.34).

From eq. (4.29) we note in particular that the continuous $\mathcal{N}_{\text{ws}} = 4$ representations which contribute to the Mordell integral have conformal weight in the range $h = \frac{x^2}{2} + \frac{3}{8} > \frac{1}{4}$, which falls within the $h > \frac{\kappa}{4}$ bound imposed by unitarity on non-BPS representations (see appendix B.1). Now, the integral $M(0|\tau)$ has been shown by Mordell to be S-invariant, which follows from rewriting it as [91]:

$$\frac{1}{2} M(\tau) = h_3(\tau) + h_3(-1/\tau), \quad (4.30)$$

with the function h_3 given by:

$$h_3(\tau) = \frac{1}{\vartheta_3(0|\tau)\eta(\tau)} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}(n-\frac{1}{2})(n+\frac{1}{2})}}{1 + q^{n-\frac{1}{2}}} = \frac{iq^{\frac{1}{8}} A_1(-\frac{\tau+1}{2}, -\frac{\tau+1}{2}|\tau)}{\vartheta_3(0|\tau)\eta(\tau)}. \quad (4.31)$$

By spectral flow of the Appell function, we may define two other such functions:

$$h_2(\tau) = \frac{iA_1(-\frac{1}{2}, -\frac{1}{2}|\tau)}{\vartheta_2(0|\tau)\eta(\tau)}, \quad h_4(\tau) = \frac{q^{\frac{1}{8}} A_1(-\frac{\tau}{2}, -\frac{\tau}{2}|\tau)}{\vartheta_4(0|\tau)\eta(\tau)}, \quad (4.32)$$

for which there is a relation to the Mordell integral analogous to (4.31):

$$\frac{1}{2} M(\tau) = h_4(\tau) + h_2(-1/\tau). \quad (4.33)$$

Using the functions h_i , the localised part of the elliptic genus (4.25) can in particular be rewritten in three different ways:

$$\Phi_2(\nu|\tau) = \left(\frac{\vartheta_i(\nu|\tau)}{\vartheta_i(\tau)} \right)^2 + h_i(\tau) \left(\frac{\vartheta_1(\nu|\tau)}{\eta(\tau)} \right)^2, \quad i = 2, 3, 4 \quad (4.34)$$

Then, combining the three above expressions, we can reformulate the regularized elliptic genus $\widehat{\Phi}_2$ as follows:

$$\widehat{\Phi}_2(\nu|\tau) = \frac{1}{24} \text{ch}_{K3}(\nu|\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{12} \widehat{F}(\tau) \frac{\vartheta_1(\nu|\tau)^2}{\eta(\tau)^3} \quad (4.35)$$

in terms of the twisted Ramond character corresponding to the elliptic genus of the $K3$ surface:

$$\text{ch}_{K3}(\nu|\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8 \left[\left(\frac{\vartheta_3(\nu|\tau)}{\vartheta_3(\tau)} \right)^2 + \left(\frac{\vartheta_4(\nu|\tau)}{\vartheta_4(\tau)} \right)^2 + \left(\frac{\vartheta_2(\nu|\tau)}{\vartheta_2(\tau)} \right)^2 \right] \quad (4.36)$$

This character can for example be determined by CFT methods from its $T^4/\mathbb{Z}_2$ orbifold limit [92].¹⁸

The function $\widehat{F}$ that appears in eq. (4.35) is a weak Maaß form of weight $1/2$, which decomposes as follows:

$$\begin{aligned} \widehat{F}(\tau) &= F(\tau) - 6R(0|\tau) \\ &= 4\eta(\tau) \sum_{i=2,3,4} h_i(\tau) - 12 \sum_{n=0}^{\infty} (-)^n \text{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \end{aligned} \quad (4.37)$$

where $\text{erfc}(x)$ is the complementary error function (3.39). Its holomorphic part has the following Fourier expansion:

$$q^{1/8} F(\tau) = 1 - 45q - 231q^2 - 770q^3 - 2277q^4 + \mathcal{O}(q^5). \quad (4.38)$$

This Mock modular form F clearly has the same shadow as $12\widehat{\mu}(\nu|\tau)$, since its completion $\widehat{F}$ satisfies the holomorphic anomaly differential equation:

$$\frac{\partial \widehat{F}(\tau)}{\partial \bar{\tau}} = \frac{3\sqrt{2}i}{\sqrt{\tau_2}} \frac{1}{\eta(\tau)^3} \quad (4.39)$$

The Fourier coefficients (4.37) actually appear in the Rademacher expansion of the elliptic genus of (non-) compact $K3$ surfaces [93], and were in particular shown to be relevant to the counting of half BPS states for string theory compactified on such surfaces and hence to a microscopic determination of the black hole entropy for these configurations [94]. They also found a more recently application in the derivation of BPS saturated one-loop amplitudes with external legs stemming from half BPS short multiplets for type II string theory compactified on $T^2 \times K3$ [95]. Here we see a novel occurrence of the Rademacher expansion of $\mathcal{N}_{\text{ws}} = 4$ characters, where the Fourier coefficients of F now encode the contribution of three-form flux to gauge threshold correction (4.27).

Then, by spectral flowing expression (4.35), one obtains for the sectors with even spin-structure:

$$\widehat{\Phi}_2 \begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau) = \frac{1}{24} \text{ch}_{K3}(\nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{12} \widehat{F}(\tau) \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau)^2}{\eta(\tau)^3}, \quad (4.40)$$

¹⁸It reproduce in particular $\chi(K3) = \text{ch}_{K3}^{\text{R}}(0|\tau) = 24$, as expected.

with the $K3$ characters given by expressions:

$$\text{ch}_{K3}(\nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} = 8 \left[(-1)^{a+1} \left(\frac{\vartheta \left[\begin{smallmatrix} 1-a \\ 1-b \end{smallmatrix} \right] (\nu|\tau)}{\vartheta_3(\tau)} \right)^2 + \left(\frac{\vartheta \left[\begin{smallmatrix} 1-a \\ b \end{smallmatrix} \right] (\nu|\tau)}{\vartheta_4(\tau)} \right)^2 + (-1)^{a+1} \left(\frac{\vartheta \left[\begin{smallmatrix} a \\ 1-b \end{smallmatrix} \right] (\nu|\tau)}{\vartheta_2(\tau)} \right)^2 \right] \quad (4.41)$$

Using (4.40) the $\text{SO}(28)$ threshold correction for $\mathcal{Q}_5 = 1$ (4.27) can be recast as follows:

$$\Lambda_{\text{SO}(28)}[1] = \frac{1}{48} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \frac{1}{12\eta^{24}} \left(- \left[\widehat{E}_2 E_{10} - \frac{2}{3} E_6^2 - \frac{1}{3} E_4^3 \right] + \eta^3 \widehat{F} (\widehat{E}_2 E_8 - E_{10}) \right) \quad (4.42)$$

We would like to make the following comments on the structure of the threshold correction (4.42) and its relationship with $K3$ characters.

- i)* The first contribution on the r.h.s. of (4.42), stemming from localised states, reproduces the gauge threshold corrections (3.16) for an $\text{SO}(32)$ heterotic compactification on $T^2 \times (T^4/\mathbb{Z}_2)$, with an orbifold action determined by the shift vector $\vec{v} = (1, 1, 0^{14})$. Upon blowing up the singularity, one obtains a $T^2 \times K3$ compactification with $\text{SU}(2)$ instanton number $t = 4$ (see eq. (3.19) and above), which we can explicitly read off the first term in (4.42). The $\text{SU}(2)$ background breaks the gauge group symmetry to $\text{SO}(28) \times \text{SU}(2)$,¹⁹ as is also the case for the non-compact model considered here, where however the breaking is due to the presence of $\text{U}(1)$ instantons in the background. The hypermultiplet spectrum for this $T^2 \times K3$ compactification is given for instance in [96] and reads $10(\mathbf{28}, \mathbf{2}) + 65(\mathbf{1}, \mathbf{1})$. For the $T^2 \times \widetilde{\text{EH}}$ background under scrutiny, the hypermultiplet multiplicities are instead $(\mathbf{28}, \mathbf{2}) + 2(\mathbf{1}, \mathbf{1})$, as given by table 1. This reduction results, on the one hand, from considering a *single* resolved A_1 singularity and, on the other hand, from having five-brane flux supported by $\text{U}(1)$ gauge instantons threading the geometry, which is featured in the second term in expression (4.42).
- ii)* The second contribution on the r.h.s. of (4.42), originating from both localised and bulk states, is both the sign that we are dealing with a non-compact space, and that we are considering a non-Kähler geometry with three-form flux, characterised by non-zero fivebrane charge $\mathcal{Q}_5$ at infinity; these two aspects are tied together, since the net fivebrane charge on a compact manifold has to vanish. The appearance of the Maaß form $\widehat{F}$, and in particular its decomposition (4.37) into a Mock modular form and its shadow function, can be understood as follows. The elliptic genus that we computed for the $T^2 \times \widetilde{\text{EH}}$ contains a contribution localised on the blown-up $\mathbb{P}^1$, due to the presence of flux; it is precisely encoded in the Mock modular form F (4.38). This expression alone would be anomalous under modular transformations. However the non-compact CFT at hand displays, alongside localised states, a continuous spectrum of bulk modes which cancel this holomorphic anomaly, at the price of introducing an extra non-holomorphic contribution in the elliptic genus (4.35). This feature, peculiar

¹⁹In the $T^2 \times K3$ model, the gauge group may be further Higgsed and broken down to the terminal group $\text{SO}(8)$.

to non-compact models, explains the appearance of the Maaß form $\widehat{F} = F - 6R$ in the threshold (4.42) with a contribution of the (transform) shadow function R corresponding to an infinite tower of non-localised massive non-BPS states (D.9). In a compact $T^2 \times \widetilde{K3}$ model, we instead expect the extra contribution of localised states due to the flux to be cancelled by a contribution from the bulk of the globally tadpole-free compactification, without spoiling the holomorphicity of the genus.

iii) It is also worth rediscussing the polar structure of the $\widetilde{\text{EH}}$ modified elliptic genus (4.42), since it exhibits some differences with respect to the bulk contribution, compared to the $k > 2$ case discussed previously. But first, we note that in the localised part of the modified elliptic genus the q^{-1} pole coming from the $K3$ and the localised flux contributions exactly compensate, as can be shown from the following Fourier expansion:

$$\begin{aligned}\hat{A}_{\text{SO}(28)}^{\text{d}}[1] &= -\frac{1}{72\eta^{24}} \left[\widehat{E}_2 E_{10} - \frac{2}{3} E_6^2 - \frac{1}{3} E_4^3 - \eta^3 F(\widehat{E}_2 E_8 - E_{10}) \right], \\ &= 8 - \frac{29}{\pi\tau_2} + \left(6960 - \frac{7955}{\pi\tau_2} \right) q + \mathcal{O}(q).\end{aligned}\tag{4.43}$$

Analysing the contribution from bulk states, encoded in the shadow function $R(\tau)D_8E_8/\eta^{21}$, is even simpler as for the $k > 2$ cases discussed in (4.17). We first consider the sum $q^{1/8}R(\tau)$. The terms of this sum (3.39) are bounded, for any $n \in \mathbb{N}$ and for $\tau_2 > 0$, by:

$$\left| (-)^n \text{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{2}n(n+1)} \right| \leq e^{-2\pi(n+\frac{1}{2})^2\tau_2} \left| e^{-i\pi n(n+1)\tau} \right| = e^{-\pi(n(n+1)+\frac{1}{2})\tau_2}.\tag{4.44}$$

All these terms are exponentially suppressed for $\tau_2 \rightarrow \infty$. Since $\frac{D_8E_8}{q^{1/8}\eta^{21}} = \frac{4}{\pi q\tau_2} - 960 + \frac{2004}{\pi\tau_2} + \dots$, the contribution $R(\tau)D_8E_8/\eta^{21}$ also has a 'dressed' pole of order one, with an exponentially decaying regulator characteristic of bulk states, as we already emphasised for the $k > 2$ cases. In particular, for $n = 0$, the real part of this pole diverges as $\tau_2^{-1}e^{\frac{3}{2}\pi\tau_2}$ for $\tau_2 \rightarrow \infty$, while it is completely suppressed for all terms with $n > 0$. Since $\hat{A}_{\text{SO}(28)}^{\text{d}}[1]$ is regular at $\tau_2 \rightarrow \infty$, we observe that $\mathcal{R}^c[1]$ contains the only 'dressed' pole related to an unphysical tachyon, with a 'dressing' acting as regulator for both the IR divergence stemming from the Casimir $\text{Tr } Q_{\text{SO}(28)}^2$ and the infinite volume divergence.

4.4 The $U(1)_R$ and $SU(2)$ gauge threshold corrections

Having determined the regularised elliptic genera (4.14) for warped Eguchi-Hanson CFT, we can compute the threshold corrections corresponding to the $U(1)$ or $SU(2)$ gauge coupling depending on whether we consider an arbitrary value $\ell \in 2\mathbb{N}^* + 1$ of the Abelian magnetic charge or the particular value $\ell = 1$. This is made easy by the fact that this gauge symmetry corresponds to the left $U(1)$ R-symmetry of the $SL(2, \mathbb{R})/U(1)$ coset, or has its Cartan generator determined by it in the $\ell = 1$ case. Hence the elliptic variable ν in the genus $\widehat{\Phi}_k \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\nu)$ keeps precisely track of its charges; the corresponding (regularised) Casimir

operator then acts as a derivative with respect to this variable. The Kac-Moody levels, that enter into the regularisation of the Casimir in (3.8), are in this case:

$$k_{\text{U}(1)} = 1 + \frac{2}{k} = 1 + \frac{1}{\ell^2}, \quad k_{\text{SU}(2)} = 2. \quad (4.45)$$

The threshold corrections to the U(1) and SU(2) gauge couplings are then given by descendants of the genera (4.14) as:

$$\Lambda_A[\mathcal{Q}_5] = -\frac{1}{32\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \sum_{(a,b) \neq (1,1)} \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (0|\tau)^{14}}{\eta(\tau)} \frac{1}{\eta^4} \left[\partial_\nu^2 + \frac{\pi(\ell^2 + 1)}{\ell^2 \tau_2} \right] \widehat{\Phi}_k \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\nu|\tau) \Big|_{\nu=0}, \quad (4.46)$$

with $A = \text{U}(1)$ for $\ell \in 2\mathbb{N}^* + 1$ and $A = \text{SU}(2)$ for $\ell = 1$.

Working out expression (4.46) explicitly, one obtains for $\ell \in 2\mathbb{N}^* + 1$:

$$\begin{aligned} \Lambda_{\text{U}(1)}[\mathcal{Q}_5] &= \frac{1}{4k} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) q^{-\frac{1}{k}(J - \frac{\ell+1}{2})^2} \times \\ &\times \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{i\pi \frac{m}{\ell}} \sum_{(a,b) \neq (1,1)} (-)^{a+b} \left[\frac{\eta \vartheta \left[\begin{smallmatrix} a+1 \\ 0 \end{smallmatrix} \right] \left((\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right) \vartheta \left[\begin{smallmatrix} 0 \\ b+1 \end{smallmatrix} \right] \left((\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right)}{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left((\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right)^3} + \right. \\ &\quad \left. + \left(\frac{k+2}{24} \right) \frac{\widehat{\mu} \left(\left(\frac{\ell a+1}{2} - J \right) \frac{\tau}{\ell} + \frac{b-1}{2} + \frac{m}{\ell}, \frac{(a-1)\tau + (b-1)}{2} \middle| \tau \right)}{\bar{\eta}} \times \right. \\ &\quad \left. \times \frac{(\widehat{E}_2 + (-)^b \vartheta \left[\begin{smallmatrix} a+b+1 \\ a \end{smallmatrix} \right]^4 - (-)^a \vartheta \left[\begin{smallmatrix} b \\ a+b+1 \end{smallmatrix} \right]^4)}{\eta^4} \right] \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^{16}}{\eta^{16}}. \quad (4.47) \end{aligned}$$

In particular, the second line of the above expression comes from the second derivative²⁰

$$\partial_\nu^2 \mu \left(\frac{\nu}{\ell} + \left(\frac{\ell a+1}{2} - J \right) \frac{\tau}{\ell} + \frac{b-1}{2} + \frac{m}{\ell}, \frac{\nu}{\ell} + \frac{(a-1)\tau + (b-1)}{2} \middle| \tau \right) \Big|_{\nu=0},$$

which is computed in appendix E.2. Moreover, the contribution from bulk states is entirely captured in the last two lines of (4.47) and is given by:

$$\begin{aligned} k_{\text{U}(1)} \mathcal{R}^c[\mathcal{Q}_5] &= \left(\frac{k+2}{8k} \right) \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) \sum_{n=0}^{\infty} (-)^n \text{erfc} \left(\frac{1}{2\ell} (nk + 2J - 1) \sqrt{2\pi\tau_2} \right) \times \\ &\times q^{-\frac{1}{4k}(nk+2J-1)^2} \frac{D_8 E_8}{\eta^{21}} \quad (4.48) \end{aligned}$$

where $\mathcal{R}_k^c$ is non-holomorphic completion (4.16) also appearing in the $\Lambda_{\text{SO}(28)}[\mathcal{Q}_5]$ threshold. This is in accordance with the general form taken by the modified elliptic genera that we outlined in (3.22). In particular, it shows that for this class of models the contribution to the gauge threshold corrections coming from bulk states is independent of the gauge group and only depends on the five-brane charge.

²⁰Note that in (4.46) all (mixed) terms containing simple derivatives of theta functions vanish since $\partial_\nu \vartheta_{2,3,4}(0|\tau) = 0$.

The $\mathcal{Q}_5 = 1$ case and $\mathcal{N} = 4$ symmetry. As previously, we consider in more detail the $\mathcal{Q}_4 = 1$ case, which has enhanced $\mathcal{N}_{\text{ws}} = 4$ left superconformal symmetry and second gauge factor $\text{SU}(2)$. By using expressions (4.40), equation (4.46) yields in this case

$$\Lambda_{\text{SU}(2)}[1] = \frac{1}{48} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \frac{1}{6\eta^{24}} \left(- \left[\widehat{E}_2 E_{10} + \frac{4}{3} E_6^2 - \frac{7}{3} E_4^3 \right] + \eta^3 \widehat{F} (\widehat{E}_2 E_8 - E_{10}) \right). \quad (4.49)$$

Again, the first term in (4.49) is the gauge threshold correction (3.16) for a $T^2 \times K3$ compactification, with this time $t = -44$.

Universality properties. By comparing $\Lambda_{\text{SO}(28)}[1]$ and $\Lambda_{\text{SU}(2)}[1]$, given in eqs. (4.42) and (4.49), we observed that these threshold corrections satisfy some universality properties when the underlying CFT exhibits enhanced $\mathcal{N}_{\text{ws}} = 4$ left super-conformal symmetry.

