Naturalness made easy: two-loop naturalness bounds on minimal SM extensions

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ABSTRACT: The main result of this paper is a collection of conservative naturalness bounds on minimal extensions of the Standard Model by (vector-like) fermionic or scalar gauge multiplets. Within, we advocate for an intuitive and physical concept of naturalness built upon the renormalisation group equations. In the effective field theory of the Standard Model plus a gauge multiplet with mass M, the low scale Higgs mass parameter is a calculable function of $\overline{\text{MS}}$ input parameters defined at some high scale $\Lambda_h > M$. If the Higgs mass is very sensitive to these input parameters, then this signifies a naturalness problem. To sensibly capture the sensitivity, it is shown how a sensitivity measure can be rigorously derived as a Bayesian model comparison, which reduces in a relevant limit to a Barbieri-Giudice-like fine-tuning measure. This measure is fully generalisable to any perturbative EFT. The interesting results of our two-loop renormalisation group study are as follows: for $\Lambda_h = \Lambda_{\text{Pl}}$ we find "10% fine-tuning" bounds on the masses of various gauge multiplets of $M < \mathcal{O}(1\text{–}10)\,\text{TeV}$, with bounds on fermionic gauge multiplets significantly weaker than for scalars; these bounds remain finite in the limit $\Lambda_h \to M^+$, weakening to $M < \mathcal{O}(10\text{–}100)\,\text{TeV}$; and bounds on coloured multiplets are no more severe than for electroweak multiplets, since they only directly correct the Higgs mass at three-loop.

Contents

1 Introduction

The Standard Model (SM) appears to represent a very good effective field theory (EFT) for energies at least $\lesssim$ TeV. Still, it has several well known theoretical and phenomenological shortcomings. Many of these can be addressed with minimal extensions of the SM by heavy fermionic and/or scalar gauge multiplets (GMs). However, the Higgs mass parameter $\mu^2(m_Z) \approx -(88 \text{ GeV})^2$, appearing in the SM potential $\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$, is sensitive to such heavy new physics; GMs couple (at the very least) at loop level to the SM Higgs, thereby inducing corrections to the Higgs mass and potentially introducing a naturalness problem.

The subject of naturalness in the modern literature is rife with various (and often conflicting) definitions. Let us therefore, at the outset, state the definition used in this paper: *a parameter in a quantum field theory is "natural" if its measured value at low scale is (sufficiently) insensitive to details of the physics at high scale*. Plainly, then, to examine naturalness of the Higgs mass parameter we require: (1) a description of the low scale physics; (2) a description of the high scale physics; (3) a map which relates them; and (4) a measure which quantifies sensitivity of $\mu^2(m_Z)$ to the high scale physics.

In this paper we confront the question, *at what mass does a heavy GM introduce a physical Higgs naturalness problem?* Vector-like fermionic and scalar GMs of various charges are studied. We advocate a renormalisation group (RG) approach to naturalness. The description of the low (high) scale physics is provided by the $\overline{\text{MS}}$ Lagrangian parameters of the SM (SM+GM) EFT defined at the scale m_Z (Λ_h), and the map which relates them is the set of RG equations (RGEs). We employ a sensitivity measure which can be interpreted as a Bayesian model comparison. Several Bayesian approaches to naturalness have previously been considered in the literature [1–13]. Inspired by the approach of ref. [9], we propose a particular model comparison which captures the "naturalness price" paid for promoting the Higgs mass parameter from a purely phenomenological input parameter at low scale to a high scale input parameter of the model. We show that this sensitivity measure then reduces in a well-motivated limit to a Barbieri-Giudice-like [14, 15] fine-tuning measure. Quantifying and bounding this sensitivity results in naturalness bounds on the masses of the various GMs.

There exist many phenomenologically motivated extensions to the SM which involve just a single GM. Such models can address a wide variety of shortcomings of the SM including neutrino mass [16–22], dark matter [23, 24], baryogenesis [25] and the strong CP problem [26–29]. The naturalness bounds we derive can be applied to these models and many others besides. Our results can also be used to provide a qualitative, conservative bound even in extended models. Moreover, the framework we present is completely general and could in principle be applied to any model.

The paper is organised as follows. In section 2 we describe an intuitive and physical concept of naturalness built upon the RGEs, and the sensitivity measure as a Bayesian model comparison. In section 3 we describe how this concept is applied to the SM+GM EFTs. Our main result is a list of naturalness bounds presented in section 4. These results are discussed in section 5, and we conclude in section 6.

