Pinching parameters for open (super) strings

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ABSTRACT: We present an approach to the parametrization of (super) Schottky space obtained by sewing together three-punctured discs with strips. Different cubic ribbon graphs classify distinct sets of pinching parameters; we show how they are mapped onto each other. The parametrization is particularly well-suited to describing the region within (super) moduli space where open bosonic or Neveu-Schwarz string propagators become very long and thin, which dominates the IR behaviour of string theories. We show how worldsheet objects such as the Green's function converge to graph theoretic objects such as the Symanzik polynomials in the $\alpha' \rightarrow 0$ limit, allowing us to see how string theory reproduces the sum over Feynman graphs. The (super) string measure takes on a simple and elegant form when expressed in terms of these parameters.

KEYWORDS: Bosonic Strings, Superstrings and Heterotic Strings

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1 Introduction

In this article, we lay out all the details of the approach introduced in [1]. There we suggested a new set of ‘pinching parameter’ co-ordinates for (super) Schottky space of multi-loop open string worldsheets.

Schottky groups are a classical way to construct genus $h \geq 1$ Riemann surfaces, and thus they can be used to describe multi-loop string worldsheets. The elements of Schottky groups are Möbius maps $z \mapsto \gamma(z) = \frac{az+b}{cz+d}$ that have one attractive fixed point u and one repulsive fixed point v , called *hyperbolic* Möbius maps. Such a map γ takes circles $\mathcal{C}$ around v to circles $\mathcal{C}' = \gamma(\mathcal{C})$ around u . Quotienting the Riemann sphere by the action of γ (after removing u and v) yields the genus one Riemann surface obtained by identifying $\mathcal{C}$ with $\mathcal{C}'$ and discarding the discs inside both circles. The resulting surface doesn't depend on the choice of $\mathcal{C}$, only on γ .

To get a genus h surface, this process can be repeated h times. Let us take h hyperbolic Möbius maps γ_i and h pairs of non-overlapping circles $\{\mathcal{C}_i, \mathcal{C}'_i = \gamma_i(\mathcal{C}_i)\}$ around the fixed points $u_i, v_i, i = 1, \dots, h$. The group freely generated by the γ_i 's is called the Schottky group G . By cutting out the interiors of the $2h$ discs bounded by the circles and then quotienting the remaining region by G , we identify the circles in pairs, yielding a genus h surface as required. The resulting surface does not depend on the choice of the circles $\mathcal{C}_i$, just on the group G .

Therefore the space of Schottky groups (called *Schottky space*) can be used to parametrize the moduli space of genus $h \geq 1$ Riemann surfaces. G can be defined by a set of h generators γ_i , and each generator can be described by three complex parameters: the fixed points u_i, v_i and the *multiplicier* k_i defined by the cross-ratio $k_i \equiv \frac{\gamma_i(z)-u_i}{\gamma_i(z)-v_i} \frac{z-v_i}{z-u_i}$ which is independent of z . To eliminate redundancy three of the fixed points can be set to $\{0, \infty, 1\}$ with a global change of coordinates. The resulting *normalized* Schottky space has complex dimension $3g - 3$, matching the dimension of moduli space. In this paper we deal with open string worldsheets, meaning that all of the Schottky group parameters are constrained to be real and Schottky space has real dimension $3g - 3$.

What we describe in this paper is a new coordinatization of normalized Schottky space, which is more symmetric than the canonical one in that all of the ‘pinching parameter’ coordinates are on the same footing as each other, rather than some of the coordinates being fixed points and some of them being multipliers. This greater symmetry is useful in a number of ways.

One benefit of the pinching parameters is that they clarify how various Feynman diagram topologies with arbitrary numbers of loops emerge from the string theory measure on (super) moduli space, in the zero slope limit. It has been known since the early days of string theory [2, 3] that as the string tension goes to infinity (i.e. $\alpha' \rightarrow 0$), string scattering amplitudes reproduce those of a quantum field theory as the stringy states become infinitely massive and decouple. In the case of bosonic strings or type I/II superstrings, the open string sector yields gauge theories in the $\alpha' \rightarrow 0$ limit while the closed string sector gives gravitation theories.

This fact has been used to gain insight about the computation of particle scattering amplitudes (without assuming any position on the validity of superstring theory itself as an accurate description of the world). The added complication due to the replacement of particles with extended objects is offset by a number of simplifications, for example, traces over colour factors and integrals over loop momenta are done for free in the string-based approach, with Schwinger-parametrized Feynman integrands arising naturally as the limits of the string measure on moduli space. An analysis of how this works in terms of tropical geometry was proposed in [4].

Starting in the late 1980's, string-inspired approaches to the computation of one-loop amplitudes were developed for gauge theories [5–9] and gravitation [10]. It was shown in [8] how to relate the one-loop string-inspired rules to the Feynman graph based approach — but only if the Feynman diagrams were computed using a particular non-Hermitian gauge condition for the Yang-Mills gauge fields, identified by Gervais and Neveu in [11]. The reasons for this gauge preference have recently been explained [12]. In [13] it was established which regions of the moduli space contributed to which Feynman diagram topologies.

Going beyond one loop, it becomes important to choose a useful set of worldsheet moduli to integrate over. The benefits of using Schottky groups for string-inspired QFT calculations were laid out in [14]. Schottky groups had arisen as a parametrization of the moduli space of multi-loop string worldsheets in the earliest days of string theory, emerging automatically from an approach to amplitudes based on sewing together string interaction vertices [15–20]. The use of Schottky groups in string theory was put on a surer footing in the 1980's with the use of methods such as BRST quantization [21–25], conformal field theory [26] and the ‘group theoretic approach’ [27, 28], and a super-geometric extension of Schottky groups [29] allowed the analysis to be generalized to the Neveu-Schwarz (NS) sector of the superstring.

Detailed two-loop scalar QFT computations using Schottky groups have been presented in [30] and [31]. Similar techniques have been used to compute multi-loop open string amplitudes with twisted periodicity conditions, allowing the analysis of states between D-branes at an angle or with a constant background gauge field switched on, allowing the computation of the two-loop Euler-Heisenberg effective action [32–34]. In fact, with such approaches, it is possible to examine the various terms in the string moduli space integrand and identify not only separate Feynman graph *topologies*, but to see which fields are propagating through which edges in the graphs, allowing string-based computations to be used on a diagram-by-diagram basis [35, 36].

The extension of these analyses to graphs with more loops was hindered by the lack of a mapping from the Schottky parameters $\{u_i, v_i, k_i\}$ to the Feynman graph Schwinger parameters t_i . The complication was due in large part to the asymmetric separation of Schottky parameters into fixed points and multipliers. This problem had been circumvented for the ‘sunset’ two-loop vacuum Feynman graph in [35, 36]. At two loops there is a symmetry between three of the Schottky group multipliers [22] which allows us to write down the mapping onto Schwinger times straightforwardly. But for Feynman graphs of more complicated topologies, there are no longer enough constraints to obtain a set of ‘pinching parameter’-type moduli from graph symmetries alone.

Our approach is based on the idea of building open string worldsheets by sewing together three-punctured discs with plumbing fixtures. For the bosonic string, the ‘lengths’ of the plumbing fixtures are the only parameters, and they become the moduli of the resulting surfaces. The construction turns out to be equivalent to a particular parametrization of Schottky groups. At two loops, our approach precisely reproduces the pinching parameters obtained by symmetry in [35, 36], but our approach has the advantage of being defined for general Feynman graph topologies, with arbitrary numbers of loops and external edges.

The basic idea of our analysis resembles a number of other approaches which have appeared in the Physics and Mathematics literature. Fock gives an approach very similar to ours in spirit, in which the geometry of a ribbon graph is translated into Möbius maps between its vertices, although the mapping is not the same [37]. The idea of building string Feynman diagrams by joining cubic vertices with propagators was used in Witten’s open string field theory [38] which is known to give a single cover of moduli space [39]. But the Witten vertex yields a complicated mapping between strip lengths and Koba-Nielsen variables [40], whereas in our approach the Koba-Nielsen variables are rational functions of the pinching parameters. Recently, Sen has constructed a gauge-invariant 1PI effective action for superstring theory by separating the contribution from string diagrams which can be obtained by gluing lower-order diagrams with plumbing fixtures [41–43]. Penner described how to use trivalent ribbon graphs to give a cell decomposition of moduli space [44], and Marden gave a coordinatization of Teichmüller space derived from the starting point of sewing three-punctured spheres, although the transition functions are not all Möbius maps (unlike ours) [45].

The use of the pinching parameters provides a link between worldsheet objects defined in terms of Riemann surfaces such as the period matrix and the Green’s function, and graph theoretic objects such as the Symanzik polynomials (cf. sections 3.4 and 4.1 of [4]). To be precise, we can express pinching parameters p_i in terms of dimensionful Schwinger parameters t_i by setting $p_i = e^{-t_i/\alpha'}$, and then in taking the $\alpha' \rightarrow 0$ limit we find that geometric objects asymptote to graph theoretic objects. We focus on the worldline Green’s function, which is used in string-inspired calculations of scalar QFT amplitudes. Dai and Siegel described in an elegant paper how the worldline Green’s function could be found for arbitrary multi-loop diagrams using methods from electric circuit theory [46]. We show how the building blocks used in their analysis can be arrived at from the limiting behaviour of Riemann surface objects. This illustrates, in particular, how Feynman graphs for scalar QFTs can arise from the asymptotics of string theory diagrams.

Our approach also defines coordinates on the super-moduli space of super Riemann surfaces (SRS), near corners corresponding to NS open string degenerations. In recent years there has been a resurgence of interest in SRS and super moduli space [47–52]. Superstring perturbation theory contains integrals which are only conditionally convergent in the IR region, and there can be ambiguities arising from the choice of which even supermoduli to hold fixed when integrating over the odd supermoduli. For example, consider the integral $\int_{y=\epsilon}^{1-\epsilon} d\theta d\phi d\log y$, where y is an even supermodulus and θ, ϕ are odd supermoduli. This seems to vanish on the face of it, but after making the change of variables $y = 1 - u - \theta\phi$ this is no longer the case. The choice of which even supermoduli to hold fixed while integrating over the odd supermoduli is thus non-arbitrary, and a complete formulation of perturbative superstring theory requires a prescription of which even supermoduli to use. Witten shows that the correct even supermoduli to fix near the corners of supermoduli space are the so-called *canonical parameters* (see, for example, section 6.3 of [47] and section 2.4 of [49]). The pinching parameters that we have written down for the NS sector are defined by gluing pairs of punctures *à la* eq. (6.31) of [47], which means they constitute a set of canonical parameters (canonical parameters, in general, are only defined modulo coordinate choices around the punctures). Thus our pinching parameters allow the correct computation of conditionally convergent integrals. The NS pinching parameters are, roughly, like square roots of the bosonic pinching parameters. The three-punctured discs we use for the construction only have NS punctures. Such discs have a Grassmann-odd super-projective invariant; these invariants become the odd supermoduli.

One benefit of the pinching parameterization is that the string measure on (super) Schottky space is given a very elegant and symmetric form. In particular, there is a direct correspondence between poles in the measure and edges in the corresponding Feynman graph. This differs from the canonical coordinatization of Schottky space for which the geometric meaning of the poles can be unintuitive. Furthermore, it is interesting that the measure includes a product over the borders of the worldsheet, meaning that, for example, the measure for a planar two-loop vacuum diagram (which has three borders) is qualitatively different from the measure for the non-planar two-loop vacuum diagram (which has just one border) — a distinction that is completely obscured with the canonical coordinates.

The construction given in section 2 requires the use of a cyclic ordering of the three punctures on the discs that are used to construct the worldsheets. This is appropriate for open strings because they have a fixed ordering at interactions. Conversely, closed strings do not have a fixed ordering at interactions, and so they cannot be described by these pinching parameters in a natural way. It would be interesting to find an extension of the pinching parametrization that exhibits the symmetries of closed string amplitudes.

The organization of the paper is as follows. In section 2 we present the construction of the pinching parametrization, first in the bosonic case in section 2.1 and then for the Neveu-Schwarz sector in section 2.2. In section 3 we describe how the graph period matrix and worldline Green’s function arise as the $\alpha' \rightarrow 0$ limit of objects on moduli space, given a straightforward mapping between pinching parameters and Schwinger parameters. Section 4 describes the relations between sets of pinching parameters that correspond to pairs of graphs differing by a duality transformation on one of the internal edges. In

Figure 1. A graph with $E = 11$ internal edges labelled with Schwinger parameters t_i , $n = 2$ external edges labelled X_j and $g = 4$ loops with a basis $\{\ell_i\}$.

section 5 we give the leading part of the (super) string measure in terms of the pinching parameters, and show how it can be proven. In appendix A we provide proofs of several claims in the paper: we derive the leading behaviour of pinching parameter expressions for the (semi) multiplier in section A.1, for the period matrix in section A.2 and for the Green's function in section A.3.

2 Construction of the pinching parametrization

Our goal is to construct a set of (super) moduli for string worldsheets that have a simple geometric interpretation in terms of degenerating surfaces, and thus can be used gainfully to analyse the $\alpha' \rightarrow 0$ limit of string amplitudes.

2.1 Bosonic strings

First we consider the simpler bosonic case. In section 2.1.1 we describe how sewing discs can lead to an expression for Koba-Nielsen variables in terms of pinching parameters, and then in section 2.1.2 we show how the construction can be extended to incorporate loops with the use of Schottky groups.

2.1.1 Local coordinates at the punctures

Let the *target graph* Γ be a connected ribbon graph (i.e. the edges connect to vertices with a fixed cyclic ordering) whose vertices are all either three-valent or one-valent (we call edges joining a 1-valent vertex *external edges*). Let all edges whose endpoints are both three-valent vertices (possibly the same one) be called *internal edges* and be labelled with a *Schwinger parameter* t_k . If the graph has g loops and n external edges, then we can label the external edges X_i ($i = 1, \dots, n$), label the cubic vertices v_j ($j = 1, \dots, 2g + n - 2$), and label the internal edges E_k ($k = 1, \dots, 3g + n - 3$). Furthermore, we choose a homology basis of g loops in Γ , i.e. a set of g closed paths ℓ_i in Γ which are independent in the sense that no non-trivial composition of the ℓ_i is homologous to a trivial path. An example target graph with 4 loops, 11 internal edges and 2 external edges is shown in figure 1.

Given such a graph, we can build a corresponding bordered Riemann surface Σ parametrized by $3g - 3 + n$ real moduli $\{p_i\}$ (i.e. coordinates on Schottky space). When we take all of the $p_i \rightarrow 0$, the surface degenerates into Σ^0 : a union of 3-punctured discs joined at nodes, so we call the p_i 's *pinching parameters*. The topology of Σ^0 is classified by the target graph Γ .

We build Σ from two ingredients: 3-punctured discs and open-string plumbing fixtures. On each 3-punctured disc, we put three standard coordinate charts: one centred at each puncture. If we label the punctures a_1, a_2 and a_3 with a clockwise ordering, then we want three coordinate charts $z_i, i = 1, 2, 3$ satisfying

$$z_i(a_i) = 0. \quad (2.1)$$

The image of the disc under z_i is the upper-half plane with the projective real line as its boundary, so let us fix the z_i coordinates of the other two coordinates to 1 and ∞ :

$$z_i(a_{i+1}) = \infty \quad z_i(a_{i-1}) = 1, \quad (2.2)$$

where the indices are mod 3. Then there is a unique Möbius map ρ which serves as a transition function cycling the three charts. In order to satisfy

$$z_i = \rho(z_{i+1}), \quad (2.3)$$

we need to have

$$\rho(0) = \infty; \quad \rho(\infty) = 1; \quad \rho(1) = 0. \quad (2.4)$$

This is satisfied by

$$\rho(z) \equiv 1 - \frac{1}{z}, \quad (2.5)$$

which can be written as the PSL(2) matrix

$$\rho \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.6)$$

The inverse transformation is given by

$$\rho^{-1}(z) = \frac{1}{1-z}; \quad \rho^{-1} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \quad (2.7)$$

Note, of course, that $\rho^3 = 1$.

The other ingredient is the ‘open string plumbing fixture’. Suppose a Riemann surface Σ has two charts w and z , both of which map its border to the real line, and whose images each include a neighbourhood around 0 (figure 2a). Then for a given pinching parameter $p \in (0, 1)$, we can *insert a plumbing fixture* by cutting out two semi-discs $|w| < \epsilon_w$ and $|z| < \epsilon_z$ and imposing the relation

$$zw = -p \quad (2.8)$$

Figure 2. Given a bordered Riemann surface Σ with two charts z and w which map the border of Σ onto the real line and whose images include a neighbourhood of 0 (figure 2a), we can add an *open string plumbing fixture* to Σ (figure 2b) by imposing eq. (2.8) on a suitable domain.

for $\epsilon_w < |w| \leq p/\epsilon_z$ and $\epsilon_z < |z| \leq p/\epsilon_w$ (figure 2b). The effect of the plumbing fixture is to add a ‘strip’ between the two charts on Σ , either by adding a loop or by joining together two connected components.

Once a plumbing fixture is in place, we can use it to get a transition function between the two charts we used to define it: we can say

$$w = \sigma_p(z) \quad (2.9)$$

where

$$\sigma_p(z) \equiv -\frac{p}{z}; \quad \sigma_p \equiv \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix}. \quad (2.10)$$

The two types of transition function we’ve defined, eq. (2.6) and eq. (2.10), are the two ‘building blocks’ we’ll use to build the Riemann surface Σ . In order to define Σ , we need to specify the transition functions between any pair of coordinate charts, as well as the positions of all the punctures. To achieve this, we lay out the three-punctured discs and plumbing fixtures in the arrangement defined by the target graph Γ . Each disc comes with three local coordinate systems as defined by eq. (2.1) and eq. (2.2), related to each other by eq. (2.3). Furthermore, two coordinate systems w and z which lie at opposite ends of the same open string plumbing fixture can be related to each other by eq. (2.9). Thus, by finding a path between two charts, we can write down a transition function as an alternating product of σ_{p_i} ’s and $\rho^{\pm 1}$ ’s.

Figure 3. Three 3-punctured discs sewn to give a surface with $n = 5$ punctures. The z_i 's label coordinate charts centred at the indicated points. We can use the pinching parametrization to find transition functions between any pair of them in terms of the p_i 's. Note that the centres of $z_2, \dots, z_5$ get excised from the surface by the sewing procedure.

Figure 4. $\hat{\Gamma}$ is obtained from Γ by first blowing up all the cubic vertices into triangles (figure 4a) and then deleting any external edges but reusing their labels (figure 4b).

For example, consider the surface indicated by the diagram in figure 3; suppose we want to find the transition function between the charts z_1 and z_6 . We can build it piece-by-piece: first we note that since z_1 and z_2 are on the same disc with the centre of z_1 situated *anti-clockwise* from the centre of z_2 , we can use eq. (2.3) to see $z_1 = \rho(z_2)$. Next, since z_2 and z_3 are at opposite ends of the open string plumbing fixture labelled by the pinching parameter p_1 , we have $z_2 = \sigma_{p_1}(z_3)$. We can combine this with the previous equation to get a transition function straight from z_3 to z_1 : $z_1 = (\rho \cdot \sigma_{p_1})(z_3)$. The next two pairs of consecutive coordinate systems are related to each other by $z_3 = \rho(z_4)$ and $z_4 = \sigma_{p_2}(z_5)$. Lastly, because z_5 is on the same disc as z_6 but in this case centred *clockwise* from it, we have $z_5 = \rho^{-1}(z_6)$. Putting this all together we find $z_1 = (\rho \cdot \sigma_{p_1} \cdot \rho \cdot \sigma_{p_2} \cdot \rho^{-1})(z_6)$.

Clearly, it was not necessary to write down the intermediate coordinate charts, and in fact, the transition function depended only on the path taken to get from the puncture at $z_6 = 0$ to the puncture at $z_1 = 0$. This is the case in general: the procedure for getting a surface Σ from a target graph Γ can be defined in terms of a mapping from *paths* on Γ into the space $\text{PSL}(2)$ of Möbius maps.

In order to define the parametrization precisely, instead of using paths on Γ we must construct a new graph $\hat{\Gamma}$ in the following way: firstly, we ‘blow up’ every 3-valent vertex into a triangle whose corners are 3-valent vertices as shown in figure 4a. Secondly, we delete every external edge, leaving a now 2-valent vertex which we label with the label X_j that belonged to the deleted external edge, as shown in figure 4b. Let us distinguish the edges in triangles and the edges inherited from Γ as *short edges* and *long edges*, respectively. The triangles in $\hat{\Gamma}$ correspond to the three-punctured discs used to construct Σ ; the edges correspond to the open string plumbing fixtures.

Figure 5. A path in $\hat{\Gamma}$ can be decomposed into three elementary types of path: traversal of a long edge E_k (T_k) (figure 5a), clockwise moves around a triangle (CW) (figure 5b) and anti-clockwise moves around a triangle (ACW) (figure 5c).

Every vertex in $\hat{\Gamma}$ corresponds to a local coordinate chart on Σ , and every path in $\hat{\Gamma}$ corresponds to a transition function between the charts at its two ends. There is a homomorphism ϕ from the groupoid of paths on $\hat{\Gamma}$ to $\text{PSL}(2)$; we can define it as follows.

Every path P can be decomposed into three elementary types of paths. The first type of elementary path is the traversal of one of the long edges E_k that was inherited from Γ with Schwinger parameter t_k . Let us denote this type of path by (T_k) . The second type of elementary path is simply a ‘rotation’ from one vertex to another vertex on the same triangle. We can distinguish further between rotations depending on whether the end vertex is immediately clockwise or anti-clockwise of the start vertex: let us call them (CW) and (ACW) respectively. Thus any path in $\hat{\Gamma}$ can be written as a sequence composed of (CW), (ACW) and (T_k) .

Once decomposed in this way, a path P can be written as a Möbius map $\phi(P) \in \text{PSL}(2)$ with the homomorphism ϕ where:

$$\phi(\text{ACW}) = \rho, \quad \phi(\text{CW}) = \rho^{-1}, \quad \phi(T_k) = \sigma_{p_k}, \quad (2.11)$$

in terms of the $\text{PSL}(2)$ maps defined in eq. (2.6), eq. (2.7) and eq. (2.10).

Note that to do this for a graph Γ we have implicitly introduced a set $\{p_k\}$ of pinching parameters with one associated to each long edge E_k . These will become the coordinates on Schottky space.

Also note that because $\rho^3 = \sigma_{p_k}^2 = \text{id}$, the mapping ϕ is insensitive to sub-paths which traverse an edge then reverse, or which visit all three nodes at a cubic vertex in succession.

Given a graph Γ , let us choose a vertex in $\hat{\Gamma}$ and fix it as the ‘base point’ B . We’ll use the coordinate chart defined by B as a canonical chart and write down all the other charts with respect to it. In particular, we want to find the transition functions to the charts which vanish at the punctures on the surface Σ (corresponding to the external edges on Γ). To achieve this, for each external edge X_j , we find a path P_{X_j} from X_j to B on the blown-up graph $\hat{\Gamma}$. Then the n sets of local coordinates at the external punctures are given by $V_{X_j}^{-1}$, where

$$V_{X_j} \equiv \phi(P_{X_j}), \quad (2.12)$$

Figure 6. Figure 6a shows an $n = 7$ tree graph Γ ; the external edges are labelled X_i , $i = 1, \dots, 7$ and the internal edges are labelled E_j , $j = 1, \dots, 4$. The blown-up graph $\hat{\Gamma}$ (figure 6b) has a base point at $B = X_1$ (indicated with a dot); a path P_{X_5} from X_5 to B is indicated with the blue dotted line.

in terms of the mapping ϕ defined by eq. (2.11). The Koba-Nielsen variables corresponding to the positions of the external punctures as seen in the chart defined by the base point B are given as

$$z_j = V_{X_j}(0). \quad (2.13)$$

For example, consider the $n = 7$ tree graph Γ shown in figure 6a and its blow-up $\hat{\Gamma}$ in figure 6b. A path P_{X_5} from X_5 to the base point $B = X_1$ is indicated on $\hat{\Gamma}$, and it can be decomposed into the basic moves of figure 5 as

$$P_{X_5} = \text{CW} \cdot T_1 \cdot \text{CW} \cdot T_2 \cdot \text{ACW} \cdot T_3 \cdot \text{CW} \cdot T_4 \cdot \text{ACW}. \quad (2.14)$$

Using eq. (2.11), this gives us

$$V_5 = \phi(P_{X_5}) = \rho^{-1} \cdot \sigma_{p_1} \cdot \rho^{-1} \cdot \sigma_{p_2} \cdot \rho \cdot \sigma_{p_3} \cdot \rho^{-1} \cdot \sigma_{p_4} \cdot \rho; \quad (2.15)$$

plugging this in eq. (2.13) we find that the Koba-Nielsen variable for the corresponding point on Σ is given in terms of the pinching parameters as

$$z_5 = \frac{1 + p_3}{1 + p_1 + p_3 + p_1 p_3 + p_1 p_2 p_3}. \quad (2.16)$$

Since the base point is at $B = X_1$, we have $V_1 = \text{id}$ and so the Koba-Nielsen variable corresponding to the first puncture is at $z_1 = 0$. The Koba-Nielsen variables for the other five punctures $z_i(p_j)$ can be computed similarly.