More generically, we would like to consider $\mathcal{N}_{\text{TS}} = 2$ six-dimensional local models with non-zero five-brane, based on T^2 times a smooth geometry corresponding to the warped resolution of a $\mathbb{C}^2/\mathbb{G}$ singularity, the action $\mathbb{G}$ leaving the gauge group $\prod_a \text{G}_a$ unbroken. If their CFT description displays $\mathcal{N}_{\text{ws}} = 4$ left super-conformal symmetry and allows for non-localised bulk states, we propose that such theories have threshold corrections Λ_a (3.11) to the couplings of the various gauge factors G_a determined by the four-dimension modified elliptic genus:

$$\hat{\mathcal{A}}_a = \frac{k_a}{6} \left(6(2 - t_a) + \frac{1}{20\eta^{24}} \left[D_{10} E_{10} - 528\eta^{24} + c_r g \widehat{F} D_8 E_8 \right] \right) \quad (4.50)$$

where $\widehat{F}$ is a Maaß form of weight r , g its shadow of weight $2 - r$, and c_r is a weight dependent constant. In the particular warped $\mathbb{C}^2/\mathbb{Z}_2$ resolution considered until now, $\widehat{F}$ is the weight $1/2$ Maaß form (4.37) with shadow function $g = -\frac{1}{2\sqrt{2}}\eta^3$, which is a weight $3/2$ holomorphic Jacobi form, and $c_{1/2} = 5\sqrt{2}$. In this particular case, expression (4.50) yields an alternative formulation of expressions (4.42) and (4.49). We thus observe that the three first terms in (4.50) reproduce $4/\chi(K3)$ of the $K3$ modified elliptic genus, see (3.13)–(3.14), with β -functions $b_a = 3k_a(2 - t_a)$. Mind that these are not the full β -functions for the torsional local models under consideration, which receive an additional contribution from the constant part of the flux induced term $\widehat{F} D_8 E_8 / \eta^{24}$. Also the normalisation factor $4/\chi(K3)$ comes from considering a *single* resolved A_1 singularity, instead of the global $K3$ geometry. It is in keep with hypermultiplet counting for a double-scaled geometry as currently investigated in [97]

For a general local model with five-brane charge which satisfies the above conditions, the four-dimension modified elliptic genus is then completely determined by a certain linear combination of three modular forms of weight 12: the quasi-holomorphic modular form $D_{10} E_{10}$, the cusp form η^{24} and the non-holomorphic modular form $g \widehat{F} D_8 E_8$, where $\widehat{F}$ is the weak Maaß form capturing the effects due to NSNS three-form flux, g is its shadow function and $D_8 E_8$ is a universal contribution. The coefficients of this linear combination are fixed by the absence of charged tachyons in the spectrum and the tadpole equation (2.12) with now non-vanishing charge $\mathcal{Q}_5$. Consequently, the difference of two such gauge thresholds

satisfies the relation

$$\frac{\Lambda_a[\mathcal{Q}_5]}{k_a} - \frac{\Lambda_b[\mathcal{Q}_5]}{k_b} = \frac{3(t_b - t_a)}{24} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U), \quad (4.51)$$

which, interestingly enough, is $4/\chi(K3)$ times what is expected for gauge threshold corrections for $T^2 \times K3$ models (3.13), which generically satisfy the relations (3.15). The fact that this universal feature of $\mathcal{N}_{\text{ST}} = 2$ heterotic compactifications carries over to the local non-compact models under consideration is clearly ascribable in this case to their displaying enhanced $\mathcal{N}_{\text{WS}} = 4$ left-moving superconformal symmetry, as for $T^2 \times K3$ compactifications with the standard embedding.

In particular, formula (4.51) holds for the $T^2 \times \widetilde{\text{EH}}$ background with $\mathcal{Q}_5 = 1$. In this case, the difference between the $\Lambda_{\text{SU}(2)}[1]$ and $\Lambda_{\text{SO}(28)}[1]$ thresholds yields a factor 6 multiplying the integral on the r.h.s. of eq. (4.51), see footnote below.

In the higher $\mathcal{Q}_5 > 1$ cases, the underlying CFT only has $\mathcal{N}_{\text{WS}} = 2$ left-moving superconformal symmetry, hence the difference between the U(1) and SO(28) threshold corrections is more complicated.²¹

$$\begin{aligned} \frac{\Lambda_{\text{U}(1)}[\mathcal{Q}_5]}{k_R} - \frac{\Lambda_{\text{SO}(28)}[\mathcal{Q}_5]}{k_{\text{SO}(28)}} &= \frac{1}{4(k+2)} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{k/2-J}) q^{-\frac{1}{k}(J-\frac{\ell+1}{2})^2} \times \\ &\times \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{i\pi \frac{m}{\ell}} \sum_{(a,b) \neq (1,1)} (-)^{a+b} \frac{\vartheta \left[\begin{smallmatrix} a+1 \\ 0 \end{smallmatrix} \right] \left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right) \vartheta \left[\begin{smallmatrix} 0 \\ b+1 \end{smallmatrix} \right] \left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right) \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^{16}}{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right)^3} \frac{1}{\eta^{15}}, \end{aligned} \quad (4.52)$$

and in particular does not abide by the rule (3.15) characterizing toroidal orbifold compactifications, as $\text{SU}(2)_{k-2}$ right-moving characters now intermingle with characters of a compact $\mathcal{N}_{\text{WS}} = 2$ CFT with $c = 1 + \frac{k}{2}$. Nonetheless, since the contribution of bulk states is, up to a multiplicative Kac-Moody level, gauge group independent, the difference of thresholds (4.52) shares the common feature with the $\mathcal{Q}_5 = 1$ case of being a purely localised effect, and could thus in principle be compared with corresponding expressions for $\mathcal{N}_{\text{ST}} = 2$ heterotic compactifications.

5 The moduli dependence

In order to determine the explicit dependence of the SO(28) and SU(2) or U(1) threshold corrections on the T^2 moduli, we have to carry out the integrals (4.15) and (4.47) or (4.49) over of the fundamental domain $\mathcal{F}$ of the modular group. Since both integrands $\tau_2 \Gamma_{2,2}(T, U) \hat{\mathcal{A}}_a$ are invariant under the full modular group Γ , we are entitled to compute these integrals by unfolding the T^2 lattice sum, a method pioneered by Dixon-Kaplunovsky-Louis (DKL) to evaluate threshold corrections for heterotic $\mathcal{N}_{\text{ST}} = 2$ compactifications [98].²²

²¹The $\mathcal{Q}_5 = 1$ case (4.51) with $\frac{3(t_{\text{SO}(28)} - t_{\text{SU}(2)})}{24} = 6$ can be recovered from expression (4.52) by setting $\ell = 1$, replacing the $\text{SU}(2)_{k-2}$ contributions by the identity and using the identity for ϑ -functions (C.5).

²²This method actually goes back to the study of string thermodynamics [99–101].

More recently, an alternative method [102, 103] has been developed to evaluate these integrals, which keeps manifest the T-duality invariance of the result under the $O(2, 2; \mathbb{Z})$ group of the Narain lattice. Generalising an idea developed in [104–106] which proposes to unfold the integral domain against the (modified) elliptic genus rather than against the torus lattice sum, these authors have shown how this procedure could be extended to any BPS-saturated amplitudes in string theory compactifications of the form $\int_{\mathcal{F}} \frac{d\tau^2}{\tau_2^2} \tau_2^{d/2} \Gamma_{d+k,d}(G, B, Y) \hat{\mathcal{A}}$, by rephrasing $\hat{\mathcal{A}}$ in terms of a certain class of non-analytic Poincaré-type series.

Given that $\hat{\mathcal{A}}$ generically includes non-holomorphic terms such as $(\hat{E}_2)^g \Phi_{12-g}$, with Φ_{12-g} a combination of products of holomorphic Eisenstein series of weight $12 - g$, and may exhibit poles in q related to unphysical tachyons, the authors of [103] have shown that all modified elliptic genera of interest can be appropriately rewritten as a linear combination of Niebur-Poincaré series [107, 108] $\mathcal{F}(s, \alpha, r)$, with $\text{Re}(s) > 1$ lying within the radius of absolute convergence of the series. Considering at first a genus $\mathcal{A}$ which can be any weakly holomorphic modular form, these authors have shown that by specialising to Niebur-Poincaré series with $s = 1 - \frac{r}{2}$ and $r < 0$ and taking a suitable linear combination of those series whose coefficients are determined by the principal part of $\mathcal{A}$, one can reproduce $\mathcal{A}$ exactly, even though $\mathcal{F}(1 - \frac{r}{2}, \alpha, r)$ taken individually are generically weak harmonic Maaß forms. This analysis extends to genera $\hat{\mathcal{A}}$ which are weak *almost* holomorphic modular form, by consider a combination of Niebur-Poincaré series with $s = 1 - \frac{r}{2} + n$ and $n \in \mathbb{N}$. This applies in particular to gauge threshold corrections for heterotic $\mathcal{N}_{\text{ST}} = 2$ compactifications (3.14) discussed previously. Then absolute convergence of the Niebur-Poincaré series for these specific values of its weight r allows to properly unfold them against integration domain $\mathcal{F}$ and compute the integral in a way that keeps manifest the $O(k + d, d; \mathbb{Z})$ invariance inherited from the Narain lattice.

Since for $s = 1 - \frac{r}{2}$ with $r < 0$ in particular the Niebur-Poincaré series belong to the space $\hat{\mathbb{M}}_r$ of weak harmonic Maaß forms (3.29), we can in principle consider rephrasing the four-dimensional genera for warped Eguchi-Hanson, eq. (4.15) and (4.47), by use of this method. However we will prefer evaluating these integrals by the traditional ‘orbit method’ of DKL, since the result is more directly interpretable, for large T_2 , in terms of perturbative and Euclidean brane instanton corrections in the type I S-dual theory. The procedure [102, 103] albeit yielding a compact and elegant result for string amplitudes, is less suited to study this corner of the moduli space.

Nevertheless, the non-compactness of the heterotic background we are considering, manifested in the contribution (4.16) of non-localised bulk states to the gauge threshold corrections, will entail novel results for these integrals for each class of orbits of the modular group. For the zero orbit piece, in particular we will even have to resort to results established in [103] by the procedure we elaborated on above, in order to exactly determine the flux contribution to the tree level correction to the heterotic gauge couplings.

In the following, we will restrict ourselves to working out explicitly the gauge threshold corrections (4.42) and (4.49) in the model with $\mathcal{Q}_5 = 1$ unit of flux. The dependence of the resulting threshold corrections on the (T, U) moduli of the two-torus will be qualitatively the same as in the general $\mathcal{Q}_5 = k/2 > 1$ case. At first sight a discrepancy might arise as

one considers the seemingly more involved structure of the bulk modes contribution in the general $\mathcal{Q}_5 > 1$ case. However, by comparing expressions (4.16) and (4.37), it appears that despite a mixing with $SU(2)_{k-2}$ characters the contribution of continuous $SL(2, \mathbb{R})_k/U(1)$ representations is very similar to the non-holomorphic completion (4.37) for the $\mathcal{Q}_5 = 1$ case, with a sum shifted by the $SU(2)_{k-2}$ spin.

5.1 The orbit method

The orbit method allows to compute integrals over the fundamental domain $\mathcal{F}$ by trading the sum over the winding modes in the T^2 partition function (2.22) for an unfolding of $\mathcal{F}$.

Following DKL [98], we decompose the set of matrices A in the T^2 lattice sum (2.22), encoding the maps from the worldsheet to the target space, into orbits of the modular group $\Gamma \cong PSL(2, \mathbb{Z})$, characterised as follows:

i) *Invariant* or zero orbit:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (5.1)$$

ii) *Degenerate orbits*: $\det A = 0$ and $A \neq 0$, parametrised by:

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix} \quad (5.2)$$

with $(j, p) \sim (-j, -p)$ and $AV = AV'$ iff $V = T^n V'$, for some $n \in \mathbb{N}$ and $V, V' \in \Gamma$.

iii) *Non-degenerate orbits*: $\det A \neq 0$:

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \quad (5.3)$$

with $d > j \geq 0$, $p \neq 0$ and $AV = AV'$, for $V, V' \in \Gamma$.

Since distinct elements of degenerate and non-degenerate orbits are in one-to-one correspondence with modular transformations mapping the $PSL(2, \mathbb{Z})$ fundamental domain $\mathcal{F}$ inside, respectively, the strip $\mathcal{S} = \{\tau \in \mathcal{H} \mid -\frac{1}{2} \leq \tau_1 < \frac{1}{2}, \tau_2 \geq 0\}$, and the double cover of the upper half-plane $\mathcal{H}$, the gauge threshold corrections (3.11) can be expressed as follows:

$$\begin{aligned} \Lambda_a &= \Lambda_a^0 + \Lambda_a^{\text{deg}} + \Lambda_a^{\text{non-deg}} \\ &= \frac{T_2}{8} \left[\int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \hat{\mathcal{A}}_a + \int_{\mathcal{S}} \frac{d^2 \tau}{\tau_2^2} \sum_{(j,p) \neq (0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2} |j+pU|^2} \hat{\mathcal{A}}_a \right. \\ &\quad \left. + 2 \int_{\mathcal{H}} \frac{d^2 \tau}{\tau_2^2} \sum_{k>j \geq 0} \sum_{p \neq 0} e^{2\pi k p i T - \frac{\pi T_2}{\tau_2 U_2} |k\tau - j - pU|^2} \hat{\mathcal{A}}_a \right]. \end{aligned} \quad (5.4)$$

If the modified elliptic genus $\hat{\mathcal{A}}_a$ exhibits a q^{-1} pole, which is typically the case for expressions (4.42), (4.49), (4.15) and (4.47) as was pointed out in (4.19) and (4.44), the unfolding

procedure (5.4) is subject to a caveat. When such a pole is present, convergence of the original threshold integral dictates a prescription for its evaluation, namely that we integrate first over τ_1 , discarding all Fourier modes of $\hat{\mathcal{A}}_a$ except the zero modes, and only then over τ_2 . In general, the modular transformations γ_i that bring the matrix A into the forms (5.2) and (5.3) characteristic of degenerate and non-degenerate orbits translate the latter into a highly a complicated, γ_i dependent prescription for the integration domains of the unfolded threshold integral, which usually invalidates the decomposition (5.4). Then, when unphysical tachyons are present in $\hat{\mathcal{A}}_a$ the identity (5.4) only holds when the integral over $\mathcal{F}$ on the l.h.s. is independent of the integration order, which is the case whenever the integration of the $(n_1, n_2) \neq (0, 0)$ terms in the Lagrangian lattice sum (2.22) is absolutely convergent. If $\hat{\mathcal{A}}_a$ contains a q^{-1} pole, this is the case when $T_2 > 1$, so that expression (5.4) is only valid in this regime.

5.2 Moduli dependence of the SO(28) threshold corrections

We give hereafter the threshold corrections to the SO(28) gauge coupling for the model (4.42) with $\mathcal{Q}_5 = 1$. The details of the evaluation of the integrals corresponding to the three classes of orbits of Γ are given in appendix F. We will nonetheless discuss later on some salient features of how the moduli dependence of the flux contributions can be established, as it is an interesting novel result. The following expression is valid in the region $T_2 > 1$ as discussed before:²³

$$\begin{aligned} \Lambda_{\text{SO}(28)}[1] = & \frac{29\pi}{144} T_2 - (\log |\eta(U)|^4 + \log(\mu^2 T_2 U_2) + \gamma) - \frac{29\pi}{360} \frac{E(U, 2)}{T_2} - \\ & - \frac{1}{8} \sum_{n=0}^{\infty} (-)^n \left[\frac{\pi}{18} d_1 \left(\frac{n(n+1)}{2} \right) \mathcal{E}_{(n+\frac{1}{2})\sqrt{2T_2}}(U, 1) - \right. \\ & - \frac{\sqrt{2}\zeta(3)}{\pi^2} \left(n + \frac{1}{2} \right) d_2 \left(\frac{n(n+1)}{2} \right) \frac{\mathcal{E}_{(n+\frac{1}{2})\sqrt{2T_2}}(U, 3/2)}{\sqrt{T_2}} \\ & \left. - \frac{\pi}{90} d_2 \left(\frac{n(n+1)}{2} \right) \frac{\mathcal{E}_{(n+\frac{1}{2})\sqrt{2T_2}}(U, 2)}{T_2} \right] \\ & + \frac{1}{4} \left(\sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i T} \left[\hat{\mathcal{A}}_{\text{SO}(28)}(\mathcal{U}) + \frac{\hat{\mathcal{A}}_K(\mathcal{U})}{T_2} \right. \right. \\ & \left. \left. + \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \frac{1}{(T_2)^{r+2}} (-iD)^{r+2} (u_2^2 \bar{\partial}_{\mathcal{U}})^r \hat{\mathcal{A}}_H(\mathcal{U}) \right] + \text{c.c.} \right). \end{aligned} \quad (5.5)$$

In the following, we will discuss the physical implication of the various terms appearing in the above result, after giving proper definitions of the expressions entering into it.

The zero orbit contribution. We first observe that in accordance with the double-scaling limit (2.15), the expression (5.5) does not depend on the blow-up modulus a , which

²³Note however that the continuous state contributions in the last two lines of (5.5) only take this form in the large volume limit of the T^2 , while for finite volume these expressions are more involved, as we will see later on.

can be rescaled away in the near-horizon geometry (2.16). We are however aware of the possibility for worldsheet instantons to wrap the blown-up $\mathbb{P}^1$ and to contribute accordingly to the threshold correction (5.5). As we will show, these terms can actually be found in the zero orbit contribution to $\Lambda_{\text{SO}(28)}[1]$, i.e. the first expression on the first line of (5.5) proportional to T_2 .

To identify these worldsheet instanton contributions, it is more handy to reason in terms of the type I S-dual theory on $T^2 \times \widetilde{\text{EH}}$, which has space-time filling D9 branes supporting the unbroken gauge group. When the singularities in the background geometry are resolved, the gauge kinetic functions of the gauge factors receive a tree-level (disk) contribution of the type [84]:

$$\sim \sum_{\text{two-cycles}} \sqrt{\det(P[g + \mathcal{F}])} T, \quad (5.6)$$

with $P[\dots]$ the pull-back to the blown-up two-cycles. For smooth $K3$ models, such contributions to the gauge kinetic functions typically arise from $\text{SU}(2)$ gauge instantons attached to the blown-up $\mathbb{P}^1$'s, see discussion following eq. (3.19). Here, in contrast, we have $\text{U}(1)$ gauge instantons (2.17b) instead of non-Abelian ones, living on the unique two-cycle of the warped Eguchi-Hanson space. In the blow-down limit, these typically give rise to small Abelian instantons sitting at the singularities, a phenomenon which also occurs at the orbifold fixed points of the singular limit of Bianchi-Sagnotti-Gimon-Polchinski models [109, 110]. This indicates that the corrections due to worldsheet instantons wrapping the blown-up $\mathbb{P}^1$ we are looking for are summed up in the $\frac{\pi}{64}$ coefficient of the zero orbit contribution in expression (5.5). If we were to determine this constant for the theory (4.15) at arbitrary five-brane charge $\mathcal{Q}_5$ we would see these instanton contribution appear explicitly as e^{-kn} corrections.

To conclude the discussion on the zero orbit piece, let us remark on some technicalities in the determination of the contribution related to the flux, i.e. from the second term on the r.h.s. of eq. (4.42). From appendix F.1, this part of the zero orbit contribution, which is separately modular invariant, reads:

$$\Lambda_{\text{flux}}^0 = \frac{T_2}{(24)^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\hat{F}(\hat{E}_2 E_8 - E_{10})}{\eta^{21}}. \quad (5.7)$$

Using Stokes' theorem (F.7) for modular integrals over $\mathcal{F}$, then integrating by parts and remembering that the weight $1/2$ Maaß form $\hat{F}$ (4.37) has shadow $g = -3\sqrt{2}\eta^3$, see (F.17), we may rewrite it,²⁴ according to (F.18):

$$\begin{aligned} \Lambda_{\text{flux}}^0 &= -\frac{\pi i T_2}{3(24)^2} \int_{\mathcal{F}} d^2\tau \hat{F} \frac{\partial}{\partial \bar{\tau}} \left(\frac{\hat{E}_2(\hat{E}_2 E_8 - 2E_{10})}{\eta^{21}} \right) \\ &= \frac{\pi T_2}{(24)^2} \left(172 - \sqrt{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (\sqrt{\tau_2} \eta \bar{\eta})^3 \left(\frac{\hat{E}_2(\hat{E}_2 E_8 - 2E_{10})}{\eta^{24}} \right) \right). \end{aligned} \quad (5.8)$$

²⁴We are very grateful to J. Manschot for invaluable help tackling this integral.