2 Physical Naturalness

In this section we describe a physical way to understand the Higgs naturalness problem, especially pertinent in the context of bottom-up extensions of the SM. To frame the discussion, let us appeal to an illuminating toy model.

2.1 Toy model

Consider a perturbative EFT consisting of the SM plus a heavy particle of mass M (whose mass is obtained independently of electroweak symmetry breaking). The μ^2 RGE valid at a renormalisation scale $\mu_R > M$ takes the form

$$\frac{d}{d\log \mu_R}\mu^2(\mu_R) = C_1(\mu_R)\mu^2(\mu_R) + C_2(\mu_R)M^2(\mu_R)\,, \tag{2.1}$$

where $C_1(\mu_R) \simeq 6y_t(\mu_R)/(4\pi)^2$, with y_t the top quark Yukawa coupling. The quantity $C_2(\mu_R)$ might be comprised of SM and/or beyond-SM couplings. The RGEs allow μ^2 (and other low scale parameters) to be extrapolated to a high scale Λ_h, at most up to the scale at

which the EFT is no longer valid. At Λ_h, these renormalised parameters can be interpreted as "input parameters" which might be derived from even higher scale physics. The input parameters are, by construction, connected with the low energy parameters via the RGEs. If the low energy parameters are very sensitive to these input parameters, then this signifies a naturalness problem.

Let us now, under this paradigm, try to understand when a heavy particle introduces a Higgs naturalness problem. One can fully solve eq. (2.1) in the limit where C_1, C_2, and M^2 have no scale dependence. Including a possible threshold correction, $\mu_+^2(M) = \mu_-^2(M) - C_T M^2$, when the SM EFT parameters $(-)$ are matched onto the full EFT parameters $(+)$ at the threshold M, and in the limit $C_1 \log(\Lambda_h/m_Z) \ll 1$,

$$\mu^2(m_Z) \simeq \mu^2(\Lambda_h) - \Theta(\Lambda_h - M) \left[C_2 M^2 \log\left(\frac{\Lambda_h}{M}\right) - C_T M^2 \right], \qquad (2.2)$$

where Θ is the step function. It is now easy to see when a naturalness problem arises. If either of $C_2 M^2$ or $C_T M^2$ is $\gg \mu^2(m_Z)$, then the input parameter $\mu^2(\Lambda_h)$ must be finely tuned against a very large contribution in order to realise the observed Higgs mass. A small change in $\mu^2(\Lambda_h)$ ruins this cancellation, and thus the Higgs mass is unnatural, i.e. it is sensitive to details of the high scale physics. Note that the $C_2 M^2$ piece captures a steep $\mu^2(\mu_R)$ RG trajectory, whereby only a very particular input $\mu^2(\Lambda_h)$ will lead to the observed low scale $\mu^2(m_Z)$; a small change in this value leads to significant over- or under-shooting.

In this picture, the Higgs naturalness problem is cast in terms of a potential sensitivity between *measurable* parameters, connected by fully calculable (in a perturbative theory) RG trajectories and matching conditions. The picture shares aspects with other discussions which have appeared in the literature, e.g. refs. [30–39]. In particular, the large cancellation between the unmeasurable bare mass and the cutoff regulator contribution is considered an unphysical artifact of the regularisation and renormalisation procedure for the EFT. The naturalness bounds we will derive within this picture ostensibly most resemble those in ref. [33], however they differ in a few key respects which we highlight here. First, we include the full effect of RGE running and matching, rather than simply bounding the finite loop corrections to μ^2. Second, since the parameters which appear in the RGEs and matching conditions are treated equally in our framework, the sensitivity arising from all parameters (and not just the gauge terms) are automatically included. Third, the appearance of apparently natural "throats," or miraculous cancellations among these corrections, are argued to be generically erased by the RG evolution and matching conditions. Last, we include (for the first time as far as we are aware) the leading three-loop correction for vector-like fermions. The inclusion of these effects leads, in most cases, to weaker naturalness bounds on the masses of heavy GMs than those obtained in ref. [33].

2.2 Sensitivity measure

To actually quantify the sensitivity to high scale physics can seem somewhat arbitrary and subjective. There are many approaches in the literature. It is important to appreciate that assumptions about the unknown high scale physics necessarily enter into all of

these approaches. However these assumptions are seldom explicitly stated, and the fine-tuning or sensitivity measures they motivate are often only written down intuitively. These shortcomings are overcome in the Bayesian approach to naturalness which we utilise here. In this approach, assumptions about the high scale physics can be explicitly and clearly stated, appearing as a complete set of prior densities for the high scale parameters. The requirement that our result be insensitive to units or parameter rescalings defines a set of agnostic priors, and the resulting naturalness bounds turn out to be rather insensitive to departures from these priors (this will be discussed in more detail in section 5). Further, instead of being written down directly, the sensitivity measure is rigorously derived from an underlying framework, having a well-defined interpretation as a Bayesian model comparison. The measure also reduces (in the cases we consider) to a Barbieri-Giudice-like [14, 15] fine-tuning measure, which can be easily understood intuitively. We provide below a short description; more details can be found in appendix A.