2.1.2 The Schottky group

We've introduced a mapping ϕ from the set of paths in $\hat{\Gamma}$ into $\text{PSL}(2)$, and said that it can be used to find transition functions between charts on Σ . But for multiply-connected graphs Γ , there are many independent paths joining any pair of vertices in $\hat{\Gamma}$. This would seem to violate the cocycle condition, but in fact is not a problem once we understand all the charts as being defined modulo a *Schottky group* G .

Figure 7. The graph Γ with $n = 0$, $g = 2$ comes with a homology basis of two loops ℓ_1 and ℓ_2 (figure 7a); these can be used to define paths P_{ℓ_1} and P_{ℓ_2} on the blown-up graph $\hat{\Gamma}$ which start and end at the base point B (marked as a black dot on figure 7b).

Recall that Γ came with a homology basis $\{\ell_i\}_{i=1}^g$; each loop ℓ_i can be used to define a closed path P_{ℓ_i} on $\hat{\Gamma}$ which starts and ends at a chosen base point B . See figure 7 for a two-loop example.

Then we define g Möbius maps γ_i with

$$\gamma_i \equiv \phi(P_{\ell_i}). \quad (2.17)$$

The γ_i 's are hyperbolic Möbius maps — that is to say, they are conjugate to dilatations $z \mapsto k_i z$, and thus they can be parametrized by their *multiplier* k_i with $|k_i| < 1$, and their *attractive* and *repulsive* fixed points u_i and $v_i \in \mathbf{R} \cup \{\infty\}$. The γ_i 's are related to their parameters via

$$\frac{\gamma_i(z) - u_i}{\gamma_i(z) - v_i} = k_i \frac{z - u_i}{z - v_i}. \quad (2.18)$$

From another point of view, the vectors $|u_i\rangle \equiv (u_i, 1)^t$ and $|v_i\rangle \equiv (v_i, 1)^t$ are eigenvectors of the matrix γ_i , satisfying

$$\gamma_i|u_i\rangle = \frac{1}{\sqrt{k_i}}|u_i\rangle \quad \gamma_i|v_i\rangle = \sqrt{k_i}|v_i\rangle; \quad (2.19)$$

these equations can be solved to express the parameters k_i , u_i and v_i as functions of the pinching parameters $\{p_j\}$.

The rank- g group G freely generated by the γ_i 's is a *Schottky group* of genus g . Schottky groups give a classical construction of Riemann surfaces as quotient spaces; in the early days of string perturbation theory they arose naturally by sewing n -string vertices [17–20]. Let us briefly recall the Schottky group construction for open strings; a more detailed discussion is given, for example, in section 2.1 of [53].

The construction uses the fact that Möbius maps take circles to circles. In particular, any semicircle $\mathcal{C}$ that is centred on the real axis and that separates the two fixed points u, v of a hyperbolic Möbius map γ is mapped by γ to another such semicircle $\mathcal{C}'$ (see figure 8a). This cuts the upper-half plane into three regions: one containing u , one containing v , and one (say $\mathcal{F}$) that contains neither fixed point. Quotienting the closure $\overline{\mathcal{F}}$ by γ has the effect of identifying the two semicircles on its boundary $\mathcal{C} \sim \mathcal{C}'$; the resulting surface is an annulus.

Figure 8. The Schottky group construction (figure 8a) of a planar $g = 2$ surface with $b = 3$ borders (figure 8b). The construction induces a loop basis $\{\ell_1, \ell_2\}$ on the surface.

The construction makes sense with any number g of Möbius maps γ_i as long as we can find an appropriate set of $2g$ semicircles $\{\mathcal{C}_i, \mathcal{C}'_i\}$ such that $\gamma_i(\mathcal{C}_i) = \mathcal{C}'_i$ and that all lie on the boundary of a single region $\mathcal{F}$ (see figure 8a for an example with $g = 2$). Then quotienting the closure $\overline{\mathcal{F}}$ by G has the effect of identifying pairs of semicircles $\mathcal{C}_i \sim \mathcal{C}'_i$, where G is the group generated by the g γ_i 's. The resulting surface is a bordered Riemann surface of genus $h = (g + 1 - b)/2$ where b is the number of border components (note that $h = 0$ for planar topologies, which have $b = g + 1$ borders). $\mathcal{F}$ can be treated as a single chart covering the surface. The example in figure 8b has $g = 2$, $h = 0$.

The surface doesn't depend on the choice of semicircles; any set of semicircles with empty intersection that satisfies $\gamma_i(\mathcal{C}_i) = \mathcal{C}'_i$ gives the same surface Σ . It's not strictly necessary for $\mathcal{C}_i$ and $\gamma_i(\mathcal{C}_i)$ to be geometric semicircles, only that they have the right topology; there exist *non-classical* Schottky groups such that the $\mathcal{C}_i$ can't all be chosen as geometric semicircles, but the Schottky construction remains valid.

The g Möbius maps $\{\gamma_i\}$ thus define a surface, but there is some redundancy because two Schottky groups that are related by a $\text{PSL}(2)$ change of coordinates define the same surface. We can choose three specific parameters, say u_1, v_1 and v_2 , to 'gauge fix' to some chosen values; this saturates the reparametrization symmetry. The remaining $3g - 3$ parameters $\{u_i\}_{i=2}^g \cup \{v_i\}_{i=3}^g \cup \{k_i\}_{i=1}^g$ can be used as moduli. The space described by these parameters is called *Schottky space*.

The $3g - 3$ Schottky parameters can themselves be expressed in terms of pinching parameters via eq. (2.17). Note that for graphs Γ with no external edges $n = 0$, the number of internal edges of Γ is $|E| = 3g - 3$, so the pinching parameters can be used as a local coordinate system on Schottky space near the corresponding degeneration.

Recall that for graphs Γ with $n \geq 1$ external edges, we need to find paths P_{X_i} from the external edges to the base point B on $\widehat{\Gamma}$. But with $g \geq 1$ loops, there isn't a unique way to choose such a path. Two different paths $P_{X_i} \neq \tilde{P}_{X_i}$ will give two different inverse charts

$V_{X_i} = \phi(P_{X_i})$ and $\tilde{V}_{X_i} = \phi(\tilde{P}_{X_i})$, and therefore two different values for the Koba-Nielsen variable of X_i : one will give $z_{X_i} = V_{X_i}(0)$ and the other will give $\tilde{z}_{X_i} = \tilde{V}_{X_i}(0)$. But z_{X_i} is defined only modulo the Schottky group G , so this isn't a problem. We can find a path $P_{\ell_\alpha} \equiv P_{X_i} \cdot (\tilde{P}_{X_i})^{-1}$; because this path starts and ends at B , it can be expressed in terms of the homology basis $\{P_{\ell_i}\}$. Thus the Möbius map $\gamma_\alpha \equiv \phi(P_{\ell_\alpha})$ can be written as a product of the $\gamma_i = \phi(P_{\ell_i})$, so it is a member of the Schottky group, $\gamma_\alpha \in G$. So $V_{X_i} = \gamma_\alpha \cdot \tilde{V}_{X_i}$, and thus $z_{X_i} = \gamma_\alpha(\tilde{z}_{X_i})$; in other words, the procedure defines the Koba-Nielsen variable unambiguously modulo the Schottky group G .

So the g Schottky group generators and the n Koba-Nielsen variables, which can be used as a set of moduli for the surface, can be written as functions of the $3g + n - 3$ gluing parameters p_k , and thus the p_k 's can be used as coordinates on a suitable region of Schottky space.

2.1.3 Properties of the parametrization

1. Since ϕ is a homomorphism, the Schottky group G doesn't depend on the choice of homology basis $\{\ell_i\}$, i.e. on which of its elements are treated as generators, but only on the graph Γ .
2. Changing the base node $B \mapsto \hat{B}$ is equivalent to a global change of coordinates: let $P_{B\hat{B}}$ be a path from B to $\hat{B}$ then the parametrization with the base node $\hat{B}$ is given by taking

$$\hat{\gamma}_i = \phi(P_{B\hat{B}}) \cdot \gamma_i \cdot \phi(P_{B\hat{B}})^{-1}; \quad \hat{V}_{X_j} = \phi(P_{B\hat{B}}) \cdot V_{X_j}. \quad (2.20)$$

3. Let us use the term *loop* to refer to a closed path in Γ defined modulo cyclic permutations, i.e. the path $E_{j_n} \cdot \dots \cdot E_{j_2} \cdot E_{j_1}$ defines the same loop ℓ as $E_{j_1} \cdot E_{j_n} \cdot \dots \cdot E_{j_2}$ (see figure 9 for an illustration). Every closed path P_α in $\hat{\Gamma}$ corresponds to a loop ℓ_α in Γ . To obtain ℓ_α , we first list the edges traversed by P_α . If the first and last edges listed are the same then we can use invariance under cyclic permutations to delete them from the list, since they must be traversed in opposite directions. This can be repeated until the first and last edges that remain are distinct.

If ℓ_α is a loop corresponding to a closed path P_α then the multiplier of the Schottky group element $\gamma_\alpha \equiv \phi(P_{\ell_\alpha})$ is given to leading order as

$$k_\alpha = \left(\prod_i p_i^{n_\alpha^i} \right) (1 + \mathcal{O}(p_j)), \quad (2.21)$$

where n_α^i counts the number of times that the loop ℓ_α crosses the edge E_i .

4. If the path P_α is conjugate to a loop ℓ_α that consists solely of either clockwise or anticlockwise turns at the vertices, then the multiplier is precisely the product of the gluing parameters of the edges traversed by ℓ_α :

$$k_\alpha = \prod_i p_i^{n_\alpha^i}. \quad (2.22)$$

Figure 9. We distinguish between closed paths P_α in the graph (figure 9a) and the corresponding loops ℓ_α (figure 9b), which are equivalence classes of closed paths defined modulo conjugation, and which are independent of the base point.

Figure 10. Some examples of graphs Γ with paths corresponding to loops (figure 10a–figure 10c) and local coordinates at punctures (figure 10d–figure 10e). The base points B are marked as large dots.

5. The moduli p_i characterize some of the boundaries of the Deligne-Mumford compactification of moduli space: the boundary corresponding to the pinching of an edge with gluing parameter p_i is given by taking $p_i \rightarrow 0$.
6. Our construction is only suitable for open strings, not closed strings, because the transition functions on the three-punctured discs can only be written down by assuming a cyclic ordering of the punctures, meaning that strings must be assumed to interact with a fixed cyclic ordering.

2.1.4 Examples

Some illustrative examples are given in figure 10. Note that we have drawn the paths P_i on the target graphs Γ instead of on the blown-up graphs $\hat{\Gamma}$, because the target graphs Γ are simpler but we can unambiguously represent paths on $\hat{\Gamma}$ in this way, as long as we indicate the base point. To indicate the base point (which is a vertex of $\hat{\Gamma}$) on Γ , we draw a black dot near the appropriate vertex on the appropriate incident edge.

1. Figure 10a corresponds to a path which can be expressed as follows:

- Traverse the edge labelled E_1 .
- Move anticlockwise around the next vertex onto the edge labelled E_2 .
- Traverse the edge labelled E_2 .
- Having arrived at the starting vertex, move *anticlockwise* around the vertex to arrive back at the base point.

More concisely, the path is $P_1 = \text{ACW} \cdot T_2 \cdot \text{ACW} \cdot T_1$, so the Möbius map $\gamma_1 = \phi(P_1)$ is given by

$$\gamma_1 = \rho \sigma_2 \rho \sigma_1 = \frac{1}{\sqrt{p_1} \sqrt{p_2}} \begin{pmatrix} 1 & p_1(1+p_2) \\ 0 & p_1 p_2 \end{pmatrix}. \quad (2.23)$$

2. The loop shown in figure 10b is the same as the one in figure 10a except that the base point B has been moved to a different leg of the same vertex. In this case, the base point is on a different edge from the one the path starts on, so the first step must be a clockwise move about the vertex:

- Starting from the base point B , move clockwise about the vertex onto the edge labelled E_1 .
- Traverse the edge labelled E_1 .
- Move anticlockwise around the next vertex onto the edge labelled E_2 .
- Traverse the edge labelled E_2 .
- Having arrived at the starting vertex, move clockwise around the vertex to arrive back at the base point.

More concisely, the path can be written right-to-left as $P_2 = \text{CW} \cdot T_2 \cdot \text{ACW} \cdot T_1 \cdot \text{CW}$, so the Möbius map $\gamma_2 = \phi(P_2)$ corresponding to this path is given as

$$\gamma_2 = \rho^{-1} \sigma_2 \rho \sigma_1 \rho^{-1} = \frac{1}{\sqrt{p_1} \sqrt{p_2}} \begin{pmatrix} p_1 & -(1+p_1) \\ p_1(1+p_2) & -(1+p_1(1+p_2)) \end{pmatrix}. \quad (2.24)$$

Note that this can be obtained from the map in eq. (2.23) after a conjugation by ρ . This means, in particular, that their multipliers are the same.

3. The loop depicted in figure 10c is almost the same as the one in figure 10b except that the path must cross the second vertex by making a clockwise turn, not an anticlockwise turn. So in this case, we must change the third step to

- Move *clockwise* around the next vertex onto the edge labelled E_2 .

while the other steps are unchanged. The path is now $P_3 = \text{CW} \cdot T_2 \cdot \text{CW} \cdot T_1 \cdot \text{CW}$ and thus the Möbius map $\gamma_3 = \phi(P_3)$ corresponding to this path is given as

$$\gamma_3 = \rho^{-1} \sigma_2 \rho^{-1} \sigma_1 \rho^{-1} = \frac{1}{\sqrt{p_1} \sqrt{p_2}} \begin{pmatrix} 0 & 1 \\ -p_1 p_2 & 1 + p_2(1+p_1) \end{pmatrix}. \quad (2.25)$$

4. The path shown in figure 10d corresponds to a local coordinate at the external puncture. The path, as always, starts at the external puncture X_1 and finishes at the base point B . It is composed of the following steps:

- Move anticlockwise about the vertex from the external puncture X_1 onto the edge labelled E_1 .
- Traverse the edge labelled E_1 .
- Move anticlockwise about the vertex onto the base point B .

The path can be written as $P_4 = \text{ACW} \cdot T_1 \cdot \text{ACW}$, and thus the corresponding local coordinate system $V_4 = \phi(P_4)$ is given by

$$V_4 = \rho \sigma_1 \rho = \frac{1}{\sqrt{p_1}} \begin{pmatrix} -1 - p_1 & 1 \\ -p_1 & 0 \end{pmatrix}. \quad (2.26)$$

The Koba-Nielsen variable z_4 corresponding to this puncture is given by

$$z_4 = V_4(0) = \infty. \quad (2.27)$$

5. Similarly, the path shown in figure 10e is an alternative choice of local coordinates for the same puncture, equivalent modulo the action of the Schottky group. In this case the path is given by $P_5 = \text{CW} \cdot T_2 \cdot \text{CW}$, so the associated coordinate chart $V_5 = \phi(P_5)$ is given by the Möbius map

$$V_5 = \rho^{-1} \sigma_2 \rho^{-1} = \frac{1}{\sqrt{p_2}} \begin{pmatrix} 0 & 1 \\ -p_2 & 1 + p_2 \end{pmatrix}, \quad (2.28)$$

so the Koba-Nielsen variable is given by

$$z_5 = V_5(0) = \frac{1}{1 + p_2}. \quad (2.29)$$

The two coordinate choices in the last two examples are related to each other by the action of the Schottky generator γ_2 given in eq. (2.24) (since these examples share a base point with the graph in figure 10b). We have

$$V_5 = \gamma_2 \cdot V_4 \quad (2.30)$$

and in particular

$$z_5 = \gamma_2(z_4). \quad (2.31)$$

In general, alternative choices of path P_a which give different Koba-Nielsen variables for the same puncture actually just correspond to different representatives of the same Schottky group equivalence class.

2.2 Neveu-Schwarz superstrings

The description of Riemann surfaces by Schottky groups can be extended to describe non-split super Riemann surfaces, with the restriction that only 2^g of the 2^{2g} possible spin-structures are admissible — roughly, those in which the odd coordinate is single-valued on the Schottky covering space, i.e. the a_i components of the ϑ -characteristics are fixed. In physics terms, these *super Schottky groups* [26, 29] can be used to describe RNS superstring worldsheets near degenerations in which all the pinched string channels belong to the NS sector.

See section 2.2 of [53] for a discussion of super Schottky groups; we will use the conventions and notation of that reference. Then to extend the approach we used in section 2.1, we need to describe how to give g super Schottky generators γ_i and n sets of local coordinates V_j , now from $\text{OSp}(1|2)$ instead of $\text{PSL}(2)$.

As in section 2.1, we give a homomorphism ϕ from the semigroup of paths between nodes in Γ into $\text{OSp}(1|2)$. To do this, we endow each of the $(2g + n - 2)$ 3-valent vertices v_i in the target graph Γ with a Grassmann-odd parameter θ_i . Also, we must pick a direction (arbitrarily) for each edge in Γ . Each path must be decomposed into *four* types of building block: now the traversal of an edge with Schwinger parameter t_k must take into account whether its sense is the same as, or opposite from, the edge's direction marking (denoted E_k and E_k^{-1} , respectively), and the two types of turn, ACW_i and CW_j , must also be labelled by the corresponding 3-valent vertex.

Then ϕ is given by

$$\phi(\text{ACW}_i) = \rho_{\theta_i} \equiv \left(\begin{array}{cc|c} -1 & 1 & -\theta_i \\ -1 & 0 & 0 \\ \hline -\theta_i & 0 & 1 \end{array} \right) \quad (2.32)$$

$$\phi(\text{CW}_i) = \rho_{\theta_i}^{-1} = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & -1 & \theta_i \\ \hline 0 & -\theta_i & 1 \end{array} \right) \quad (2.33)$$

$$\phi(E_j^{-1}) = \sigma_{\varepsilon_j}^{-1} \equiv \left(\begin{array}{cc|c} 0 & \varepsilon_j & 0 \\ -\varepsilon_j^{-1} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \quad (2.34)$$

$$\phi(E_j) = \sigma_{\varepsilon_j} \equiv \left(\begin{array}{cc|c} 0 & -\varepsilon_j & 0 \\ \varepsilon_j^{-1} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \quad (2.35)$$

(note that $\sigma_{\varepsilon_k} \in \text{OSp}(1|2)$ is not its own inverse, unlike $\sigma_{p_k} \in \text{PSL}(2)$). Just as in section 2.1, we choose a base node B and then write down an appropriate path for each loop and each external edge. Now using ϕ , we can write each path as an element of $\text{OSp}(1|2)$, giving us the g super Schottky group generators γ_i and the n local coordinates V_j .

The derivation of the parametrization is similar to the bosonic case, but some changes need to be made. Firstly, instead of 3-punctured discs, the surfaces are to be built from SRS discs with three NS punctures (NNN discs). Whereas 3-punctured discs have no

moduli, NNN discs have one odd supermodulus: if the three punctures have superconformal coordinates $\mathbf{x}_i = x_i|\xi_i$, then

$$\Theta_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3} \equiv \pm \frac{\xi_1(\mathbf{x}_2 \dot{-} \mathbf{x}_3) + \xi_2(\mathbf{x}_3 \dot{-} \mathbf{x}_1) + \xi_3(\mathbf{x}_1 \dot{-} \mathbf{x}_2) + \xi_1\xi_2\xi_3}{\sqrt{(\mathbf{x}_1 \dot{-} \mathbf{x}_2)(\mathbf{x}_2 \dot{-} \mathbf{x}_3)(\mathbf{x}_3 \dot{-} \mathbf{x}_1)}} \quad (2.36)$$

is a superprojective (pseudo)-invariant. Here $\dot{-}$ denotes the NS difference of two superpoints: if $\mathbf{a} = a|\alpha$ and $\mathbf{b} = b|\beta$ then

$$\mathbf{a} \dot{-} \mathbf{b} \equiv a - b - \alpha\beta. \quad (2.37)$$

$\Theta_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3}$ can be used as the odd supermodulus of the NNN disc. We can fix a global superconformal chart so the three punctures have homogeneous coordinates

$$(0, 1|0)^t, \quad (1, 0|0)^t, \quad (1, 1|\Theta_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3})^t. \quad (2.38)$$

Then eq. (2.32) and eq. (2.33) can be used to cycle between the three charts on an NNN disc whose pseudo-invariant is θ_i , in the same way that eq. (2.6) and eq. (2.7) cycle between the three charts on a 3-punctured disc.

The second change that needs to be made to the bosonic derivation is that the plumbing fixture eq. (2.8) needs to be replaced with a NS plumbing fixture. If $\mathbf{z} = z|\zeta$ and $\mathbf{w} = w|\psi$ are two superconformal charts with small discs cut out around $0|0$, then we can sew them together by identifying

$$z w = -\varepsilon^2, \quad z \psi = \varepsilon \zeta, \quad w \zeta = -\varepsilon \psi, \quad \psi \zeta = 0 \quad (2.39)$$

(eq. (6.31) of [47]). Here ε is a Grassmann-even parameter which behaves roughly as the square root of a bosonic pinching parameter; we call it the “NS pinching parameter” of the plumbing fixture. Eq. (2.39) is satisfied by taking $\mathbf{w} = \sigma_\varepsilon(\mathbf{z})$, where σ_ε is the superconformal function

$$\sigma_\varepsilon(\mathbf{z}) \equiv -\varepsilon^2/z \mid \varepsilon \zeta/z; \quad (2.40)$$

this function is equivalent to the $\text{OSp}(1|2)$ matrix in eq. (2.35).

The $(3g + n - 3)$ even parameters ε_i and the $(2g + n - 2)$ odd parameters θ_j can be used as local coordinates for super Schottky space.

2.2.1 Properties of the parametrization of supermoduli

The obvious analogues of properties 1 and 2 from section 2.1.3 also hold in the NS case. For hyperbolic $\text{OSp}(1|2)$ maps (such as super Schottky group elements), it is more useful to talk about *semimultipliers* than multipliers: each such map is conjugate to one of the form $z|\zeta \mapsto q^2 z|q\zeta$, then q is the semimultiplier (see section 2.2.3 of [53]). Thus properties 3 and 4 need to be replaced with expressions for the *semi*-multipliers of loops.

- Just as in the bosonic case, every super Schottky group element γ_α can be associated to a closed path $P_\alpha \in \pi_1(\Gamma, B)$ in the fundamental group of Γ . Furthermore, we can reduce the closed path P_α to a *loop* ℓ_α by cancelling out any consecutive pairs

of edges which are inverses of each other. The semimultiplier q_α of γ_α is given to leading order as

$$q_\alpha = (-1)^{n_\alpha} \left(\prod_{E_i \in \Gamma} \varepsilon_i^{m_\alpha^i} \right) (1 + \mathcal{O}(\varepsilon_j)), \quad (2.41)$$

where

$$\begin{aligned} n_\alpha &= \# \text{ of clockwise turns in } \ell_\alpha \\ &+ \# \text{ of edges that } \ell_\alpha \text{ crosses against the marked direction} \end{aligned} \quad (2.42)$$

and

$$m_\alpha^i = \# \text{ of times that } \ell_\alpha \text{ crosses the edge } E_i. \quad (2.43)$$

This formula is proved in appendix A.1.