The first term on the last line of (5.8) comes from evaluating the integral (F.19) using the standard formula (F.8). The second integral inside the parenthesis, called I''_{flux}/π in the appendix eq. (F.21), is more involved. Using Poisson resummation, we can reexpress:

$$\begin{aligned} (\sqrt{\tau_2} \eta \bar{\eta})^3 &= \frac{1}{4\sqrt{2}} \sum_{m,n} (-)^{(m+1)(n+1)} \frac{|m+n\tau|^2}{\tau_2} e^{-\frac{\pi}{2\tau_2} |m+n\tau|^2} \\ &= -\frac{1}{8\pi} \frac{\partial}{\partial R} \left(\frac{1}{R} \Gamma_{1,1}(2R) - \Gamma_{1,1}(R) \right) \Big|_{R=\frac{1}{\sqrt{2}}} . \end{aligned} \quad (5.9)$$

As the rest of the integrand on the last line of expression (5.8) exhibits a $\frac{1}{q}$ pole coming from the cusp form η^{24} and since the radius of the second $\Gamma_{1,1}$ in (5.9) is fixed at $R = \frac{1}{\sqrt{2}}$, the integration of the $n \neq 0$ terms in the Lagrangian lattice sum (5.9) are not absolutely convergent, so that one cannot unfold the integral against it. This is where the novel approach to modular integrals developed in [102, 103] comes into play: by considering the Niebur-Poincaré series [107, 108]²⁵

$$\mathcal{F}(s, \alpha, r) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(c\tau+d)^r} \mathcal{M}_{s,r} \left(\frac{-\alpha\tau_2}{|c\tau+d|^2} \right) e^{\frac{a}{c} - \frac{c\tau_1+d}{c|c\tau+d|^2}} \quad (5.10)$$

which for the weight values $r \leq 0$ of interest are defined by:

$$\mathcal{M}_{s,r}(-y) = (4\pi y)^{s-\frac{r}{2}} e^{-2\pi y} {}_1F_1 \left(s + \frac{r}{2}; 2s; 4\pi y \right) , \quad (5.11)$$

we may rewrite the second part of the integrand in (5.8) as the linear combination [103]:

$$\frac{\widehat{E}_2(\widehat{E}_2 E_8 - 2E_{10})}{\eta^{24}} = \frac{1}{5} \mathcal{F}(3, 1, 0) - 6 \mathcal{F}(2, 1, 0) + 23 \mathcal{F}(1, 1, 0) + 432 . \quad (5.12)$$

By reexpressing the integral I''_{flux} in terms of expressions (5.9) and (5.12), and by unfolding the fundamental domain against the absolutely convergent Niebur-Poincaré series in (5.12), we obtain, from eq. (4.18) in [103] the result $I''_{\text{flux}} = 40$. Putting the two pieces of eq. (5.8) together, we thus arrive at (F.22):

$$\Lambda_{\text{flux}}^0 = \frac{53\pi}{144} T_2 . \quad (5.13)$$

Adding up the flux contribution and the $1/6$ $K3$ contribution appearing in (4.42), yields the zero orbit piece (F.23) on the first line of the r.h.s. of expression (5.5). We refer the reader to the detailed calculation in (F.11).

Contributions from degenerate orbits. The first, second and third lines of the r.h.s. of expression (5.5) capture the contributions from degenerate orbits of Γ , determined in appendix F.2. These are expressed in terms of generalised Eisenstein series:²⁶

$$\mathcal{E}_A(U, s) = \frac{1}{2\zeta(2s)} \sum_{(j,p) \neq (0,0)} \frac{U_2^s}{|j+pU|^{2s}} e^{-2\pi A \frac{|j+pU|}{\sqrt{U_2}}} , \quad \text{Re } A \in R_+ , \quad (5.14)$$

²⁵More recent works on the subject can be found in [111–114].

²⁶In another context, a close relative of these generalised Eisenstein series with $A \neq 0$ and for $s = 3/2$ appears in [115, 116], where it captures corrections to the metric of the hypermultiplet moduli space of type IIB compactifications on CY threefolds due to E(-1) instantons, and to E1 instantons wrapping two-cycles in the geometry. Here however the interpretation is different as we will see.

which reduce to the well-known real analytic Eisenstein series for $A = 0$ (see eq. (F.30)):

$$\mathcal{E}_0(U, s) \equiv E(U, s). \quad (5.15)$$

In particular, upon patching together 16 local models such as (2.2) into a full-fledged heterotic compactification with flux, the sums over (j, p) in the (generalised) Eisenstein series (5.14) are expected, like for smooth $T^2 \times K3$ models, to correctly reproduce the double sum over Kaluza-Klein momenta in the open-string channel of the corresponding type I compactification [117].

To be more specific, the second and third term on the first line of (5.5) are the degenerate orbit contribution coming from states localising on the blown-up $\mathbb{P}^1$ which, in accordance with results for compact heterotic models, are expressed in terms of the real analytic Eisenstein series (5.15). In particular, the first contribution proportional to $E(U, 1)$ exhibits a logarithmic IR divergence which can be regularised as follows:

$$E(U, 1) = -\frac{3}{\pi} (\log |\eta(U)|^4 + \log (\mu^2 T_2 U_2)) + \gamma \quad (5.16)$$

with γ a renormalisation scheme depend constant. In particular, if we adopt the regularisation [98], which introduces a $(1 - e^{-\frac{N}{\tau_2}})$ regulator in the integrand of (F.30) for $r = 1$, and eliminate the IR cutoff by sending $N \rightarrow \infty$, which corresponds to sending $\mu \rightarrow 1$ in expression (5.16), we get:

$$\gamma = 1 + \log \frac{8\pi}{3\sqrt{3}} - \gamma_E, \quad (5.17)$$

with γ_E being the Euler-Mascheroni constant. Furthermore, the constant multiplying the second term on the first line of (5.5) is $-\frac{b_{\text{SO}(28)}}{4}$ (see discussion about β -functions for both gauge factors below). Since expression (5.16) contains the regulator $-\log \mu^2$ which is an IR effect, we indeed expect only massless modes to contribute to this constant factor. In our local model, only localised states (from discrete $\text{SL}(2, \mathbb{R})_2/\text{U}(1)$ representations) give rise to BPS massless modes, and it can be checked from (F.26), (F.29) and (F.30) that these states alone and non non-localised states contribute to the coefficient in front of $E(U, 1)$, in accordance with the fact that $b_{\text{SO}(28)}$ counts the number of massless hypermultiplets (3.20).

The second and third line of (5.5) are the contributions from non-localised states carrying continuous $\text{SL}(2, \mathbb{R})_2/\text{U}(1)$ representations. Their coefficients are determined by the Fourier expansion of the following holomorphic (quasi-)modular forms (F.27):

$$\frac{E_2 E_8 - E_{10}}{q^{1/8} \eta^{21}} = \sum_{n=0}^{\infty} d_1(n) q^n, \quad \frac{E_8}{q^{1/8} \eta^{21}} = \sum_{n=-1}^{\infty} d_2(n) q^n. \quad (5.18)$$

To see how the generalised Eisenstein series (5.14) come about, we single out the contribution from bulk states, which is given according to the second line of (F.29) by the following

simple integral:

$$\begin{aligned} \Lambda_{\text{deg}}^c[1] &= -\frac{T_2}{48} \sum_{(j,p) \neq (0,0)} \int_0^\infty \frac{d\tau_2}{\tau_2^2} e^{-\frac{\pi T_2}{\tau_2 U_2} |j+pU|^2} \sum_{n=0}^\infty (-)^n \left(d_1 \left(\frac{n(n+1)}{2} \right) - \frac{3}{\pi \tau_2} d_2 \left(\frac{n(n+1)}{2} \right) \right) \times \\ &\times \operatorname{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi \tau_2} \right). \end{aligned} \quad (5.19)$$

We now exploit the fact that the series $R(0|\tau)$ (4.37) converges absolutely and uniformly for $\tau_2 > 0$ [47] to invert the integral and the sum over n , and we then use the following integrals:

$$\begin{aligned} I_r(a, b) &\equiv \int_0^\infty \frac{dx}{x^{2+r}} \operatorname{erfc}(a\sqrt{x}) e^{-\frac{b}{x}}, \quad r \geq 0 \\ &= \left(-\frac{\partial}{\partial b} \right)^r \int_0^\infty \frac{dx}{x^2} \operatorname{erfc}(a\sqrt{x}) e^{-\frac{b}{x}} = \left(-\frac{\partial}{\partial b} \right)^r \frac{e^{-2a\sqrt{b}}}{b} \\ &= \frac{1}{b^{r+1}} \left(\Gamma(r+1) - \frac{2a\sqrt{b}}{\sqrt{\pi}} \Gamma\left(r + \frac{1}{2}\right) {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{1}{2} - r; a^2 b\right) + \right. \\ &\quad \left. + (-)^r \frac{a^{2r+2} b^{r+1} \sqrt{\pi}}{(r+1)\Gamma(r + \frac{3}{2})} {}_1F_2\left(r+1; r + \frac{3}{2}, r+2; a^2 b\right) \right). \end{aligned} \quad (5.20)$$

In particular, expression (5.19) is given in terms of the following functions:

$$I_0(a, b) = \frac{e^{-2a\sqrt{b}}}{b}, \quad I_1(a, b) = \frac{(1 + a\sqrt{b})}{b^2} e^{-2a\sqrt{b}} \quad (5.21)$$

by identifying

$$a = \sqrt{2\pi} \left(n + \frac{1}{2} \right), \quad b = \frac{\pi T_2}{U_2} |j + pU|^2, \quad (5.22)$$

so that $\Lambda_{\text{deg}}^c[1]$ precisely gives the second and third line of (5.5). For degenerate orbits, we observe that bulk contributions distinguish themselves from the contributions of localised states by an exponential suppression in $\mathcal{E}_{(n+\frac{1}{2})\sqrt{T_2}}(U, s)$ in the large volume limit of the T^2 , a novel feature with respect to compact models which appear to be peculiar to local models with non-localised bulk states in their spectrum.

To try and grasp the physical significance of the degenerate orbit contributions, let us first concentrate on localised states. These give rise the $E(U, 1)$ and $E(U, 2)/T_2$ terms in the first line of (5.5) and are on every count similar to contributions from degenerate orbits for heterotic $T^2 \times K3$ compactifications (3.16). By using the correspondence [118, 119], they can be shown to map on the type I side to (perturbative) higher-genus contributions correcting the Kähler potential and the $SO(28)$ gauge kinetic function. The particulars of this mapping can be understood as follows: under S-duality, the heterotic / type I string couplings and the T^2 Kähler modulus transform (in the σ -model frame) as:

$$\lambda_{\text{het}} = \frac{1}{\lambda_I}, \quad T_2^{\text{het}} = \frac{T_2^I}{\lambda_I}, \quad T_1^{\text{het}} = T_1^I, \quad (5.23)$$

with λ_{het} given by the double-scaling parameter (2.15). Then, the expansion in inverse powers of T_2 characteristic of the degenerate orbit contribution is translated to a higher genus expansion on the type I side [67, 117], with λ_I playing the rôle of the open-string loop-counting parameter. Thus, the zero orbit contribution, proportional to T_2 , corresponds to the leading, $\chi = 1$ diagram contribution in the dual type I theory, coming from the disk amplitude, while the $E(U, 1)$ term is mapped to $\chi = 0$ subleading corrections corresponding to a combination of the annulus and the Möbius strip amplitudes, and finally the $E(U, 2)/T_2$ term is translated to a $\chi = -1$ two-loop diagram, such as the disk with two holes.

From this angle, we expect the second and third line of the threshold (5.5) to encode higher perturbative corrections on the type I side due to non-localised bulk states. This should be verified by carrying out in the large T_2 limit the appropriate expansion of the exponential factor. In this regard, the observed mixing of the U and T moduli in the $e^{-2\pi(n+\frac{1}{2})\sqrt{\frac{2T_2}{U_2}}|pU_2+j|}$ factors of the generalised Eisenstein series (5.14) seems at first sight puzzling. Nevertheless a similar mixing which occurs in the $\log(\mu^2 T_2 U_2)$ term regularising the IR divergence in $E(U, 1)$ (5.16) sheds some light on this issue. Since non-localised state contributions to the threshold corrections act as regulator of the infinite volume divergence of the underlying CFT target-space, we can understand this mixing as a distinctive feature of compensating for the holomorphic anomaly of the modified elliptic genus. This however remains an analogy since in the first case we regularise an IR divergence in the effective field theory with a scale dependent regulator, while in the second case we renormalise the infinite volume divergence of the transverse space, where no scale is present to fix a cutoff on the massive non-localised modes of the theory.

Contributions from non-degenerate orbits. The contribution corresponding to degenerate orbits of Γ appear in the last two lines of expression (5.5), as computed in appendix F.3. On the type I side, these terms map to non-perturbative corrections due to E1 instantons wrapping the T^2 , since they are function of the induced Kähler and complex structure moduli:

$$\mathcal{T} = kpT, \quad \mathcal{U} = \frac{j + pU}{k}. \quad (5.24)$$

In particular, by using the heterotic / type I map (5.23), we see that the factor $e^{2\pi\mathcal{T}}$ in the fourth line of expression (5.5) is precisely the exponential of the Nambu-Goto action of an E1 string wrapped $N = kp$ times around the T^2 . The non-degenerate orbit contribution is also written as an expansion in the inverse volume of the T^2 obtained by acting on the elliptic genus with the modular invariant operator $(-iD)^r(\tau_2\bar{\partial}_\tau)^r$ which annihilates holomorphic modular forms. Then, this expansion can be elegantly expressed in terms of two descendants of the modified elliptic genus, namely: the weight 0 weak harmonic Maaß

form:

$$\begin{aligned}\hat{\mathcal{A}}_K(\tau) &\doteq (-iD)\tau_2^2\bar{\partial}_\tau\hat{\mathcal{A}}_{\text{SO}(28)}(\tau) \\ &= -\frac{1}{144\pi\eta^{24}}\left(\hat{E}_2E_{10}+2E_6^2+3E_4^3-\eta^4\left[\hat{E}_2E_8+4E_{10}+3E_8D_0\right]\left(\frac{\hat{F}}{\eta}\right)-\right. \\ &\quad \left.-\frac{\pi}{\sqrt{2}}\left[\hat{E}_2^2E_8+17\hat{E}_2E_{10}-8E_6^2-10E_4^3\right](\sqrt{\tau_2}\bar{\eta}\eta)^3\right)\end{aligned}\quad (5.25)$$

where the covariant derivative reduces to $D_0(\hat{F}/\eta) = \pi^{-1}i\partial_\tau(\hat{F}/\eta)$; and the weight -4 weak harmonic Maaß form:

$$\begin{aligned}\hat{\mathcal{A}}_H(\tau) &\doteq (\tau_2^2\bar{\partial}_\tau)^2\hat{\mathcal{A}}_{\text{SO}(28)}(\tau) \\ &= \frac{1}{48\sqrt{2}\pi\eta^{24}}\left((\pi\tau_2)^2\bar{\hat{E}}_2(\hat{E}_2E_8-E_{10})+12E_8\right)(\sqrt{\tau_2}\bar{\eta}\eta)^3.\end{aligned}\quad (5.26)$$

The interpretation of the non-degenerate orbit contribution as E1 multi-instanton corrections on the type I side becomes more manifest if we reexpress it by use of the Hecke operator H_N , which acts on a modular form of weight r as follows:

$$H_N[\Phi_r](\tau) = N^{r-1} \sum_{\substack{k,p>0 \\ kp=N}} \sum_{k>j\geq 0} \frac{1}{k^r} \Phi_r\left(\frac{j+p\tau}{k}\right) \quad (5.27)$$

and thus preserves the space of modular forms of a given weight. Adopting a compact notation for the differential operator appearing in (5.5):

$$\begin{aligned}\mathfrak{D} : \hat{\mathbb{M}}_r &\longrightarrow \hat{\mathbb{M}}_{r-2} \\ \Phi_r(\tau) &\mapsto \tau_2^2\bar{\partial}_\tau\Phi_r(\tau)\end{aligned}\quad , \quad (5.28)$$

the last two lines of the threshold corrections (5.5) can be expressed in terms of a sum over Hecke operators (5.27): applied to weight 0 weak harmonic Maaß forms:

$$\begin{aligned}\Lambda_{\text{SO}(28)}^{\text{non-deg}} &= \frac{1}{4} \sum_{N=1}^{\infty} e^{2\pi iNT} H_N \left[\hat{\mathcal{A}}_{\text{SO}(28)} + \frac{1}{NT_2} \hat{\mathcal{A}}_{\text{SO}(28)}^K \right] (U) + \\ &+ \frac{1}{4} \sum_{N=1}^{\infty} e^{2\pi iNT} \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \frac{1}{(NT_2)^{r+2}} H_N \left[(-iD)^{r+2} \mathfrak{D}^r \hat{\mathcal{A}}_{\text{SO}(28)}^H \right] (U) + \text{c.c.}.\end{aligned}\quad (5.29)$$

This rewriting makes particularly manifest the non-perturbative nature of these contributions on the type I side, where they map to multi-instanton corrections, due to E1 instantons wrapping N times the T^2 . Anti-instanton contributions are also taken into account in the complex conjugate of this expression, which corresponds to terms with $p < 0$ in the sum on the last line of (5.4). Expressing the instanton sum in terms of a sum over Hecke operators, as in (5.29), which by construction preserve the weight and modular properties of the forms, has the virtue of making apparent the invariance of (5.5) under $\text{SL}(2, \mathbb{Z})_U$, in keep with the corresponding global symmetry of the background.

The bulk state contributions: finite and large volume expression. In the following, we will elaborate on some subtleties in the derivation of the non-degenerate orbit contribution coming from non-localised bulk states. The details on how to evaluate the integral (5.4) over the double cover of $\mathcal{H}$ can be found in appendix F.3. In particular, from the last two lines of (F.35), we obtain the contribution of bulk states by using (F.40):

$$\begin{aligned} & \Lambda_{\text{non-deg}}^c[1] \\ &= -\frac{1}{24} \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} \sum_{n,m} (-)^n \left(e^{2\pi i \mathcal{T}} \left[d_1(m) - \frac{3}{\pi \mathcal{U}_2} d_2(m) - \frac{3}{\pi \mathcal{T}_2} \left(m + \frac{1}{2\pi \mathcal{U}_2} \right) d_2(m) \right] \times \right. \\ & \quad \times e^{2\pi i (m - \frac{1}{2}n(n+1))\mathcal{U}} - 2e^{2\pi i \mathcal{T}_1} \sum_{l=0}^{\infty} e_{n,l} \left[d_1(m) (\mathcal{T}_2 \mathcal{U}_2)^{l+\frac{1}{2}} W_{l,n,m}(\mathcal{T}, \mathcal{U}) - \right. \\ & \quad \left. \left. - \frac{3}{\pi} d_2(m) (\mathcal{T}_2 \mathcal{U}_2)^{l-\frac{1}{2}} W_{l-1,n,m}(\mathcal{T}, \mathcal{U}) \right] \right) + \text{c.c.}, \end{aligned} \quad (5.30)$$

with $d_i(n)$, $i = 1, 2$ the coefficients of the Fourier expansions (5.18), while the coefficients:

$$e_{n,m} = \frac{\pi^m}{m!} \frac{(\sqrt{2}(n + \frac{1}{2}))^{2m+1}}{m + \frac{1}{2}} \quad (5.31)$$

are obtained by expanding the complementary error function, which is an odd entire series:

$$\text{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi \tau_2} \right) = 1 - \sum_{m=0}^{\infty} (-)^m e_{n,m} \tau_2^{m+\frac{1}{2}}. \quad (5.32)$$

We also use the compact notation:

$$W_{l,n,m}(\mathcal{T}, \mathcal{U}) \doteq \frac{K_l \left(2\pi \left| \mathcal{T}_2 + (m - \frac{1}{2}n(n+1))\mathcal{U}_2 \right| \right)}{\left| \mathcal{T}_2 + (m - \frac{1}{2}n(n+1))\mathcal{U}_2 \right|^l} \quad (5.33)$$

given in terms of the modified Bessel functions of the second kind:

$$K_\alpha(z) = \int_0^\infty dt e^{-z \cosh t} \cosh \alpha t, \quad \text{Re } z > 0, \quad (5.34)$$

which are clearly even in α :

$$K_{-\alpha}(z) = K_\alpha(z). \quad (5.35)$$

Now, for $z \in \mathbb{R}$ and $z \gg \left| \alpha^2 - \frac{1}{4} \right|$ the Bessel functions (5.34) have the series expansion:

$$K_\alpha(z) \cong \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n + \frac{1}{2})}{n! \Gamma(\alpha - n + \frac{1}{2})} \frac{1}{(2z)^n}, \quad (5.36)$$

since for $z > 0$ it can be shown that by considering an asymptotic expansion where we truncate (5.36) at $n = N$, the remainder R of this series is bounded by [120]:

$$|R| < \left| \frac{\Gamma(\alpha + 2n + \frac{1}{2})}{(2n)! \Gamma(\alpha - 2n + \frac{1}{2}) (2z)^{2n}} \right|, \quad n > \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \quad (5.37)$$

which is less than the absolute value of the first discarder terms in (5.36). Then, when in particular $z \gg |\alpha^2 - \frac{1}{4}|$, we obtain the infinite power series expansion (5.36).