Assuming a flat prior belief in the high scale input parameters $\mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$, and a perfectly measured set of $m \leq n$ independent observables $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_m)$, the Bayesian evidence B for a model $\mathcal{M}$ is a function of the unconstrained input parameters $\mathcal{I}' = (\mathcal{I}_{m+1}, \ldots, \mathcal{I}_n)$:

$$B(\mathcal{M}; \mathcal{I}') \propto \frac{1}{\sqrt{|JJ^T|}} \bigg|_{\substack{\mathcal{O}_{ex} \\ \mathcal{I}'}}, \qquad (2.3)$$

where J is the $m \times n$ matrix defined by $J_{ij} = \partial \mathcal{O}_i / \partial \mathcal{I}_j$ [9]. Let us take, for model $\mathcal{M}$, $\mathcal{I}_1 = \log \mu^2(\Lambda_h)$ and $\mathcal{O}_1 = \log \mu^2(m_Z)$. The logarithms here ensure that our result is independent with respect to units or parameter rescalings (absolute values are implied for the argument of any log and dimensionful parameters can be normalised by any unit). Our Higgs mass sensitivity measure arises from a particular Bayesian model comparison: we compare to a model $\mathcal{M}_0$ in which we instead take $\mathcal{I}_1 = \mathcal{O}_1 = \log \mu^2(m_Z)$, i.e. the Higgs mass parameter is considered as an input parameter at scale m_Z. The sensitivity measure can then be written as a function of the unconstrained parameters,

$$\Delta(\mathcal{M}; \mathcal{I}') = \frac{B(\mathcal{M}_0; \mathcal{I}')}{B(\mathcal{M}; \mathcal{I}')} . \qquad (2.4)$$

This measure captures the "naturalness price" paid for promoting the Higgs mass parameter to a high scale input parameter of the model as opposed to a purely phenomenological input parameter at low scale. In our context, a large value of Δ essentially tells us that, given a flat prior density in $\log \mu^2(\Lambda_h)$, the observed value $\mu^2(m_Z)$ is unlikely [specifically with respect to a flat probability density in $\log \mu^2(m_Z)$], i.e. $\mu^2(m_Z)$ is sensitive to the realised input parameters. In the special case that the low scale observables, except for possibly $\mu^2(m_Z)$, are approximately insensitive to the unconstrained inputs, $B(\mathcal{M}_0; \mathcal{I}')$ becomes independent of $\mathcal{I}'$ and eq. (2.4) reduces to

$$\Delta(\mathcal{M}; \mathcal{I}') \simeq \sqrt{\left(\frac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)}\right)^2 + \sum_{j \geq m+1} \left(\frac{\partial \log \mu^2(m_Z)}{\partial \mathcal{I}_j}\right)^2} \Bigg|_{\substack{\mathcal{O}_{ex} \\ \mathcal{I}'}} . \qquad (2.5)$$

In the absence of unconstrained inputs ($n = m$) the summation disappears and the equality is exact. This is clearly reminiscent of the Barbieri-Giudice fine-tuning measure. A value of $\Delta = 10$ can now be interpreted as the onset of strong Bayesian evidence (for $\mathcal{M}_0$ over $\mathcal{M}$) on the Jeffreys scale [40], or a 10% fine-tuning from the Barbieri-Giudice perspective.

Notice here that we have a sensitivity measure which depends on unconstrained inputs. It might be that we want to "project out" some of these nuisance parameters. In this paper we will minimise over them, which picks out a conservative best case naturalness scenario in the model. Our SM+GM models $\mathcal{M}$ are defined by $\overline{\text{MS}}$ inputs at the high scale Λ_h, with the renormalised mass parameter $M^k(\Lambda_h) \subset \mathcal{I}'$ [and $k = 1$ (2) in the fermionic (scalar) case]. We minimise over all unconstrained parameters apart from $M^k(\Lambda_h)$ to obtain a sensitivity measure which depends only on M and Λ_h:

$$\Delta(M, \Lambda_h) = \min_{\mathcal{I}' \setminus \{M^k(\Lambda_h)\}} \left[\Delta(\mathcal{M}; \mathcal{I}') \right] . \tag{2.6}$$

In practice we minimise over eq. (2.5), which is now valid under the looser criterion that the low scale observables, except for possibly $\mu^2(m_Z)$, are approximately insensitive to the unconstrained inputs in the vicinity of the minimum.