- For paths P_α which are composed solely of CW or ACW turns, q_α is given exactly by the leading term of eq. (2.41):

$$q_\alpha = (-1)^{n_\alpha} \prod_{E_i \in \Gamma} \varepsilon_i^{m_\alpha^i}. \quad (2.44)$$

- The supermoduli ε_i characterize some of the boundaries of the Deligne-Mumford compactification of supermoduli space: the boundary corresponding to the pinching of an edge with gluing parameter ε_i is given by taking $\varepsilon_i \rightarrow 0$.

3 Graph polynomials from Riemann surfaces

Dai and Siegel [46] showed how Feynman graph integrands for scalar field theories can be computed in terms of geometric properties of the skeleton graph. In this section we show how the objects they used can be obtained as the $\alpha' \rightarrow 0$ limit of objects defined on (super) Riemann surfaces in terms of the parametrization given above, so long as we make the following identification between Schwinger times and gluing parameters:

$$p_i = e^{-t_i/\alpha'}. \quad (3.1)$$

Let Γ be a g -loop Schwinger-parametrized cubic Feynman graph with a chosen homology basis ℓ_i , then we can define the $g \times g$ graph period matrix (θ_{ij}) of Γ . First define an inner product measuring the degree of intersection of two paths:

$$\langle P, \hat{P} \rangle^k \equiv \begin{cases} 1 & \text{if } P \text{ and } \hat{P} \text{ cross } E_k \text{ in the same direction} \\ -1 & \text{if } P \text{ and } \hat{P} \text{ cross } E_k \text{ in opposite directions} \\ 0 & \text{if } P \text{ and } \hat{P} \text{ do not both cross } E_k. \end{cases} \quad (3.2)$$

Then the graph period matrix is given by

$$\theta_{ij} = \sum_k \langle \ell_i, \ell_j \rangle^k t_k, \quad (3.3)$$

where $i, j = 1, \dots, g$. Let X_a and X_b be two external edges on Γ , then choose a path P_{ab} from X_b to X_a . Then define a vector $\vec{v}$ by

$$v_i = \sum_k \langle \ell_i, P_{ab} \rangle^k t_k, \quad (3.4)$$

where the sum runs over all the edges. Finally, define a scalar s by

$$s = \sum_k \langle P_{ab}, P_{ab} \rangle^k t_k. \quad (3.5)$$

Dai and Siegel compute the Green's function as

$$G(X_a, X_b) = -\frac{1}{2}s + \frac{1}{2}\vec{v}^t \cdot \theta^{-1} \cdot \vec{v}. \quad (3.6)$$

This formula is analogous to the worldsheet Green's function commonly used in string theory [13]

$$\mathcal{G}(x_1, x_2) = \log |E(x_1, x_2)| - \frac{1}{2} \left(\int_{x_1}^{x_2} \vec{\omega} \right) \cdot (2\pi \text{Im}\tau)^{-1} \cdot \left(\int_{x_1}^{x_2} \vec{\omega} \right). \quad (3.7)$$

Here $E(x_1, x_2)$ is the Schottky-Klein prime form, given by

$$E(x_1, x_2) \equiv \frac{x_1 - x_2}{\sqrt{dx_1}\sqrt{dx_2}} \prod_{\alpha}' \frac{x_1 - \gamma_{\alpha}(x_2)}{x_1 - \gamma_{\alpha}(x_1)} \frac{x_2 - \gamma_{\alpha}(x_1)}{x_2 - \gamma_{\alpha}(x_2)}, \quad (3.8)$$

with the Schottky group product including one from each pair of inverse elements $\{\gamma_{\alpha}, \gamma_{\alpha}^{-1}\}$. ω_i , $i = 1, \dots, g$ is a basis of Abelian differentials and τ is the period matrix, defined by

$$\frac{1}{2\pi i} \oint_{a_j} \omega_i = \delta_{ij}; \quad \tau_{ij} = \frac{1}{2\pi i} \oint_{b_j} \omega_i. \quad (3.9)$$

where a_i and b_j are a basis of homology cycles normalized in the standard way [23]. Using the Schottky parametrization, $\omega_i(z)$ are given by the formula:

$$\omega_i(z) = \sum_{\alpha}^{(i)} \left(\frac{1}{z - \gamma_{\alpha}(u_i)} - \frac{1}{z - \gamma_{\alpha}(v_i)} \right) dz \quad (3.10)$$

where the sum is over all Schottky group elements whose right-most factor is not $\gamma_i^{\pm n}$. There is a Schottky group formula for the period matrix:

$$\tau_{ij} = \frac{1}{2\pi i} \left(\delta_{ij} \log k_i - \sum_{\gamma_{\alpha}}^{(i)} \log \frac{u_i - \gamma_{\alpha}(v_j)}{u_i - \gamma_{\alpha}(u_j)} \frac{v_i - \gamma_{\alpha}(u_j)}{v_i - \gamma_{\alpha}(v_j)} \right) \quad (3.11)$$

where the summation is over all Schottky group elements γ_{α} whose left-most factor is not $\gamma_i^{\pm n}$ and whose right-most factor is not $\gamma_j^{\pm n}$; the identity is excluded for $i = j$.

When using the pinching parametrization described in section 2, and choosing the paths from the external edges to the base node such that

$$P_{X_b} = P_{X_a} P_{ab}, \quad (3.12)$$

Figure 11. A non-planar graph Γ with $g = 2$ loops and $n = 2$ external edges (labelled X_1 and X_2). The internal edges are labelled E_i , $i = 1, \dots, 5$. A loop basis $\{\ell_1, \ell_2\}$ is indicated. The base point is marked as a black dot; the chosen paths from the external edges to the base point are marked as P_{X_1} and P_{X_2} .

we find that graph-theoretic objects defined on Γ arise as limits of objects defined on Σ :

$$s = \lim_{\alpha' \rightarrow 0} 2\alpha' \log \frac{E(z_a, z_b)}{\sqrt{V'_a(0)V'_b(0)}} \quad (3.13)$$

$$v_i = \lim_{\alpha' \rightarrow 0} \alpha' \int_{z_a}^{z_b} \vec{\omega}_i \quad (3.14)$$

$$\theta_{ij} = \lim_{\alpha' \rightarrow 0} 2\pi\alpha' \operatorname{Im} \tau_{ij}, \quad (3.15)$$

if we make the identification eq. (3.1).

This makes clear how Feynman graph integrands arise as limits of the string measure on moduli space (at least for scalar fields).

3.1 Example

Let us work out an example. We will pick a Feynman graph and show first of all in section 3.1.1 how its period matrix and worldline Green's function can be written down by inspecting the graph *à la* Dai and Siegel, and then in section 3.1.2 how the same quantities arise asymptotically from objects on moduli space.

3.1.1 By inspection

Consider the $n = 2$, $g = 2$ non-planar graph Γ shown in figure 11. The paths P_{X_1} , P_{X_2} from the two external edges X_1 and X_2 to the base point are indicated; the paths P_{ℓ_1} and P_{ℓ_2} corresponding to the loop basis can be taken as the natural choice (since both loops pass through the vertex at which the base point lies).

First of all, let us compute the graph period matrix (θ_{ij}) . The diagonal entries θ_{ii} are just the sums of the Schwinger parameters of the edges in the loops ℓ_i , so we have

$$\theta_{11} = t_1 + t_3 + t_4, \quad \theta_{22} = t_2 + t_3 + t_5. \quad (3.16)$$

Figure 12. The Feynman graph in figure 11 can be used to construct a Riemann surface with pinching parameters arranged in such a way that the graph is recovered in the $\alpha' \rightarrow 0$ limit, after matching pinching parameters to Schwinger parameters according to eq. (3.1).

The two loops ℓ_1 and ℓ_2 intersect on one edge, E_3 , and they both cross it in the same direction so we have

$$\theta_{12} = \theta_{21} = t_3. \quad (3.17)$$

The first Symanzik polynomial can be computed as

$$\det \theta = (t_1 + t_4)(t_2 + t_5) + t_3(t_1 + t_2 + t_4 + t_5). \quad (3.18)$$

Next let us compute the Green's function *à la* Dai and Siegel [46]. We need to pick a path from X_2 to X_1 ; let us use $P_{X_1 X_2} \equiv P_{X_1}^{-1} \cdot P_{X_2}$. Then since $P_{X_1 X_2}$ crosses the edges E_2 and E_1 , we have

$$s = t_1 + t_2. \quad (3.19)$$

$P_{X_1 X_2}$ intersects ℓ_1 along the edge E_1 , which they both traverse in the same direction, while $P_{X_1 X_2}$ intersects ℓ_2 along the edge E_2 which they both traverse in opposite directions. Therefore the vector $\vec{v}$ is given by

$$v_1 = t_1, \quad v_2 = -t_2. \quad (3.20)$$

Plugging eqs. (3.16)–(3.20) into eq. (3.6), we get the following expression for the worldline Green's function:

$$G = -\frac{t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4 + t_1 t_2 t_5 + t_1 t_3 t_5 + t_2 t_3 t_5 + t_1 t_4 t_5 + t_2 t_4 t_5}{2 \det \theta}. \quad (3.21)$$

3.1.2 From the zero-slope limit

Let us now verify that our approach gives the same answer. We can use the graph Γ in figure 11 to construct a bordered Riemann surface figure 12 parametrized by pinching parameters $\{p_i\}$ corresponding to the graph edges E_i . Then we can write down expressions for the period matrix and the terms in the Green's function, use eq. (3.1) to take the zero-slope limit, and verify that eqs. (3.13) – (3.15) hold.

The paths in figure 11 can be expressed as

$$P_{X_1} = \text{CW} \cdot \text{T}_1 \cdot \text{CW} \quad (3.22)$$

$$P_{X_2} = \text{T}_2 \cdot \text{CW} \quad (3.23)$$

$$P_{\ell_1} = \text{ACW} \cdot \text{T}_3 \cdot \text{ACW} \cdot \text{T}_4 \cdot \text{CW} \cdot \text{T}_1 \cdot \text{ACW} \quad (3.24)$$

$$P_{\ell_2} = \text{ACW} \cdot \text{T}_3 \cdot \text{CW} \cdot \text{T}_5 \cdot \text{CW} \cdot \text{T}_2. \quad (3.25)$$

Then according to our prescription, the corresponding bordered Riemann surface (figure 12) can be parameterized by the Möbius maps $V_i = \phi(P_{X_i})$, $\gamma_j = \phi(P_{\ell_j})$, giving

$$V_1 = \rho^{-1} \cdot \sigma_1 \cdot \rho^{-1}, \quad (3.26)$$

$$V_2 = \sigma_2 \cdot \rho^{-1}, \quad (3.27)$$

$$\gamma_1 = \rho \cdot \sigma_3 \cdot \rho \cdot \sigma_4 \cdot \rho^{-1} \cdot \sigma_1 \cdot \rho, \quad (3.28)$$

$$\gamma_2 = \rho \cdot \sigma_3 \cdot \rho^{-1} \cdot \sigma_5 \cdot \rho^{-1} \cdot \sigma_2. \quad (3.29)$$

The six Schottky group moduli can be computed from γ_1 and γ_2 ; they are given in terms of the p_i 's by

$$k_1 = \frac{F_{413} - \sqrt{F_{413}^2 - 4p_1 p_3 p_4}}{F_{413} + \sqrt{F_{413}^2 - 4p_1 p_3 p_4}} = p_1 p_3 p_4 + \mathcal{O}(p_i^2), \quad (3.30)$$

$$k_2 = \frac{F_{352} - \sqrt{F_{352}^2 - 4p_2 p_3 p_5}}{F_{352} + \sqrt{F_{352}^2 - 4p_2 p_3 p_5}} = p_2 p_3 p_5 + \mathcal{O}(p_i^2), \quad (3.31)$$

$$u_1 = \frac{1}{1+p_1} + \frac{F_{413} + \sqrt{F_{413}^2 - 4p_1 p_3 p_4}}{2(1+p_1)p_3 p_4} = \frac{1}{p_3 p_4} (1 + \mathcal{O}(p_i)), \quad (3.32)$$

$$v_1 = \frac{1}{1+p_1} + \frac{F_{413} - \sqrt{F_{413}^2 - 4p_1 p_3 p_4}}{2(1+p_1)p_3 p_4} = 1 + \mathcal{O}(p_i), \quad (3.33)$$

$$u_2 = -\frac{p_2 p_5}{1+p_5} + \frac{F_{352} + \sqrt{F_{352}^2 - 4p_2 p_3 p_5}}{2p_3(1+p_5)} = \frac{1}{p_3} (1 + \mathcal{O}(p_i)), \quad (3.34)$$

$$v_2 = -\frac{p_2 p_5}{1+p_5} + \frac{F_{352} - \sqrt{F_{352}^2 - 4p_2 p_3 p_5}}{2p_3(1+p_5)} = -p_2 p_3 p_5 + \mathcal{O}(p_i^2), \quad (3.35)$$

where we've used the notation

$$F_{abc\dots} \equiv 1 + p_a F_{bc\dots}; \quad F_a \equiv 1 + p_a. \quad (3.36)$$

The Koba-Nielsen variables $z_i = V_i(0)$ are given by

$$z_1 = \frac{1}{1+p_1}, \quad z_2 = -p_2. \quad (3.37)$$

We also have

$$V_1'(0) = \frac{p_1}{(1+p_1)^2}, \quad V_2'(0) = p_2. \quad (3.38)$$

First of all we can compute the period matrix using eq. (3.11). At leading order, the diagonal elements come only from the Schottky group multipliers: from eq. (3.30) and eq. (3.31) we get

$$\tau_{11} = \frac{1}{2\pi i} \log(p_1 p_3 p_4) + \mathcal{O}(p_i), \quad \tau_{22} = \frac{1}{2\pi i} \log(p_2 p_3 p_5) + \mathcal{O}(p_i). \quad (3.39)$$

To compute the diagonal entries $\tau_{12} = \tau_{21}$, we use the second term in eq. (3.11). For this graph, the only non-vanishing contribution to the sum at leading order in the p_i comes from the identity element of the Schottky group. After substituting the expressions for the fixed points we computed in eq. (3.32)–eq. (3.35) we get

$$\begin{aligned} \tau_{12} &= -\frac{1}{2\pi i} \log\left(\frac{u_1 - v_2}{u_1 - u_2} \frac{v_1 - u_2}{v_1 - v_2}\right) + \mathcal{O}(p_i) = -\frac{1}{2\pi i} \log\left(-\frac{1}{p_3}\right) + \mathcal{O}(p_i) \\ &= \frac{1}{2\pi i} \log p_3 + n + \frac{1}{2} + \mathcal{O}(p_i), \end{aligned} \quad (3.40)$$

where we've written $\log(-1) = (2n+1)\pi$. Then we can check that eq. (3.15) holds. We have

$$2\pi \operatorname{Im}(\tau_{ij}) = \begin{pmatrix} -\log(p_1 p_3 p_4) & -\log p_3 \\ -\log p_3 & -\log(p_2 p_3 p_5) \end{pmatrix} + \mathcal{O}(p_i). \quad (3.41)$$

After making the substitution in eq. (3.1) and multiplying by α' , we get

$$2\pi \alpha' \operatorname{Im}(\tau_{ij}) = \begin{pmatrix} t_1 + t_3 + t_4 & t_3 \\ t_3 & t_2 + t_3 + t_5 \end{pmatrix} + \mathcal{O}(e^{-t_i/\alpha'}). \quad (3.42)$$

In the $\alpha' \rightarrow 0$ limit, the correction vanishes and we arrive at the graph period matrix (θ_{ij}) given in eq. (3.16) and eq. (3.17), showing that eq. (3.15) holds in this case.

Next let's check eq. (3.13), for the 'scalar' term s in Dai and Siegel's approach to the worldline Green's function. First we can compute the numerator. To leading order in the p_i 's the only contribution to the Schottky-Klein prime form eq. (3.8) comes from the first factor, so we have

$$\log E(z_1, z_2) = \log(z_1 - z_2) + \mathcal{O}(p_i). \quad (3.43)$$

The Koba-Nielsen variables z_i can be substituted from eq. (3.37) giving

$$\log E(z_1, z_2) = \log\left(\frac{1}{1+p_1} + p_2\right) + \mathcal{O}(p_i) = \mathcal{O}(p_i), \quad (3.44)$$

i.e. the numerator vanishes at leading order and the only contribution comes from the weights in the denominators. Substituting from eq. (3.38), we get

$$\log \frac{1}{\sqrt{V'_1(0)V'_2(0)}} = \log \frac{1+p_1}{(p_1 p_2)^{1/2}} + \mathcal{O}(p_i) \quad (3.45)$$

$$= -\frac{1}{2} \log p_1 p_2 + \mathcal{O}(p_i). \quad (3.46)$$

Combining eq. (3.44) and eq. (3.46), making the substitution in eq. (3.1) and multiplying by $2\alpha'$ we get

$$2\alpha' \log \frac{E(z_1, z_2)}{\sqrt{V_1'(0)V_2'(0)}} = t_1 + t_2 + \mathcal{O}(e^{-t_i/\alpha'}). \quad (3.47)$$

Taking $\alpha' \rightarrow 0$, the correction vanishes and we recover s from eq. (3.19) as expected, verifying eq. (3.13) for this example.

Last of all, we need to check that eq. (3.14) holds. We can integrate eq. (3.10) to get

$$\int_{z_2}^{z_1} \omega_i(z) = \sum_{\alpha}^{(i)} \left[\log \frac{z - \gamma_{\alpha}(u_i)}{z - \gamma_{\alpha}(v_i)} \right]_{z_2}^{z_1}. \quad (3.48)$$

At leading order in the p_i 's, the only Schottky group element that gives a non-vanishing contribution to the sum is the identity, so we obtain

$$\int_{z_2}^{z_1} \omega_i(z) = \log \frac{z_1 - u_i}{z_1 - v_i} \frac{z_2 - v_i}{z_2 - u_i} + \mathcal{O}(p_i). \quad (3.49)$$

Substituting the Koba-Nielsen variables from eq. (3.37) and the fixed points from eq. (3.32)–eq. (3.35), we find

$$\int_{z_2}^{z_1} \vec{\omega}(z) = (-\log p_1, \log p_2 + (2n+1)i\pi)^t + \mathcal{O}(p_i). \quad (3.50)$$

Multiplying by α' and replacing the pinching parameters with Schwinger parameters according to eq. (3.1), we get

$$\alpha' \int_{z_2}^{z_1} \vec{\omega}(z) = (t_1, -t_2 + i\pi\alpha'(2n+1))^t + \mathcal{O}(e^{-t_i/\alpha}). \quad (3.51)$$

Taking the limit $\alpha' \rightarrow 0$, we recover $\vec{v}$ as given in eq. (3.20), so we've also seen that eq. (3.14) holds too as expected. This completes the computation demonstrating that eqs. (3.13) – (3.14) all hold for our example.

There is nothing special about our choice of example. A proof that this type of computation would give the expected results in the general case is given in sections A.2 and A.3.

4 Duality transformations

We say two cubic skeleton graphs are ‘dual’ to each other (in the sense of dual resonance models) if one can be obtained from the other by (repeatedly) cutting out a subgraph of the form of figure 13a and replacing it with one of the form of figure 13b, respecting the labels of the cut edges. This is sometimes known as a *Whitehead move*.

If two graphs are dual to each other, then their corresponding pinching parametrizations can be mapped onto each other with a change of variables, which we compute in this section. We work out the change of variables needed in the super-geometric case, since the bosonic case follows as a corollary by ignoring the odd parameters and taking $p_i = \epsilon_i^2$.

Figure 13. Two four-point tree graphs which are dual to each other (in the sense of dual-resonance models).

Figure 14. An example cubic graph with $n = 5$, $g = 2$ (Figure 14a) and the intermediate graphs (figure 14b, figure 14c) obtained via duality transformations in order to put it in the canonical form (figure 14d). In figures 14a–14c, edges drawn with red dots are those which are duality-transformed to arrive at the successive graph.

Any two planar cubic graphs with the same number of loops and the same number of external edges labelled in the same order are dual to each other. This means that any planar cubic graph is dual to one of the form in figure 19. Figure 14 shows an example with $n = 5$, $g = 2$, illustrating the duality transformations needed to get from the graph in figure 14a to the canonical one in figure 14d.

As described in section 2.1, each g -loop, n -point skeleton graph induces a set of (super) moduli for a worldsheet with the same number of loops and n punctures. Thus, the notion of duality between graphs gives a natural notion of duality between sets of pinching moduli p_i . We can derive the algebraic relationship between the two sets of moduli.

4.1 Duality on internal edges

First of all, let us consider the case where all of the edges involved in the duality transformation are internal edges with associated Schwinger times t_i . Consider a subgraph of a target graph, as shown in figure 15a, and the subgraph it is dual to, shown in figure 15b.

We are considering the supergeometric case, so we have labelled the vertices involved in the duality transformation with odd supermoduli $\varphi_a, \varphi_b, \varphi_1$ and φ_2 , and we have given the involved edges an orientation.

We have indicated three paths P_α, P_β and P_γ on figure 15a. Any path which traverses this subgraph may be written in terms of these three paths. On figure 15b, we have indicated three paths $P_{\tilde{\alpha}}, P_{\tilde{\beta}}$ and $P_{\tilde{\gamma}}$, which correspond to the matching ones on figure 15a in the sense that they join the same pairs of vertices. If we can find a relationship between the dual supermoduli which solves

$$\phi(P_\alpha) = \phi(P_{\tilde{\alpha}}), \quad \phi(P_\beta) = \phi(P_{\tilde{\beta}}), \quad \phi(P_\gamma) = \phi(P_{\tilde{\gamma}}), \quad (4.1)$$

then the super Schottky group generators γ_ℓ and local co-ordinate charts V_i induced by the two dual graphs will be identical, and thus describe the same surface. Thus, eq. (4.1), gives us a mapping between dual sets of pinching supermoduli.