Therefore, by considering a region of the moduli space where T_2 is large enough, we can expand (5.30) as follows:

$$\begin{aligned}
 & \Lambda_{\text{non-deg}}^c[1] \\
 &= -\frac{1}{24} \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i \mathcal{T}} \sum_{n,m} (-)^n \left(\left[d_1(m) - \frac{3}{\pi \mathcal{U}_2} d_2(m) - \frac{3}{\pi \mathcal{T}_2} \left(m + \frac{1}{2\pi \mathcal{U}_2} \right) d_2(m) \right] - \right. \\
 & \quad \left. - \sum_{l=0}^{\infty} e_{n,l} \mathcal{U}_2^{l+\frac{1}{2}} \sum_{s,r=0}^{\infty} \frac{(-1)^r}{(4\pi)^s} \frac{1}{s!r!} \frac{1}{\mathcal{T}_2^{r+s}} \left[\frac{\Gamma(l+s+r+\frac{1}{2})}{\Gamma(l-s+\frac{1}{2})} d_1(m) - \frac{\Gamma(l+s+r-\frac{1}{2})}{\Gamma(l-s-\frac{1}{2})} \frac{3d_2(m)}{\pi \mathcal{U}_2} \right] \times \right. \\
 & \quad \left. \times \left(\left[m - \frac{1}{2}n(n+1) \right] \mathcal{U}_2 \right)^r \right) e^{2\pi i(m-\frac{1}{2}n(n+1))\mathcal{U}} + \text{c.c.} \quad (5.38)
 \end{aligned}$$

Then, since for half integer argument the Γ -function has an expression in terms of double factorials:

$$\Gamma\left(k + \frac{1}{2}\right) = \frac{(2k-1)!!\sqrt{\pi}}{2^k}, \quad \Gamma\left(\frac{1}{2} - k\right) = (-)^k \frac{2^k\sqrt{\pi}}{(2k-1)!!} \equiv \frac{(-)^k\pi}{\Gamma\left(k + \frac{1}{2}\right)}, \quad k \in \mathbb{N}, \quad (5.39)$$

we can elegantly express (5.38) by means of the modular invariant operator [67]

$$\square \equiv \mathcal{U}_2^2 \partial_{\mathcal{U}} \bar{\partial}_{\mathcal{U}} \quad (5.40)$$

which annihilates holomorphic modular forms, by observing that:

$$\begin{aligned}
 e^{-2\pi i n \mathcal{U}} \mathcal{U}_2^{-(l\pm\frac{1}{2})} \frac{1}{\pi} \square \mathcal{U}_2^{l\pm\frac{1}{2}} e^{2\pi i n \mathcal{U}} &= -\frac{\Gamma(l+1\pm\frac{1}{2})}{\Gamma(l\pm\frac{1}{2})} n \mathcal{U}_2 + \frac{1}{4\pi} \frac{\Gamma(l+1\pm\frac{1}{2})}{\Gamma(l-1\pm\frac{1}{2})} \\
 e^{-2\pi i n \mathcal{U}} \mathcal{U}_2^{-(l\pm\frac{1}{2})} \frac{1}{\pi^2} \left(\square^2 - \frac{1}{2} \square \right) \mathcal{U}_2^{l\pm\frac{1}{2}} e^{2\pi i n \mathcal{U}} &= \frac{\Gamma(l+2\pm\frac{1}{2})}{\Gamma(l\pm\frac{1}{2})} (n \mathcal{U}_2)^2 - \frac{1}{2\pi} \frac{\Gamma(l+2\pm\frac{1}{2})}{\Gamma(l-1\pm\frac{1}{2})} n \mathcal{U}_2 + \\
 & \quad + \frac{1}{(4\pi)^2} \frac{\Gamma(l+2\pm\frac{1}{2})}{\Gamma(l-2\pm\frac{1}{2})} \\
 & \quad \vdots
 \end{aligned} \quad (5.41)$$

Then, in the large T_2 region, the contribution from bulk states (5.38) can be written as follows:

$$\begin{aligned}
 \Lambda_{\text{non-deg}}^c[1] &= -\frac{1}{24} \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i \mathcal{T}} \left[1 + \frac{1}{\pi \mathcal{T}_2} \square + \frac{1}{2!} \frac{1}{(\pi \mathcal{T}_2)^2} \left(\square^2 - \frac{1}{2} \square \right) + \dots \right] \times \\
 & \quad \times \sum_{n=0}^{\infty} (-)^n \text{erfc}\left(\left(n + \frac{1}{2} \right) \sqrt{2\pi \mathcal{U}_2} \right) e^{-\pi i(n+\frac{1}{2})^2 \mathcal{U}} \frac{\hat{E}_2(\mathcal{U}) E_8(\mathcal{U}) - E_{10}(\mathcal{U})}{\eta(\mathcal{U})^{21}} + \text{c.c.} \quad (5.42)
 \end{aligned}$$

The expansion in $\square$ can alternatively be reorganised as a power expansion in the covariant derivative $\mathfrak{D}$ (5.28), so that by putting together the contribution of localised states (F.35) containing the Mock modular form F (4.38) and the bulk state contribution (5.41), we

can reconstitute an expression in terms of the Maaß form $\widehat{F}(\mathcal{U})$ (4.37), which is therefore manifestly $\text{SL}(2, \mathbb{Z})_U$ invariant as already discussed:

$$\Lambda_{\text{non-deg}}^{\text{flux}}[1] = -\frac{1}{384} \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i \tau} \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{\mathcal{T}_2^r} (-iD)^r (\mathcal{U}_2^2 \bar{\partial}_{\mathcal{U}})^r \left[\frac{\widehat{F}(\mathcal{U}) D_8 E_8(\mathcal{U})}{\eta(\mathcal{U})^{21}} \right] + \text{c.c.} \quad (5.43)$$

Then, by using formulæ (C.17)–(C.20), the instanton contributions on the two last lines of (5.5) can be compactly expressed in terms of the modified elliptic genus in (4.42) and its descendants (5.25) and (5.26).

In this respect, since the operators $(-iD)^r \mathfrak{D}^r$ annihilate holomorphic modular forms and $\mathcal{U}_2^{-s}$ terms for $s > r$ positive, the contribution of localised states to the non-degenerate orbit integral stops at the $O(\mathcal{T}_2^{-1})$ term $\hat{\mathcal{A}}_{\text{SO}(28)}^K$, so that expansion in powers of $\mathcal{U}_2^{-1}$ is finite and governed by g_{max} (3.22), whose value is dictated by the number of unbroken supercharges [67]. In contrast, because they enter into the threshold corrections through the combination gg^* of the shadow function and its transform (see last term in eq. (4.50) with $\widehat{F} = F + g^*$), bulk state contribute an infinite number of descendants of the modified elliptic genus to the non-degenerate orbit integral, such in particular as the $(-iD)^{r+2} \mathfrak{D}^r \hat{\mathcal{A}}_{\text{SO}(28)}^H$ terms in eq. (5.5). The $\mathcal{T}_2^{-1}$ expansion in the large volume limit (5.38) indicates that this is not in contradiction with space-time supersymmetry, as the highest power of $\mathcal{U}_2^{-1}$ is in this case still $g_{\text{max}} = 1$, as expected from a background preserving $\mathcal{N}_{\text{ST}} = 2$ in four dimensions.

Since the modular invariant descendants of the elliptic genus $(-iD)^r \mathfrak{D}^r \hat{\mathcal{A}}_{\text{SO}(28)}$ actually determine corrections to various dimension-eight operators in the effective theory [67], this analysis tends to suggest that non-localised bulk modes in non-compact heterotic models entail an infinite number of such corrections to the effective action.

Let us make one final comment about the finite / large volume issue in determining instanton threshold corrections. The large T^2 volume limit used to derive the bulk state contributions on the last two lines of expression (5.5) makes manifest, on the type I side, the exponential $e^{-S_{\text{E1}}}$ of $S_{\text{E1}} = 2\pi N \left(\frac{T_1^2}{\lambda_1} - iT_1^1 \right)$ which is the classical action of an E1 instanton wrapping N times the T^2 . In the finite $T_2 > 1$ regime, we have in contrast to use the more involved expression (5.30). There, the exponential of the topological part of the action $\text{Im } S_{\text{E1}} = -\frac{1}{2\pi\alpha'} \int \widehat{B}_1$ is still apparent, while the part of the action depending on the pull-back of the metric on the instanton worldvolume $\text{Re } S_{\text{E1}} = \frac{1}{2\pi\alpha'\lambda_1} \int d\sigma^2 \sqrt{|\widehat{G}_1|}$ is now apparently entangled with the complex structure modulus, a peculiarity that calls for further investigation, possibly of the the S-dual type I model.

5.3 Moduli dependence of the $\text{SU}(2)$ threshold corrections

The $\text{SU}(2)$ threshold correction can now be determined very economically by using the difference (4.51), whose r.h.s. can be readily integrated:

$$\begin{aligned} \Lambda_{\text{SU}(2)}[1] - 2\Lambda_{\text{SO}(28)}[1] &= 12 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \\ &= 4\pi T_2 - 12 \left(\log |\eta(U)|^4 + \log(\mu^2 T_2 U_2) + \gamma \right) + 24 \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{-2\pi\mathcal{T}_2} \cos(2\pi\mathcal{T}_1). \end{aligned} \quad (5.44)$$

Using the $\text{SO}(28)$ threshold just computed (5.5), we get a general formula for the β -functions:

$$b_a = k_a \left(6 - \frac{t_a}{2} \right). \quad (5.45)$$

As previously seen: $t_{\text{SO}(28)} = 4$ and $t_{\text{SU}(2)} = -44$, leading to the following β -functions for the $\mathcal{Q}_5 = 1$ model:

$$b_{\text{SO}(28)} = 4, \quad b_{\text{SU}(2)} = 56. \quad (5.46)$$

This is in perfect agreement with the field theory results (3.21) obtained from the hypermultiplet counting in table 1.

6 The dual type I model

As discussed in the previous section, the gauge thresholds (5.5) and (5.44) translate as perturbative and instanton corrections to the Kähler and gauge kinetic functions in the S-dual type I model. In the case under scrutiny, this theory only has space-time filling D9-branes, which support the gauge group $\text{SO}(28) \times \text{SU}(2)$. Half D5-branes at singularities which are usually necessary in orbifold models for anomaly cancellation [121] are absent here, since the A_1 singularity is resolved and the tadpole equation (2.12) is satisfied by $\text{U}(1)$ gauge instantons on the blown-up $\mathbb{P}^1$ two-cycle.

A microscopic description of the type I theory dual to the warped Eguchi-Hanson heterotic background can still be hard to come by. Nevertheless, by using the field theory dictionary [118, 119] mapping heterotic gauge threshold corrections to type I one-loop corrections to the gauge kinetic functions:

$$\begin{aligned} \frac{4\pi^2}{g_a^2(\mu^2)} \Big|_{1\text{-loop}} &= \text{Ref}_a(M)|_{1\text{-loop}} - \frac{b_a}{4} \left[\log(-iS + i\bar{S}) - \log \frac{M_{\text{Planck}}^2}{\mu^2} \right] + \\ &+ \frac{1}{4} \left[c_a K(M, \bar{M}) - 2 \sum_{\mathbf{R}} T_a(\mathbf{R}) \log \det C_{\mathbf{R}}(M, \bar{M}) \right], \end{aligned} \quad (6.1)$$

these corrections can be extracted from the heterotic result, even when the corresponding type I model is unknown. In particular, on the r.h.s. of eq. (6.1), $K(M, \bar{M})$ is the tree-level Kähler potential, $\det C_r(M, \bar{M})$ the determinant of the tree-level Kähler metric for the matter multiplets in the representation $\mathbf{R}$ of the gauge group factor G_a , and the model dependent constants:

$$c_a = \sum_{\mathbf{R}} n_{\mathbf{R}} T_a(\mathbf{R}) - T_a(\text{Adj}_a), \quad (6.2)$$

which for $\mathcal{N}_{\text{ST}} = 2$ theories are equal to $b_a/2$, see (3.3).

For our purpose, we alternatively express the heterotic one-loop gauge thresholds (3.1) in terms of the axio-dilaton multiplet S and the universal function $\Delta_{\text{univ}}(M, \bar{M})$, as in (3.2), and decompose the latter into harmonic functions and a non-harmonic remainder:

$$\Delta_{\text{univ}}(M, \bar{M}) = V(M, \bar{M}) + H(M) + \overline{H}(\bar{M}). \quad (6.3)$$

Then, by using the tree-level identification of the imaginary part of S :

$$\text{Im } S = \frac{M_{\text{Planck}}^2}{M_s^2}, \quad (6.4)$$

we recover the corrected Kähler potential and gauge kinetic functions in terms of heterotic one-loop quantities [119, 122, 123]:

$$\begin{aligned} \text{Re } f_a|_{1\text{-loop}} = k_a \text{Im } S + \frac{1}{4} \left(-c_a K_{\text{tree}}(M, \overline{M}) + 2 \sum_{\mathbf{R}} T_a(\mathbf{R}) \log \det C_{\mathbf{R}}(M, \overline{M}) + \right. \\ \left. + \Delta(M, \overline{M}) - k_a V(M, \overline{M}) \right), \end{aligned} \quad (6.5a)$$

$$K|_{1\text{-loop}} = K_{\text{tree}}(M, \overline{M}) - \log \left(-iS + i\bar{S} - \frac{1}{2} V(M, \overline{M}) \right). \quad (6.5b)$$

As seen in the previous section, these corrections have a natural interpretation in the type I S-dual model in terms of perturbative string-loop corrections and non-perturbative corrections due to E1 instantons wrapping the T^2 .

It has been shown [84] that for a general $\mathcal{N}_{\text{ST}} = 2$ heterotic orbifold compactification, the second and third term on the l.h.s. of the first line of (6.5a) conspire to cancel the $b_a \log(T_2 U_2)$ term in the threshold correction (5.5) coming from the IR regulator in (5.16), due to the correlated way the Casimirs $T(\mathbf{R})$ enter into the contribution from the Kähler metric of the matter fields and into the functions c_a (6.2). Thus, the corrections to the gauge kinetic functions for both $a = \{\text{SO}(28), \text{SU}(2)\}$ gauge factors are given by the harmonic contributions in (5.5) and (5.44):

$$f_a = -ik_a S - \frac{k_a(53 - 6t_a)\pi}{144} iT - b_a \log \eta(U) + \frac{1}{2} \sum_{k>j \geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i T} \mathcal{A}_a(\mathcal{U}). \quad (6.6)$$

In particular, they receive perturbative corrections from the disk and a combination of the annulus and the Möbius strip diagrams, along with E1 instanton contributions, which are given in terms of the holomorphic part of the modified elliptic genera (4.42) and (4.49):

$$\mathcal{A}_a(\tau) = -\frac{k_a}{72} \left(E_2 E_{10} - \frac{12 + t_a}{24} E_6^2 - \frac{12 - t_a}{24} E_4^3 - \eta^3 G_1(E_2 E_8 - E_{10}) \right), \quad (6.7)$$

where we have defined the holomorphic function:

$$G_1(\tau) = F(\tau) - 12 \sum_{n=0}^{\infty} (-1)^n q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \quad (6.8)$$

and the gauge factor dependent coefficients previously determined are summarised in the table:

	k_a	t_a	b_a
SO(28)	1	4	4
SU(2)	2	-44	56

(6.9)

In contrast, the corrections to the tree level Kähler potential of the effective type I theory is given by the non-harmonic real analytic part of (5.5), which for convenience we split into a perturbative and non-perturbative contribution:

$$K = -\log[(T-\bar{T})(\bar{U}-U)] - \log(a^2) - \log\left(-iS + i\bar{S} - \frac{V_{\text{pert.}}(T, U) + V_{\text{E1}}(T, U)}{2}\right), \quad (6.10)$$

These corrections originate from higher string-loop and multi-instanton corrections and yield:

$$\begin{aligned} V_{\text{pert.}}(T, U) = & -\frac{29\pi}{90} \frac{E(U, 2)}{T_2} - \frac{1}{2} \sum_{n=0}^{\infty} (-)^n \left[\frac{\pi}{18} d_1 \left(\frac{n(n+1)}{2} \right) \mathcal{E}_{(n+\frac{1}{2})\sqrt{2T_2}}(U, 1) - \right. \\ & - \frac{\sqrt{2}\zeta(3)}{\pi^2} \left(n + \frac{1}{2} \right) d_2 \left(\frac{n(n+1)}{2} \right) \frac{\mathcal{E}_{(n+\frac{1}{2})\sqrt{2T_2}}(U, 3/2)}{\sqrt{T_2}} - \\ & \left. - \frac{\pi}{90} d_2 \left(\frac{n(n+1)}{2} \right) \frac{\mathcal{E}_{(n+\frac{1}{2})\sqrt{2T_2}}(U, 2)}{T_2} \right], \end{aligned} \quad (6.11a)$$

$$\begin{aligned} V_{\text{E1}}(T, U) = & \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i \mathcal{T}} \left[\frac{1}{6\pi\eta^{24}(\mathcal{U})} \left(\frac{E_{10}(\mathcal{U}) - \eta^3(\mathcal{U})\hat{F}(\mathcal{U})E_8(\mathcal{U})}{\mathcal{U}_2} - \right. \right. \\ & \left. \left. - 3\pi\eta^3(\mathcal{U})\hat{G}_2(\mathcal{U})D_8E_8(\mathcal{U}) \right) + \right. \\ & \left. + \frac{\hat{\mathcal{A}}_K(\mathcal{U})}{\mathcal{T}_2} + \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \frac{1}{(\mathcal{T}_2)^{r+2}} (-iD)^{r+2} (\mathcal{U}_2^2 \bar{\partial}\mathcal{U})^r \hat{\mathcal{A}}_H(\mathcal{U}) \right] + \text{c.c.} \end{aligned} \quad (6.11b)$$

where we have defined the non-holomorphic function:

$$\hat{G}_2(\tau) = \sum_{n=0}^{\infty} (-)^n E\left(\left(n + \frac{1}{2}\right) \sqrt{2\tau_2}\right) q^{-\frac{1}{2}(n+\frac{1}{2})^2}. \quad (6.12)$$

Note in particular that in these expressions $\text{SL}(2, \mathbb{Z})_T$ modular transformations mix perturbative and instanton corrections, in accordance with the fact that T-duality is not a symmetry of type I string theory.