This all may sound rather abstract. Let us now check that, in the relevant cases, the sensitivity measure eq. (2.6) captures the sensitivity we expect in our toy model when $C_2 M^2, C_T M^2 \gg \mu^2(m_Z)$.

2.3 Fermion-like case

In the minimal fermionic SM+GM there are no new dimensionless parameters; C_2 is fully constrained by experiment so that $\mathcal{O}_i = \left\{ \log \mu^2(m_Z), \log C_1, \log C_2 \right\}$ and $\mathcal{I}_j = \left\{ \log \mu^2(\Lambda_h), \log C_1, \log C_2, \log M \right\}$. It is easy to show directly from eq. (2.4) that, even allowing for possible $C_{1,2}$ correlation $\partial \log C_1 / \partial \log C_2 \neq 0$,

$$\Delta(M, \Lambda_h) = \sqrt{\left(\frac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)} \right)^2 + \left(\frac{\partial \log \mu^2(m_Z)}{\partial \log M} \right)^2}. \tag{2.7}$$

This is just a Barbieri-Giudice-like fine-tuning measure comparing percentage changes in the low scale Higgs mass parameter to those in the input parameters. In the limit $C_1 \log(\Lambda_h/m_Z) \ll 1$ and taking $\Lambda_h > M$, we see that $\Delta(\Lambda_h)$ is made up of two pieces:

$$\left| \frac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)} \right| = \left| 1 + \frac{C_2 M^2}{\mu^2(m_Z)} \log\left(\frac{\Lambda_h}{M} \right) - \frac{C_T M^2}{\mu^2(m_Z)} \right|,$$
$$k \left| \frac{\partial \log \mu^2(m_Z)}{\partial \log M^k} \right| = \left| \frac{C_2 M^2}{\mu^2(m_Z)} \left[2 \log\left(\frac{\Lambda_h}{M} \right) - 1 \right] - 2 \frac{C_T M^2}{\mu^2(m_Z)} \right|. \tag{2.8}$$

The "1" piece is the SM contribution, the $\log(\Lambda_h/M)$ pieces reflect sensitivity to the RG trajectory (with slope $C_2 M^2$), the $C_T M^2$ piece is due to the finite threshold correction, and the log-independent $C_2 M^2$ piece arises because a variation in $\log M$ results in a shift in the matching scale, which reintroduces a small amount of RG evolution. Clearly $\Delta \gg 1$ if $C_2 M^2$ or $C_T M^2$ is $\gg \mu^2(m_Z)$, as expected. This even holds in the limit where the high scale approaches the heavy particle mass, $\Lambda_h \to M^+$.

In the fermionic SM+GM EFT at two-loop order with one-loop matching, we have

$$C_2 \sim \frac{g^4}{(4\pi)^4}, \qquad\qquad\qquad C_T = 0, \qquad\qquad (2.9)$$

where g is a placeholder for a gauge coupling(s). It is interesting to note that in the limit $\Lambda_h \to M^+$, eq. (2.7) just becomes

$$\Delta(M^+) = \sqrt{1 + \left(\frac{C_2(M)M^2}{\mu^2(m_Z)}\right)^2} \,. \qquad\qquad (2.10)$$

If we bound this sensitivity measure by $\Delta_{\max}$, this is almost equivalent to simply bounding the contribution to the $\mu^2(\mu_R)$ RGE in eq. (2.1) at the scale $\mu_R = M$, i.e. $C_2(M)M^2 \lesssim \Delta_{\max}\mu^2(m_Z)$. This is not an uncommon practice as a zeroth-order naturalness bound for M.

There is one case we wish to comment on here: the special case where $C_2(M)$ happens to vanish, somewhat reminiscent of the Veltman condition [41]. In this case there is plainly no naturalness bound on M from the $\Delta(M^+)$ measure we have written above, no matter the size of M. So it appears that there is a fine-tuning which is not captured by our framework in this limit. Is this indeed the case? One can show that extending this toy model to include RG evolution of C_2 is not enough to reintroduce the naturalness bound (we will see this in our numerical analysis). Instead, it turns out that this apparent "Veltman throat" is only a limitation of the order to which we are working. In the fermionic SM+GM EFT with *two*-loop matching, C_T becomes a non-zero function of the gauge couplings. In general $C_T(M) \neq 0$ when $C_2(M) = 0$, and thus a sensitivity proportional to M^2 and powers of gauge couplings is recaptured at this special value of M. In any case, we do not attribute much physical significance to this special case; in the full model, even if $C_2(M) = 0$, RG effects reinstate $C_2(\mu_R) \neq 0$ at $\mu_R > M$, and the sensitivity of $\mu^2(m_Z)$ to M^2 is *rapidly* recaptured in the realistic case with $\Lambda_h > M$.