According to the prescription in section 2.2, eq. (4.1) we have

$$\phi(P_\alpha) = \sigma_2^{-1} \rho_a^{-1} \sigma_3 \rho_b^{-1} \sigma_5 \quad (4.2)$$

$$= \frac{1}{\varepsilon_2 \varepsilon_3 \varepsilon_5} \left(\begin{array}{cc|c} p_2(1+p_3-\varepsilon_3\varphi_a\varphi_b) & p_2p_3p_5 & p_2\varepsilon_3\varepsilon_5(\varphi_a-\varepsilon_3\varphi_b) \\ -1 & 0 & 0 \\ \varepsilon_2(\varphi_a-\varepsilon_3\varphi_b) & 0 & \varepsilon_2\varepsilon_3\varepsilon_5 \end{array} \right) \quad (4.3)$$

$$\phi(P_{\tilde{\alpha}}) = \sigma_d^{-1} \rho_2^{-1} \sigma_e \quad (4.4)$$

$$= \frac{1}{\varepsilon_d \varepsilon_e} \left(\begin{array}{cc|c} -p_d & -p_d p_e & p_d \varepsilon_e \varphi_2 \\ 1 & 0 & 0 \\ -\varepsilon_d \varphi_2 & 0 & \varepsilon_d \varepsilon_e \end{array} \right) \quad (4.5)$$

$$\phi(P_\beta) = \sigma_5^{-1} \rho_b^{-1} \sigma_4 \quad (4.6)$$

$$= \frac{1}{\varepsilon_4 \varepsilon_5} \left(\begin{array}{cc|c} -p_5 & -p_4 p_5 & p_5 \varepsilon_4 \varphi_b \\ 1 & 0 & 0 \\ -\varepsilon_5 \varphi_b & 0 & \varepsilon_4 \varepsilon_5 \end{array} \right) \quad (4.7)$$

$$\phi(P_{\tilde{\beta}}) = \sigma_e^{-1} \rho_2^{-1} \sigma_c \rho_1^{-1} \sigma_b \quad (4.8)$$

$$= \frac{1}{\varepsilon_b \varepsilon_c \varepsilon_e} \left(\begin{array}{cc|c} p_e(1+p_c+\varepsilon_c\varphi_1\varphi_2) & p_b p_c p_e & p_e \varepsilon_b \varepsilon_c (\varphi_2 - \varepsilon_c \varphi_1) \\ -1 & 0 & 0 \\ \varepsilon_e(\varphi_2 - \varepsilon_c \varphi_1) & 0 & \varepsilon_b \varepsilon_c \varepsilon_e \end{array} \right) \quad (4.9)$$

$$\phi(P_\gamma) = \sigma_4^{-1} \rho_b^{-1} \sigma_3^{-1} \rho_a^{-1} \sigma_1 \quad (4.10)$$

$$= \frac{1}{\varepsilon_1 \varepsilon_3 \varepsilon_4} \left(\begin{array}{cc|c} -p_4(1+p_3-\varepsilon_3\varphi_a\varphi_b) & -p_1 p_3 p_4 & p_4 \varepsilon_1 \varepsilon_3 (\varphi_b + \varepsilon_3 \varphi_a) \\ 1 & 0 & 0 \\ -\varepsilon_4(\varphi_b + \varepsilon_3 \varphi_a) & 0 & \varepsilon_1 \varepsilon_3 \varepsilon_4 \end{array} \right) \quad (4.11)$$

$$\phi(P_{\tilde{\gamma}}) = \sigma_b^{-1} \rho_1^{-1} \sigma_a \quad (4.12)$$

$$= \frac{1}{\varepsilon_a \varepsilon_b} \left(\begin{array}{cc|c} -p_b & -p_a p_b & p_b \varepsilon_a \varphi_1 \\ 1 & 0 & 0 \\ -\varepsilon_b \varphi_1 & 0 & \varepsilon_a \varepsilon_b \end{array} \right), \quad (4.13)$$

Figure 15. Two dual sub-graphs of a cubic graph. We can find the relationship between the two sets of supermoduli $(t_1, \dots, t_5 | \varphi_a, \varphi_b)$ for figure 15a and $(t_a, \dots, t_e | \varphi_1, \varphi_2)$ for figure 15b) by identifying the corresponding paths as described in the text.

where we have written $p_i \equiv \varepsilon_i^2$. Using eq. (4.3)–eq. (4.13), we see that eq. (4.1) requires the simultaneous solution of 15 equations (9 Grassmann even and 6 Grassmann odd); since we have only 5|2 supermoduli to solve for, this is a non-trivial check of the consistency of our approach. The solution exists and is given by

$$\begin{aligned} \varepsilon_a &= \pm \frac{\varepsilon_1 \varepsilon_3}{M}, & \varepsilon_b &= \pm \varepsilon_4 M, & \varepsilon_c &= \pm \frac{1}{\varepsilon_3}, & \varepsilon_d &= \varepsilon_2 M, & \varepsilon_e &= -\frac{\varepsilon_3 \varepsilon_5}{M}, \\ \varphi_1 &= \pm \frac{\varphi_b + \varepsilon_3 \varphi_a}{M}, & \varphi_2 &= \frac{\varphi_a - \varepsilon_3 \varphi_b}{M}, \end{aligned} \quad (4.14)$$

where M is one of the two roots of

$$M^2 = 1 + p_3 - \varepsilon_3 \varphi_a \varphi_b, \quad (4.15)$$

giving four branches to the solution (the $\pm$ signs must be chosen consistently). Thus, once we have found a set of pinching moduli $(\varepsilon_i | \varphi_j)$ corresponding to some skeleton graph Γ , we can compute the pinching supermoduli $(\hat{\varepsilon}_i | \hat{\varphi}_j)$ for a dual graph $\hat{\Gamma}$ with the (repeated) use of eq. (4.14).

We have assumed in this calculation that all of the edges involved in the duality transformation are distinct from each other. In section 4.3 we describe the case in which two edges are identified such that there is a loop (as in the transformation from figure 14c to figure 14d), which has a slightly different solution.

4.2 Duality with external edges

If some of the edges involved in the duality transformation are external edges, then a modified approach is needed. Recall that each external edge on the skeleton graph Γ is associated to a puncture on the surface Σ located at $z_i = \mathbf{V}_i(0|0)$, where $\mathbf{V}_i$ is computed as described in section 2.2. Thus, we want to find a relationship between the dual sets of pinching supermoduli $(\varepsilon_i | \varphi_j)$, $(\hat{\varepsilon}_i | \hat{\varphi}_j)$ such that we leave unchanged all paths traversing the subgraph and also such that the positions of the punctures on Σ are unchanged, $\mathbf{V}_i(0|0) = \hat{\mathbf{V}}_i(0|0)$.

Figure 16. Two dual sub-graphs of a target graph with an external edge in common. The subgraphs here each have one less even supermodulus than those in figure 15. To find the duality relation we need to fix z_1 by satisfying $\hat{P}_\gamma(0|0) = \hat{P}_{\tilde{\gamma}}(0|0)$.

For example, consider a duality transformation when one of the involved edges is an external edge, as in figure 16. The paths P_α and P_β in figure 16a and $P_{\tilde{\alpha}}$, $P_{\tilde{\beta}}$ in figure 16b are identical to those in figure 15. Again, solving the equations

$$\phi(P_\alpha) = \phi(P_{\tilde{\alpha}}), \quad \phi(P_\beta) = \phi(P_{\tilde{\beta}}) \quad (4.16)$$

will ensure that any path traversing the subgraphs is unaffected by the duality transformation.

But now there is a new constraint of a different type: we need the position of z_1 — the position of the puncture corresponding to the external edge X_1 — to be fixed by the duality transformation. Recall that $z_1 = \mathbf{V}_1(0|0)$, where $\mathbf{V}_1 = \phi(P_1)$ with P_1 some path which starts at z_1 and ends at the base point. Any path which starts at z_1 on figure 16a can be written with $\hat{P}_\gamma$ as its right-most factor. For figure 16b, we can use the same path, but with the right-most factor changed from $\hat{P}_\gamma$ to $\hat{P}_{\tilde{\gamma}}$ and any factors of P_α and P_β changed to $P_{\tilde{\alpha}}$ and $P_{\tilde{\beta}}$, respectively.

Then

$$\phi(\hat{P}_\gamma)(0|0) = \phi(\hat{P}_{\tilde{\gamma}})(0|0) \quad (4.17)$$

is a sufficient condition to guarantee that z_1 is fixed by duality (as long as eq. (4.16) also holds). The two matrices in eq. (4.17) are given by

$$\phi(\hat{P}_\gamma) = \sigma_4^{-1} \rho_b^{-1} \sigma_3^{-1} \rho_a^{-1} \quad (4.18)$$

$$= \left(\begin{array}{cc|c} \varepsilon_3 \varepsilon_4 & -\varepsilon_4 M^2 / \varepsilon_3 & \varepsilon_3 \varepsilon_4 \varphi_a + \varepsilon_4 \varphi_b \\ 0 & 1 / \varepsilon_3 \varepsilon_4 & 0 \\ 0 & -\varphi_a - \varphi_b / \varepsilon_3 & 1 \end{array} \right) \quad (4.19)$$

$$\phi(\hat{P}_{\tilde{\gamma}}) = \sigma_b^{-1} \rho_1^{-1} \quad (4.20)$$

$$= \left(\begin{array}{cc|c} \varepsilon_b & -\varepsilon_b & \varepsilon_b \varphi_1 \\ 0 & 1 / \varepsilon_b & 0 \\ 0 & -\varphi_1 & 1 \end{array} \right) \quad (4.21)$$

and then the images of $0|0$ are computed as

$$\phi(\hat{P}_\gamma) \cdot (0 \ 1|0)^t = (-\varepsilon_4 M^2/\varepsilon_3, \ 1/\varepsilon_3 \varepsilon_4 | -\varphi_a - \varphi_b/\varepsilon_3)^t \quad (4.22)$$

$$\sim -p_4 M^2 | -\varepsilon_3 \varepsilon_4 \varphi_a - \varepsilon_4 \varphi_b \quad (4.23)$$

and

$$\phi(\hat{P}_{\tilde{\gamma}}) \cdot (0 \ 1|0)^t = (-\varepsilon_b, \ 1/\varepsilon_b | -\varphi_1)^t \quad (4.24)$$

$$\sim -p_b | -\varepsilon_b \varphi_1. \quad (4.25)$$

It can be easily checked that eq. (4.14) satisfies eq. (4.17).

This follows automatically from the fact that eq. (4.14) solves $\phi(P_\gamma) = \phi(P_{\tilde{\gamma}})$. To see why this is true, note that we have

$$\phi(P_\gamma) = \phi(\hat{P}_\gamma) \cdot \sigma_1, \quad \phi(P_{\tilde{\gamma}}) = \phi(\hat{P}_{\tilde{\gamma}}) \cdot \sigma_a, \quad (4.26)$$

so

$$\phi(\hat{P}_{\tilde{\gamma}}) = \phi(\hat{P}_\gamma) \cdot \sigma_1 \cdot \sigma_a^{-1}. \quad (4.27)$$

But $0|0$ is fixed by $\sigma_1 \cdot \sigma_a^{-1}$, and thus eq. (4.14) solves eq. (4.17).

Note that since ε_1 and ε_a are not parameters of figure 16a and figure 16b, respectively, the first equation in eq. (4.14) ($\varepsilon_a = \pm \varepsilon_1 \varepsilon_3 / M$) is to be simply ignored.

For similar reasons, the relevant equations in eq. (4.14) continue to give the relationship between dual sets of pinching supermoduli when two, three or four of the involved edges are external (the edges labelled t_3 and t_c must be internal edges). Furthermore, the solutions in section 4.3 also reduce to give solutions in the cases where one or both of the unconnected edges are taken to be external.

4.3 Duality involving loops

4.3.1 Loop connecting adjacent edges

It is also possible to compute dual pinching supermoduli when a loop is involved in the duality transformation, in the sense that one of the involved edges connects one of the involved vertices back to itself or to the other involved vertex as in figure 17. The paths P_β and $P_{\tilde{\beta}}$ in figure 17a and figure 17b, respectively, are identical to those in figure 15; thus we still need to solve

$$\phi(P_\beta) = \phi(P_{\tilde{\beta}}) \quad (4.28)$$

where the matrices for $\phi(P_\beta)$ and $\phi(P_{\tilde{\beta}})$ are given in eq. (4.7) and eq. (4.9), respectively. But now we have the paths P_δ in figure 17a and $P_{\tilde{\delta}}$ in figure 17b so we need to solve

$$\phi(P_\delta) = \phi(P_{\tilde{\delta}}) \quad (4.29)$$

These paths are represented by the matrices

$$\phi(P_\delta) = \sigma_4^{-1} \rho_b^{-1} \sigma_3^{-1} \rho_a^{-1} \sigma_1 \rho_a^{-1} \sigma_3 \rho_b^{-1} \sigma_5 \quad (4.30)$$

Figure 17. Two dual subgraphs in which a loop is involved in the duality.

and

$$\phi(P_{\tilde{\delta}}) = \sigma_b^{-1} \rho_1^{-1} \sigma_a \rho_2^{-1} \sigma_e. \quad (4.31)$$

As always, $\phi(P_\delta)$ is of the form

$$\phi(P) = \left(\begin{array}{c|cc} x & y & \xi \\ -1/y & 0 & 0 \\ \hline \xi/y & 0 & 1 \end{array} \right) \quad (4.32)$$

with

$$(\phi(P_\delta))_1^1 = \frac{\varepsilon_3(1 + p_1 p_3 - \varepsilon_1(1 + p_3))\varphi_a \varphi_b - (1 + p_3)(1 + p_1 p_3)}{\varepsilon_1 \varepsilon_3^2 \varepsilon_5} \varepsilon_4 \quad (4.33)$$

$$(\phi(P_\delta))_1^2 = -\varepsilon_1 \varepsilon_4 \varepsilon_5 p_3 \quad (4.34)$$

$$(\phi(P_\delta))_1^3 = (1 - \varepsilon_1)\varepsilon_3 \varepsilon_4 \varphi_a + (1 + \varepsilon_1 p_3)\varepsilon_4 \varphi_b. \quad (4.35)$$

$\phi(P_{\tilde{\delta}})$ is of the same form with

$$(\phi(P_{\tilde{\delta}}))_1^1 = \frac{1 + p_a - \varepsilon_a \phi_1 \phi_2}{\varepsilon_a \varepsilon_e} \varepsilon_b \quad (4.36)$$

$$(\phi(P_{\tilde{\delta}}))_1^2 = \varepsilon_a \varepsilon_b \varepsilon_e \quad (4.37)$$

$$(\phi(P_{\tilde{\delta}}))_1^3 = \varepsilon_b \varphi_1 - \varepsilon_a \varepsilon_b \varphi_2. \quad (4.38)$$

Eq. (4.28) and eq. (4.29) are solved by:

$$\varepsilon_a = \mp \varepsilon_1 \varepsilon_3, \quad \varepsilon_b = \pm M \varepsilon_4, \quad \varepsilon_c = \mp \frac{1}{\varepsilon_3}, \quad \varepsilon_e = \frac{\varepsilon_3 \varepsilon_5}{M}, \quad (4.39)$$

$$\varphi_1 = \pm \frac{\varepsilon_3 \varphi_a + \varphi_b}{M} \quad \varphi_2 = -\frac{\varphi_a - \varepsilon_3 \varphi_b}{M}, \quad (4.40)$$

where M is defined in eq. (4.15). Notice that we have $\varepsilon_1 = \varepsilon_a \varepsilon_c$, corresponding to the multiplier of the loop. There is another solution

$$\varepsilon_a = \mp \varepsilon_3, \quad \varepsilon_b = \pm \hat{M} \varepsilon_4, \quad \varepsilon_c = \mp \frac{1}{\varepsilon_1 \varepsilon_3}, \quad \varepsilon_e = \frac{\varepsilon_1 \varepsilon_3 \varepsilon_5}{\hat{M}}, \quad (4.41)$$

$$\varphi_1 = \mp \frac{\varepsilon_1 \varepsilon_3 \varphi_a - \varphi_b}{\hat{M}}, \quad \varphi_2 = \frac{\varphi_a + \varepsilon_1 \varepsilon_3 \varphi_b}{\hat{M}}, \quad (4.42)$$

Figure 18. Two dual subgraphs with a pair of opposite edges joined to form a loop.

where

$$\hat{M}^2 = 1 + p_1 p_3 + \varepsilon_1 \varepsilon_3 \varphi_a \varphi_b. \quad (4.43)$$

However, this solution has $\varepsilon_a \varepsilon_c = 1/\varepsilon_1$, so it is not relevant to the pinching limit we want, since both sides of this equation should get small simultaneously as the Schwinger time length of the loop gets large.

4.3.2 Loop connecting opposite edges

It is also possible to compute the duality relation in the case that two opposite edges are connected in a loop, as depicted in figure 18. To find the duality relation, we need to solve

$$\phi(P_\xi) = \phi(P_{\tilde{\xi}}), \quad \phi(P_\zeta) = \phi(P_{\tilde{\zeta}}). \quad (4.44)$$

The first two matrices here are given by

$$\begin{aligned} \phi(P_\xi) &= \sigma_5^{-1} \rho_b \sigma_3^{-1} \rho_a^{-1} \sigma_1 \\ &= \left(\begin{array}{cc|c} \varepsilon_3 \varepsilon_5 / \varepsilon_1 & \varepsilon_1 \varepsilon_3 \varepsilon_5 & -\varepsilon_3 \varepsilon_5 \varphi_a \\ M^2 / \varepsilon_1 \varepsilon_3 \varepsilon_5 & -\varepsilon_1 \varepsilon_3 / \varepsilon_5 & (\varepsilon_3 \varphi_a + \varphi_b) / \varepsilon_5 \\ -(\varphi_a - \varepsilon_3 \varphi_b) / \varepsilon_1 & \varepsilon_1 \varepsilon_3 \varphi_b & 1 + \varepsilon_3 \varphi_a \varphi_b \end{array} \right) \end{aligned} \quad (4.45)$$

$$\begin{aligned} \phi(P_{\tilde{\xi}}) &= \sigma_e^{-1} \rho_2^{-1} \sigma_c \rho_1 \sigma_a \\ &= \left(\begin{array}{cc|c} -\varepsilon_e / \varepsilon_a \varepsilon_c & -(1 + p_c + \varepsilon_c \varphi_1 \varphi_2) \varepsilon_a \varepsilon_e / \varepsilon_c & (\varphi_1 + \varepsilon_c \varphi_2) \varepsilon_e / \varepsilon_c \\ 1 / \varepsilon_a \varepsilon_c \varepsilon_e & \varepsilon_a / \varepsilon_c \varepsilon_e & -\varphi_1 / \varepsilon_c \varepsilon_e \\ -\varphi_2 / \varepsilon_a \varepsilon_c & (\varepsilon_c \varphi_1 - \varphi_2) \varepsilon_a / \varepsilon_c & 1 - \varphi_1 \varphi_2 / \varepsilon_c \end{array} \right). \end{aligned} \quad (4.46)$$

The second two are given by

$$\phi(P_\zeta) = \sigma_1^{-1} \rho_a^{-1} \sigma_2 \rho_b^{-1} \sigma_3^{-1} \rho_a^{-1} \sigma_1, \quad (4.47)$$

$$\phi(P_{\tilde{\zeta}}) = \sigma_a^{-1} \rho_1^{-1} \sigma_c^{-1} \rho_2^{-1} \sigma_b^{-1} \rho_1^{-1} \sigma_a, \quad (4.48)$$

which are of the form eq. (4.32) (since P_{ζ} and $P_{\bar{\zeta}}$ contain only clockwise turns), where

$$(\phi(P_{\zeta}))_1^1 = \frac{1 + p_2(1 + p_3) - \varepsilon_2(1 + \varepsilon_2 \varepsilon_3)\varphi_a \varphi_b}{\varepsilon_2 \varepsilon_3} \quad (4.49)$$

$$(\phi(P_{\zeta}))_1^2 = p_1 \varepsilon_2 \varepsilon_3 \quad (4.50)$$

$$(\phi(P_{\zeta}))_1^3 = \varepsilon_1(1 - \varepsilon_2 \varepsilon_3)\varphi_a - \varepsilon_1 \varepsilon_2 \varphi_b \quad (4.51)$$

and

$$(\phi(P_{\bar{\zeta}}))_1^1 = -\frac{1 + p_c(1 + p_b) + \varepsilon_c(1 - \varepsilon_b \varepsilon_c)\varphi_1 \varphi_2}{\varepsilon_b \varepsilon_c} \quad (4.52)$$

$$(\phi(P_{\bar{\zeta}}))_1^2 = -\varepsilon_a^2 \varepsilon_b \varepsilon_c \quad (4.53)$$

$$(\phi(P_{\bar{\zeta}}))_1^3 = \varepsilon_a(1 + \varepsilon_b \varepsilon_c)\varphi_1 + \varepsilon_a \varepsilon_c \varphi_2. \quad (4.54)$$

Then eq. (4.44) is solved by

$$\varepsilon_a = \mp \frac{\varepsilon_1 \varepsilon_3}{M}, \quad \varepsilon_b = \mp M^2 \varepsilon_2, \quad \varepsilon_c = \pm \frac{1}{\varepsilon_3}, \quad \varepsilon_e = \frac{\varepsilon_3 \varepsilon_5}{M}, \quad (4.55)$$

$$\varphi_1 = \mp \frac{\varepsilon_3 \varphi_a + \varphi_b}{M}, \quad \varphi_2 = -\frac{\varphi_a - \varepsilon_3 \varphi_b}{M}. \quad (4.56)$$

where M satisfies eq. (4.15).

5 The string measure

The Schottky space measure for an n -point, h -loop bosonic string theory amplitude is given by [13]

$$\begin{aligned} [dm]_g^n &= \frac{1}{dV_{abc}} \prod_{i=1}^n \frac{dz_i}{V'_i(0)} \prod_{j=1}^g \left[\frac{dk_j du_j dv_j}{k_j^2 (u_j - v_j)^2} (1 - k_j)^2 \right] \\ &\quad \times (\det \operatorname{Im} \tau)^{-d/2} \prod_{\alpha}' \left[\prod_{i=1}^{\infty} (1 - k_{\alpha}^n)^{-d} \prod_{j=2}^{\infty} (1 - k_{\alpha}^n)^2 \right] \\ &\quad \times \text{factors coming from external states.} \end{aligned} \quad (5.1)$$

dV_{abc} is the projective-invariant volume element given by

$$dV_{abc} = \frac{d\rho_a d\rho_b d\rho_c}{(\rho_a - \rho_b)(\rho_b - \rho_c)(\rho_a - \rho_c)}. \quad (5.2)$$

where the three ρ_i s have been chosen from among the Koba-Nielsen points z_j and the Schottky fixed points u_{ℓ}, v_{ℓ} to be fixed (usually to 0, 1 and ∞) to eliminate the projective invariance. Let us write the first line of eq. (5.1) as $[dm_0]_g^n$. Using the parametrization described above, it takes the elegant form

$$[dm_0]_g^n = \prod_{\text{edges } i} \frac{dp_i}{p_i^2} \prod_{\text{cl.bord. } \beta} (1 - k_{\beta}) \quad (5.3)$$

Figure 19. An example planar skeleton graph with g loops and n external edges. The large dot indicates our choice of base point.

where the second product is over all closed borders. We define a *border path* in a graph Γ to be a path P whose decomposition includes solely either CW or ACW turns. A *closed border* of Γ is a closed path P_β that is a border path. The multiplier k_β of the corresponding Möbius map $\phi(P_\beta)$ is simply the product of the gluing parameters p_k of the edges E_k traversed by P_β .

5.1 Planar surfaces

We will prove eq. (5.3) for planar surfaces by the following method: first we will compute it explicitly for some particular choice of parametrization of a planar surface with g loops and n external edges. Then we will show that its form is preserved by ‘duality transformations’ which change the topology of the target skeleton graph Γ , as in section 4.

Let us choose to parametrize our surface in such a way as to correspond with the skeleton graph in figure 19. Then the surface is parametrized by the $3g - 3 + n$ variables $\{p_{i_s}, p_{j_s}, p_{k_s}, p_{m_t}; s = 1, \dots, g; t = 1, \dots, n - 3\}$.