Some comments about E1 instanton contributions: the analysis in terms of Hecke operators (5.27) gave us an understanding of the non-perturbative contributions in (6.10) and (6.6) as coming from E1 instantons wrapping N times the T^2 , so as multi-instanton corrections. Since the A_1 singularity is resolved, all potential E1 instantons initially sitting at the fixed point in the orbifold limit have been moved away from it. Thus all such instantons present in the blowup regime are either localised on the resolved two cycle or at some position in the bulk. Thus they all carry $\text{SO}(r)$ Chan-Paton factors. These E1 instantons are characterised by the following uncharged massless modes:

- in the antisymmetric representation $\frac{\mathbf{r}(\mathbf{r}-1)}{2}$: bosonic zero modes z and $\bar{z}$ and the corresponding fermionic ones $\lambda^{\alpha,a}$ and $\lambda^{\dot{\alpha},a}$,

- in the symmetric representation $\frac{\mathbf{r}(\mathbf{r}+1)}{2}$: bosonic zero modes x_μ , ρ , θ , ϕ and ψ and fermionic zero modes $\chi^{\alpha,a}$ and $\chi^{\dot{\alpha},a}$.

The strings extending between an E1 instanton and a stack of n D9-branes produce in addition a bosonic zero mode σ in the bifundamental representations $(\mathbf{r}, \bar{\mathbf{n}}) + (\bar{\mathbf{r}}, \mathbf{n})$.

As a final remark, since for instantons to contribute to the gauge kinetic functions they should possess four neutral massless zero modes [124–126], we expect most of the zero modes listed above to acquire a mass through the Scherk-Schwarz mechanism. To determine the surviving zero-modes, one should analyse the subspace of the multi-instanton moduli space for the N -instanton contributions (5.29), corresponding to deformations of the instantons along the warped Eguchi-Hanson space, in order to find out when all the components of the multi-instantons coincide [127, 128], for instance on the resolved $\mathbb{P}^1$. This we will not attempt here.

7 Perspectives

In order to generalise the results presented in sections 5 and 6, it would be interesting to compute explicitly the modular integrals (4.15) and (4.47) for generic five-brane charge Q_5 . This would allow to distinguish explicitly the contributions from worldsheet instantons wrapped around the $\mathbb{P}^1$, by isolating e^{-kn} factors in the tree-level contribution to the heterotic gauge thresholds. Then, one could repeat in this case the analysis of section 6 and explicitly extract perturbative and non-perturbative corrections to the Kähler potential and the gauge kinetic functions for arbitrary five-brane charge Q_5 .

In this perspective it would be interesting to be able to cross-check the results we obtain on the type I side from S-duality by direct string amplitude computations, along the line of [75], and have an explicit derivation of Chan-Paton factors attached to the E1-brane instantons, as in [24]. This would be particularly attractive in the present models which allow to go beyond the large volume limit commonly considered for type II (orientifold) models. An explicit description of the lifting of fermionic zero modes by instanton effects in the torsional local models considered here would also be appealing. This analysis would call for a microscopic understanding of multi-instanton effects originating from E1-branes by adapting the approach [127–130] to smooth local non-Kähler geometries with non-vanishing five-brane charge.

A very important follow-up of this paper would be to consider the situations where the fibration of the two-torus over the base is non-trivial, i.e. geometries of the type $T^2 \hookrightarrow \mathcal{M} \rightarrow \widehat{\text{EH}}$ [19]. The main novelty in these cases is that the topology of the solution is modified, namely the $\mathbb{P}^1 \times T^2$ is replaced by $S^3/\mathbb{Z}_N \times S^1$, the first factor being a Lens space. Therefore the E1 instantons can only wrap a *torsional* two-cycle, meaning that the instanton sums should terminate at wrapping number $N - 1$. This would be a very interesting new effect. Another interesting novelty with non-trivial fibration is that part of the torus moduli are restricted to a set of discrete values [131]. The explicit computation of gauge threshold corrections can be done straightforwardly as the worldsheet CFT is also known in these cases.

Finally, a most challenging extension of this work is the computation of threshold corrections for genuinely *compact* torus fibrations $T^2 \hookrightarrow \mathcal{M} \rightarrow \widetilde{K3}$. There, the worldsheet CFT is not known; one should then rather use the gauged linear sigma model description given in [7], or a purely geometrical approach extending ref. [132] to generalised CY geometries. This would open an exciting avenue to tackle phenomenological issues such as moduli stabilisation due to instantonic corrections to the Kähler potential, by extending the analysis [43, 44] to type I compactifications on smooth non-Kähler spaces supporting line bundles, without the restriction of considering a large volume limit thereof. Other applications come from instanton corrections to gauge kinetic functions, which generically modify gaugino masses and gauge couplings, and might thus affect the phenomenology of the effective theory. Constructing an exact CFT description for a full-fledged compactification of the local heterotic torsional models examined in the present work would prove particularly relevant to these questions. We expect to come back to these issues in the next future.

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A $\mathcal{N} = 2$ characters and useful identities

A.1 $\mathcal{N} = 2$ minimal models

The characters of the $\mathcal{N} = 2$ minimal models, i.e. the supersymmetric $SU(2)_k/U(1)$ gauged wzw model, are conveniently defined through the characters $C_m^{j(s)}$ [133] of the $[SU(2)_{k-2} \times U(1)_2]/U(1)_k$ bosonic coset, obtained by splitting the Ramond and Neveu-Schwartz sectors according to the fermion number mod 2. Defining $q = e^{2\pi i\tau}$ for the complex structure $\tau \in \mathcal{H}$ and $z = e^{2\pi i\nu}$ for the elliptic variable $\nu \in \mathbb{C}$, these characters are determined implicitly through the identity:

$$\chi_{k-2}^j(\nu|\tau)\Theta_{s,2}(\nu - \nu'|\tau) = \sum_{m \in \mathbb{Z}_{2k}} C_m^{j(s)}(\nu'|\tau)\Theta_{m,k}\left(\nu - \frac{2\nu'}{k}|\tau\right), \quad (\text{A.1})$$

in terms of the theta functions of $\widehat{\mathfrak{su}(2)}_k$:

$$\Theta_{m,k}(\tau, \nu) = \sum_n q^{k(n + \frac{m}{2k})^2} z^{k(n + \frac{m}{2k})}, \quad m \in \mathbb{Z}_{2k} \quad (\text{A.2})$$

and χ_{k-2}^j the characters of the affine algebra $\widehat{\mathfrak{su}(2)}_{k-2}$:

$$\chi_{k-2}^j(\nu|\tau) = \frac{\Theta_{2j+1,k}(\nu|\tau) - \Theta_{-(2j+1),k}(\nu|\tau)}{i\vartheta_1(\nu|\tau)}. \quad (\text{A.3})$$

We also mention an identity on $\widehat{\mathfrak{su}(2)}_k$ theta functions, which we use in the present work:

$$\Theta_{m/p, k/p}(\nu|\tau) = \sum_{n \in \mathbb{Z}_p} \Theta_{m+2kn, pk} \left(\frac{\nu}{p} \middle| \tau \right). \quad (\text{A.4})$$

and another way of writing the $\text{SU}(2)_{k-2}$ characters for $\nu = 0$:

$$\chi_{k-2}^j(0|\tau) = \frac{1}{\eta^3(\tau)} \sum_{n \in \mathbb{Z}} (2kn + 2j + 1) q^{k(n + \frac{2j+1}{2k})^2} = \frac{\Theta'_{2j+1,k}(0|\tau)}{\pi i \eta^3(\tau)}, \quad (\text{A.5})$$

where $' \equiv \partial_\nu$.

Highest-weight representations are labeled by (j, m, s) , corresponding primaries of $\text{SU}(2)_{k-2} \times \text{U}(1)_k \times \text{U}(1)_2$. The following identifications apply:

$$(j, m, s) \sim (j, m + 2k, s) \sim (j, m, s + 4) \sim \left(\frac{k}{2} - j - 1, m + k, s + 2 \right) \quad (\text{A.6})$$

as the selection rule $2j + m + s = 0 \pmod{2}$. The spin j is restricted to $0 \leq j \leq \frac{k}{2} - 1$. The conformal weights of the superconformal primary states are:

$$\begin{aligned} \Delta &= \frac{j(j+1)}{k} - \frac{m^2}{4k} + \frac{s^2}{8} && \text{for } -2j \leq m - s \leq 2j \\ \Delta &= \frac{j(j+1)}{k} - \frac{m^2}{4k} + \frac{s^2}{8} + \frac{m - s - 2j}{2} && \text{for } 2j \leq m - s \leq 2k - 2j - 4 \end{aligned} \quad (\text{A.7})$$

and their R -charge reads:

$$Q_R = \frac{s}{2} - \frac{m}{k} \pmod{2}. \quad (\text{A.8})$$

Chiral primary states: they are obtained for $m = 2(j+1)$ and $s = 2$ (thus odd fermion number). Their conformal dimension reads:

$$\Delta = \frac{Q_R}{2} = \frac{1}{2} - \frac{j+1}{k}. \quad (\text{A.9})$$

Anti-chiral primary states: they are obtained for $m = 2j$ and $s = 0$ (thus even fermion number). Their conformal dimension reads:

$$\Delta = -\frac{Q_R}{2} = \frac{j}{k}. \quad (\text{A.10})$$

Finally we have the following modular S-matrix for the $\mathcal{N} = 2$ minimal-model characters:

$$S_{j'm's'}^{jms} = \frac{1}{2k} \sin \pi \frac{(1+2j)(1+2j')}{k} e^{i\pi \frac{mm'}{k}} e^{-i\pi ss'/2}. \quad (\text{A.11})$$

The usual Ramond and Neveu-Schwarz characters, that we use in the bulk of the paper, are obtained as:

$$C_m^j \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau) = e^{\frac{i\pi ab}{2}} \left[C_m^{j(a)}(\nu|\tau) + (-)^b C_m^{j(a+2)}(\nu|\tau) \right], \quad (\text{A.12})$$

where $a = 0$ (resp. $a = 1$) denote the $\widehat{\text{NS}}$ (resp. $\widehat{\text{R}}$) sector, and characters with $b = 1$ are twisted by $(-)^F$. They are related to $\widehat{\mathfrak{su}(2)}_k$ characters through:

$$\chi^j(\nu|\tau) \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau) = \sum_{m \in \mathbb{Z}_{2k}} C_m^j \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau) \Theta_{m,k}(\nu|\tau). \quad (\text{A.13})$$

In terms of those one has the reflexion symmetry:

$$C_m^j \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau) = (-)^b C_{m+k}^{\frac{k}{2}-j-1} \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau). \quad (\text{A.14})$$

A.2 Supersymmetric $\text{SL}(2, \mathbb{R})/\text{U}(1)$

The characters of the $\text{SL}(2, \mathbb{R})/\text{U}(1)$ super-coset at level k come in different categories corresponding to irreducible unitary representations of $\text{SL}(2, \mathbb{R})$.

Continuous representations: they correspond to spin $J = \frac{1}{2} + ip$ and continuous momentum $p \in \mathbb{R}^+$ states. Their characters are denoted by $\text{ch}_c(\frac{1}{2} + ip, M) \begin{bmatrix} a \\ b \end{bmatrix}$, where the $\text{U}(1)_R$ charge of the primary is $Q_R = 2M/k$. They read:

$$\text{ch}_c\left(\frac{1}{2} + ip, M; \nu|\tau\right) \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\eta^3(\tau)} q^{\frac{p^2+M^2}{k}} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau) z^{\frac{2M}{k}}. \quad (\text{A.15})$$

Discrete representations: their characters $\text{ch}_d(J, r) \begin{bmatrix} a \\ b \end{bmatrix}$, have a real $\text{SL}(2, \mathbb{R})$ spin in the range $\frac{1}{2} < J < \frac{k+1}{2}$. Their $\text{U}(1)_R$ charge reads $Q_R = 2(J + r + \frac{a}{2})/k$, $r \in \mathbb{Z}$. Their characters are given by

$$\text{ch}_d(J, r; \nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} = \frac{q^{\frac{-(J-1/2)^2 + (J+r+a/2)^2}{k}} z^{\frac{2J+2r+a}{k}} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau)}{1 + (-)^b z q^{1/2+r+a/2}} \frac{1}{\eta^3(\tau)}. \quad (\text{A.16})$$

Chiral primaries: they are obtained for $r = 0$, i.e. $M = J$, in the NS sector (with even fermion number). Their conformal dimension reads

$$\Delta = \frac{Q_R}{2} = \frac{J}{k}. \quad (\text{A.17})$$

Anti-chiral primaries: they are obtained for $r = -1$ (with odd fermion number), and their conformal dimension reads

$$\Delta = -\frac{Q_R}{2} = \frac{1}{2} - \frac{J-1}{k}. \quad (\text{A.18})$$

Identity representation: this representation corresponds to the vacuum of the level k super-Liouville theory, which is both a highest and lowest weight representation and a chiral and anti-chiral primary with spin and $U(1)_R$ charge $J = 0 = Q_R$. Its characters are labeled by a discrete charge $r \in \mathbb{Z}$:

$$\text{ch}_{\text{Id}}(r; \nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} = \frac{q^{\frac{(r+\frac{a+1}{2})(r+\frac{a-1}{2})}{k} + r + \frac{a-1}{2}} z^{\frac{2(r+\frac{a}{2})}{k} + 1}}{\left(1 + (-)^b z q^{r+\frac{a+1}{2}}\right) \left(1 + (-)^b z q^{r+\frac{a-1}{2}}\right)} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau, \nu)}{\eta^3(\tau)}. \quad (\text{A.19})$$

The identity representation in $SL(2, \mathbb{R})_k/U(1)$ is non-normalisable.

Extended characters. Extended characters are defined for k integer by summing over k units of spectral flow [134].²⁷ For instance, the extended discrete characters of charge $r \in \mathbb{Z}_k$ read:

$$\begin{aligned} \text{Ch}_d(J, r; \nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} &= \sum_{w \in \mathbb{Z}} \text{ch}_d(J, r + kw; \nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \sum_{w \in \mathbb{Z}} \frac{\left(q^{kw+r+\frac{a+1}{2}} z\right)^{\frac{2J-1}{k}}}{1 + (-)^b z q^{kw+r+\frac{a+1}{2}}} q^{\frac{(kw+r+\frac{a+1}{2})^2}{k}} z^{\frac{2(kw+r+\frac{a+1}{2})}{k}} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau)}{\eta^3(\tau)} \\ &= \sum_{w \in \mathbb{Z}} \frac{q^{\frac{(kw+J+r+\frac{a}{2})^2 - (J-\frac{1}{2})^2}{k}} z^{\frac{2(kw+J+r)+a}{k}}}{1 + (-)^b z q^{kw+r+\frac{a+1}{2}}} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau)}{\eta^3(\tau)}. \end{aligned} \quad (\text{A.20})$$

and the extended continuous characters:

$$\begin{aligned} \text{Ch}_c\left(\frac{1}{2} + ip, M; \nu|\tau\right) \begin{bmatrix} a \\ b \end{bmatrix} &= \sum_{w \in \mathbb{Z}} \text{ch}_c\left(\frac{1}{2} + ip, M + kw; \nu|\tau\right) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{q^{\frac{p^2}{k}}}{\eta^3(\tau)} \Theta_{2M, k}\left(\tau, \frac{2\nu}{k}\right) \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau), \end{aligned} \quad (\text{A.21})$$

where discrete $\mathcal{N} = 2$ R-charges are chosen: $2M \in \mathbb{Z}_{2k}$.

Finally we can also define by the same procedure extended characters for the identity representation, with discrete charge $r \in \mathbb{Z}_k$:

$$\text{Ch}_{\text{Id}}(r; \nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{w \in \mathbb{Z}} \text{ch}_{\text{Id}}(r + kw; \nu|\tau) \begin{bmatrix} a \\ b \end{bmatrix} \quad (\text{A.22})$$

Extended characters close under the action of the modular group. It is worthwhile noting however that although all three kinds of extended characters separately close among

²⁷One can extend their definition to the case of rational k , which is not useful here.

themselves under a T-transformation, only continuous extended characters (A.21) do so under S-transformation:

$$\begin{aligned} & \text{Ch}_c \left(\frac{1}{2} + ip, M; -\frac{1}{\tau} \right) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{1}{2k} \int_0^\infty dp' \cos \frac{4\pi pp'}{k} \sum_{2M' \in \mathbb{Z}_{2k}} e^{-\frac{4i\pi MM'}{k}} \text{Ch}_c \left(\frac{1}{2} + ip', M'; \tau \right) \begin{bmatrix} b \\ -a \end{bmatrix}. \end{aligned} \quad (\text{A.23})$$

Extended discrete / identity characters in contrast S-transform in a more involved way into combination of extended discrete / identity characters and extended continuous characters (see [134, 135]).

Finally we mention that the characters of continuous representations in the limit $p \rightarrow 0^+$ branch into a linear combination of characters of discrete boundary representations [135]:

$$\lim_{p \rightarrow 0^+} \text{Ch}_c \left(\frac{1}{2} + ip, r + \frac{1}{2}; \nu | \tau \right) \begin{bmatrix} a \\ b \end{bmatrix} = \text{Ch}_d \left(\frac{1}{2}, r; \nu | \tau \right) \begin{bmatrix} a \\ b \end{bmatrix} + \text{Ch}_d \left(\frac{k+1}{2}, r; \nu | \tau \right) \begin{bmatrix} a \\ b \end{bmatrix},$$

$r \in \mathbb{Z}_k$. (A.24)

B $\mathcal{N} = 4$ characters

B.1 Classification of unitary representations

Unitary representations exist for discrete values of the central charge $c = 6\kappa$ ($\kappa \in \mathbb{N}$), for which the $\mathcal{N} = 4$ super-conformal algebra contains a $\widehat{\mathfrak{su}(2)}_\kappa$ affine subalgebra. The highest weight states are distinguished by their eigenvalue with respect to L_0 : h , and their spin $\widehat{\mathfrak{su}(2)}_\kappa$ I . Unitarity imposes a bound on h : namely $h \geq I$ in the NS sector and $h \geq \kappa/4$ in the R sector. Their characters are discussed in [89, 136, 137]. We summarised hereafter their distinguishing features.

Massive representations: these representations have an equal number of bosons and fermions in their ground states, and thus vanishing Witten index. Their characters are defined in terms of two parameters, ν and μ , related, respectively, to the spin I and the fermion quantum numbers of a given representation. In the following, we denote $y = e^{2\pi i \mu}$ and $z = e^{2\pi i \nu}$.

Ramond sector: in the R sector, massive representations exist for $h > \frac{\kappa}{4}$ and in the range $\frac{1}{2} \leq I \leq \frac{\kappa}{2}$:

$$\text{ch}_{\kappa, h, I}^R(\nu, \mu | \tau) = q^{h - \frac{I^2}{\kappa+1} + \frac{1}{8} - \frac{c}{24}} F^R(\nu, \mu | \tau) \chi_{\kappa-1}^{I-\frac{1}{2}}(2\nu | \tau). \quad (\text{B.1})$$

with $\chi_{\kappa-1}^I$ the bosonic $\text{SU}(2)_{\kappa-1}$ characters for I spin representation defined in eq. (A.3) and the elliptic function:

$$\begin{aligned} F^R(\nu, \mu | \tau) &= z \prod_{n=1}^{\infty} \frac{(1 + yzq^n)(1 + y^{-1}zq^n)(1 + yz^{-1}q^{n-1})(1 + y^{-1}z^{-1}q^{n-1})}{(1 - q^n)} \\ &= q^{-\frac{1}{8}} \frac{\vartheta_2(\nu + \mu | \tau) \vartheta_2(\nu - \mu | \tau)}{\eta(\tau)^3}. \end{aligned} \quad (\text{B.2})$$

Neveu-Schwarz sector: in the NS sector, we have the bound $h > I$ and the spin is defined in the range: $0 \leq I \leq \frac{\kappa-1}{2}$. The characters read:

$$\text{ch}_{\kappa,h,I}^{\text{NS}}(\nu, \mu|\tau) = q^{h - \frac{(I+1/2)^2}{\kappa+1} + \frac{1}{8} - \frac{c}{24}} F^{\text{NS}}(\nu, \mu|\tau) \chi_{\kappa-1}^I(2\nu|\tau). \quad (\text{B.3})$$

with:

$$\begin{aligned} F^{\text{NS}}(\nu, \mu|\tau) &= \prod_{n=1}^{\infty} \frac{(1 + yzq^{n-\frac{1}{2}})(1 + y^{-1}zq^{n-\frac{1}{2}})(1 + yz^{-1}q^{n-\frac{1}{2}})(1 + y^{-1}z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)} \\ &= q^{\frac{1}{8}} \frac{\vartheta_3(\nu + \mu|\tau) \vartheta_3(\nu - \mu|\tau)}{\eta(\tau)^3} \end{aligned} \quad (\text{B.4})$$

Massless representations: these representations saturate the unitary bounds: $h = \kappa/4$ for the R sector and $h = I$ for the NS sector, and preserve $\mathcal{N} = 4$ worldsheet supersymmetry. Their ground states have non-vanishing Witten index. These representations have been proposed as CFT T-dual description of (non)-compact manifolds with $c_1(\mathcal{M}) = 0$ and produce massless supergravity multiplets.