2.4 Scalar-like case

Let us first consider the SM plus scalar GM case with only one portal quartic $\lambda_{H1}(H^\dagger H)(\Phi^\dagger \Phi)$ and one self quartic $\lambda_\Phi(\Phi^\dagger \Phi)(\Phi^\dagger \Phi)$. This occurs whenever the scalar is an SU(2) singlet. At two-loop with one-loop matching, we have

$$C_2 = C_2^{SM} + 2Q_3 \frac{\lambda_{H1}}{(4\pi)^2} + \ldots, \qquad\qquad C_T = Q_3 \frac{\lambda_{H1}}{(4\pi)^2}, \qquad (2.11)$$

where $C_2^{SM} \sim g^4/(4\pi)^4$, and (Q_1, Q_2, Q_3) are the $(\mathrm{U}(1)_Y, \mathrm{SU}(2), \mathrm{SU}(3))$ charges of the GM. Assuming for simplicity no RG evolution of these parameters, we have $\mathcal{O}_i = \{\log \mu^2(m_Z), \log C_1, \log C_2^{SM}\}$ and $\mathcal{I}_j = \{\log \mu^2(\Lambda_h), \log C_1, \log C_2^{SM}, \log \lambda_{H1}, \log \lambda_\Phi, \log M^2\}$. The sensitivity measure, assuming C_1 and C_2^{SM} are insensitive to changes in λ_{H1} and λ_Φ, is given by

$$\Delta(M, \Lambda_h) = \min_{\lambda_{H1}} \left[\sqrt{\left(\frac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)}\right)^2 + \left(\frac{\partial \log \mu^2(m_Z)}{\partial \log M^2}\right)^2 + \left(\frac{\partial \log \mu^2(m_Z)}{\partial \log \lambda_{H1}}\right)^2 \Bigg|_{\lambda_{H1}}} \right],$$
$$(2.12)$$

where we have ignored the subdominant $\partial/\partial \log \lambda_\Phi$ term for clarity. Before minimisation, the contribution of the first two terms under the square root are exactly those in eq. (2.8). Note that the $C_2 M^2$ contributions are removed if λ_{H1} takes the fortuitous value $-(4\pi)^2 C_2^{SM}/(2Q_3)$, however the $C_T M^2$ contributions remain. Conversely, $\lambda_{H1} = 0$ removes the threshold correction contributions, leaving non-vanishing $C_2 M^2$ contributions. Thus it seems that, even with the extra freedom granted by λ_{H1}, one cannot remove the naturalness problem. Indeed, the minimisation over λ_{H1} can be performed analytically in this toy model. The result is rather lengthy, and we do not reproduce it here. It is anyway not terribly illuminating, since this toy model is too strong of an oversimplification to reflect the full scalar SM+GM case when $\Lambda_h \gg M$; pure gauge contributions in the $\lambda_{H1}(\mu_R)$ RGE which destabilise any fortuitous cancellation for $C_2(\mu_R) = 0$ must be taken into account. Nevertheless, the toy model result can serve as an argument for the existence of a finite naturalness bound even in the limit $\Lambda_h \to M^+$, where we obtain

$$\Delta(M^+) = \sqrt{\frac{1}{12}\left[10 + 4\frac{C_2^{SM}M^2}{\mu^2(m_Z)} + \left(\frac{C_2^{SM}M^2}{\mu^2(m_Z)}\right)^2\right]}, \qquad (2.13)$$

if $C_1 \log(\Lambda_h/m_Z) \ll 1$. Again, clearly $\Delta \gg 1$ when $C_2^{SM}M^2 \gg \mu^2(m_Z)$, as expected. If we bound this sensitivity measure by $\Delta(M^+) < \Delta_{\max}$, this is approximately equivalent to $C_2^{SM}(M)M^2/\sqrt{12} \lesssim \Delta_{\max}\mu^2(m_Z)$.