The Schottky group generators γ_s are given by

$$\gamma_s = U_s \sigma_{j_s} \rho \sigma_{k_s} \rho \sigma_{j_s} U_s^{-1} \quad s = 1, \dots, g, \quad (5.4)$$

where

$$U_s = U_{s-1} \rho^{-1} \sigma_{i_s} \rho^{-1}, \quad U_0 = \text{Id}. \quad (5.5)$$

Since γ_s is conjugate to $\rho^{-1} \sigma_{k_s}$, its multiplier is

$$k_s = p_{k_s}. \quad (5.6)$$

Its fixed points can be computed as

$$u_s = \frac{1 + f_{2s} p_{i_s} (1 + p_{j_s} p_{k_s})}{1 + f_{1s} p_{i_s} (1 + p_{j_s} p_{k_s})}; \quad v_s = \frac{1 + f_{2s} p_{i_s} (1 + p_{j_s})}{1 + f_{1s} p_{i_s} (1 + p_{j_s})}, \quad (5.7)$$

where

$$f_{rs} \equiv \begin{cases} 0 & \text{if } s < r \\ 1 + p_{i_{s-1}} f_{r(s-1)} & \text{if } s \geq r, \end{cases} \quad (5.8)$$

for $r = 1, 2$. The local coordinates at the punctures, V_t , are given by

$$V_1 = U_g \rho^{-1} \sigma_{m_1} \rho^{-1}, \quad (5.9)$$

$$V_t = V_{t-1} \rho^{-1} \sigma_{m_t} \rho^{-1} \quad (5.10)$$

for $t = 2, \dots, n-3$,

$$V_{n-2} = V_{n-3} \rho^{-1}, \quad (5.11)$$

$$V_{n-1} = \rho, \quad (5.12)$$

$$V_n = \text{Id}. \quad (5.13)$$

As holomorphic functions, the coordinate charts V_t are given in the form

$$V_t(z) = \frac{1 + (1-z) g_{2t} p_{m_t}}{1 + (1-z) g_{1t} p_{m_t}} \quad (5.14)$$

for $t = 1, \dots, n-3$, where we've extended the definition of f_{rs} from eq. (5.8) with

$$g_{rt} = \begin{cases} f_{r(g+1)} & \text{if } t = 1 \\ 1 + p_{m_t} g_r(z_{t-1}) & \text{if } t > 1. \end{cases} \quad (5.15)$$

For these values of t , this gives

$$z_t = V_t(0) = \frac{1 + g_{2t} p_{m_t}}{1 + g_{1t} p_{m_t}}; \quad V'_t(0) = \frac{\prod_{s=1}^g p_{i_s} \prod_{r=1}^t p_{m_r}}{(1 + g_{1t} p_{m_t})^2}, \quad (5.16)$$

where we've used

$$g_{1t} - g_{2t} = \prod_{s=1}^g p_{i_s} \prod_{r=1}^{t-1} p_{m_r}. \quad (5.17)$$

Note that with this choice of base point, the coordinates of three particular punctures $z_t = V_t(0)$ are ‘gauge fixed’, i.e. independent of the moduli:

$$V_n(0) = 0, \quad V_{n-1}(0) = \infty, \quad V_{n-2}(0) = 1. \quad (5.18)$$

To keep track of the infinity, let us temporarily include a small ϵ in V_{n-1} :

$$V_{n-1}^{(\epsilon)}(0) \equiv 1 - \frac{1}{z + \epsilon}, \quad (5.19)$$

so $V_{n-1}^{(\epsilon)}(z) = 1 - 1/\epsilon$. Since the three Koba-Nielsen variables z_{n-2} , z_{n-1} and z_n are independent of the moduli, they must be inserted in the projective-invariant volume element dV_{abc} eq. (5.2) as ρ_a , ρ_b and ρ_c , giving

$$dV_{abc} = \frac{dz_{n-2} dz_{n-1} dz_n}{(z_{n-2} - z_{n-1}^{(\epsilon)})(z_{n-1}^{(\epsilon)} - z_n)(z_{n-2} - z_n)} = -\frac{dz_{n-2} dz_{n-1} dz_n}{(1 - \epsilon)/\epsilon^2}. \quad (5.20)$$

To write down the measure eq. (5.1) in terms of our new moduli, we need to compute the Jacobian determinant. Note that u_s , v_s and k_s are independent of p_{i_r} , p_{j_r} and p_{k_r}

for $r > s$; similarly z_t is independent of p_{m_r} for $r > t$ and $t \leq n - 3$. This means that the Jacobian matrix is in upper-triangular block form: so to compute its determinant we need to evaluate only the entries of the blocks on the diagonal. We can write the Jacobian matrix as $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ where A is a $3g \times 3g$ matrix, also in upper-triangular block form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1g} \\ 0 & A_{22} & \cdots & A_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{gg} \end{pmatrix}; \quad A_{rs} = \begin{pmatrix} \partial u_s / \partial p_{i_r} & \partial v_s / \partial p_{i_r} & 0 \\ \partial u_s / \partial p_{j_r} & \partial v_s / \partial p_{j_r} & 0 \\ \partial u_s / \partial p_{k_r} & \partial v_s / \partial p_{k_r} & \partial k_s / \partial p_{k_r} \end{pmatrix}, \quad (5.21)$$

D is an $(n - 3) \times (n - 3)$ upper-triangular matrix,

$$D = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1,(n-3)} \\ 0 & D_{22} & \cdots & D_{2,(n-3)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{(n-3),(n-3)} \end{pmatrix}; \quad D_{rs} = \frac{\partial z_s}{\partial p_{m_r}}, \quad (5.22)$$

and the $3g \times (n - 3)$ matrix B is not needed to compute the determinant. The determinant can be evaluated with the use of the following partial derivatives:

$$\frac{\partial u_s}{\partial p_{i_s}} = -\frac{(1 + p_{j_s} p_{k_s}) \prod_{r=1}^{s-1} p_{i_r}}{(1 + f_{1s} p_{i_s} (1 + p_{j_s} p_{k_s}))^2} \quad (5.23)$$

$$\frac{\partial u_s}{\partial p_{j_s}} = -\frac{p_{k_s} \prod_{r=1}^s p_{i_r}}{(1 + f_{1s} p_{i_s} (1 + p_{j_s} p_{k_s}))^2} \quad (5.24)$$

$$\frac{\partial v_s}{\partial p_{i_s}} = -\frac{(1 + p_{j_s}) \prod_{r=1}^{s-1} p_{i_r}}{(1 + f_{1s} p_{i_s} (1 + p_{j_s}))^2} \quad (5.25)$$

$$\frac{\partial v_s}{\partial p_{j_s}} = -\frac{\prod_{r=1}^s p_{i_r}}{(1 + f_{1s} p_{i_s} (1 + p_{j_s}))^2} \quad (5.26)$$

$$\frac{\partial k_s}{\partial p_{k_s}} = 1 \quad (5.27)$$

$$\frac{\partial z_t}{\partial p_{m_t}} = -\frac{\prod_{s=1}^g p_{i_s} \prod_{r=1}^{t-1} p_{m_r}}{(1 + g_{1t} p_{m_t})^2} \quad (5.28)$$

($\partial u_s / \partial p_{k_s}$ and $\partial v_s / \partial p_{k_s}$ are also unneeded to compute the determinant). The minors can be evaluated as

$$\begin{vmatrix} \partial u_s / \partial p_{i_s} & \partial u_s / \partial p_{j_s} \\ \partial v_s / \partial p_{i_s} & \partial v_s / \partial p_{j_s} \end{vmatrix} = \frac{(1 - p_{k_s}) p_{i_s} \prod_{r=1}^{s-1} p_{i_r}^2}{(1 + f_{1s} p_{i_s} (1 + p_{j_s}))^2 (1 + f_{1s} p_{i_s} (1 + p_{j_s} p_{k_s}))^2}, \quad (5.29)$$

which can be combined with

$$(u_s - v_s)^2 = \frac{(1 - p_{k_s})^2 p_{j_s}^2 \prod_{r=1}^s p_{i_r}^2}{(1 + f_{1s} p_{i_s} (1 + p_{j_s}))^2 (1 + f_{1s} p_{i_s} (1 + p_{j_s} p_{k_s}))^2} \quad (5.30)$$

to give

$$\frac{(1 - k_s)^2}{k_s^2 (u_s - v_s)^2} \begin{vmatrix} \partial u_s / \partial p_{i_s} & \partial u_s / \partial p_{j_s} \\ \partial v_s / \partial p_{i_s} & \partial v_s / \partial p_{j_s} \end{vmatrix} = \frac{1 - p_{k_s}}{p_{i_s} p_{j_s}^2 p_{k_s}^2}. \quad (5.31)$$

Similarly, $V'_t(0)$ from eq. (5.16) can be combined with eq. (5.28) to give

$$\frac{1}{V'_t(0)} \frac{\partial z_t}{\partial p_{m_t}} = -\frac{1}{p_{m_t}}, \quad (5.32)$$

for $t = 1, \dots, n-3$. We also have

$$\frac{1}{V'_{n-2}(0)} = \prod_{s=1}^g p_{i_s} \prod_{t=1}^{n-3} p_{m_t}, \quad \frac{1}{V_{n-1}^{(\epsilon)'}(0)} = \epsilon^2, \quad \frac{1}{V'_n(0)} = 1. \quad (5.33)$$

Combining all the factors from eq. (5.20), eq. (5.31), eq. (5.32) and eq. (5.33) and letting $\epsilon \rightarrow 0$, we get (up to a sign)

$$\frac{1}{dV_{abc}} \prod_{t=1}^n \frac{dz_t}{V'_t(0)} \prod_{s=1}^g \frac{du_s dv_s dk_s}{(u_s - v_s)^2} \frac{(1 - k_s^2)}{k_s^2} = \prod_{t=1}^n \frac{dp_{m_t}}{p_{m_t}^2} \prod_{s=1}^g \frac{dp_{i_s}}{p_{i_s}^2} \frac{dp_{j_s}}{p_{j_s}^2} \frac{dp_{k_s}}{p_{k_s}^2} (1 - p_{k_s}). \quad (5.34)$$

This is exactly of the form of eq. (5.3), because the closed borders of the diagram in figure 19 are precisely the g loops whose multipliers are p_{k_s} .

5.2 Invariance under duality transformations

We can prove that the pole structure of the Schottky group string measure, eq. (5.3), is invariant under the duality transformations of the pinching moduli p_i as described in section 4. This implies, in particular, that our calculation of the string measure for the graph in figure 19 proves the validity of eq. (5.3) for all planar graphs.

5.2.1 Invariance under transformations without external edges

For a duality transformation which doesn't involve external edges, we must simply show that eq. (5.3) is invariant under the change of variables in eq. (4.14). For the bosonic case we can write eq. (4.14) in terms of the p 's and set the odd supermoduli to 0 to get

$$p_a = \frac{p_1 p_3}{1 + p_3} \quad p_b = p_4 (1 + p_3) \quad p_c = \frac{1}{p_3} \quad p_d = p_2 (1 + p_3) \quad p_e = \frac{p_3 p_5}{1 + p_3}. \quad (5.35)$$

The factor

$$\prod_{\text{cl.bord.} \beta} (1 - k_\beta) \quad (5.36)$$

is unchanged. To see why this true, let us consider an example. Supposed the path P_α in figure 15a is part of a closed border, i.e. that there is some path $P_{\alpha'}$ including only CW turns such that $P_\alpha \cdot P_{\alpha'}$ is a closed path in some graph Γ . Its multiplier will be

$$k = p_2 p_3 p_5 \prod_{E_i \in P_{\alpha'}} p_i. \quad (5.37)$$

Then $P_{\bar{\alpha}} \cdot P_{\alpha'}$ will be a closed path in the dual graph $\hat{\Gamma}$, where $P_{\bar{\alpha}}$ is the path depicted in figure 15b. The multiplier of this path will be given by

$$\hat{k} = p_d p_e \prod_{E_i \in P_{\alpha'}} p_i. \quad (5.38)$$

But from the last two equation of eq. (5.35) we get

$$p_d p_e = p_2 p_3 p_5, \quad (5.39)$$

which implies that k in eq. (5.37) is equal to $\hat{k}$ in eq. (5.38), and thus the contribution to eq. (5.36) is unchanged.

Similarly, from eq. (5.35) we have

$$p_a p_b = p_1 p_3 p_4, \quad p_b p_c p_e = p_4 p_5, \quad p_a p_c p_d = p_1 p_2, \quad (5.40)$$

so any closed border in Γ will correspond to a closed border in $\hat{\Gamma}$ with the same multiplier.

The Jacobian determinant of eq. (5.35) is given by

$$\left| \frac{\partial(p_a, \dots, p_e)}{\partial(p_1, \dots, p_5)} \right| = \begin{vmatrix} p_3/(1+p_3) & 0 & p_1/(1+p_3)^2 & 0 & 0 \\ 0 & 0 & p_4 & 1+p_3 & 0 \\ 0 & 0 & -1/p_3^2 & 0 & 0 \\ 0 & 1+p_3 & p_2 & 0 & 0 \\ 0 & 0 & p_5/(1+p_3)^2 & 0 & p_3/(1+p_3) \end{vmatrix} \quad (5.41)$$

$$= 1. \quad (5.42)$$

Since

$$p_a p_b p_c p_d p_e = p_1 p_2 p_3 p_4 p_5, \quad (5.43)$$

the form of the measure eq. (5.3) is unchanged by the duality transformation, which is what we wanted to show.

If the duality transformation includes a loop as in figure 17 or figure 18 then it is necessary to use the slightly different transformations as given in sections 4.3.1 and 4.3.2.

5.2.2 Invariance under transformations involving external edges

For a duality transformation which does involve external edges, we need to do more work.

The reason for this is that although our prescription for duality transformations *does* ensure that $V_t(0) = z_t = \hat{V}_t(0)$ where V_t and $\hat{V}_t$ are local coordinates around a puncture, it does *not* in general give $V'_t(0) = \hat{V}'_t(0)$. Therefore since $V'_t(0)^{-1}$ appears on the left-hand-side of eq. (5.1), the measure needs to be multiplied by $V'_t(0)/\hat{V}'_t(0)$ for each external edge involved in the duality transformation.

So to check that eq. (5.1) is invariant under duality transformations, we need to show that

$$\left| \prod_{\text{ext. edges } i} \frac{V'_i(0)}{\hat{V}'_i(0)} \right| = \left| \prod_{\text{int. edges } j, k} \frac{p_j}{\hat{p}_k} \right|^2 \times \left| \frac{\partial \hat{p}_k}{\partial p_j} \right| \quad (5.44)$$

holds for all combinations of external edges, where $p_j \in \{p_a, \dots, p_e\}$ and $\hat{p}_k \in \{p_1, \dots, p_5\}$. The cases in which there are 1, 2, 3 or 4 external edges must each be checked separately; furthermore in the case of 2 external edges, there are two sub-cases which differ in whether the edges are adjacent or opposite.

All of these cases can be checked by direct computation; let us give one in detail so that the approach is clear. Suppose there is one external edge involved in the duality transformation; let us use the labels from figure 16. First of all, let us note that since we no longer have p_1 and p_a as moduli in figure 16a and figure 16b, respectively, the Jacobian determinant will not be as simple as it was in eq. (5.42). We get

$$\left| \frac{\partial(p_b, \dots, p_e)}{\partial(p_2, \dots, p_5)} \right| = \begin{vmatrix} 0 & p_4 & 1+p_3 & 0 \\ 0 & -1/p_3^2 & 0 & 0 \\ 1+p_3 & p_2 & 0 & 0 \\ 0 & p_5/(1+p_3)^2 & 0 & p_3/(1+p_3) \end{vmatrix} = \frac{1+p_3}{p_3}. \quad (5.45)$$

The product of the moduli is given by

$$p_b p_c p_d p_e = p_2 (1+p_3) p_4 p_5. \quad (5.46)$$

To find the ratio $V'_1(0)/\hat{V}'_1(0)$, we use the fact that we can write

$$V_1 = V_0 \cdot \phi(\hat{P}_\gamma), \quad \hat{V}_1 = V_0 \cdot \phi(\hat{P}_{\tilde{\gamma}}). \quad (5.47)$$

where V_0 is some map which is unchanged by the duality transformation, so that

$$V_1 = \hat{V}_1 \cdot \phi(\hat{P}_{\tilde{\gamma}})^{-1} \cdot \phi(\hat{P}_\gamma). \quad (5.48)$$

Using the chain rule, we have

$$V'_1(0) = \hat{V}'_1(\phi(\hat{P}_{\tilde{\gamma}}^{-1} \hat{P}_\gamma)(0)) \cdot \phi(\hat{P}_{\tilde{\gamma}}^{-1} \hat{P}_\gamma)'(0) \quad (5.49)$$

but $\phi(\hat{P}_{\tilde{\gamma}}^{-1} \hat{P}_\gamma)(0) = 0$ by construction, so

$$\frac{V'_1(0)}{\hat{V}'_1(0)} = \phi(\hat{P}_{\tilde{\gamma}}^{-1} \hat{P}_\gamma)'(0) = \frac{p_3}{1+p_3}. \quad (5.50)$$

We can then combine eq. (5.45), eq. (5.46) and eq. (5.50) to get

$$V'_1(0) \frac{dp_b}{p_b^2} \frac{dp_c}{p_c^2} \frac{dp_d}{p_d^2} \frac{dp_e}{p_e^2} = \frac{\hat{V}'_1(0)}{\phi(\hat{P}_{\tilde{\gamma}}^{-1} \hat{P}_\gamma)'(0)} \left| \frac{\partial(p_b, \dots, p_e)}{\partial(p_2, \dots, p_5)} \right| \frac{dp_1}{p_2^2} \frac{dp_2}{(1+p_3)^3} \frac{dp_3}{p_4^2} \frac{dp_4}{p_5^2} \quad (5.51)$$

$$= \hat{V}'_1(0) \frac{dp_2}{p_2^2} \frac{dp_3}{p_3^2} \frac{dp_4}{p_4^2} \frac{dp_5}{p_5^2}, \quad (5.52)$$

so eq. (5.3) is preserved by duality transformations involving one external edge.

By similar computations, it is straightforward to show that eq. (5.3) is invariant under a duality transformation involving two, three, or four external edges.

5.3 For non-planar diagrams

The proof in section 5.1 only establishes eq. (5.3) for planar diagrams. For non-planar diagrams, we can prove the result by induction on the number of loops ℓ , by showing that the formula eq. (5.3) is respected by the process of sewing together a pair of legs on an $(\ell-1)$ -loop, $(n+2)$ -point diagram to obtain an ℓ -loop, n -point diagram.

Figure 20. Cutting the edge E_j in figure 20a leaves a connected, but not closed, border cycle joining two external edges marked X_1 and X_2 in figure 20b. To prove eq. (5.3) inductively we compare the string measures of the two diagrams.

5.3.1 Sewing to form a loop with a closed border

The inductive step is performed differently depending on whether or not the sewing creates a closed border path on the diagram (recall that this means that this means a closed path whose turns are either all clockwise or all anti-clockwise). Let us deal first of all with the case in which a closed border cycle is created.

This case can be further divided into two sub-cases: in the first, the loop being sewn includes at least one edge E_j which also belongs to another, distinct border path (this is not the case, for example, for non-planar diagrams that have a single border cycle). When this is the case, cutting E_j leaves the border open but still with a single component, as in figure 20.

To perform the inductive step, we compute the ratio of the string measures $[dm_0]_\ell^n$ and $[dm_0]_{\ell-1}^{n+2}$ for the diagrams in figure 20a and figure 20b respectively and show that they differ by the correct factor required by eq. (5.3), i.e.

$$[dm_0]_\ell^n = [dm_0]_{\ell-1}^{n+2} \times \frac{dp_j}{p_j^2} (1 - k_{\text{border}}) \quad (5.53)$$

where p_j is the pinching parameter for the edge which is cut and k_{border} is the multiplier of the closed border cycle in figure 20a, which is given by

$$k_{\text{border}} = p_j \times \prod_{s=1}^r p_{i_s}. \quad (5.54)$$

The factors in the measures corresponding to the loops that aren't cut and to the external edges that aren't sewn together will clearly give an identical contribution to both sides of eq. (5.53), so we need to compute only the factors that differ. Thus we need to show

$$\frac{du_\ell dv_\ell dk_\ell (1 - k_\ell)^2}{(u_\ell - v_\ell)^2 k_\ell^2} = \frac{dz_1 dz_2}{V'_1(0)V'_2(0)} \times \frac{dp_j}{p_j^2} (1 - k_{\text{border}}) \quad (5.55)$$

where u_ℓ , v_ℓ and k_ℓ are the parameters of a Schottky generator corresponding to the newly sewn loop.

The base point (marked by the dot in figure 20a) lies at the position of the external edge marked X_1 , so the chart V_1 is just the identity:

$$V_1 = \text{Id}, \quad z_1 = V_1(0) = 0, \quad V'_1(0) = 1. \quad (5.56)$$

The chart V_2 can be found by tracing the path from X_2 to the base point indicated in figure 20b by the dotted line and following the procedure described in section 2.1 to get

$$V_2 = \rho^{-1} \sigma_{i_1} \rho^{-1} \sigma_{i_2} \rho^{-1} \dots \rho^{-1} \sigma_{i_n} \quad (5.57)$$

or as a holomorphic function,

$$V_2(z) = \frac{1}{y - xz}, \quad (5.58)$$

where

$$x = p_{i_1} p_{i_2} \dots p_{i_n}, \quad y = 1 + p_{i_1}(1 + p_{i_2}(1 + \dots (1 + p_{i_m}) \dots)). \quad (5.59)$$

This gives

$$z_2 = V_2(0) = 1/y, \quad V_2'(0) = x z_2^2. \quad (5.60)$$

We need to add a generator γ_ℓ to the Schottky group to account for the loop being added; we can use the Schottky group element defined by the closed border cycle:

$$\gamma_\ell = V_2 \sigma_j \quad (5.61)$$

or, as a holomorphic map,

$$\gamma(z) = \frac{z}{y z + p_j x}. \quad (5.62)$$

Its multiplier and fixed points are given by

$$k_\ell = p_j x = k_{\text{border}}, \quad u_\ell = \frac{1 - k_\ell}{y}, \quad v_\ell = 0. \quad (5.63)$$

Since v_ℓ and z_1 are both gauge-fixed to 0, we can delete dv_ℓ and dz_1 from eq. (5.55) as they are both cancelled by the projective-invariant volume element dV_{abc} in eq. (5.1). Then using

$$u_\ell = z_2(1 - x p_j) \quad k_\ell = x p_j \quad (5.64)$$

we find

$$\left| \frac{\partial(u_\ell, k_\ell)}{\partial(z_2, p_j)} \right| = \left| \begin{array}{cc} 1 - p_j x & 0 \\ -z_2 x & x \end{array} \right| = x(1 - p_j x). \quad (5.65)$$

Inserting this in

$$du_\ell dk_\ell = \left| \frac{\partial(u_\ell, k_\ell)}{\partial(z_2, p_j)} \right| dz_2 dp_j, \quad (5.66)$$

it is easy to verify with simple algebra that eq. (5.55) holds.

Figure 21. To prove eq. (5.3) inductively for diagrams belonging to a certain class, we can show that the measures $[dm_0]_{\ell-1}^{n+2}$ and $[dm_0]_{\ell}^n$ for the diagrams in figure 21a and figure 21b have the desired ratio.

Figure 22. A diagram that contains a loop with many edges attached to both its inside and its outside (an example is shown in figure 22a) is dual to a diagram (e.g. figure 22b) where the corresponding loop has one edge attached to its inside and two edges attached to its outside.

5.3.2 Sewing to form a loop without a closed border

In section 5.3.1 we gave the details of the inductive step to prove eq. (5.3) in the case that an ℓ -loop, n -point diagram with B closed border cycles had an internal edge removed to give an $(\ell - 1)$ -loop, $(n + 2)$ -point diagram with $(B - 1)$ closed border cycles. In this section we deal with the case that both diagrams have the same number of border cycles B .

The loop being cut must include both clockwise and anti-clockwise turns (or it would have a closed border cycle). Without loss of generality, let us assume that the loop being cut consists of two clockwise turns and one anti-clockwise turn as in figure 21b. A diagram with a loop consisting of a complicated sequence of clockwise and anti-clockwise turns (such as figure 22a) is dual to another diagram (such as figure 22b) where the corresponding loop is of this sort, and we have proven that eq. (5.3) is unchanged by duality transformations. So then we can take the local coordinates of the punctures (before sewing the loop) to be given by:

$$V_1 = \rho^{-1} \cdot \sigma_{i_1} \cdot \rho^{-1}, \quad (5.67)$$

$$V_2 = V_1 \cdot \rho^{-1} \cdot \sigma_{i_2} \cdot \rho^{-1}, \quad (5.68)$$

$$V_3 = V_2 \cdot \rho^{-1} \cdot \sigma_{i_3} \cdot \rho^{-1}, \quad (5.69)$$

while the Schottky generator for the loop in figure 22b (after sewing) is given by

$$\gamma = V_3 \cdot \sigma_j \cdot V_1^{-1}. \quad (5.70)$$

We find

$$\frac{du dv dk}{(u-v)^2} \frac{(1-k)^2}{k^2} \frac{dz_2}{V_2'(0)} = \prod_{m=1}^3 \frac{dp_{i_m}}{p_{i_m}^2} \times \frac{dp_j}{p_j^2} \quad (5.71)$$

and

$$\prod_{m=1}^3 \frac{dz_m}{V_m'(0)} = \prod_{m=1}^3 \frac{dp_{i_m}}{p_{i_m}^2} \quad (5.72)$$

Eq. (5.72) and eq. (5.71) differ by a factor of dp_j/p_j^2 , so eq. (5.3) has been shown by induction.

5.4 The superstring case

The measure for the NS sector has a leading factor that is analogous to the first line of eq. (5.1); it is given by

$$[d\mathbf{m}_0]_g^n = \frac{1}{d\mathbf{V}_{abc}} \prod_{t=1}^n \frac{dz_t}{(D\mathbf{V}_t^\zeta)(0|0)} \prod_{s=1}^g \frac{d\mathbf{u}_s d\mathbf{v}_s d(q_s^2)}{\mathbf{u}_s \div \mathbf{v}_s} \frac{(1+q_s)^2}{q_s^3} \quad (5.73)$$

where

$$\frac{1}{d\mathbf{V}_{abc}} = \frac{\sqrt{(\mathbf{a} \div \mathbf{b})(\mathbf{b} \div \mathbf{c})(\mathbf{c} \div \mathbf{a})}}{d\mathbf{a} d\mathbf{b} d\mathbf{c}} d\Theta_{\mathbf{a}\mathbf{b}\mathbf{c}}, \quad (5.74)$$

where $\Theta_{\mathbf{a}\mathbf{b}\mathbf{c}}$ and $\mathbf{a} \div \mathbf{b}$ have been defined in eq. (2.36) and eq. (2.37), respectively. Furthermore, $\mathbf{V}_t^\zeta$ is the odd part of $\mathbf{V}_t$ and D is the superderivative. The full measure is given by eq. (30) of [24] (for example).