Ramond sector: in the R sector, massless representations saturate the bound $h = \frac{\kappa}{4}$ and exist in the range $0 \leq I \leq \frac{\kappa}{2}$:

$$\text{ch}_{\kappa,h,I}^{\text{R}}(\nu, \mu|\tau) = q^{h - \frac{I^2}{\kappa+1} + \frac{1}{8} - \frac{c}{24}} F^{\text{R}}(\nu, \mu|\tau) \chi_{\kappa-1}^{(\text{R})I-\frac{1}{2}}(2\nu|\tau), \quad (\text{B.5})$$

where $\chi_{\kappa}^{(\text{R})I}$ are modified $\text{SU}(2)_{\kappa}$ characters for the spin I representation, defined as follows:

$$\begin{aligned} \chi_{\kappa}^{(\text{R})I}(2\nu|\tau) &= \frac{zq^{-\frac{1}{8}}}{i\vartheta_1(2\nu|\tau)} \sum_{m \in \mathbb{Z}} q^{(\kappa+2)\left(m + \frac{2I+1}{2(\kappa+2)}\right)^2} \\ &\quad \times \left(\frac{z^{2(\kappa+2)\left(m + \frac{I}{\kappa+2}\right)}}{(1 + yzq^{-m})(1 + y^{-1}zq^{-m})} - \frac{z^{-2(\kappa+2)\left(m + \frac{I+1}{\kappa+2}\right)}}{(1 + yz^{-1}q^{-m})(1 + y^{-1}z^{-1}q^{-m})} \right) \end{aligned} \quad (\text{B.6})$$

Neveu-Schwarz sector: in the NS sector, we have the bound $h = I$ and the spin is defined in the range: $0 \leq I \leq \frac{\kappa}{2}$. The characters read:

$$\text{ch}_{\kappa,h,I}^{\text{NS}}(\nu, \mu|\tau) = q^{I - \frac{(I+1/2)^2}{\kappa+1} + \frac{1}{8} - \frac{c}{24}} F^{\text{NS}}(\nu, \mu|\tau) \chi_{\kappa-1}^{(\text{NS})I}(\tau, 2\nu), \quad (\text{B.7})$$

with the modified $SU(2)_\kappa$ characters:

$$\chi_\kappa^{(\text{NS})I}(2\nu|\tau) = \frac{zq^{-\frac{1}{8}}}{i\vartheta_1(2\nu|\tau)} \sum_{m \in \mathbb{Z}} q^{(\kappa+2)\left(m + \frac{2I+1}{2(\kappa+2)}\right)^2} \times \left(\frac{z^{2(\kappa+2)\left(m + \frac{I}{\kappa+2}\right)}}{(1 + yzq^{(m+\frac{1}{2})})(1 + y^{-1}zq^{(m+\frac{1}{2})})} - \frac{z^{-2(\kappa+2)\left(m + \frac{I+1}{\kappa+2}\right)}}{(1 + yz^{-1}q^{(m+\frac{1}{2})})(1 + y^{-1}z^{-1}q^{(m+\frac{1}{2})})} \right) \quad (\text{B.8})$$

Boundary representations: similar to what happens for $\mathcal{N}_{\text{ws}} = 2$ characters (A.24), we observe a reducibility of continuous representations when h reaches its unitary bound:

$$\lim_{h \rightarrow I^+} \text{ch}_{\kappa,h,I}^{\text{NS}}(\nu, \mu|\tau) = \text{ch}_{\kappa,I,I}^{\text{NS}}(\nu, \mu|\tau) + (y + y^{-1}) \text{ch}_{\kappa,I+\frac{1}{2},I+\frac{1}{2}}^{\text{NS}}(\nu, \mu|\tau) + \text{ch}_{\kappa,I+1,I+1}^{\text{NS}}(\nu, \mu|\tau), \quad (\text{B.9})$$

with $y = e^{2\pi i \mu}$. A similar relations holds for characters in the Ramond sector:

$$\lim_{h \rightarrow \frac{\kappa}{4}^+} \text{ch}_{\kappa,h,I}^{\text{R}}(\nu, \mu|\tau) = \text{ch}_{\kappa,\frac{\kappa}{4},I}^{\text{R}}(\nu, \mu|\tau) + (y + y^{-1}) \text{ch}_{\kappa,\frac{\kappa}{4},I-\frac{1}{2}}^{\text{R}}(\nu, \mu|\tau) + \text{ch}_{\kappa,\frac{\kappa}{4},I-1}^{\text{R}}(\nu, \mu|\tau). \quad (\text{B.10})$$

B.2 $\mathcal{N} = 4$ characters at level $\kappa = 1$, with $c = 6$

Massive representations: in this case, the spin takes only two values: $I = 0$ in the NS sector and $I = \frac{1}{2}$ in the R sector, which label representations of the $\widehat{\mathfrak{su}(2)}_1$ subalgebra characterising the $\mathcal{N}_{\text{ws}} = 4$ super-conformal algebra at level $\kappa = 1$. The corresponding characters are:

$$\begin{aligned} \text{ch}_{1,h,0}^{\text{NS}}(\nu, \mu|\tau) &= q^{h-\frac{1}{4}} F^{\text{NS}}(\nu, \mu|\tau), & h > 0. \\ \text{ch}_{1,h,\frac{1}{2}}^{\text{R}}(\nu, \mu|\tau) &= q^{h-\frac{1}{4}} F^{\text{R}}(\nu, \mu|\tau), & h > \frac{1}{4}. \end{aligned} \quad (\text{B.11})$$

In particular, when setting $\mu = 0$ we have:

$$\begin{aligned} \text{ch}_{1,h,0}^{\text{NS}}(\nu|\tau) &= q^{h-\frac{1}{8}} \frac{\vartheta_3(\nu|\tau)^2}{\eta(\tau)^3}, & \text{ch}_{1,h,\frac{1}{2}}^{\text{R}}(\nu|\tau) &= q^{h-\frac{3}{8}} \frac{\vartheta_2(\nu|\tau)^2}{\eta(\tau)^3}, \\ \text{ch}_{1,h,0}^{\widetilde{\text{NS}}}(\nu|\tau) &= q^{h-\frac{1}{8}} \frac{\vartheta_4(\nu|\tau)^2}{\eta(\tau)^3}, & \text{ch}_{1,h,\frac{1}{2}}^{\widetilde{\text{R}}}(\nu|\tau) &= -q^{h-\frac{3}{8}} \frac{\vartheta_1(\nu|\tau)^2}{\eta(\tau)^3}. \end{aligned} \quad (\text{B.12})$$

Massless representations: the spin has two values $I = 0, \frac{1}{2}$ for both the R and NS sector. For $\mu = 0$, the $\mathcal{N} = 4$ characters simplify considerably.

Normalisable states:

$$\begin{aligned} \text{ch}_{1,\frac{1}{2},\frac{1}{2}}^{\text{NS}}(\nu|\tau) &= \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}(n-\frac{1}{2})(n+\frac{1}{2})} z^n}{1 + zq^{n-\frac{1}{2}}} \frac{\vartheta_3(\nu|\tau)}{\eta(\tau)^3}, \\ \text{ch}_{1,\frac{1}{4},0}^{\text{R}}(\nu|\tau) &= \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}n(n+1)} z^{n+\frac{1}{2}}}{1 + zq^n} \frac{\vartheta_2(\nu|\tau)}{\eta(\tau)^3}, \\ \text{ch}_{1,\frac{1}{2},\frac{1}{2}}^{\widetilde{\text{NS}}}(\nu|\tau) &= \sum_{n \in \mathbb{Z}} (-)^n \frac{q^{\frac{1}{2}(n-\frac{1}{2})(n+\frac{1}{2})} z^n}{1 - zq^{n-\frac{1}{2}}} \frac{\vartheta_4(\nu|\tau)}{\eta(\tau)^3}, \\ \text{ch}_{1,\frac{1}{4},0}^{\widetilde{\text{R}}}(\nu|\tau) &= - \sum_{n \in \mathbb{Z}} (-)^n \frac{q^{\frac{1}{2}n(n+1)} z^{n+\frac{1}{2}}}{1 - zq^n} \frac{i\vartheta_1(\nu|\tau)}{\eta(\tau)^3}. \end{aligned} \quad (\text{B.13})$$

Non-normalisable states:

$$\begin{aligned}
 \text{ch}_{1,0,0}^{\text{NS}}(\nu|\tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})(n+\frac{1}{2})} z^n \frac{zq^{n-\frac{1}{2}} - 1}{zq^{n-\frac{1}{2}} + 1} \frac{\vartheta_3(\tau, \nu)}{\eta(\tau)^3}, \\
 \widetilde{\text{ch}}_{1,0,0}^{\text{NS}}(\nu|\tau) &= \sum_{n \in \mathbb{Z}} (-)^n q^{\frac{1}{2}(n-\frac{1}{2})(n+\frac{1}{2})} z^n \frac{zq^{n-\frac{1}{2}} + 1}{zq^{n-\frac{1}{2}} - 1} \frac{\vartheta_4(\tau, \nu)}{\eta(\tau)^3}, \\
 \text{ch}_{1,\frac{1}{4},\frac{1}{2}}^{\text{R}}(\nu|\tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n(n+1)} z^{n+\frac{1}{2}} \frac{zq^n - 1}{zq^n + 1} \frac{\vartheta_2(\tau, \nu)}{\eta(\tau)^3}, \\
 \widetilde{\text{ch}}_{1,\frac{1}{4},\frac{1}{2}}^{\text{R}}(\nu|\tau) &= - \sum_{n \in \mathbb{Z}} (-)^n q^{\frac{1}{2}n(n+1)} z^{n+\frac{1}{2}} \frac{zq^n + 1}{zq^n - 1} \frac{i\vartheta_1(\tau, \nu)}{\eta(\tau)^3}.
 \end{aligned} \tag{B.14}$$

C Some useful material on modular forms

Jacobi ϑ functions.

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-\frac{a}{2})^2} e^{2\pi i(n-\frac{a}{2})(\nu-\frac{b}{2})}, \quad a, b \in \mathbb{R} \tag{C.1}$$

with $q = e^{2\pi i\tau}$. In the Jacobi-Erderlyi notation one has: $\vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vartheta_1$, $\vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vartheta_2$, $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vartheta_3$ and $\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vartheta_4$.

Their modular transformations read:

$$\begin{aligned}
 \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|\tau + 1) &= e^{-\frac{i\pi}{4}a(a-2)} \vartheta \begin{bmatrix} a \\ a+b-1 \end{bmatrix} (\nu|\tau), \\
 \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \left(\frac{\nu}{\tau} - \frac{1}{\tau} \right) &= \sqrt{-i\tau} e^{\frac{i\pi}{2}ab} e^{i\pi \frac{\nu^2}{\tau}} \vartheta \begin{bmatrix} b \\ -a \end{bmatrix} (\nu|\tau).
 \end{aligned} \tag{C.2}$$

Some useful identities:

$$\vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau) = 2\eta^3(\tau). \tag{C.3}$$

$$\text{Jacobi identity:} \quad \vartheta_1^4(\nu|\tau) - \vartheta_2^4(\nu|\tau) + \vartheta_3^4(\nu|\tau) - \vartheta_4^4(\nu|\tau) = 0. \tag{C.4}$$

$$\vartheta_2(0|\tau)^{12} - \vartheta_3(0|\tau)^{12} + \vartheta_4(0|\tau)^{12} = -48\eta(\tau)^{12} \tag{C.5}$$

We now give explicit expressions for the derivatives of the theta-functions with respect to the variable ν .

- *First derivatives in ν :*

$$\partial_\nu \vartheta_1(\nu|\tau)|_{\nu=0} = \vartheta_1'(\tau) = 2\pi\eta^3(\tau) \tag{C.6}$$

Also

$$\begin{aligned}
 \frac{1}{\pi} \frac{\partial}{\partial \nu} \left(\frac{\vartheta_1(\nu|\tau)}{\vartheta_4(\nu|\tau)} \right) &= \vartheta_4(0|\tau)^2 \frac{\vartheta_2(\nu|\tau)\vartheta_3(\nu|\tau)}{\vartheta_4(\nu|\tau)^2}, \\
 \frac{1}{\pi} \frac{\partial}{\partial \nu} \left(\frac{\vartheta_2(\nu|\tau)}{\vartheta_4(\nu|\tau)} \right) &= -\vartheta_3(0|\tau)^2 \frac{\vartheta_1(\nu|\tau)\vartheta_3(\nu|\tau)}{\vartheta_4(\nu|\tau)^2}, \\
 \frac{1}{\pi} \frac{\partial}{\partial \nu} \left(\frac{\vartheta_3(\nu|\tau)}{\vartheta_4(\nu|\tau)} \right) &= -\vartheta_2(0|\tau)^2 \frac{\vartheta_1(\nu|\tau)\vartheta_2(\nu|\tau)}{\vartheta_4(\nu|\tau)^2}.
 \end{aligned} \tag{C.7}$$

From which we deduce:

$$\begin{aligned}\frac{1}{\pi} \frac{\partial}{\partial \nu} \left(\frac{\vartheta_1(\nu|\tau)}{\vartheta_2(\nu|\tau)} \right) &= \vartheta_2(0|\tau)^2 \frac{\vartheta_3(\nu|\tau)\vartheta_4(\nu|\tau)}{\vartheta_2(\nu|\tau)^2}, \\ \frac{1}{\pi} \frac{\partial}{\partial \nu} \left(\frac{\vartheta_1(\nu|\tau)}{\vartheta_3(\nu|\tau)} \right) &= \vartheta_3(0|\tau)^2 \frac{\vartheta_2(\nu|\tau)\vartheta_4(\nu|\tau)}{\vartheta_3(\nu|\tau)^2}.\end{aligned}\tag{C.8}$$

• *Second derivatives in ν :*

$$\begin{aligned}\vartheta_2''(\tau) &= 4\pi i \partial_\tau \vartheta_2(\tau) = -\frac{\pi^2}{3} (\hat{E}_2 + \vartheta_3^4 + \vartheta_4^4), \\ \vartheta_3''(\tau) &= 4\pi i \partial_\tau \vartheta_3(\tau) = -\frac{\pi^2}{3} (\hat{E}_2 + \vartheta_2^4 - \vartheta_4^4), \\ \vartheta_4''(\tau) &= 4\pi i \partial_\tau \vartheta_4(\tau) = -\frac{\pi^2}{3} (\hat{E}_2 - \vartheta_2^4 - \vartheta_3^4).\end{aligned}\tag{C.9}$$

Eisenstein series. An example of weight $2k > 2$ holomorphic modular forms is given by the Eisenstein series:

$$E_{2k}(\tau) = -\frac{(2k)!}{(2\pi i)^{2k} B_{2k}} \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^{2k}},\tag{C.10}$$

where B_{2k} are the Bernoulli numbers. The holomorphic Eisenstein series E_2 diverges and is quasi-modular under S -transformation, since it is alternatively given by the following first derivative:

$$E_2(\tau) = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.\tag{C.11}$$

It can nonetheless be regularised by a non-holomorphic deformation:

$$\hat{E}_2(\tau) = \frac{3}{\pi^2} \lim_{s \rightarrow 0} \sum_{(m,n) \neq (0,0)} \frac{1}{|m\tau + n|^s (m\tau + n)^2}.\tag{C.12}$$

In the language of Mock modular forms, this corresponds to a non-holomorphic completion of E_2 into the weight 2 Maaß form $\hat{E}_2$, whose shadow function $g(\tau) = 12/\pi$ is a constant:

$$\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2}.\tag{C.13}$$

One can express the Eisenstein series in terms of Jacobi functions:

$$\begin{aligned}E_4 &= \frac{1}{2}(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \\ E_6 &= \frac{1}{2}(\vartheta_2^4 + \vartheta_3^4)(\vartheta_3^4 + \vartheta_4^4)(\vartheta_4^4 - \vartheta_2^4) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}, \\ E_8 &\doteq E_4^2 = \frac{1}{2}(\vartheta_2^{16} + \vartheta_3^{16} + \vartheta_4^{16}) = 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n}, \\ E_{10} &\doteq E_4 E_6 = -\frac{1}{2}(\vartheta_2^{16}(\vartheta_3^4 + \vartheta_4^4) + \vartheta_3^{16}(\vartheta_2^4 - \vartheta_4^4) - \vartheta_4^{16}(\vartheta_2^4 + \vartheta_3^4)) = 1 - 264 \sum_{n=1}^{\infty} \frac{n^9 q^n}{1-q^n}.\end{aligned}\tag{C.14}$$

Finally, the unique weight 12 cusp form and the Klein invariant are:

$$\eta^{24} = \frac{E_4^3 - E_6^2}{26 \cdot 3^3} = q - 24q^2 + 252q^3 + \dots, \quad j = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + 196884q + \dots.\tag{C.15}$$

Modular covariant derivative. We define the covariant derivative D_r which maps a weight r modular form Φ_r to a weight $r + 2$ modular form as:

$$D_r \Phi_r(\tau) \doteq \left(\frac{i}{\pi} \frac{\partial}{\partial \tau} + \frac{r}{2\pi\tau_2} \right) \Phi_r(\tau), \quad (\text{C.16})$$

and satisfies a Leibniz rule: $D_{r+s}(\Phi_r \Phi_s) = \Phi_s D_r \Phi_r + \Phi_r D_s \Phi_s$, for $\Psi_{r+s} = \Phi_r \Phi_s$ a modular form of weight $r + s$. In particular we have:

$$D_2 \hat{E}_2 = \frac{1}{6} (E_4 - \hat{E}_2^2), \quad D_4 E_4 = \frac{2}{3} (E_6 - \hat{E}_2 E_4), \quad D_6 E_6 = E_4^2 - \hat{E}_2 E_6. \quad (\text{C.17})$$

Combining the above:

$$D_8 E_8 = \frac{4}{3} (E_{10} - \hat{E}_2 E_8), \quad D_{10} E_{10} = \frac{1}{3} (2E_6^2 + 3E_4^3 - 5\hat{E}_2 E_{10}), \quad D_{\frac{\alpha}{2}} \eta^\alpha = -\frac{\alpha}{12} \hat{E}_2 \eta^\alpha. \quad (\text{C.18})$$

Then, using the last of the above expressions:

$$D_{-2}((\sqrt{\tau_2} \bar{\eta})^3 \eta^{-1}) = -\frac{(\sqrt{\tau_2} \bar{\eta})^3 \hat{E}_2}{12\eta}, \quad \text{since } D_{-\frac{3}{2}}(\sqrt{\tau_2} \bar{\eta})^3 = 0. \quad (\text{C.19})$$

Also applying the Cauchy-Riemann operator the weight $(-\frac{3}{2}, 0)$ modular form $(\sqrt{\tau_2} \bar{\eta})^3$ we get:

$$\begin{aligned} \bar{\partial}(\sqrt{\tau_2} \bar{\eta})^3 &= \bar{\partial}(\sqrt{\tau_2}^3) \bar{\eta}^3 + \sqrt{\tau_2}^3 \bar{\partial} \bar{\eta}^3 \\ &= \frac{3i}{4} \sqrt{\tau_2} \bar{\eta}^3 + i\pi \sqrt{\tau_2}^3 \left(\bar{D}_{\frac{3}{2}} \bar{\eta}^3 - \frac{3\bar{\eta}^3}{4\pi\tau_2} \right) \\ &= -\frac{i\pi}{4} (\sqrt{\tau_2} \bar{\eta})^3 \bar{\hat{E}}_2, \end{aligned} \quad (\text{C.20})$$

which is a weight $(-\frac{3}{2}, 2)$ modular form.