The scalar SU(2) doublet or triplet SM+GM EFT case with two portal quartic couplings is more delicate. At two-loop with one-loop matching we have

$$C_2 = C_2^{SM} + 2Q_3 Q_2 \frac{\lambda_{H1}}{(4\pi)^2} - 24Q_3 \frac{\lambda_{H2}^2}{(4\pi)^4} + \dots, \qquad C_T = Q_3 Q_2 \frac{\lambda_{H1}}{(4\pi)^2}, \qquad (2.14)$$

where $\lambda_{H1,2}$ will be defined in section 3.2. There is now enough freedom for a minimisation analogous to eq. (2.12) to select $\lambda_{H1,2}$ such that $C_2 = 0$ and $C_T = 0$ simultaneously, removing the sensitivity of $\mu^2(m_Z)$ to M^2. Still, in the realistic case including RG effects beyond our toy model, $C_2 \neq 0$ and $C_T \neq 0$ will be reinstated at $\mu_R > M$, and $\Delta(\Lambda_h)$ with $\Lambda_h > M$ will sensibly capture the $\mu^2(m_Z)$ sensitivity to M^2. A question remains as to whether $\Delta(M^+)$ acts sensibly in this scenario. Is it possible to choose $\lambda_{H1,2}(M)$ such that $C_2(M) = 0$ and $C_T(M) = 0$ and the relevant terms $\propto M^2$ in eq. (2.8) vanish? Indeed, it is possible. However, once RG effects are included, this is not sufficient to minimise $\Delta(M^+)$. In particular, $dC_T/d\log\mu_R$ will generally be non-zero, leading to an extra term in the $\partial/\partial \log M^2$ sensitivity measure:

$$\lim_{\Lambda_h \to M^+} \left|\frac{\partial \log \mu^2(m_Z)}{\partial \log M^2}\right| = \frac{1}{2}\left|\frac{M^2}{\mu^2(m_Z)}\left(C_2(M) + 2C_T(M) + \frac{dC_T}{d\log\mu_R}\bigg|_{\mu_R=M}\right)\right|. \quad (2.15)$$

The extra term can be thought of as arising from a shift in the matching scale. If $\lambda_{H1}(M) = 0$ is chosen so that $C_T(M) = 0$ in order to minimise the $\partial/\partial \log \mu^2(\Lambda_h)$ sensitivity, the full sensitivity measure is no longer minimised for $C_2(M) = 0$. Instead, one would like to set $[C_2(M) + dC_T/d\log\mu_R|_{\mu_R=M}] = 0$. However, $dC_T/d\log\mu_R|_{\mu_R=M}$ is itself a function of $\lambda_{H1,2}$ (and gauge couplings), thus it is not guaranteed that this is possible. Indeed, in

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the cases we explore, it is not; remarkably, $dC_T/d\log\mu_R|_{\mu_R=M} \supset +24Q_3\lambda_{H2}^2/(4\pi)^4$, which cancels the negative contribution in eq. (2.14) and leaves $[C_2(M) + dC_T/d\log\mu_R|_{\mu_R=M}]$ positive for any value of $\lambda_{H2}(M)$ when $\lambda_{H1}(M) = 0$.[1] Our numerical study captures this, and we always recover a sensible value for $\Delta(M^+)$.

In any case, the possibility of a miraculous cancellation is clearly not the generic case, and such cancellations are anyway quickly violated in the realistic scenario with RG effects and $\Lambda_h > M$. Nevertheless we find it interesting that, in this framework (and even in the limit $\Lambda_h \to M^+$ limit), there is a certain amount of $\mu^2(m_Z)$ sensitivity which cannot be made to go away by a judicious choice of quartic couplings.

Much of the discussion here has only been of technical interest since we have chosen to project out our unknowns by minimising the sensitivity measure over them. The reason for discussing possible regions of cancellation in such detail is to highlight the fact that, with this sensitivity measure, there appears to be no place in parameter space to "hide". Where it appears that there might be somewhere to hide, we have argued that apparent "throats" disappear at higher loop orders, or that sensitivity is always captured by some subcomponent of the sensitivity measure. It is important to establish this fact, since any violation would suggest that the sensitivity measure is incomplete, and until one can establish that the sensitivity measure is always capturing the fine-tuning in these most simple of cases, then why would one trust it on more complicated models?

2.5 Naturalness bounds

Naturalness bounds can be derived simply by bounding the sensitivity measure eq. (2.6). In figure 1 we show the $\Delta(\Lambda_h) = 10$ contours for $\Lambda_h = M^+$, or $\Lambda_{\rm Pl} \sim 10^{19}\,{\rm GeV}$ in our fermion-like toy model eq. (2.7). Points in parameter space below these lines can be considered natural, and points above increasingly unnatural.