When expressed in terms of the pinching parameters of section 2.2, $[d\mathbf{m}_0]_g^n$ take the following form (cf. eq. (5.3)):

$$[d\mathbf{m}_0]_g^n = \prod_{\text{edges } i} \prod_{\text{vertices } j} \prod_{\text{cl.bord.}\beta} \frac{dp_i}{\varepsilon_i^3} d\varphi_j (1+q_\beta) \quad (5.75)$$

Note that as well as the products over edges and borders, there is now a product over graph vertices, which we have put in 1–1 correspondence with the odd supermoduli ϕ_j .

This formula could be proven by applying the same approach as in the previous sections; instead we will show how the computation can be done for one particular diagram.

5.4.1 The Mercedes-Benz diagram

Consider the graph Γ depicted in figure 23. Let us compute the super Schottky measure $[d\mathbf{m}_0]_{g=3}^{n=0}$ eq. (5.73) in terms of the pinching parameters $(\varepsilon_i|\varphi_j)$ corresponding to this graph. We get

$$\gamma_1 = \Sigma_{d,-6} \Sigma_{c,1} \Sigma_{b,5}, \quad (5.76)$$

$$\gamma_2 = \rho_d^{-1} \Sigma_{d,-4} \Sigma_{a,2} \Sigma_{c,6} \rho_d, \quad (5.77)$$

$$\gamma_3 = \rho_d \Sigma_{d,-5} \Sigma_{b,3} \Sigma_{a,4} \rho_d^{-1}, \quad (5.78)$$

Figure 23. The symmetric $g = 3$, $n = 0$ tetrahedral graph.

where

$$\Sigma_{s,\pm n} \equiv \rho_s^{-1} \sigma_n^{\pm 1}. \quad (5.79)$$

The semimultipliers are given by

$$q_1 = \varepsilon_1 \varepsilon_5 \varepsilon_6, \quad q_2 = \varepsilon_2 \varepsilon_4 \varepsilon_6, \quad q_3 = \varepsilon_3 \varepsilon_4 \varepsilon_5. \quad (5.80)$$

The fixed points can be written as eigenvectors satisfying

$$\gamma_i U_i = q_i^{-1} U_i, \quad \gamma_i V_i = q_i V_i, \quad (5.81)$$

in the form

$$U_i = \rho_d^{1-i} \cdot \begin{pmatrix} 1 - p_i \\ A_i \\ (1 + x_i) \Psi_i \end{pmatrix}, \quad V_i = \rho_d^{1-i} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (5.82)$$

where

$$\Psi_{\alpha\beta\gamma}^{ij} \equiv \varphi_a + \varepsilon_i \varphi_b - \varepsilon_i \varepsilon_j \varphi_\gamma, \quad (5.83)$$

$$A_{\alpha\beta\gamma}^{ij} \equiv 1 + p_i(1 + p_j) + \varphi_a \Psi_{\alpha\beta\gamma}^{ij} - p_i \varepsilon_j \varphi_\beta \varphi_\gamma. \quad (5.84)$$

The 3 super-points that are gauge-fixed are

$$\mathbf{v}_1 = 0|0, \quad \mathbf{v}_2 = 1|\varphi_d, \quad \mathbf{v}_3 = \infty|0; \quad (5.85)$$

their odd super-projective invariant is

$$\Theta_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3} = \varphi_d. \quad (5.86)$$

Thus the standard $(3g-3|2g-2) = (6|4)$ super Schottky moduli used for this worldsheet are

$$(q_1, q_2, q_3, u_1, u_2, u_3 | \theta_1, \theta_2, \theta_3, \Theta_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3}). \quad (5.87)$$

We can express these in terms of the pinching supermoduli

$$(\varepsilon_1, \dots, \varepsilon_6 | \varphi_a, \dots, \varphi_d). \quad (5.88)$$

To rewrite the super-Schottky measure in terms of the pinching supermoduli we need to compute the Berezinian of the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where

$$A = \begin{pmatrix} \partial q_1 / \partial \varepsilon_1 & \dots & \partial u_3 / \partial \varepsilon_1 \\ \vdots & \ddots & \vdots \\ \partial q_1 / \partial \varepsilon_6 & \dots & \partial u_3 / \partial \varepsilon_6 \end{pmatrix}; \quad B = \begin{pmatrix} \partial \theta_1 / \partial \varepsilon_1 & \dots & \partial \theta_3 / \partial \varepsilon_1 \\ \vdots & \ddots & \vdots \\ \partial \theta_1 / \partial \varepsilon_6 & \dots & \partial \theta_3 / \partial \varepsilon_6 \end{pmatrix}; \quad (5.89)$$

$$C = \begin{pmatrix} \partial q_1 / \partial \varphi_a & \dots & \partial u_3 / \partial \varphi_a \\ \vdots & \ddots & \vdots \\ \partial q_1 / \partial \varphi_d & \dots & \partial u_3 / \partial \varphi_d \end{pmatrix}; \quad D = \begin{pmatrix} \partial \theta_1 / \partial \varphi_a & \dots & \partial \theta_3 / \partial \varphi_a \\ \vdots & \ddots & \vdots \\ \partial \theta_1 / \partial \varphi_d & \dots & \partial \theta_3 / \partial \varphi_d \end{pmatrix}; \quad (5.90)$$

We can find (with the use of *Mathematica*) that this matrix has

$$Ber = \frac{\det(A - BD^{-1}C)}{\det(D)} = -8 \frac{(1+q_4) \varepsilon_4^2 \varepsilon_5^2 \varepsilon_6^2 u_1 (u_2 - v_2 - \theta_2 \phi_2)}{(1+q_1)(1+q_2)(1+q_3)} \quad (5.91)$$

where $q_4 \equiv \varepsilon_1 \varepsilon_2 \varepsilon_3$. Multiplying this by

$$\prod_{r=1}^2 \frac{1}{\mathbf{u}_r \div \mathbf{v}_r} \prod_{s=1}^2 \left(\frac{1+q_s}{q_s} \right)^2, \quad (5.92)$$

we get

$$[d\mathbf{m}_0]_3^0 = \frac{1}{d\mathbf{V}_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3}} \prod_{s=1}^g \frac{d\mathbf{u}_s d\mathbf{v}_s d(q_s^2)}{\mathbf{u}_s \div \mathbf{v}_s} \frac{(1+q_s)^2}{q_s^3} \quad (5.93)$$

$$= d\varphi_d \prod_{r=1}^2 \frac{1}{\mathbf{u}_r \div \mathbf{v}_r} \prod_{s=1}^g d\mathbf{u}_s d(q_s^2) \frac{(1+q_s)^2}{q_s^3} \quad (5.94)$$

$$= \prod_{i=a,b,c,d} d\varphi_i \prod_{j=1}^6 \frac{d\varepsilon_j}{\varepsilon_j^3} \prod_{\ell=1}^4 (1-q_\ell). \quad (5.95)$$

This is exactly of the form given in eq. (5.75). In section 4.2 of [1] a similar computation was given for the $g=2, n=2$ diagram obtained by cutting one of the edges of figure 23.

6 Directions for further research

6.1 Feynman graphs with gluons

The pinching parametrization allows us to arrive at the worldline Green's function as the $\alpha' \rightarrow 0$ limit of the worldsheet Green's function, and thus we can see exactly how the

various Feynman graphs for scalar Φ^3 theory arise from the infrared dynamics of string theories with scalar fields in their spectrum. In [36], it was shown how to isolate not only different Feynman graph topologies within the string measure, but even further to distinguish propagators belonging to different fields (such as gluons, ghosts and scalars). Whereas the results of [35, 36] were worked out at length only for the two loop vacuum diagram, the use of pinching parameters would allow a similar analysis to be extended to Feynman graphs of arbitrary topology. However, our proof in section 3 that the pinching parameters give a diagram-by-diagram matching between string theory and QFT is only valid for the scalar sector. We do know that at two loops the pinching parameters allow us to correctly obtain Feynman graphs involving gluons, because the pinching parametrization is a generalization of the parametrization used in [35, 36]. However, it is possible that for graphs with more complicated topologies the correspondence ceases to hold for gluons.

It would be interesting, then, to investigate whether Feynman diagrams with gluons can be reproduced with the use of the pinching parametrization, possibly with modification.

6.2 Ramond pinching parameters

The construction described in section 2.2 provides a pinching parametrization for superstring worldsheets with states from the NS sector — which correspond to spacetime bosons — propagating along the plumbing fixtures. To describe a general superstring theory we would have to allow states from both the Ramond (R) and NS sectors to propagate. The proper inclusion of the R sector — necessary for the description of spacetime fermions — would require a different type of plumbing fixture to be used (see section 6.2.1 of [47]). The inclusion of R plumbing fixtures would require the introduction of a number of novel features. First of all, R punctures are best described by non-superconformal coordinates $z|\zeta$ in which the superderivative takes the form $D_\zeta^* = \partial_\zeta + \zeta f(z) \partial_z$, where $f(z)$ is some polynomial vanishing at the puncture. Next, an SRS disc with three punctures can have zero or two R punctures, and unlike the NNN disc, the NRR disc has no supermoduli (odd or even). Lastly, a plumbing fixture between two R punctures must have one Grassmann-even parameter and one Grassmann-odd parameter, whereas an NS plumbing fixture just has an even one (ε_i in our case). Whereas an NS plumbing fixture is an $\text{OSp}(1|2)$ mapping between the charts at its two ends, this is not the case for the R plumbing fixture, and so the transition functions obtained will not belong to $\text{OSp}(1|2)$.

One possible pinching parametrization that implements these constraints is the following. Let us put three charts on the NRR disc, $z_i|\zeta_i$, $i = 0, 1, 2$. Let $z_0|\zeta_0$ be a superconformal chart centred on the NS puncture at $z_0|\zeta_0 = 0|0$. Then we can define the other two charts by

$$z_0|\zeta_0 = \rho(z_1) \Big| \frac{\zeta_1}{\sqrt{z_1}} = \rho^{-1}(z_2) \Big| \frac{\sqrt{z_2}}{z_2 - 1} \zeta_2, \quad (6.1)$$

where ρ and ρ^{-1} are defined in eq. (2.5) and eq. (2.7), respectively. It follows from the chain rule that the superderivative takes on the following form:

$$\partial_{\zeta_0} + \zeta_0 \partial_{z_0} = z_1^{1/2} (\partial_{\zeta_1} + \zeta_1 z_1 \partial_{z_1}) = \left(z_2^{1/2} - z_2^{-1/2} \right) (\partial_{\zeta_2} + \zeta_2 z_2 \partial_{z_2}), \quad (6.2)$$

so the charts $z_1|\zeta_1$ and $z_2|\zeta_2$ define standard R punctures at their origins (cf. eq. (4.4) of [47]). Sewing two such R punctures can be effected with the map

$$\sigma_{p|\kappa}^\pm(z|\zeta) \equiv -p \frac{1 + \kappa \zeta}{z} \Big| \pm i(\zeta + \kappa), \quad (6.3)$$

which preserves the superderivative. Here p is a Grassmann-even parameter of the sewing and κ is a Grassmann-odd parameter. Note that the inverse of $\sigma_{p|\kappa}^\pm$ is given by

$$\left(\sigma_{p|\kappa}^\pm\right)^{-1} = \sigma_{p|\mp i\kappa}^\mp. \quad (6.4)$$

Of course, the purely bosonic parts of eq. (6.1) and eq. (6.3) reproduce eq. (2.6) and eq. (2.10). Unfortunately, the pinching parametrization defined by these equations does not enjoy the nice properties of the parametrization defined in section 2.2. For example, if two target graphs are related by a duality transformation as in section 4, the two corresponding pinching parametrizations may no longer be obtained from each other simply by changing variables.

It could be very valuable to construct a pinching parametrization for the R sector (whether the one defined by eq. (6.1) and eq. (6.3) or otherwise), and to write down, for example, the superstring measure in terms of these supermoduli.

6.3 Integration region

Since we have been primarily concerned with the asymptotic behaviour of string theory as the worldsheet degenerates (or as $\alpha' \rightarrow 0$), the integrals we consider are assumed to be localized to small regions around the corners of (super) Schottky space, which are connected to each other by the ‘duality transformations’ of section 4. Therefore we have left for future work the question of determining the full integration region for the pinching parameters, and whether it can be chosen in such a way that the set of ribbon graphs defines a decomposition of open string (super) Schottky space.

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A Proofs of formulae

A.1 (Semi)multiplicator

In this section we prove eq. (2.21) and eq. (2.41), i.e. that to leading order the (semi)multiplicator of a (super) Schottky group element is simply the (signed) product of the (NS) pinching parameters associated to the edges in the graph that are included in the corresponding loop.

We will write out the proof for the NS case eq. (2.41); the bosonic case eq. (2.21) follows as a corollary because any Schottky group can be embedded in a split super Schottky group (i.e. one with odd constants θ_j set to 0), and then the multipliers are the squares $k_\alpha = q_\alpha^2$ of the corresponding semimultipliers.

Any super-Schottky group element γ_α written down according to the procedure above is conjugate to one of the form

$$\gamma_N = \sigma_{\varepsilon_{i_1}}^{r_1} \rho_{\theta_{j_1}}^{s_1} \sigma_{\varepsilon_{i_2}}^{r_2} \rho_{\theta_{j_2}}^{s_2} \cdots \sigma_{\varepsilon_{i_N}}^{r_N} \rho_{\theta_{j_N}}^{s_N}, \quad (\text{A.1})$$

where the r_n 's and s_n 's are all ± 1 and the ε_{i_n} 's and θ_{j_n} 's are not necessarily distinct (in case the loop traverses an edge or vertex more than once). Since the semimultiplier is a conjugacy class invariant, we can use eq. (A.1) to compute it without loss of generality.

Let us define

$$\Pi_{\gamma_N} = \prod_{n=1}^N r_n s_n \varepsilon_{i_n}. \quad (\text{A.2})$$

Then eq. (2.41) can be restated as

$$q_{\gamma_N} = \Pi_{\gamma_N} (1 + \mathcal{O}(\varepsilon_{i_n})). \quad (\text{A.3})$$

The overall sign $\prod_n r_n s_n$ in eq. (A.3) matches the sign we described geometrically in eq. (2.41) because each edge E_{i_n} traversed against the marked direction corresponds to a factor of $\sigma_{\varepsilon_{i_n}}^{-1}$ in eq. (A.1) so $r_n = -1$, and each clockwise move around a vertex V_{j_n} corresponds to a factor of $\sigma_{\theta_{j_n}}^{-1}$ in eq. (A.1) so $s_n = -1$.

Note that to invert γ_N , as well as reversing the order of the factors in eq. (A.1), we would multiply each of the exponents by (-1) , so eq. (A.2) implies $\Pi_{\gamma_N^{-1}} = \Pi_{\gamma_N}$ because $(-1)^{2N} = 1$.

To compute the semimultiplier, we first show that γ is always of the form

$$\gamma_N = \begin{cases} \Pi_{\gamma_N}^{-1} \left(\begin{array}{cc|c} \Pi_{\gamma_N}^2 + P_{2N-1} & P_{2N-2} & P_{2N-1} \Omega \\ -1 + P_{2N-2} & 1 + P_{2N-2} & P_{2N-3} \Omega - \theta_N \\ \hline P_{2N-1} \Omega & P_{2N-3} \Omega & \Pi_{\gamma_N} + P_{2N-2} \Omega^2 \end{array} \right) & \text{if } s_N = 1 \\ \Pi_{\gamma_N}^{-1} \left(\begin{array}{cc|c} \Pi_{\gamma_N}^2 + P_{2N-1} & -\Pi_{\gamma_N}^2 + P_{2N-1} & P_{2N-1} \Omega + \Pi_{\gamma_N}^2 \theta_N \\ P_{2N-2} & 1 + P_{2N-2} & P_{2N-2} \Omega \\ \hline P_{2N-1} \Omega & P_{2N-1} \Omega & \Pi_{\gamma_N} + P_{2N-1} \Omega^2 \end{array} \right) & \text{if } s_N = -1, \end{cases} \quad (\text{A.4})$$

where the notation P_n has been used to stand for any arbitrary polynomial in the ε_i 's of degree at most n with constant term zero, whose coefficients are valued in the Grassmann algebra generated by the θ_i 's. None of the various polynomials P_n necessarily coincide with each other, and $P_n \equiv 0$ if $n \leq 0$. Ω denotes the odd-parity subspace of the Grassmann algebra generated by the θ_i 's.

We can prove eq. (A.4) by induction on N . We can take all of the r_n 's to be $+1$ without loss of generality since $\sigma_{\varepsilon_{i_n}}^{\pm 1} = \sigma_{\pm \varepsilon_{i_n}}$ so the exponents may be absorbed into the signs of the ε_{i_n} 's. In the $N = 1$ case, eqs. (A.1) and (2.32) – (2.35) give us

$$\gamma_1 = \begin{cases} \varepsilon_{i_1}^{-1} \left(\begin{array}{cc|c} \varepsilon_{i_1}^2 & 0 & 0 \\ -1 & 1 & -\theta_{j_1} \\ \hline -\varepsilon_{i_1} & \theta_{j_1} & 0 \\ \varepsilon_1 \end{array} \right) & \text{if } s_1 = 1 \\ (-\varepsilon_{i_1})^{-1} \left(\begin{array}{cc|c} \varepsilon_{i_1}^2 & -\varepsilon_{i_1}^2 & \varepsilon_{i_1}^2 \theta_1 \\ 0 & 1 & 0 \\ \hline 0 & \varepsilon_{i_1} \theta_{j_1} & -\varepsilon_{i_1} \end{array} \right) & \text{if } s_1 = -1, \end{cases} \quad (\text{A.5})$$

which is of the form eq. (A.4) with $\Pi_{\gamma_1} = s_1 \varepsilon_1$.

For the inductive step, we need to multiply each of the matrices in eq. (A.4) on the right by both of the matrices in eq. (A.5) (after replacing $\varepsilon_{i_1}, \theta_{j_1} \mapsto \varepsilon_{i_{N+1}}, \theta_{j_{N+1}}$). It can be verified that the resulting matrices are still of the form in eq. (A.4), with the replacement $N \mapsto N + 1$. To check this, note that $P_m \varepsilon_{i_n}$ may be written as P_{n+1} , that any polynomial P_m can also be written as P_n if $m \leq n$, and that we can write $P_n + P_n = P_n$.

Now we know the general form of an $\text{OSp}(1|2)$ element given by eq. (A.1), we can compute the semimultiplier. The *supertrace* of an $\text{OSp}(1|2)$ matrix $\gamma = (\gamma_{ij})$ is defined to be

$$\text{sTr}(\gamma) = \gamma_{11} + \gamma_{22} - \gamma_{33}. \quad (\text{A.6})$$

Any hyperbolic $\text{OSp}(1|2)$ map γ satisfies

$$\text{sTr}(\gamma) = q_\gamma + q_\gamma^{-1} - 1, \quad (\text{A.7})$$

where q_γ is the semimultiplier. To compute the semimultiplier of a hyperbolic $\text{OSp}(1|2)$ map, we can compute the supertrace and solve eq. (A.7) for q_γ .

Both matrices in eq. (A.4) have supertraces of the form

$$\text{sTr}(\gamma_N) = \Pi_{\gamma_N} + \frac{1 + P_{2N-1}}{\Pi_{\gamma_N}} - 1, \quad (\text{A.8})$$

(after simplifying $P_{2N-1} + P_{2N-2} - P_{2N-2}\Omega^2 \mapsto P_{2N-1}$ in the first case and $P_{2N-1} + P_{2N-2} - P_{2N-1}\Omega^2 \mapsto P_{2N-1}$ in the second case).

Equating the right-hand-sides of eq. (A.7) and eq. (A.8), we get a quadratic equation for q_{γ_N} , with solution

$$q_{\gamma_N} = \frac{\Pi_{\gamma_N}^2 + P_{2N-1} + 1}{2\Pi_{\gamma_N}} \left(1 - \sqrt{1 - \left(\frac{2\Pi_{\gamma_N}}{\Pi_{\gamma_N}^2 + P_{2N-1} + 1} \right)^2} \right) \quad (\text{A.9})$$

(the other root is $q_{\gamma_N}^{-1}$). Then expanding the square root as a power series in Π_{γ_N} ,

$$q_{\gamma_N} = \frac{\Pi_{\gamma_N}^2 + P_{2N-1} + 1}{2\Pi_{\gamma_N}} \left(1 - 1 + \frac{1}{2} \left(\frac{2\Pi_{\gamma_N}}{\Pi_{\gamma_N}^2 + P_{2N-1} + 1} \right)^2 + \mathcal{O}(\Pi_{\gamma_N}^3) \right) \quad (\text{A.10})$$

$$= \frac{\Pi_{\gamma_N}}{\Pi_{\gamma_N}^2 + P_{2N-1} + 1} + \mathcal{O}(\Pi_{\gamma_N}^3), \quad (\text{A.11})$$

and lastly we can expand the denominator in the pinching parameters yielding

$$q_{\gamma_N} = \Pi_{\gamma_N}(1 + \mathcal{O}(\varepsilon_i)), \quad (\text{A.12})$$

which is what we wanted to show.

Of course, the proof of the bosonic case eq. (2.21) is analogous, and can be obtained by setting all Grassmann-odd parameters to 0 and considering only the $\text{PSL}(2)$ subgroup of $\text{OSp}(1|2)$.

A.2 Period matrix

We want to prove the period matrix formula

$$(\tau_{ij}) = \frac{1}{2\pi i} \sum_k \langle \ell_i, \ell_j \rangle^k \log(\pm p_k) + \mathcal{O}(p_n). \quad (\text{A.13})$$

Our starting point is the Schottky group formula for the period matrix, eq. (3.11). To compute the cross-ratio in eq. (3.11), we can use

$$\gamma_\alpha(u_j) = u_{\alpha j \bar{\alpha}} \quad \gamma_\alpha(v_j) = v_{\alpha j \bar{\alpha}} \quad (\text{A.14})$$

where $u_{\alpha j \bar{\alpha}}$ (respectively $v_{\alpha j \bar{\alpha}}$) is the attractive (repulsive) fixed point of $\gamma_{\alpha j \bar{\alpha}} \equiv \gamma_\alpha \gamma_j \gamma_\alpha^{-1}$. If we write $\eta_{\alpha\beta}$ to denote the following cross-ratio of the fixed points of two Schottky group elements $\gamma_\alpha, \gamma_\beta$:

$$\eta_{\alpha\beta} \equiv \frac{u_\alpha - v_\beta}{u_\alpha - u_\beta} \frac{v_\alpha - u_\beta}{v_\alpha - v_\beta}, \quad (\text{A.15})$$

then eq. (3.11) can be rewritten as

$$\tau_{ij} = \frac{1}{2\pi i} \left(\delta_{ij} \log k_i - {}^{(i)}\sum_{\gamma_\alpha} {}^{(j)}\log \eta_{i, \alpha j \bar{\alpha}} \right). \quad (\text{A.16})$$

Now we will show how the cross-ratio $\eta_{i, \alpha j \bar{\alpha}}$ on the right-hand-side of eq. (A.16) can be expressed in terms of multipliers, which can ultimately be expressed in terms of pinching parameters depending on the geometry of the graph Γ , thanks to the results of section A.1.

Let us note that the multiplier $k_{\alpha\beta}$ of the product of two Schottky group elements $\gamma_\alpha, \gamma_\beta$ is given by:

$$k_{\alpha\beta} = \frac{(1 - \eta_{\alpha\beta})^2}{\eta_{\alpha\beta}^2} k_\alpha k_\beta + \mathcal{O}(k_\alpha)^2 + \mathcal{O}(k_\beta)^2. \quad (\text{A.17})$$

Figure 24. Ignoring all the other loops and external edges, if two loops ℓ_α, ℓ_β don't cross each other as in figure 24a then $\eta_{\alpha\beta} > 0$; if they do cross as in figure 24b then $\eta_{\alpha\beta} < 0$.