D Elliptic genus of the $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ cft

We summarise here the computation of the elliptic genus for the $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ Kazama-Suzuki model (or equivalently $\mathcal{N} = (2, 2)$ super-Liouville theory), that was done in the work [46]. This elliptic genus is defined as usual by the trace:

$$\hat{\mathcal{Z}}_k(\nu|\tau) = \text{Tr}_{\mathcal{H}_R \otimes \mathcal{H}_{\bar{R}}} \left(e^{i\pi F} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} z^{J^R} \right). \quad (\text{D.1})$$

where the trace is over the Ramond sector of the Hilbert space weighted by the worldsheet fermion number operator $F = J_R + J_{\bar{R}}$, defined from both left- and right-U(1) R-charge currents of the theory.

For a non-compact CFT, we expect the elliptic genus to receive contribution from both localised and non-localised states, and we split (D.1) accordingly

$$\hat{\mathcal{Z}}_k(\nu|\tau) = \mathcal{Z}_k^d(\nu|\tau) + \mathcal{Z}_k^c(\nu|\tau) \quad (\text{D.2})$$

where again c and d refer to continuous and discrete $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ representations.

Discrete representations. The contribution of discrete representation can be straightforwardly computed either by a free field calculation or by the algebraic method used in the bulk of this work. By this latter method, we obtain the result by summing all *extended* discrete characters (as all spectrally flowed Hilbert spaces must be taken into account for consistency) in the twisted Ramond ground state with $r = -1$ over all possible spin values $1/2 \leq J \leq k/2$.²⁸

$$\mathcal{Z}_k^d(\nu|\tau) = \sum_{2J=1}^k \text{Ch}_d(J, -1; \tau, \nu) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\mathcal{K}_{2k} \left(0, \frac{\nu}{k} | \tau \right) \frac{i\vartheta_1(\nu|\tau)}{\eta(\tau)^3}. \quad (\text{D.3})$$

Note that because of supersymmetry, $\mathcal{Z}_k^d$ only depends on left-moving states. We have also repackaged the result into the higher level Appell function, as is usually done [90]. This function is define for $\tau \in \mathcal{H}$ and $u, v \in \mathbb{C}$ with $u + v \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$:

$$\mathcal{K}_K(u, v|\tau) = \sum_{n \in \mathbb{Z}} \frac{q^{\frac{K}{2}n^2} b^{Kn}}{1 - a b q^n}, \quad \text{with } a = e^{2\pi i u}, \quad b = e^{2\pi i v}. \quad (\text{D.4})$$

By resorting to the following identity for geometric series:

$$\frac{1}{1 - z^{\frac{1}{k}} q^m} = \sum_{n=1}^{k-1} \frac{(z q^{km})^{\frac{n}{k}}}{1 - z q^{km}}. \quad (\text{D.5})$$

we can further cast this result into the well known Appell-Lerch sum of level $2k$ seen previously (3.31)

$$\mathcal{Z}_k^d(\nu|\tau) = -z^{-1} A_{2k} \left(\frac{\nu}{k}, 2\nu - k\tau | \tau \right) \frac{i\vartheta_1(\tau, \nu)}{\eta(\tau)^3}. \quad (\text{D.6})$$

At this stage, it becomes quite obvious, from the mathematical perspective, that any proper derivation of the elliptic genus (D.1) necessary leads to completing (D.6) into a Maaß form. This computation has been carried out [46] and will be briefly presented in the following.

Continuous representations: as shown [46], we can reformulate the elliptic genus (D.1) in terms of a path integral, consisting in a Ray-Singer torsion, a twisted fermion partition functions together with twisted bosonic zero modes. This path integral reads, in both Lagrangian and (after Poisson resummation) Hamiltonian formulation:

$$\begin{aligned} \widehat{\mathcal{Z}}_k(\nu|\tau) &= \sum_{m,w} \int_0^1 ds_1 \int_0^1 ds_2 \frac{\vartheta_1 \left(\tau, s_1\tau + s_2 - \frac{k+1}{k}\nu \right)}{\vartheta_1 \left(\tau, s_1\tau + s_2 - \frac{1}{k}\nu \right)} e^{-\frac{\pi}{k\tau_2} |(m+ks_2)+(w+ks_1)\tau|^2} z^{\frac{n}{k}} \\ &= \sqrt{k\tau_2} \sum_{m,w} \int_0^1 ds_1 \int_0^1 ds_2 \frac{\vartheta_1 \left(\tau, s_1\tau + s_2 - \frac{k+1}{k}\nu \right)}{\vartheta_1 \left(\tau, s_1\tau + s_2 - \frac{1}{k}\nu \right)} q^{\frac{1}{4k}(kw-n-ks_1)^2} \bar{q}^{\frac{1}{4k}(kw+m+ks_1)^2} z^{\frac{m}{k}} e^{-2\pi i k s_2 w} \end{aligned} \quad (\text{D.7})$$

In particular, the twist in the fermion partition function not only depends on the R -charge but is also due the holonomies s_i , $i = 1, 2$ of the gauge fields on the torus, which shift

²⁸Note that boundary representations must be included: we choose here to include the $J = \frac{1}{2}$ representation with weight 1 which is equivalent to summing over both $J = \{\frac{1}{2}, \frac{k+1}{2}\}$ with weights $\frac{1}{2}$.

the left- and right-moving momenta $(n - kw)/\sqrt{2k}$ and $(n + kw)/\sqrt{2k}$. The bosonic zero modes too are twisted by these holonomies and thereby couple to the oscillators.

At non-zero ν the path integral (D.1) exhibits poles in the ϑ -function in the denominator which are not cancelled by zeros in the numerator, due to the infinite volume divergence of target-space. This divergence was regularised in holomorphic but non-modular invariant way in the partition function (2.24), in order to recover an expression which is interpretable in terms of discrete and continuous extended characters of $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$. Following [46], the opposite choice will be made here, which is more natural from the elliptic genus perspective, as we expect this index to transform as Jacobi form.

This regularisation procedure first assumes the range $|q| < |q^{s_1} z^{-\frac{1}{k}}|$ and $|z| \sim 1$, for which we can disentangle the contribution from discrete representations from that from states with continuous momenta. Ref. [46] then shows how the path integral (D.7) splits into two pieces, the first one exactly reproducing the localised contribution (D.3). By this token, we can identify the remainder as the contribution from continuous representations in the following way. After redefining the right moment $n = m + kw$ and introducing a continuous momentum variable p to linearise the dependence in s_1 , we obtain:

$$\mathcal{Z}_k^c(\nu|\tau) \equiv -\frac{i\vartheta_1(\nu|\tau)}{\pi\eta^3} \sum_{n,w} q^{kw^2-nw} z^{-2w+\frac{n}{k}} \int_{\mathbb{R}-i\varepsilon} \frac{dp}{2ip+n} |q|^{\frac{2p^2}{k}+\frac{n^2}{2k}}. \quad (\text{D.8})$$

In this expression, the right-moving fermionic and bosonic oscillators have cancelled leaving a measure over the total right momentum. The integral in p over a continuum of states, weighted by the $\text{U}(1)_R$ -charge and conformal weights, is evidence showing that the above expressions originates from non-localised states in the spectrum. To support this claim, $\mathcal{Z}_k^c$ can indeed be rephrased in terms of a combination of extended *left*-moving continuous $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ characters:

$$\mathcal{Z}_k^c(\nu|\tau) = -\frac{i}{\pi} \sum_n \int_{\mathbb{R}-i\varepsilon} \frac{dp}{2ip+n} \text{Ch}_c\left(\frac{1}{2}+ip, \frac{n}{2}; \nu|\tau\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{q}^{\frac{p^2}{k}+\frac{n^2}{4k}}. \quad (\text{D.9})$$

Contrary to what happens in the partition function (2.24) for example, the sum over the left $\text{U}(1)_R$ charges labeling these continuous representations is now weighted differently by *right*-moving bosonic zero modes with continuous momentum, this particular asymmetric structure clearly resulting from the non-holomorphic nature of $\mathcal{Z}_k^c$.

To make final contact with non-holomorphic Appell-Lech sums (3.46), we compute explicitly the integral over continuous momenta:

$$\int_{\mathbb{R}-i\varepsilon} \frac{ds}{2ip+n} |q|^{\frac{2p^2}{k}+\frac{n^2}{2k}} = -\frac{\pi}{2} \left(\text{sgn}(n+\varepsilon) - E\left(n\sqrt{\frac{\tau_2}{k}}\right) \right), \quad (\text{D.10})$$

the first term on the RHS of the equation being the contribution from continuous representations with momentum $p \rightarrow 0$.

Plugging expression (D.10) back in (D.8), we recover the non-holomorphic completion (3.46) for the level $2k$ Appell-Lerch sum (D.6). By Putting together the contributions from both discrete (D.6) and continuous (D.8) representations and by further using the elliptic transformations (3.48), we find the neat expression for the elliptic genus of the level

k super-Liouville theory:

$$\widehat{\mathcal{Z}}_k(\nu|\tau) = -\hat{A}_{2k} \left(\frac{\nu}{k}, 2\nu|\tau \right) \frac{i\theta_1(\tau, \nu)}{\eta(\tau)^3}. \quad (\text{D.11})$$

Furthermore, a glance at the modular and elliptic properties of the non-holomorphic Appell-Lerch sums (3.47) and (3.48) shows that the elliptic genus (D.11) transforms as a Jacobi form of weight 0 and index $\frac{k^2 c}{6} = \frac{k(k+2)}{2}$:

$$\widehat{\mathcal{Z}}_k \left(\frac{\nu}{d\tau + e} \middle| \frac{a\tau + b}{d\tau + e} \right) = e^{2\pi i \frac{d}{d\tau + e} \frac{c}{6} \nu^2} \widehat{\mathcal{Z}}_k(\nu|\tau), \quad \text{for } \begin{pmatrix} a & b \\ d & e \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad (\text{D.12})$$

$$\widehat{\mathcal{Z}}_k(\nu + \lambda\tau + \mu|\tau) = e^{\pi i \frac{c}{3}(\lambda + \mu)} q^{-\frac{c}{6}\lambda^2} z^{-\frac{c}{3}\lambda} \widehat{\mathcal{Z}}_k(\nu|\tau), \quad \text{for } \lambda, \mu \in k\mathbb{Z}. \quad (\text{D.13})$$

In particular, the modular transformations of $\widehat{\mathcal{Z}}_k$ are those expected from the boundary conditions on the path integral and the factorisation of the $\text{U}(1)_R$ current algebra. Also, the fact that the elliptic transformations (D.13) hold for $\mu \in k\mathbb{Z}$ is a consequence of $1/k$ quantisation of the $\text{U}(1)_R$ charges in the NS sector of the theory, while the restriction $\lambda \in k\mathbb{Z}$ is related to the expression of the elliptic genus in terms of extended $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ characters, which are constructed by summing over k units of spectral flow (see appendix A.2). This explains why the index of this non-holomorphic Jacobi form is in fact $k^2 c/6$ rather than $c/6$ as one would naively think from the transformations (D.12)–(D.13).

E Details of $\text{SO}(28)$ and $\text{U}(1)_R$ threshold calculations for $\mathcal{Q}_5 = k/2$

E.1 Bulk state contributions to the $\text{SO}(28)$ threshold corrections

We consider the contribution to the $\text{SO}(28)$ threshold corrections coming from continuous $\text{SL}(2, \mathbb{R})_k/\text{U}(1)$ representations, cf. eq. (4.16):

$$\mathcal{R}^c[\mathcal{Q}_5] = \frac{1}{16} \sum_{J=1}^{k/2} (\chi_{k-2}^{J-1} + \chi_{k-2}^{\frac{k}{2}-J}) \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} q^{-\frac{1}{k}(J - \frac{\ell+1}{2})^2} R\left(\left(\frac{\ell+1}{2} - J\right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau\right) \frac{D_8 E_8}{\eta^{21}}. \quad (\text{E.1})$$

In the following we demonstrate how to derive the second line of formula (4.16). For $\ell \in 2\mathbb{N} + 1$ and $J = 1, \dots, \frac{k}{2}$, we have:

$$\begin{aligned} & \frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} q^{-\frac{1}{k}(J - \frac{\ell+1}{2})^2} R\left(\left(\frac{\ell+1}{2} - J\right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau\right) \\ &= \sum_{n \in \mathbb{Z}} (-)^n \left[\frac{1}{\ell} \sum_{m=0}^{\ell-1} e^{-2\pi i (n+1) \frac{m}{\ell}} \right] \left(\text{sgn}\left(n + \frac{1}{2}\right) - \right. \\ & \quad \left. - E\left(\left[n + 1 + \frac{1}{2\ell}(1 - 2J)\right] \sqrt{2\tau_2}\right) \right) q^{-\frac{1}{2}(n+1 + \frac{1}{2\ell}(1-2J))^2} \\ &= \sum_{n \in \mathbb{Z}} (-)^n \sum_{r \in \mathbb{Z}} \delta_{n, -1 \bmod \ell} \left(\text{sgn}\left(n + \frac{1}{2}\right) - \right. \\ & \quad \left. - E\left(\left[n + 1 + \frac{1}{2\ell}(1 - 2J)\right] \sqrt{2\tau_2}\right) \right) q^{-\frac{1}{2}(n+1 + \frac{1}{2\ell}(1-2J))^2} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{r \in \mathbb{Z}} \underbrace{\left(\operatorname{sgn} \left(r\ell - \frac{1}{2} \right) - \operatorname{sgn} \left(r\ell + \frac{1}{2\ell}(1-2J) \right) \right)}_{=0} + \\
 &\quad + (-)^{\ell r-1} \operatorname{sgn} \left(r\ell + \frac{1}{2\ell}(1-2J) \right) \operatorname{erfc} \left(\left| r\ell + \frac{1}{2\ell}(1-2J) \right| \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{2}(\ell r + \frac{1}{2\ell}(1-2J))^2} \\
 &= \sum_{r \in \mathbb{Z}} (-)^{r+1} \operatorname{sgn} \left(r - \frac{1}{2} \right) \operatorname{erfc} \left(\frac{1}{2\ell} |kr + 1 - 2J| \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{4k}(kr+1-2J)^2} \\
 &= \sum_{n \in \mathbb{Z}} (-)^n \operatorname{sgn} \left(n + \frac{1}{2} \right) \operatorname{erfc} \left(\frac{1}{2\ell} |kn + 1 + k - 2J| \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{4k}(kn+1+k-2J)^2}.
 \end{aligned} \tag{E.2a}$$

This is the expression appearing on the second line of eq. (4.16).

E.2 The $U(1)_R$ threshold corrections

We give the details of the computation of the second derivative

$$\partial_\nu^2 \hat{\mu} \left(\frac{\nu}{\ell} + \left(\frac{\ell a + 1}{2} - J \right) \frac{\tau}{\ell} + \frac{b-1}{2} + \frac{m}{\ell}, \frac{\nu}{\ell} + \frac{(a-1)\tau + (b-1)}{2} \middle| \tau \right) \Big|_{\nu=0} \tag{E.3}$$

appearing in expression (4.47). First we use relation (3.45), to compute the directional derivative for $u, v \notin \mathbb{Z}\tau + \mathbb{Z}$:

$$\lim_{\varepsilon \rightarrow 0} \frac{\hat{\mu}(u + \varepsilon, v + \varepsilon) - \hat{\mu}(u, v)}{\varepsilon} = -\frac{\eta^3 \vartheta_1(u+v)}{\vartheta_1^2(u) \vartheta_1^2(v)} \vartheta_1'(0) = -\frac{2\pi \eta^6 \vartheta_1(u+v)}{\vartheta_1^2(u) \vartheta_1^2(v)}. \tag{E.4}$$

For $(a, b) \neq (1, 1)$, we use (E.4) to compute:

$$\begin{aligned}
 &\frac{\partial^2}{\partial \nu^2} \hat{\mu} \left(\frac{\nu}{\ell} + \left(\frac{\ell a + 1}{2} - J \right) \frac{\tau}{\ell} + \frac{b-1}{2} + \frac{m}{\ell}, \frac{\nu}{\ell} + \frac{(a-1)\tau + (b-1)}{2} \right) \Big|_{\nu=0} \\
 &= -\frac{2\pi \eta^6}{\ell^2} \frac{\partial}{\partial u} \left[\frac{\vartheta_1(2u + (\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell} + (a-1)\tau + (b-1))}{\vartheta_1^2(u + (\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell} + \frac{a-1}{2}\tau + \frac{b-1}{2}) \vartheta_1^2(u + \frac{a-1}{2}\tau + \frac{b-1}{2})} \right] \Big|_{u=0} \\
 &= -\frac{2\pi \eta^6}{\ell^2} (-)^{a+b} \frac{\partial}{\partial u} \left[\frac{\vartheta_1(2u + (\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell})}{\vartheta^2[\frac{a}{b}](u + (\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell}) \vartheta^2[\frac{a}{b}](u)} \right] \Big|_{u=0} \\
 &= -\frac{2\pi \eta^6}{\ell^2} (-)^{a+b} \frac{1}{\vartheta^2[\frac{a}{b}](0)} \frac{\partial}{\partial u} \left[\frac{\vartheta_1(2u + (\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell})}{\vartheta^2[\frac{a}{b}](u + (\frac{\ell+1}{2} - J) \frac{\tau}{\ell} + \frac{m}{\ell})} \right] \Big|_{u=0}.
 \end{aligned} \tag{E.5}$$

In the last line we used the fact that $\vartheta_i'(0) = 0$ for $i = 2, 3, 4$.

Notice that for $\delta \notin (\frac{a-1}{2} + \mathbb{Z})\tau + \frac{b-1}{2} + \mathbb{Z}$, we have:

$$\frac{\partial}{\partial u} \left[\frac{\vartheta_1(2u + \delta)}{\vartheta[\frac{a}{b}]^2(u + \delta)} \right] \Big|_{u=0} = \frac{2}{\vartheta[\frac{a}{b}](\delta)} \frac{\partial}{\partial v} \left[\frac{\vartheta_1(v)}{\vartheta[\frac{a}{b}](v)} \right] \Big|_{v=\delta} = \frac{2\pi \vartheta^2[\frac{a}{b}](0) \vartheta[\frac{a+1}{0}](\delta) \vartheta[\frac{0}{b+1}](\delta)}{\vartheta[\frac{a}{b}]^3(\delta)}. \tag{E.6}$$

To obtain the final identity we used (C.7) and (C.8).

Plugging (E.6) into (E.5), we get for $(a, b) \neq (1, 1)$:

$$\begin{aligned} \frac{\partial^2}{\partial \nu^2} \hat{\mu} \left(\frac{\nu}{\ell} + \left(\frac{\ell a + 1}{2} - J \right) \frac{\tau}{\ell} + \frac{b-1}{2} + \frac{m}{\ell}, \frac{\nu}{\ell} + \frac{(a-1)\tau + (b-1)}{2} \middle| \tau \right) \bigg|_{\nu=0} \\ = -\frac{8\pi\eta^6}{k} (-)^{a+b} \frac{\vartheta \left[\begin{smallmatrix} a+1 \\ 0 \end{smallmatrix} \right] \left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right) \vartheta \left[\begin{smallmatrix} 0 \\ b+1 \end{smallmatrix} \right] \left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right)}{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]^3 \left(\left(\frac{\ell+1}{2} - J \right) \frac{\tau}{\ell} + \frac{m}{\ell} \middle| \tau \right)}, \end{aligned} \quad (\text{E.7})$$

which is the expression appearing on the second line of (4.47).