Figure 1 can be used to estimate naturalness bounds on the masses of fermionic particles. Consider for example a heavy fermion with a top-like coupling strength such that $C_2 = 6y_t^2/(4\pi)^2$; taking $y_t^2 = y_t^2(m_Z) \approx 0.96$ and reading across one finds a naturalness bound $M \lesssim {\rm TeV}$. For a right-handed neutrino involved in a Type I see-saw, $C_2 = 4y_\nu^2/(4\pi)^2$ with $y_\nu^2 \simeq Mm_\nu/(174\,{\rm GeV})^2$; taking $m_\nu = 0.05\,{\rm eV}$ results in a naturalness bound $M \lesssim 10^7\,{\rm GeV}$ [42, 43].[2] The reason that this naturalness bound is so large is simply because C_2 is so small. Indeed, in the limit $C_2 \to 0$ there is no naturalness bound on M. In models with gauge singlets, $C_2 \to 0$ can correspond to a technically natural limit [44, 45] associated with decoupling of the particle from the SM fields. It makes sense

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[1]The reader might wonder if this is just a convenient happenstance. It is possible. However, we note that extending to three-loop RGEs with two-loop matching, this objection becomes moot. At higher loop matching the threshold correction will generally become a function of both λ_{H1} and λ_{H2}. The $\partial/\partial\log\mu^2(\Lambda_h)$ sensitivity is minimised for $C_T(M) = 0$. However, $\partial/\partial\log\lambda_{H1,2}(\Lambda_h)$ terms also appear in the full sensitivity measure. The simultaneous vanishing of these terms is in general only guaranteed if $\lambda_{H1}(M) = \lambda_{H2}(M) = 0$. Plainly this restriction is too severe to absorb sensitivity arising elsewhere.

[2]This is not quite the correct thing to do, since the observable at low scale is m_ν and not C_2 [which was assumed to derive eq. (2.8)]. Rest assured that using the appropriate sensitivity measure derived from eq. (2.4) only marginally changes this picture.
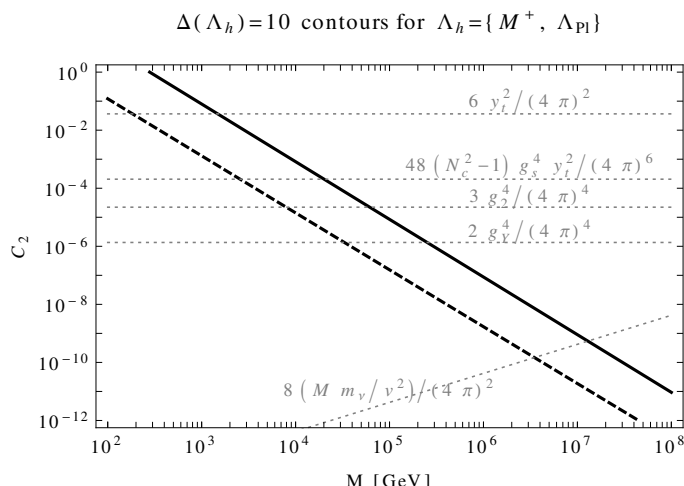
Figure 1. $\Delta(M, \Lambda_h) = 10$ contours for $\Lambda_h = M^+$ (solid), or $\Lambda_{\rm Pl} \sim 10^{19}$ GeV (dashed). Also shown as gray dotted lines are approximate C_2 contours for some benchmark heavy particles. The results for negative values of C_2 are very similar.

that there is no Higgs naturalness bound on the mass of such a particle, given that in this limit the heavy particle can no longer "talk" to the Higgs at all.

The focus of this paper will be Higgs naturalness within the EFT of the SM plus a heavy GM. For fermionic GMs with SU(3), SU(2), or U(1)$_Y$ charge Q_3, Q_2, or Q_1, the leading pure gauge contributions to C_2 are $-2Q_1^2 g_1^4/(4\pi)^4$, $-\frac{1}{2}Q_2(Q_2^2 - 1)g_2^4/(4\pi)^4$, and $+48(N_C^2 - 1)g_3^4 y_t^2/(4\pi)^6$, respectively. Taking $g_1^2 \approx 0.13$, $g_2^2 \approx 0.43$, and $g_3^2 \approx 1.48$, these correspond to rough naturalness bounds of (perhaps surprisingly to some) tens to hundreds of TeV, as sketched in figure 1. The size of the mass bounds is just a reflection of the smallness of $g^4/(4\pi)^4$. The main purpose of this paper is to derive these bounds more rigorously; we perform a full two-loop analysis to examine the effects of adding various (vector-like) fermionic and scalar GMs to the SM. The above naturalness bound approximations turn out to be quite good for the fermionic GMs, but they significantly deviate for scalar gauge multiplets, since these always couple directly to the Higgs via a quartic term(s). As already indicated, sensitivity to the RG evolved quartics must be properly taken into account.