We can use eq. (A.17) to express the cross-ratio eq. (A.15) in terms of multipliers:

$$\eta_{\alpha\beta} = \frac{\pm \sqrt{k_\alpha k_\beta / k_{\alpha\beta}}}{1 \pm \sqrt{k_\alpha k_\beta / k_{\alpha\beta}}} + \mathcal{O}(k_\alpha) + \mathcal{O}(k_\beta). \quad (\text{A.18})$$

The sign choice in eq. (A.18) depends on the ordering of the fixed points $\{u_\alpha, v_\alpha, u_\beta, v_\beta\}$ on the projective line, which is determined by the topology of the two loops ℓ_α and ℓ_β . If the two loops don't cross, as in figure 24a, then $\eta_{\alpha\beta} > 0$; if they do cross as in figure 24b) then we have $\eta_{\alpha\beta} < 0$.

With this, the period matrix can be expressed as a function of multipliers, and thus pinching parameters, by making the replacement

$$\log \eta_{i,\alpha j \bar{\alpha}} = \log \frac{\pm \sqrt{k_i k_j / k_{i\alpha j \bar{\alpha}}}}{1 \pm \sqrt{k_i k_j / k_{i\alpha j \bar{\alpha}}}} + \mathcal{O}(p_i)^2 \quad (\text{A.19})$$

in eq. (A.16), where $k_{i\alpha j \bar{\alpha}} \equiv k(\gamma_i \gamma_\alpha \gamma_j \gamma_\alpha^{-1})$ and so on, and we have used the fact that $k_{\alpha j \bar{\alpha}} = k_j$.

Only finitely many Schottky group elements give a contribution to eq. (3.11) that doesn't vanish to first order in the p_i 's. Let's compute one of the off-diagonal elements — say τ_{12} without loss of generality. Let's compute the contribution to the series coming from the identity element (which is always included in the summation). We have

$$\log \eta_{12} = \log \frac{\pm \sqrt{k_1 k_2 / k_{12}}}{1 \pm \sqrt{k_1 k_2 / k_{12}}} + \mathcal{O}(k_1) + \mathcal{O}(k_2). \quad (\text{A.20})$$

Let's suppose, first of all, that the loops ℓ_1 and ℓ_2 do not have any edges in common. Let us take the base point to be on ℓ_1 (there is no loss of generality since the period matrix elements are $\text{PSL}(2)$ invariant). Then γ_1 depends only on the pinching parameters in the loop ℓ_1 while γ_2 is of the form $\gamma_2 = V^{-1} \tilde{\gamma}_2 V$, where $\tilde{\gamma}_2$ depends only on the pinching parameters in the loop ℓ_2 and $V = \phi(P)$ is a matrix moving the base point along a path P to a point on the loop ℓ_2 . Then we have

$$\gamma_1 \gamma_2 = \gamma_1 V^{-1} \tilde{\gamma}_2 V \quad (\text{A.21})$$

so, with the results of section A.1, we have

$$k_{12} = k(\gamma_1 \gamma_2) = k_1 k_2 \prod_{E_i \in P} p_i^2 (1 + \mathcal{O}(p_j)). \quad (\text{A.22})$$

Plugging this in eq. (A.20) we get

$$\log \eta_{12} = \mathcal{O}(p_i) \quad (\text{A.23})$$

in the case of no intersecting edges. One can check that the contribution from non-identity Schottky group elements vanishes.

Next, suppose that there is a connected chain of edges $E_{i_1}, E_{i_2}, \dots, E_{i_n}$ lying on both ℓ_1 and ℓ_2 ; and that both loops cross these edges in *opposite* directions. Assume the base point is at the start of the chain as in figure 25. Then the Schottky group element $\gamma_1 \gamma_2$ corresponds to a closed path P_{12} which includes all of the edges in γ_1 and γ_2 except from $E_{i_1} \dots E_{i_n}$. Thus we get

$$k_{12} = k(\gamma_1 \gamma_2) = \frac{k_1 k_2}{p_{i_1}^2 \cdots p_{i_n}^2} (1 + \mathcal{O}(p_j)). \quad (\text{A.24})$$

Putting this in eq. (A.20), we find

$$\log \eta_{12} = \log (\pm p_{i_1} p_{i_2} \cdots p_{i_n}) + \mathcal{O}(p_j). \quad (\text{A.25})$$

Now suppose, on the other hand, that ℓ_1 and ℓ_2 cross $E_{i_1} \dots E_{i_n}$ in the *same* direction. Then ℓ_1 and ℓ_2^{-1} cross in opposite directions, so the above result has to be modified by replacing η_{12} with $\eta_{1\bar{2}}$ to get

$$\log \eta_{1\bar{2}} = \log (\pm p_{i_1} p_{i_2} \cdots p_{i_n}) + \mathcal{O}(p_j). \quad (\text{A.26})$$

We want to compute η_{12} not $\eta_{1\bar{2}}$, but since γ_2^{-1} and γ_2 can be obtained from each other by fixing the multiplier k_2 and swapping the fixed points (i.e. $u_2 = v_2, v_2 = u_2$), we have

$$\eta_{1\bar{2}} = \frac{1}{\eta_{12}}, \quad (\text{A.27})$$

and thus in this case

$$\log \eta_{12} = -\log (\pm p_{i_1} p_{i_2} \cdots p_{i_n}) + \mathcal{O}(p_j). \quad (\text{A.28})$$

Now suppose that ℓ_1 and ℓ_2 intersect more than once, i.e. that they both traverse $E_{i_1}, \dots, E_{i_{n_1}}$, then separate, then come together again to intersect on a second connected chain $E_{j_1}, \dots, E_{j_{n_2}}$, as in figure 26. Suppose that they cross $E_{j_1}, \dots, E_{j_{n_2}}$ in opposite directions. Then there is a contribution from the Schottky group identity which is the same as in eq. (A.26), but now contributions from other Schottky group elements must be added. If we follow ℓ_2 from $E_{i_{n_1}}$ to E_{j_1} and then ℓ_1 from E_{j_1} back to $E_{i_{n_1}}$, then we get a loop ℓ_α corresponding to a closed path P_α starting and ending at the base point, and a corresponding Schottky group element γ_α . The loop $\ell_{1\alpha 2\bar{\alpha}}$ crosses each edge as many times as ℓ_1 and ℓ_2 do, with the exception of the edges $E_{j_1}, \dots, E_{j_{n_2}}$. Using our expression for Schottky group multipliers in terms of pinching parameters eq. (2.21), we get

$$k_{1\alpha 2\bar{\alpha}} = k(\gamma_1 \gamma_\alpha \gamma_2 \gamma_\alpha^{-1}) = \frac{k_1 k_2}{p_{j_1}^2 \cdots p_{j_{n_2}}^2} (1 + \mathcal{O}(p_i)) \quad (\text{A.29})$$

Figure 25. Two loops ℓ_1 and ℓ_2 from the homology basis (shown as blue dotted curves) intersect along a chain of edges $E_{i_1}, \dots, E_{i_n}$. Here, their orientations are opposite where they intersect. The red dashed curve is the reduced path $P_{12} = P_1 \cdot P_2$ (where P_j is the path corresponding to the loop ℓ_j). It crosses all the edges that ℓ_1 and ℓ_2 do, with the exception of $E_{i_1}, \dots, E_{i_n}$.

Figure 26. Two loops ℓ_1 and ℓ_2 from the homology basis (shown as blue dotted curves) could intersect as in figure 25 before splitting apart and then joining together to intersect again in a chain $E_{j_1}, \dots, E_{j_{n_2}}$ of edges. The loop ℓ_α (also shown with blue dots) is formed by tracing ℓ_2 from $E_{i_{n_1}}$ to E_{j_1} and then tracing ℓ_1 back to $E_{i_{n_1}}$. The red dashed curve is the reduced path $P_{1\alpha 2\alpha} = P_1 \cdot P_\alpha \cdot P_2 \cdot P_\alpha^{-1}$. It crosses all the edges that ℓ_1 and ℓ_2 do, with the exception of $E_{j_1}, \dots, E_{j_{n_2}}$.

which we can insert into eq. (A.19) finding

$$\log \eta_{1,\alpha 2\bar{\alpha}} = \log (\pm p_{j_1} \cdots p_{j_{n_2}}) + \mathcal{O}(p_i), \quad (\text{A.30})$$

which (recalling eq. (A.16)) is the contribution to the period matrix entry τ_{12} coming from the Schottky group element γ_α .

If ℓ_1 and ℓ_2 traverse $E_{j_1}, \dots, E_{j_{n_2}}$ in the same direction then, as above, we get

$$\log \eta_{1,\alpha 2\bar{\alpha}} = -\log (\pm p_{j_1} \cdots p_{j_{n_2}}) + \mathcal{O}(p_i). \quad (\text{A.31})$$

The computation of eq. (A.30) and eq. (A.31) is valid for arbitrary numbers of intersections between ℓ_1 and ℓ_2 , with each one getting a contribution from a different Schottky group element γ_α . The only Schottky group elements γ_α which give contributions to eq. (3.11) that don't vanish at leading order are those that arise in this manner. In fact, in the remaining cases, the path $\ell_{1\alpha 2\bar{\alpha}} = \ell_1 \ell_\alpha \ell_2 \ell_\alpha^{-1}$ does not have any cancellations and eq. (2.21) gives simply

$$k_{1\alpha 2\bar{\alpha}} = k_1 k_2 k_\alpha^2 (1 + \mathcal{O}(p_i)), \quad (\text{A.32})$$

which can be substituted in eq. (A.19) to give

$$\log \eta_{1,\alpha 2\bar{\alpha}} = \log \frac{1}{1 + k_\alpha} + \mathcal{O}(p_i) = \mathcal{O}(p_i). \quad (\text{A.33})$$

Now suppose that ℓ_1 and ℓ_2 have no edges in common. Then for any loop ℓ_α , we find

$$\frac{k_{1\alpha 2\bar{\alpha}}}{k_1 k_2} = \mathcal{O}(p_i)^2, \quad (\text{A.34})$$

which can be substituted in eq. (A.19) to give

$$\log \eta_{1,\alpha 2\bar{\alpha}} = \mathcal{O}(p_i). \quad (\text{A.35})$$

The general proof of eq. (A.34) is quite cumbersome. It is easy to check in the sub-case that none of the loops (ℓ_1 , ℓ_2 and ℓ_α) have edges in common with each other. It can also be checked in the sub-case that ℓ_1 and ℓ_2 may have some edges in common with ℓ_α : it can be verified that any edges from ℓ_1 cancelled in the product $\ell_1 \ell_\alpha$ are recovered from $\ell_{\bar{\alpha}}$, and so on, such that $\ell_{1\alpha 2\bar{\alpha}}$ contains all of the edges in ℓ_1 and ℓ_2 plus more (see figure 27).

This proves that eq. (A.13) holds for the off-diagonal elements of the period matrix.

The leading contribution to the diagonal elements comes entirely from the term $\frac{1}{2\pi i} \delta_{ij} \log k_i$ in eq. (3.11), since

$$\log k_i = \sum_{E_j \in \ell_i} \log p_j + \mathcal{O}(p_j). \quad (\text{A.36})$$

The second term in eq. (3.11) vanishes at leading order for the diagonal elements because $\log \eta_{1,\alpha 1\bar{\alpha}}$ is always $\mathcal{O}(p_i)$ for all Schottky group elements γ_α . For γ_α such that the corresponding loop ℓ_α has no edges in common with ℓ_1 , this claim can be checked easily.

The proof is a bit more involved in the case where ℓ_α and ℓ_1 have at least one edge (or connected chain of edges) in common. Let us suppose the base point is at one of the ends of $\ell_1 \cap \ell_\alpha$ such that the paths corresponding to the loops are given by $P_1 = P'_1 \cdot P_I$ and $P_\alpha = P'_\alpha \cdot P_I$, where P_I is a path traversing the connected component of the intersection (see figure 28a). Then

$$P_1 P_\alpha P_1 P_{\bar{\alpha}} = P'_1 P_I P'_\alpha P_I P'_1 P'_{\bar{\alpha}}. \quad (\text{A.37})$$

Since P_I and P'_1 are each traversed twice, and together they comprise all the edges in the loop ℓ_1 (as shown in figure 28b), we see that

$$k_{1\alpha 1\bar{\alpha}} = k_1^2 \left(\prod_{E_i \in P'_\alpha} p_i^2 \right) (1 + \mathcal{O}(p_i)). \quad (\text{A.38})$$

Figure 27. A 3-loop example illustrating that if ℓ_1 and ℓ_2 have no edges in common then the reduced loop $\ell_{1\alpha 2\bar{\alpha}}$ always includes all the edges in the loops ℓ_1 and ℓ_2 even if there are some intermediate cancellations. Figure 27a shows the three paths P_1 , P_2 and P_α .

We’ve assumed here that P_α and P_1 cross P_I in the same direction, but if not then we could simply swap $P_\alpha \leftrightarrow P_{\bar{\alpha}}$ without changing the result since $\ell_{1\alpha 1\bar{\alpha}} = \ell_{1\bar{\alpha} 1\alpha}$. Eq. (A.19) gives us

$$\log \eta_{1,\alpha 1\bar{\alpha}} = \log \frac{1}{1 \pm \sqrt{k_{1\alpha 1\bar{\alpha}}/k_1}} + \mathcal{O}(p_i), \quad (\text{A.39})$$

into which eq. (A.38) can be inserted giving

$$\log \eta_{1,\alpha 1\bar{\alpha}} = - \prod_{E_i \in P'_\alpha} p_i + \mathcal{O}(p_j) = \mathcal{O}(p_j). \quad (\text{A.40})$$

So at leading order, the only contribution to the diagonal elements of the period matrix τ_{ii} comes from eq. (A.36).

This complete the proof that the graph period matrix arises as the $\alpha' \rightarrow 0$ limit of the period matrix when using the pinching parametrization along with eq. (3.1).

Figure 28. The path $\ell_{1\alpha 1\bar{\alpha}}$ always includes all of the edges in ℓ_1 at least twice. This figure illustrates why this is true in the case that ℓ_1 and ℓ_α have an edge in common.

A.2.1 An example requiring non-identity Schottky group elements

Let us consider an example where the computation of the period matrix requires summing over more Schottky group elements than just the identity, even at leading order. Consider the graph shown in figure 29. The graph period matrix $(\theta)_{ij}$ with the indicated loop basis is

$$\theta_{ij} = \begin{pmatrix} t_2 + t_3 + t_5 + t_6 + t_8 + t_9 & -t_2 - t_6 - t_9 & -t_5 & -t_8 \\ -t_2 - t_6 - t_9 & t_1 + t_2 + t_4 + t_6 + t_7 + t_9 & -t_4 & -t_7 \\ -t_5 & -t_4 & t_4 + t_5 & 0 \\ -t_8 & -t_7 & 0 & t_7 + t_8 \end{pmatrix}. \quad (\text{A.41})$$

We want to see how the Schottky group formula reproduces eq. (A.41). Let us choose the following set of Schottky generators corresponding to the loop basis and base point indicated:

$$\gamma_1 = \sigma_9 \cdot \rho^{-1} \cdot \sigma_8 \cdot \rho^{-1} \cdot \sigma_6 \cdot \rho^{-1} \cdot \sigma_5 \cdot \rho^{-1} \cdot \sigma_2 \cdot \rho^{-1} \cdot \sigma_3 \cdot \rho^{-1}, \quad (\text{A.42})$$

$$\gamma_2 = \rho^{-1} \cdot \sigma_1 \cdot \rho^{-1} \cdot \sigma_2 \cdot \rho^{-1} \cdot \sigma_4 \cdot \rho^{-1} \cdot \sigma_6 \cdot \rho^{-1} \cdot \sigma_7 \cdot \rho^{-1} \cdot \sigma_9, \quad (\text{A.43})$$

$$\gamma_3 = (\rho \sigma_6 \rho^{-1} \sigma_7 \rho^{-1} \sigma_9)^{-1} \cdot \rho^{-1} \sigma_4 \rho^{-1} \sigma_5 \cdot (\rho \sigma_6 \rho^{-1} \sigma_7 \rho^{-1} \sigma_9) \quad (\text{A.44})$$

$$\gamma_4 = (\rho \cdot \sigma_9)^{-1} \cdot \rho^{-1} \cdot \sigma_7 \cdot \rho^{-1} \cdot \sigma_8 \cdot (\rho \cdot \sigma_9). \quad (\text{A.45})$$

Figure 29. A 4-loop graph with a homology basis marked. This example shows the necessity of including contributions from non-identity Schottky group elements in the period matrix formula eq. (3.11) even at leading order.

The multipliers of these generators are

$$k_1 = p_2 p_3 p_5 p_6 p_8 p_9, \quad k_2 = p_1 p_2 p_4 p_6 p_7 p_9, \quad k_3 = p_4 p_5, \quad k_4 = p_7 p_8. \quad (\text{A.46})$$

Clearly, the diagonal elements of eq. (A.41) are given by

$$\theta_{ii} = -\alpha' \log k_i \quad (\text{A.47})$$

after making the replacement eq. (3.1). The off-diagonal entries come from the second term in eq. (3.11). We can compute all the non-vanishing contributions as

$$(-\log \hat{\eta}_{ij}) = \begin{pmatrix} 0 & -\log p_9 & 0 & -\log p_8 \\ -\log p_9 & 0 & -\log p_4 & -\log p_7 \\ 0 & -\log p_4 & 0 & 0 \\ -\log p_8 & -\log p_7 & 0 & 0 \end{pmatrix} + \mathcal{O}(p_n) \quad (\text{A.48})$$

$$(-\log \hat{\eta}_{i, \bar{4}j4}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\log p_6 & 0 & 0 & 0 \\ -\log p_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(p_n) \quad (\text{A.49})$$

$$(-\log \hat{\eta}_{i, 4j\bar{4}}) = \begin{pmatrix} 0 & -\log p_6 & -\log p_5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(p_n) \quad (\text{A.50})$$

$$(-\log \hat{\eta}_{i, 34j\bar{4}\bar{3}}) = \begin{pmatrix} 0 & -\log p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(p_n) \quad (\text{A.51})$$

$$(-\log \hat{\eta}_{i, \bar{4}\bar{3}j34}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\log p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(p_n), \quad (\text{A.52})$$

Figure 30. Starting from the red arrows, the blue curves indicate paths of the form $P_{1\alpha 2\bar{\alpha}} \equiv P_1 \cdot P_\alpha \cdot P_2 \cdot P_\alpha^{-1}$ for $\gamma_\alpha = \text{id}$ (figure 30a), $\gamma_\alpha = \gamma_4^{-1}$ (figure 30a) and $\gamma_\alpha = \gamma_4^{-1}\gamma_3^{-1}$ (figure 30a), for the marked graph in figure 29.

where we've used the short-hand notation $\hat{\eta}_{i,\alpha j\bar{\alpha}}$ to mean $\hat{\eta}_{i,\alpha j\bar{\alpha}} \equiv 1$ if $i = j$ and $\gamma_\alpha = \text{id}$ or if the left-most factor of γ_α is $\gamma_i^{\pm n}$ or the right-most factor is $\gamma_j^{\pm n}$; otherwise $\hat{\eta}_{i,\alpha j\bar{\alpha}} \equiv \eta_{i,\alpha j\bar{\alpha}}$ as defined in eq. (A.15). Clearly, when all these terms are inserted in the sum eq. (A.16) along with the diagonal elements, we can use eq. (3.1) to arrive at

$$\theta_{ij} = \lim_{\alpha' \rightarrow 0} \text{Im}(2\pi\alpha'\tau_{ij}) \quad (\text{A.53})$$

for the graph period matrix in eq. (A.41).

To see an application of the construction in figure 26 which was used to arrive at eq. (A.30) and eq. (A.31) in the proof above, let us consider one particular entry in the period matrix — say, τ_{12} . It is clear from eqs. (A.48)–(A.52) that the important contributions to τ_{12} come from

$$\log \eta_{12} \sim \log p_9, \quad \log \eta_{1,42\bar{4}} \sim \log p_6, \quad \log \eta_{1,342\bar{4}\bar{3}} \sim \log p_2. \quad (\text{A.54})$$

To see why these are the important Schottky group elements, recall from the proof above that each cross-ratio $\log \eta_{1,\alpha 2\bar{\alpha}}$ can be computed with eq. (A.19) by finding the multiplier of $k_{1\alpha 2\bar{\alpha}}$ and noting how it differs from $k_1 k_2$ at leading order.

In figure 30 we have drawn the paths $P_{1\alpha 2\bar{\alpha}} \equiv P_1 \cdot P_\alpha \cdot P_2 \cdot P_\alpha^{-1}$ for the three relevant Schottky group elements, namely $\gamma_\alpha = \text{id}$, $\gamma_\alpha = \gamma_4^{-1}$ and $\gamma_\alpha = \gamma_4^{-1}\gamma_3^{-1}$. These paths can be reduced by removing pairs of consecutive edges that are inverses of each other; the resulting reduced paths (shown in figure 31) can be used to read off the multipliers $k_{1\alpha 2\bar{\alpha}}$. These reduced paths cross the same edges that ℓ_1 and ℓ_2 do, with the exception of one edge in each case (E_9 in figure 31a, E_6 in figure 31b and E_2 in figure 31c). In accordance with the calculation in the above proof (particularly eq. (A.30)), this explains why at leading order τ_{12} gets the three non-zero contributions listed in eq. (A.54), with one corresponding to each of the three respective edges where ℓ_1 and ℓ_2 intersect.

We have seen above that the necessity of including non-identity Schottky group elements in eq. (3.11) comes from pairs of loops ℓ_i, ℓ_j whose intersections have > 1 connected

Figure 31. By deleting pairs of consecutive edges that are inverses of each other in the paths in figure 30, we arrive at the loops shown here.

components. Although ℓ_1 and ℓ_3 intersect in just one edge (E_5), the non-vanishing contribution to τ_{13} comes from $\gamma_\alpha = \gamma_4^{-1}$ in eq. (A.50), not from $\gamma_\alpha = \text{id}$. This is because we have used the same base point (indicated on figure 29) for each period matrix entry τ_{ij} , instead of using $\text{PSL}(2)$ invariance to choose a new one common to ℓ_i and ℓ_j in each case.

A.3 Green's function

In this section we prove that the worldline Green's function arises as the $\alpha' \rightarrow 0$ limit of the worldsheet Green's function, as described in section 3. In section A.3.1 and A.3.2 we show eq. (3.13), then in section A.3.3 we prove eq. (3.14).

A.3.1 Leading factor

In this section we show that when expressed in terms of pinching parameters, the Schottky-Klein prime form converges to the length of a path in the graph, as in eq. (3.13). We will begin by considering the first factor $(x_1 - x_2)/(\sqrt{dx_1}\sqrt{dx_2})$ in eq. (3.8). It turns out that this is the only factor which gives a non-vanishing contribution at leading order in the pinching parameters. The computation will be re-used in section A.3.2 to show that the rest of the factors vanish.

Choose a coordinate system at X_1 (i.e. a path from X_1 to the base point) and pick some path P from X_2 to X_1 . This determines a coordinate system at X_2 : let $W = \phi(P)$ (where ϕ is the homomorphism from paths to Möbius maps defined in section 2.1) and take

$$V_2 = V_1 \cdot W, \quad (\text{A.55})$$

then the factor we want to compute is given by

$$\frac{V_1(0) - V_2(0)}{\sqrt{V_1'(0)V_2'(0)}} \equiv \frac{V_1(0) - (V_1 \cdot W)(0)}{\sqrt{V_1'(0)(V_1 \cdot W)'(0)}} \equiv f(V_1, W). \quad (\text{A.56})$$

If we write

$$W = W_N \equiv \rho^{\pm 1} \sigma_{i_1} \rho^{\pm 1} \sigma_{i_2} \rho^{\pm 1} \dots \rho^{\pm 1} \sigma_{i_N} \rho^{\pm 1}, \quad (\text{A.57})$$

where N is the number of edges in the path P , then we can denote the sum of the logs of the pinching parameters in this path as

$$S_P = \sum_{E_i \in P} \log p_i. \quad (\text{A.58})$$

Then we claim that

$$\log |f(V_1, W)| = -\frac{1}{2} S_P + \mathcal{O}(p_{i_m}). \quad (\text{A.59})$$

We can show this by induction on N (the number of edges in the path P between X_1 and X_2). Let's start with the $N = 0$ base case in which the two external edges X_1 and X_2 must be on the same cubic vertex and thus

$$W = W_0 = \rho^{\pm 1}. \quad (\text{A.60})$$

Let us write $V_1 = V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$, so $z_1 = b/d$ and $V'_1(0) = 1/d^2$.