F SO(28) threshold corrections for $\mathcal{Q}_5 = 1$

Hereafter, we give the detailed evaluation of the threshold correction (4.42):

$$\Lambda_{\text{SO}(28)}[1] = \frac{1}{8} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U) \hat{\mathcal{A}}_{\text{SO}(28)} \quad (\text{F.1})$$

using the orbit method.

F.1 Zero orbit

The zero orbit contribution reads:

$$\Lambda_{\text{SO}(28)} = \frac{T_2}{8} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \hat{\mathcal{A}}_{\text{SO}(28)}. \quad (\text{F.2})$$

To evaluate it, we split the modified elliptic genus (4.42)

$$\hat{\mathcal{A}}_{\text{SO}(28)} = \frac{1}{6} (\hat{\mathcal{A}}_{\text{K3}}[4] + \hat{\mathcal{A}}_{\text{flux}}), \quad (\text{F.3})$$

in terms of the modified elliptic genus of $K3$ (3.16) and the flux contribution:

$$\hat{\mathcal{A}}_{\text{K3}}[t] = -\frac{1}{12\eta^{24}} \left[\hat{E}_2 E_{10} - \frac{12+t}{24} E_6^2 - \frac{12-t}{24} E_4^3 \right], \quad \hat{\mathcal{A}}_{\text{flux}} = \frac{\hat{F}(\hat{E}_2 E_8 - E_{10})}{12\eta^{21}}, \quad (\text{F.4})$$

both of which are separately modular invariant.

Stokes theorem for modular integrals: to compute integrals of the type:

$$I_r(\Phi) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (\hat{E}_2)^r \Phi(\tau) \quad (\text{F.5})$$

for Φ a holomorphic modular form of weight $w = -2r$, we use the fact that:

$$\partial_{\bar{\tau}} \hat{E}_2 = \frac{3i}{2\pi\tau_2^2} \quad (\text{F.6})$$

and that the measure on $\mathcal{H}$ can be rewritten as $d\tau_1 \wedge d\tau_2 = \frac{1}{2i} d\tau \wedge d\bar{\tau}$, to recast the integral as follows [138]:

$$\begin{aligned} I_r(\Phi) &= -\frac{\pi}{3(r+1)} \int_{\mathcal{F}} d\tau d\bar{\tau} \partial_{\bar{\tau}} ((\hat{E}_2)^{r+1} \Phi) = -\frac{\pi}{3(r+1)} \lim_{w \rightarrow \infty} \int_{\mathcal{F}_w} d((\hat{E}_2)^{r+1} \Phi d\tau) \\ &= \frac{\pi}{3(r+1)} \lim_{w \rightarrow \infty} \int_{\tau_1 = -\frac{1}{2} + iw}^{\tau_1 = \frac{1}{2} + iw} (\hat{E}_2)^{r+1} \Phi = \frac{\pi}{3(r+1)} \lim_{\tau_2 \rightarrow \infty} (\hat{E}_2)^{r+1} \Phi|_{q^0} \\ &= \frac{\pi}{3(r+1)} (\text{constant term of } (E_2)^{r+1} \Phi) \end{aligned} \quad (\text{F.7})$$

where we have used Stoke's theorem in the first line, and we have introduced a cutoff on the fundamental domain: $\mathcal{F}_w = \{|\tau_1| \leq \frac{1}{2}, |\tau| \geq 1, 0 \leq \tau_2 \leq w\}$.

For instance, for a holomorphic modular form with expansion $Q(\tau) = \sum_{n=-1}^{\infty} c_n q^n$ we have:

$$I_r(Q) = \frac{\pi}{3(r+1)} (c_0 - 24(r+1)c_{-1}). \quad (\text{F.8})$$

K3 contribution. To compute the modular integral (F.2) for the K3 contribution (F.4), we rewrite:

$$\hat{\mathcal{A}}_{K3}[4] = -\frac{1}{12} (\hat{E}_2 Q_1 - Q_2) \quad (\text{F.9})$$

in terms of modular forms with at most a pole of order one:

$$\begin{aligned} Q_1(\tau) &= \frac{E_{10}}{\eta^{24}} = \frac{1}{q} - 240 - 141444q - 8529280q^2 + \mathcal{O}(q^3), \\ Q_2(\tau) &= \frac{2E_6^2 + E_4^3}{3\eta^{24}} = \frac{1}{q} - 408 + 196884q + 21493760q^2 + \mathcal{O}(q^3). \end{aligned} \quad (\text{F.10})$$

The K3 contribution to the zero orbit integral can now be computed by formula (F.8):

$$\Lambda_0^{K3} = \frac{T_2}{48} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \hat{\mathcal{A}}_{K3}[4] = -\frac{T_2}{576} (I_1(Q_1) - I_0(Q_2)) = -\frac{\pi}{6} T_2. \quad (\text{F.11})$$

Flux contribution. Next, the flux contribution to the zero orbit integral reads:

$$\Lambda_0^{\text{flux}} = \frac{T_2}{48} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \hat{\mathcal{A}}_{\text{flux}} = \frac{T_2}{576} I_{\text{flux}}, \quad (\text{F.12})$$

in terms of the integral:

$$I_{\text{flux}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \hat{F} \left(\frac{\hat{E}_2 E_8 - E_{10}}{\eta^{21}} \right). \quad (\text{F.13})$$

The integrand is a weight 0 Maaß form, as $\hat{F}$ has weight $1/2$:

$$\hat{F}(\tau) = F(\tau) - 6R(\tau) \quad (\text{F.14})$$

with a Mock modular piece with Fourier expansion:

$$q^{1/8} F(\tau) = 4q^{1/8} \eta(\tau) \sum_{i=2}^4 h_i(\tau) = 1 - 45q - 231q^2 - 770q^3 - 2277q^4 + \mathcal{O}(q^5) \quad (\text{F.15})$$

and a non-holomorphic completion given by:

$$q^{1/8} R(\tau) = 2 \sum_{n=0}^{\infty} (-)^n \text{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) q^{-\frac{1}{2}n(n+1)}. \quad (\text{F.16})$$

The shadow of F is the same as for $12\hat{\mu}(\tau, \nu)$, the non-holomorphic completion of the Appell function. More precisely, we have:

$$\partial_{\bar{\tau}} \hat{F}(\tau) = 3\sqrt{2}i \frac{\overline{\eta(\tau)}^3}{\sqrt{\tau_2}}. \quad (\text{F.17})$$

Computation of I_f : we use the pocedure outlined above to compute the integral (F.13). We rewrite

$$I_{\text{flux}} = \frac{\pi}{3i} \int_{\mathcal{F}} d^2\tau \hat{F} \frac{\partial}{\partial \bar{\tau}} \left(\frac{(\hat{E}_2)^2 E_8 - 2\hat{E}_2 E_{10}}{\eta^{21}} \right) = I'_{\text{flux}} + I''_{\text{flux}}, \quad (\text{F.18})$$

where the I_i are obtained by integration by parts. The first one is easily computed by using the procedure (F.7):

$$I'_{\text{flux}} = \frac{\pi}{3i} \int_{\mathcal{F}} d^2\tau \frac{\partial}{\partial \bar{\tau}} \left[\hat{F} \left(\frac{(\hat{E}_2)^2 E_8 - 2\hat{E}_2 E_{10}}{\eta^{21}} \right) \right] = \frac{\pi}{6} \left[F \left(\frac{(E_2)^2 E_8 - 2E_2 E_{10}}{\eta^{21}} \right) \right]_{q^0} = 172\pi \quad (\text{F.19})$$

where we have used

$$\begin{aligned} \frac{F E_2 E_{10}}{\eta^{21}} &= \frac{1}{q} - 312 - 123180 q + 1424120 q^2 + \mathcal{O}(q^3), \\ \frac{F(E_2)^2 E_8}{\eta^{21}} &= \frac{1}{q} + 408 + 28020 q - 3068680 q^2 + \mathcal{O}(q^3). \end{aligned} \quad (\text{F.20})$$

The second integral I_2 requires more care. Using the definition of the shadow (F.17), it can be cast into the following form:

$$\begin{aligned} I''_{\text{flux}} &= -\frac{\pi}{3i} \int_{\mathcal{F}} d^2\tau \frac{\partial \hat{F}}{\partial \bar{\tau}} \left(\frac{(\hat{E}_2)^2 E_8 - 2\hat{E}_2 E_{10}}{\eta^{21}} \right) \\ &= -\sqrt{2}\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (\sqrt{\tau_2} \eta \bar{\eta})^3 \left(\frac{(\hat{E}_2)^2 E_8 - 2\hat{E}_2 E_{10}}{\eta^{24}} \right) = 40\pi, \end{aligned} \quad (\text{F.21})$$

this integral having been computed in [103], eq. (4.25) therein, by a method developed in [102]. As outlined in section 5.2, this yields the result:

$$I_{\text{flux}} = 212\pi \implies \Lambda_0^{\text{flux}} = \frac{53\pi}{144} T_2. \quad (\text{F.22})$$

Zero orbit result. Putting together (F.11) and (F.22) we obtain:

$$\Lambda_{\text{SO}(28)}^0 = \Lambda_0^{K3} + \Lambda_0^{\text{flux}} = \frac{29\pi}{144} T_2. \quad (\text{F.23})$$

F.2 Degenerate orbits

The contribution from degenerate orbits is evaluated over the strip $\mathcal{S} = \{ -\frac{1}{2} \leq \tau_1 < 1/2, \tau_2 \geq 0 \}$:

$$\Lambda_{\text{SO}(28)}^{\text{deg}} = \frac{T_2}{8} \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \sum_{(j,p) \neq (0,0)} e^{-\frac{\pi T_2}{\tau_2 \tau_2^2} |j+pU|^2} \hat{\mathcal{A}}_{\text{SO}(28)}. \quad (\text{F.24})$$

To compute this integral we now choose to decompose the modified elliptic genus according to discrete and continuous $\text{SL}(2, \mathbb{R})_k / \text{U}(1)$ representations:

$$\hat{\mathcal{A}}_{\text{SO}(28)} = \hat{\mathcal{A}}_{\text{SO}(28)}^{\text{d}} + \mathcal{R}^c. \quad (\text{F.25})$$

The contribution from the discrete spectrum of states can be expanded as:

$$\begin{aligned}\hat{\mathcal{A}}_{\text{SO}(28)}^{\text{d}} &= -\frac{1}{72\eta^{24}} \left[\hat{E}_2 E_{10} - \frac{2}{3} E_6^2 - \frac{1}{3} E_4^3 - \eta^3 F(\hat{E}_2 E_8 - E_{10}) \right], \\ &= 8 - \frac{29}{\pi\tau_2} + \mathcal{O}(q).\end{aligned}\quad (\text{F.26})$$

Notice that there is a subtle cancellation so that this expression does even have a 'dressed' pole $(\tau_2 q)^{-1}$, unlike $K3$ models. Defining the Fourier expansion:

$$\frac{E_2 E_8 - E_{10}}{q^{1/8} \eta^{21}} = \sum_{n=0}^{\infty} d_1(n) q^n, \quad \frac{E_8}{q^{1/8} \eta^{21}} = \sum_{n=-1}^{\infty} d_2(n) q^n. \quad (\text{F.27})$$

the 'zero mode' contribution in $\mathcal{R}^c$ reads:

$$\mathcal{R}^c|_{q^0} = -\frac{1}{6} \sum_{n=0}^{\infty} \text{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) \left(d_1 \left(\frac{n(n+1)}{2} \right) - \frac{3}{\pi\tau_2} d_2 \left(\frac{n(n+1)}{2} \right) \right). \quad (\text{F.28})$$

When we integrate (F.24) first over τ_1 , only the $\mathcal{O}(q^0)$ contributions (F.26) and (F.28) survive. Then using uniform and absolute convergence of the exponential sum we obtain:

$$\begin{aligned}\Lambda_{\text{SO}(28)}^{\text{deg}} &= \frac{T_2}{8} \sum_{(j,p) \neq (0,0)} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} e^{-\frac{\pi T_2}{\tau_2 U_2} |j+pU|^2} \left(8 - \frac{29}{\pi\tau_2} + \right. \\ &\quad \left. - \frac{1}{6} \sum_{n=0}^{\infty} \text{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) \left[d_1 \left(\frac{n(n+1)}{2} \right) - \frac{3}{\pi\tau_2} d_2 \left(\frac{n(n+1)}{2} \right) \right] \right).\end{aligned}\quad (\text{F.29})$$

We can evaluate the first line of (F.29) by using the integrals:

$$\sum_{(j,p) \neq (0,0)} \int_0^{\infty} \frac{d^2\tau}{\tau_2^{1+r}} e^{-\frac{\pi T_2}{\tau_2 U_2} |j+pU|^2} = \frac{2\zeta(2k)\Gamma(k)}{(\pi T_2)^r} E(U, r). \quad (\text{F.30})$$

Combining the ensuing $E(U, 1)$ and $E(U, 2)$ contributions we get from (F.29) we obtain the second and third terms on the first line of (5.5), which are the degenerate orbit contributions of localised states. Notice that the real analytic Eisenstein series $E(U, 1)$ needs to be regularised:

$$E(U, 1) = -\frac{3}{\pi} \left(\log |\eta(U)|^4 + \log (\mu^2 T_2 U_2) \right) + \gamma \quad (\text{F.31})$$

with μ^2 an IR regulator and γ a renormalisation scheme dependent constant. This is expression we use in (5.5).

The bulk state contributions on the second line of (F.30) is treated in details in (5.19)–(5.22) et seq.

F.3 Non-degenerate orbits

We compute the non-degenerate orbit contribution to the $\text{SO}(28)$ threshold:

$$\Lambda_{\text{SO}(28)}^{\text{non-deg}} = 2 \int_{\mathcal{H}} \frac{d^2\tau}{\tau_2^2} \sum_{k>j\geq 0} \sum_{p\neq 0} e^{2\pi k p i T - \frac{\pi T_2}{\tau_2 U_2} |k\tau - j - pU|^2} \hat{\mathcal{A}}_{\text{SO}(28)}. \quad (\text{F.32})$$

The holomorphic part of the above modified elliptic genus is defined in terms of the following Fourier expansions:

$$-\frac{1}{12\eta^{24}} \left[E_2 E_{10} - \frac{2}{3} E_6^2 - \frac{1}{3} E_4^3 - \eta^3 F(E_2 E_8 - E_{10}) \right] = \sum_{n=0}^{\infty} c_1(n) q^n, \quad (F.33)$$

$$\frac{E_{10} - \eta^3 F E_8}{\eta^{24}} = \sum_{n=0}^{\infty} c_2(n) q^n.$$

Expanding the complementary error function in powers of τ_2 :

$$\operatorname{erfc} \left(\left(n + \frac{1}{2} \right) \sqrt{2\pi\tau_2} \right) = 1 - \sum_{m=0}^{\infty} (-)^m e_{n,m} \tau_2^{m+\frac{1}{2}}, \quad \text{with } e_{n,m} = \frac{\pi^m}{m!} \frac{(\sqrt{2}(n+\frac{1}{2}))^{2m+1}}{m+\frac{1}{2}}, \quad (F.34)$$

and performing the Gaussian integral over τ_1 in (F.32) we get:

$$\begin{aligned} \Lambda_{\text{SO}(28)}^{\text{non-deg}} &= \frac{\sqrt{T_2 U_2}}{24} \sum_{k>j\geq 0} \frac{1}{k} \sum_{p\neq 0} e^{2\pi i k p T_1} \int_0^{\infty} \frac{d\tau_2}{\tau_2^{3/2}} \\ &\quad \left(\sum_n e^{2\pi i n (\frac{j+pU_1}{k})} \left[c_1(n) - \frac{c_2(n)}{4\pi\tau_2} \right] e^{-\frac{\pi T_2}{U_2} \left(k + \frac{n}{k} \frac{U_2}{T_2} \right)^2 \tau_2 - \frac{\pi p^2 T_2 U_2}{\tau_2}} - \right. \\ &\quad - \sum_{n=0}^{\infty} (-)^n \sum_m e^{2\pi i (m - \frac{1}{2} n(n+1)) (\frac{j+pU_1}{k})} \left[d_1(m) - \frac{3}{\pi\tau_2} d_2(m) - \right. \\ &\quad \left. \left. - \sum_{l=0}^{\infty} e_{n,l} \left(d_1(m) \tau_2^{l+\frac{1}{2}} - \frac{3}{\pi} d_2(m) \tau_2^{l-\frac{1}{2}} \right) \right] e^{-\frac{\pi T_2}{U_2} \left(k + \frac{m - \frac{1}{2} n(n+1)}{k} \frac{U_2}{T_2} \right)^2 \tau_2 - \frac{\pi p^2 T_2 U_2}{\tau_2}} \right) \end{aligned} \quad (F.35)$$

To evaluate the first two lines of the integral in (F.35), we use the integrals:

$$\begin{aligned} J_r(a, b) &= \int_0^{\infty} \frac{dx}{x^{\frac{1}{2}+r}} e^{-ax - \frac{b}{x}} = \left(-\frac{\partial}{\partial b} \right)^r J_0(a, b) \\ &= \sqrt{\pi} \left(-\frac{\partial}{\partial b} \right)^r \frac{e^{-2\sqrt{ab}}}{\sqrt{a}} = 2 \left(\frac{a}{b} \right)^{\frac{r}{2}-\frac{1}{4}} K_{r-\frac{1}{2}}(2\sqrt{ab}), \quad r \in \mathbb{N}, \quad \operatorname{Re} a, \operatorname{Re} b > 0 \end{aligned} \quad (F.36)$$

given in terms of the modified Bessel functions of the second kind $K_{\alpha}(z)$ defined (5.34).

The integrals of interest here are in particular:

$$J_1(a, b) = \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}}, \quad J_2(a, b) = \left(\frac{1}{2b} + \sqrt{\frac{a}{b}} \right) \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}}. \quad (F.37)$$

Contribution of discrete representations: the first line of (F.35) gives the localised contribution to the non-degenerate orbit integral, which can be evaluated using (F.37):

$$\begin{aligned} \Lambda_{\text{SO}(28)}^{\text{non-deg d}} &= \frac{1}{24} \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i k p T} \left(\sum_n \left[c_1(n) - \frac{k}{4\pi p U_2} c_2(n) \right] e^{2\pi n \mathcal{U}} - \right. \\ &\quad \left. - \frac{1}{kp T_2} \sum_n \frac{1}{4\pi} \left[n + \frac{k}{2\pi p U_2} \right] c_2(n) e^{2\pi n \mathcal{U}} \right) + \text{c.c.} \\ &= \frac{1}{4} \sum_{k>j\geq 0} \sum_{p>0} \frac{1}{kp} e^{2\pi i \mathcal{T}} \left(\hat{\mathcal{A}}_{\text{SO}(28)}^{\text{d}}(\mathcal{U}) + \frac{1}{\pi T_2} \square \hat{\mathcal{A}}_{\text{SO}(28)}^{\text{d}}(\mathcal{U}) \right) + \text{c.c.} \end{aligned} \quad (F.38)$$

with the modular invariant operator:

$$\square \equiv \mathcal{U}_2^2 \partial_{\mathcal{U}} \bar{\partial}_{\mathcal{U}}. \quad (\text{F.39})$$

Contribution of continuous representations: the last two lines of (F.35) are the contributions of bulk states to the non-degenerate orbit integral. They can be evaluated using (F.37) for the second line while using

$$\int_0^\infty \frac{dx}{x^{1-r}} e^{-ax - \frac{b}{x}} = 2 \left(\frac{b}{a} \right)^{\frac{r}{2}} K_r(2\sqrt{ab}) = -2 \left(\frac{\partial}{\partial a} \right)^r K_0(2\sqrt{ab}), \quad (\text{F.40})$$

for the third line of eq. (F.35). This leads to expression (5.30) discussed earlier.

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