First we can check the case with $W_0 = \rho$, which gives $z_2 = a/c$ and $V'_2(0) = 1/c^2$. Then

$$\log \frac{|z_1 - z_2|}{\sqrt{\log V'_1(0) V'_2(0)}} = \log \frac{|b/d - a/c|}{\sqrt{1/c^2 d^2}} = \log |ad - bc| = 0. \quad (\text{A.61})$$

Alternatively, if $W_0 = \rho^{-1}$ so $z_2 = \frac{a+b}{c+d}$ and $V'_2(0) = 1/(c+d)^2$, then

$$\log \frac{|z_1 - z_2|}{\sqrt{\log V'_1(0) V'_2(0)}} = \log \frac{|a/c - (a+b)/(c+d)|}{\sqrt{(1/(c+d)^2)(1/c^2)}} = 0, \quad (\text{A.62})$$

Since the sum in eq. (A.59) has no terms for $N = 0$ it should vanish, so the base case has been checked.

It remains to do the inductive step: for $m \geq 1$ let us denote by $C(m)$ the statement

$$\log \left| \frac{(V \cdot W_m)(0) - V(0)}{\sqrt{(V \cdot W_m)'(0) V'(0)}} \right| = -\frac{1}{2} \sum_{n=1}^m \log p_{i_n} + \mathcal{O}(p_{i_\ell}), \quad (\text{A.63})$$

then we want to show that $C(m) \Rightarrow C(m+1)$. This is equivalent to

$$\log \left| \frac{(V \cdot W_{m+1})(0) - V(0)}{(V \cdot W_m)(0) - V(0)} \sqrt{\frac{(V \cdot W_m)'(0)}{(V \cdot W_{m+1})'(0)}} \right| = -\frac{1}{2} \log p_{i_{m+1}} + \mathcal{O}(p_{i_n}). \quad (\text{A.64})$$

We can show this by expanding $(V \cdot W_{m+1})(0)$ and $(V \cdot W_{m+1})'(0)$ as power series in $p_{i_{m+1}}$. By factorizing $(V \cdot W_{m+1})(z)$ as

$$(V \cdot W_{m+1})(z) = (V \cdot W_m)(\sigma_{i_{m+1}} \cdot \rho^{\pm 1}(z)) \quad (\text{A.65})$$

we can power expand:

$$(V \cdot W_{m+1})(z) = (V \cdot W_m)(0) + (V \cdot W_m)'(0) (\sigma_{i_{m+1}} \cdot \rho^{\pm 1})(z) + \dots \quad (\text{A.66})$$

Similarly, using the chain rule on eq. (A.65) gives

$$(V \cdot W_{m+1})'(z) = (V \cdot W_m)'(\sigma_{i_{m+1}} \cdot \rho^{\pm 1}(z)) \times (\sigma_{i_{m+1}} \cdot \rho^{\pm 1})'(z), \quad (\text{A.67})$$

and the first factor $(V \cdot W_m)'$ can be power-expanded as above,

$$(V \cdot W_{m+1})'(z) = \left((V \cdot W_m)'(0) + (\sigma_{i_{m+1}} \cdot \rho^{\pm 1}(z)) (V \cdot W_m)''(0) + \dots \right) \times (\sigma_{i_{m+1}} \cdot \rho^{\pm 1})'(z). \quad (\text{A.68})$$

To find the dependence of $(V \cdot W_{m+1})(0)$ and $(V \cdot W_{m+1})'(0)$ on $p_{i_{m+1}}$, we can use

$$(\sigma_{i_{m+1}} \rho)(z) = p_{i_{m+1}} \frac{z}{1-z} \quad (\text{A.69})$$

and

$$(\sigma_{i_{m+1}} \rho^{-1})(z) = -p_{i_{m+1}}(1-z), \quad (\text{A.70})$$

to see

$$(\sigma_{i_{m+1}} \rho)(0) = 0, \quad (\sigma_{i_{m+1}} \rho^{-1})(0) = -p_{i_{m+1}}; \quad (\text{A.71})$$

we will use the observation that these are both of the general form

$$(\sigma_{i_{m+1}} \rho^{\pm 1})(0) = \mathcal{O}(p_{i_{m+1}}). \quad (\text{A.72})$$

In both eq. (A.69) and eq. (A.70) the derivative at $z = 0$ is the same:

$$(\sigma_{i_{m+1}} \rho^{\pm 1})'(0) = p_{i_{m+1}}. \quad (\text{A.73})$$

Now, putting $z = 0$ in eq. (A.66) and inserting eq. (A.72) we see that

$$(V \cdot W_{m+1})(0) = (V \cdot W_m)(0) + \mathcal{O}(p_{i_{m+1}}). \quad (\text{A.74})$$

Similarly, putting $z = 0$ in eq. (A.68) and inserting eq. (A.73) and eq. (A.72) we see that

$$(V \cdot W_{m+1})'(0) = p_{i_{m+1}} \left((V \cdot W_m)'(0) + \mathcal{O}(p_{i_{m+1}}) \right). \quad (\text{A.75})$$

Now, with the use of eq. (A.74), the first factor inside the log on the left-hand-side of eq. (A.64) becomes

$$\begin{aligned} \log \left| \frac{(V \cdot W_{m+1})(0) - V(0)}{(V \cdot W_m)(0) - V(0)} \right| &= \log \left| 1 + \mathcal{O}(p_{i_{m+1}}) \frac{1}{(V \cdot W_m)(0) - V(0)} \right| \\ &= \mathcal{O}(p_{i_{m+1}}). \end{aligned} \quad (\text{A.76})$$

Similarly, with the use of eq. (A.74) and eq. (A.75) the other term on the left-hand side of eq. (A.64) becomes

$$\begin{aligned} \log \left| \sqrt{\frac{(V \cdot W_m)'(0)}{(V \cdot W_{m+1})'(0)}} \right| &= \log \left| \sqrt{\frac{1}{p_{i_{m+1}}} (1 + \mathcal{O}(p_{i_{m+1}}))} \right| \\ &= -\frac{1}{2} \log p_{i_{m+1}} + \mathcal{O}(p_{i_{m+1}}). \end{aligned} \quad (\text{A.77})$$

Adding eq. (A.76) and eq. (A.77), we can check that eq. (A.64) holds for all $m \geq 0$ and thus by induction, eq. (A.59) holds for all N .

This proves that

$$\log |E(x_1, x_2)| = -\frac{1}{2} S_P + \mathcal{O}(p_i). \quad (\text{A.78})$$

A.3.2 The rest of the prime form factors

The rest of the factors in the prime form are of the form

$$F_\alpha(x_1, x_2) \equiv \frac{x_1 - \gamma_\alpha(x_2)}{x_1 - \gamma_\alpha(x_1)} \frac{x_2 - \gamma_\alpha(x_1)}{x_2 - \gamma_\alpha(x_2)}. \quad (\text{A.79})$$

These factors don't contribute at order $\mathcal{O}(p_i)$,

$$\log F_\alpha(x_1, x_2) = \mathcal{O}(p_i). \quad (\text{A.80})$$

To see why this is true, first note that since $F_\alpha(x_1, x_2) \equiv F_\alpha(V_1(0), V_2(0))$ is a cross-ratio, it is a projective invariant so we can make a change of coordinates to replace V_1 , V_2 and γ_α with

$$\hat{V}_1 = \text{id}, \quad \hat{V}_2 = V_1^{-1}V_2, \quad \hat{\gamma}_\alpha = V_1^{-1}\gamma_\alpha V_1. \quad (\text{A.81})$$

Then we can write $F_\alpha(x_1, x_2)$ in the form

$$F_\alpha(x_1, x_2) = \frac{\text{id}(0) - (\hat{\gamma}_\alpha \hat{V}_2)(0)}{\text{id}(0) - \hat{\gamma}_\alpha(0)} \cdot \frac{\hat{V}_2(0) - \hat{\gamma}_\alpha(0)}{\hat{V}_2(0) - (\hat{\gamma}_\alpha \hat{V}_2)(0)}. \quad (\text{A.82})$$

In terms of the function $f(V, W)$ defined in eq. (A.56), $F_\alpha(x_1, x_2)$ may be expressed as

$$F_\alpha(x_1, x_2) = \frac{f(\text{id}, \hat{\gamma}_\alpha \hat{V}_2)}{f(\text{id}, \hat{\gamma}_\alpha)} \frac{f(\hat{V}_2, \hat{V}_2^{-1} \hat{\gamma}_\alpha)}{f(\hat{V}_2, \hat{V}_2^{-1} \hat{\gamma}_\alpha \hat{V}_2)}. \quad (\text{A.83})$$

Using eq. (A.59) and the results of section A.3.1, we can express $F_\alpha(x_1, x_2)$ in terms of the 'lengths' of some paths in the graph:

$$\log |F_\alpha(x_1, x_2)| = -\frac{1}{2} \left| \hat{S}_{\alpha 2} - \hat{S}_{2\alpha 2} + \hat{S}_{2\alpha} - \hat{S}_\alpha \right| + \mathcal{O}(p_i), \quad (\text{A.84})$$

where $\hat{S}_{2\alpha}$ means $S_{\hat{P}_2^{-1}\hat{P}_\alpha}$ and so on, with $\hat{P}_2$ and $\hat{P}_\alpha$ being the paths corresponding to the maps $\hat{V}_2$ and $\hat{\gamma}_2$ defined in eq. (A.81).

The sum over edges on the right-hand side of eq. (A.84) vanishes. In fact, the path edge sums S_α obey the sum rule

$$S_{\beta\delta} = S_\beta + S_\delta - 2S_{\widehat{\beta\delta}}, \quad (\text{A.85})$$

where $S_{\widehat{\beta\delta}}$ means the contribution from any right-most factors of the path P_β that cancel left-most factors of the path P_δ . Then

$$\hat{S}_{\alpha 2} = \hat{S}_\alpha + \hat{S}_2 - 2\hat{S}_{\widehat{\alpha 2}} \quad (\text{A.86})$$

$$\hat{S}_{2\alpha 2} = \hat{S}_2 + \hat{S}_\alpha + \hat{S}_2 - 2\hat{S}_{\widehat{2\alpha}} - 2\hat{S}_{\widehat{\alpha 2}} \quad (\text{A.87})$$

$$\hat{S}_{2\alpha} = \hat{S}_2 + \hat{S}_\alpha - 2\hat{S}_{\widehat{2\alpha}} \quad (\text{A.88})$$

and so

$$\hat{S}_{\alpha 2} - \hat{S}_{2\alpha 2} + \hat{S}_{2\alpha} - \hat{S}_\alpha = 0. \quad (\text{A.89})$$

This shows that eq. (A.3.2) doesn't contribute at leading order.

A.3.3 The second term in the Green's function

The second term in the Green's function eq. (3.7) ensures that $\mathcal{G}(x_1, x_2)$ is single-valued on $\Sigma \times \Sigma$ [54]. For our purposes, this means that the Green's function is independent of the choice of path P from X_2 to X_1 used to define the coordinate chart V_2 . Therefore certainly when we find $\lim_{\alpha' \rightarrow 0} \alpha' \mathcal{G}(x_1, x_2)$, we must also find an expression that is independent of the path P .

We have proven in the previous section that in the $\alpha' \rightarrow 0$ limit, the first term in $\mathcal{G}(x_1, x_2)$ asymptotes to $s/2$, where s is the sum of the Schwinger parameters t_i in the path P (eq. (3.5)). This means that the second term in the Green's function should asymptote to some expression

$$f(\Gamma, P, t_i) \equiv \lim_{\alpha' \rightarrow 0} \alpha' \left(\int_{x_1}^{x_2} \vec{\omega} \right) \cdot (2\pi \text{Im}\tau)^{-1} \cdot \left(\int_{x_1}^{x_2} \vec{\omega} \right) \quad (\text{A.90})$$

such that the combination

$$-\frac{1}{2}s + \frac{1}{2}f(\Gamma, P, t_i) \quad (\text{A.91})$$

is independent of which path P is chosen between external edges X_1 and X_2 on Γ , and vanishes when the two external edges coincide.

The function which satisfies this symmetry requirement is

$$f(\Gamma, P, t_i) = \vec{v}^t \cdot \theta^{-1} \cdot \vec{v}, \quad (\text{A.92})$$

where the graph period matrix and the vector $\vec{v}$ associated to the path P are defined in eq. (3.3) and eq. (3.4), respectively.

This completes the proof of the claim that the worldline Green's function arises as the limit of the worldsheet Green's function,

$$G(X_1, X_2) = - \lim_{\alpha' \rightarrow 0} \alpha' \mathcal{G}(x_1, x_2), \quad (\text{A.93})$$

and in particular this is an indirect proof that $\int_{x_1}^{x_2} \vec{\omega}$ has the asymptotics given in eq. (3.14), since we already checked in section A.2 that the asymptotics of τ_{ij} are given by eq. (3.15).

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References

- [1] S. Playle and S. Sciuto, *Schwinger-type parametrization of open string worldsheets*, *Phys. Lett. B* **766** (2017) 317 [[arXiv:1611.08604](https://arxiv.org/abs/1611.08604)] [[INSPIRE](#)].
- [2] J. Scherk, *Zero-slope limit of the dual resonance model*, *Nucl. Phys. B* **31** (1971) 222 [[INSPIRE](#)].
- [3] J. Scherk and J.H. Schwarz, *Dual Models for Nonhadrons*, *Nucl. Phys. B* **81** (1974) 118 [[INSPIRE](#)].

- [4] P. Tourkine, *Tropical Amplitudes*, *Annales Henri Poincaré* **18** (2017) 2199 [[arXiv:1309.3551](#)] [[INSPIRE](#)].
- [5] Z. Bern and D.A. Kosower, *A New Approach to One Loop Calculations in Gauge Theories*, *Phys. Rev. D* **38** (1988) 1888 [[INSPIRE](#)].
- [6] Z. Bern and D.A. Kosower, *Efficient calculation of one loop QCD amplitudes*, *Phys. Rev. Lett.* **66** (1991) 1669 [[INSPIRE](#)].
- [7] Z. Bern and D.A. Kosower, *Color decomposition of one loop amplitudes in gauge theories*, *Nucl. Phys. B* **362** (1991) 389 [[INSPIRE](#)].
- [8] Z. Bern and D.C. Dunbar, *A Mapping between Feynman and string motivated one loop rules in gauge theories*, *Nucl. Phys. B* **379** (1992) 562 [[INSPIRE](#)].
- [9] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop corrections to five gluon amplitudes*, *Phys. Rev. Lett.* **70** (1993) 2677 [[hep-ph/9302280](#)] [[INSPIRE](#)].
- [10] Z. Bern, D.C. Dunbar and T. Shimada, *String based methods in perturbative gravity*, *Phys. Lett. B* **312** (1993) 277 [[hep-th/9307001](#)] [[INSPIRE](#)].
- [11] J.-L. Gervais and A. Neveu, *Feynman rules for massive gauge fields with dual diagram topology*, *Nucl. Phys. B* **46** (1972) 381 [[INSPIRE](#)].
- [12] I. Pesando, *On the gauge chosen by the bosonic open string*, *Nucl. Phys. B* **918** (2017) 129 [[arXiv:1701.07855](#)] [[INSPIRE](#)].
- [13] P. Di Vecchia, L. Magnea, A. Lerda, R. Russo and R. Marotta, *String techniques for the calculation of renormalization constants in field theory*, *Nucl. Phys. B* **469** (1996) 235 [[hep-th/9601143](#)] [[INSPIRE](#)].
- [14] K. Roland, *Multiloop gluon amplitudes in pure gauge theories*, *Phys. Lett. B* **289** (1992) 148 [[INSPIRE](#)].
- [15] G. Carbone and S. Sciuto, *On amplitudes involving excited particles in dual-resonance models*, *Lett. Nuovo Cim.* **3S1** (1970) 246 [[INSPIRE](#)].
- [16] J.M. Kosterlitz and D.A. Wray, *The general N-point vertex in a dual model*, *Lett. Nuovo Cim.* **3S1** (1970) 491 [[INSPIRE](#)].
- [17] C. Lovelace, *M-loop generalized Veneziano formula*, *Phys. Lett. B* **32** (1970) 703 [[INSPIRE](#)].
- [18] M. Kaku and L. Yu, *The general multi-loop veneziano amplitude*, *Phys. Lett. B* **33** (1970) 166 [[INSPIRE](#)].
- [19] V. Alessandrini, *A general approach to dual multiloop diagrams*, *Nuovo Cim. A* **2** (1971) 321 [[INSPIRE](#)].
- [20] V. Alessandrini and D. Amati, *Properties of dual multiloop amplitudes*, *Nuovo Cim. A* **4** (1971) 793 [[INSPIRE](#)].
- [21] P. Di Vecchia, R. Nakayama, J.L. Petersen, J. Sidenius and S. Sciuto, *BRST invariant N reggeon vertex*, *Phys. Lett. B* **182** (1986) 164 [[INSPIRE](#)].
- [22] P. Di Vecchia, M. Frau, A. Lerda and S. Sciuto, *A Simple Expression for the Multiloop Amplitude in the Bosonic String*, *Phys. Lett. B* **199** (1987) 49 [[INSPIRE](#)].
- [23] P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and A. Sciuto, *N Point g Loop Vertex for a Free Bosonic Theory With Vacuum Charge Q*, *Nucl. Phys. B* **322** (1989) 317 [[INSPIRE](#)].

- [24] P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda and S. Sciuto, *N String, g Loop Vertex for the Fermionic String*, *Phys. Lett. B* **211** (1988) 301 [[INSPIRE](#)].
- [25] P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and S. Sciuto, *N Point g Loop Vertex for a Free Fermionic Theory With Arbitrary Spin*, *Nucl. Phys. B* **333** (1990) 635 [[INSPIRE](#)].
- [26] E.J. Martinec, *Conformal Field Theory on a (Super)Riemann Surface*, *Nucl. Phys. B* **281** (1987) 157 [[INSPIRE](#)].
- [27] A. Neveu and P.C. West, *Group Theoretic Approach to the Open Bosonic String Multiloop S Matrix*, *Commun. Math. Phys.* **114** (1988) 613 [[INSPIRE](#)].
- [28] A. Neveu and P.C. West, *Group Theoretic Approach to the Superstring and Its Supermoduli*, *Nucl. Phys. B* **311** (1988) 79 [[INSPIRE](#)].
- [29] Y.I. Manin, *Quantum Strings and Algebraic Curves*, in *Proceedings of the International Congress of Mathematicians*, Berkeley, California, 3–11 August 1986, A.M. Gleason ed., American Mathematical Society (1988), pp. 1286.
- [30] P. Di Vecchia, L. Magnea, A. Lerda, R. Marotta and R. Russo, *Two loop scalar diagrams from string theory*, *Phys. Lett. B* **388** (1996) 65 [[hep-th/9607141](#)] [[INSPIRE](#)].
- [31] A. Frizzo, L. Magnea and R. Russo, *Scalar field theory limits of bosonic string amplitudes*, *Nucl. Phys. B* **579** (2000) 379 [[hep-th/9912183](#)] [[INSPIRE](#)].
- [32] R. Russo and S. Sciuto, *Twisted determinants on higher genus Riemann surfaces*, *Nucl. Phys. B* **669** (2003) 207 [[hep-th/0306129](#)] [[INSPIRE](#)].
- [33] R. Russo and S. Sciuto, *Twisted determinants and bosonic open strings in an electromagnetic field*, *Fortsch. Phys.* **52** (2004) 678 [[hep-th/0312205](#)] [[INSPIRE](#)].
- [34] L. Magnea, R. Russo and S. Sciuto, *Two-loop Euler-Heisenberg effective actions from charged open strings*, *Int. J. Mod. Phys. A* **21** (2006) 533 [[hep-th/0412087](#)] [[INSPIRE](#)].
- [35] L. Magnea, S. Playle, R. Russo and S. Sciuto, *Multi-loop open string amplitudes and their field theory limit*, *JHEP* **09** (2013) 081 [[arXiv:1305.6631](#)] [[INSPIRE](#)].
- [36] L. Magnea, S. Playle, R. Russo and S. Sciuto, *Two-loop Yang-Mills diagrams from superstring amplitudes*, *JHEP* **06** (2015) 146 [[arXiv:1503.05182](#)] [[INSPIRE](#)].
- [37] V.V. Fock, *Description of moduli space of projective structures via fat graphs*, [[hep-th/9312193](#)] [[INSPIRE](#)].
- [38] E. Witten, *Noncommutative Geometry and String Field Theory*, *Nucl. Phys. B* **268** (1986) 253 [[INSPIRE](#)].
- [39] B. Zwiebach, *A Proof that Witten's open string theory gives a single cover of moduli space*, *Commun. Math. Phys.* **142** (1991) 193 [[INSPIRE](#)].
- [40] S.B. Giddings, *The Veneziano Amplitude from Interacting String Field Theory*, *Nucl. Phys. B* **278** (1986) 242 [[INSPIRE](#)].
- [41] A. Sen, *Off-shell Amplitudes in Superstring Theory*, *Fortsch. Phys.* **63** (2015) 149 [[arXiv:1408.0571](#)] [[INSPIRE](#)].
- [42] A. Sen, *Gauge Invariant 1PI Effective Action for Superstring Field Theory*, *JHEP* **06** (2015) 022 [[arXiv:1411.7478](#)] [[INSPIRE](#)].

- [43] A. Sen, *Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector*, *JHEP* **08** (2015) 025 [[arXiv:1501.00988](#)] [[INSPIRE](#)].
- [44] R.C. Penner, *The Moduli Space of Punctured Surfaces*, in *Mathematical Aspects of String Theory*, *Adv. Ser. Math. Phys.* **1** (1987) 313.
- [45] A. Marden, *Geometric Complex Coordinates for Teichmüller Space*, in *Mathematical Aspects of String Theory*, *Adv. Ser. Math. Phys.* **1** (1987) 341.
- [46] P. Dai and W. Siegel, *Worldline Green Functions for Arbitrary Feynman Diagrams*, *Nucl. Phys. B* **770** (2007) 107 [[hep-th/0608062](#)] [[INSPIRE](#)].
- [47] E. Witten, *Notes On Super Riemann Surfaces And Their Moduli*, [arXiv:1209.2459](#) [[INSPIRE](#)].
- [48] E. Witten, *Superstring Perturbation Theory Revisited*, [arXiv:1209.5461](#) [[INSPIRE](#)].
- [49] E. Witten, *More On Superstring Perturbation Theory: An Overview Of Superstring Perturbation Theory Via Super Riemann Surfaces*, [arXiv:1304.2832](#) [[INSPIRE](#)].
- [50] E. Witten, *Notes On Holomorphic String And Superstring Theory Measures Of Low Genus*, [arXiv:1306.3621](#) [[INSPIRE](#)].
- [51] R. Donagi and E. Witten, *Supermoduli Space Is Not Projected*, *Proc. Symp. Pure Math.* **90** (2015) 19 [[arXiv:1304.7798](#)] [[INSPIRE](#)].
- [52] R. Donagi and E. Witten, *Super Atiyah classes and obstructions to splitting of supermoduli space*, [arXiv:1404.6257](#) [[INSPIRE](#)].
- [53] S. Playle, *Deforming super Riemann surfaces with gravitinos and super Schottky groups*, *JHEP* **12** (2016) 035 [[arXiv:1510.06749](#)] [[INSPIRE](#)].
- [54] E. D'Hoker, *String Theory*, in *Quantum fields and strings: A course for mathematicians*, P. Deligne et al. eds., vol. 2, American Mathematical Society, Providence, RI (1999), pp. 807–1011.