2.6 Comment on the Planck-weak hierarchy

Before leaving this section, we want to comment on how the Planck-weak hierarchy fits into in this picture. From eqs. (2.2) and (2.5), one can see that $\Delta \simeq 1$ in the pure SM limit, i.e. there is no enhanced sensitivity when the Higgs mass is promoted to a high scale input parameter. This should come as no surprise, since the only explicit scale in the SM is μ^2 itself: the value $\mu^2(\Lambda_h)$ is multiplicatively related to the value $\mu^2(m_Z)$ and remains electroweak scale up to high scales.[3] Indeed, the effective Higgs potential remains consistent (albeit metastable) even up to the very highest scale to which the SM can

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[3]For the SM at two-loop we find $\overline{\mu^2}(\Lambda_{\rm Pl}) \simeq -(94 \text{ GeV})^2$ and $\Delta(\Lambda_{\rm Pl}) \simeq 1$ to one part in 10^6.

be valid: $\Lambda_{\mathrm{Pl}} \sim 10^{19}\,\mathrm{GeV}$. Now, it could be that gravity introduces large and physical corrections to $\mu^2(\mu_R)$ [or some related parameter(s)] at or below this scale. However, without a complete theory of quantum gravity, we cannot calculate these corrections.[4] This picture therefore claims that the SM with inputs at Λ_{Pl} is natural, in the sense that the low-energy observable $\mu^2(m_Z)$ is not extremely sensitive to the input $\mu^2(\Lambda_{\mathrm{Pl}})$. In such a case, one could sensibly ask: why is $\mu^2(\Lambda_{\mathrm{Pl}}) \ll \Lambda_{\mathrm{Pl}}^2$? We do not address this problem. By construction our sensitivity measure remains agnostic to this input value by assuming a flat prior in $\log \mu^2(\Lambda_{\mathrm{Pl}})$. Of course, as we have argued, the presence of a heavy gauge multiplet *can introduce a calculable and physical naturalness problem irrespective of the situation with gravity*, and this is the problem that we study. In such models a flat prior belief in $\log \mu^2(\Lambda_h)$ devolves to a low scale posterior belief which favours $\mu^2(m_Z) \gtrsim C_2 M^2$. It could be that gravity behaves in a similar way, but we cannot yet perform the calculation.

3 Method

The main purpose of this paper is to derive and present naturalness bounds on the masses of GMs within SM+GM EFTs valid up to scale Λ_h. In section 2 we motivated a general procedure for determining these bounds: take the low energy observables at m_Z, evolve them under the RGEs to the scale Λ_h, then evaluate and bound the sensitivity measure eq. (2.4). Presently we detail our method. We use sets of two-loop RGEs generated using a modified version of PyR@TE [47].

The low scale observables are taken as the logarithms of SM $\overline{\mathrm{MS}}$ Lagrangian parameters at scale m_Z: $\exp(\mathcal{O}_i) = \{\mu^2(m_Z), \lambda(m_Z), g_1(m_Z), g_2(m_Z), g_3(m_Z), y_t(m_Z), y_b(m_Z), y_\tau(m_Z)\} = \{-(88\,\mathrm{GeV})^2, 0.13, 0.36, 0.66, 1.22, 0.96, 0.017, 0.010\}$. For simplicity we ignore the Higgs and the top quark thresholds. The high scale input parameters are taken as the logarithms of the minimal set of SM+GM $\overline{\mathrm{MS}}$ Lagrangian parameters at scale Λ_h (to be explicitly listed in the following subsections); by minimal we mean that terms in the SM+GM Lagrangian which can be set to zero in a technically natural way are not included. The observables are numerically evolved under the two-loop SM RGEs up to the threshold of the GM, $\mu_R = M$, where we perform one-loop matching onto the parameters of the SM+GM EFT. The mass parameter for the GM is also a renormalised $\overline{\mathrm{MS}}$ parameter, which we set equal to M at the scale $\mu_R = M$.[5] New parameters are introduced in the case of a scalar GM; these are left as free parameters which are numerically minimised over when evaluating the sensitivity measure. The two-loop SM+GM RGEs are used to evolve all parameters up to the high scale Λ_h. The approximation to the full sensitivity measure, eq. (2.5), is then evaluated numerically by varying the appropriate input parameters around their values at Λ_h, evolving all parameters back down to the scale m_Z, matching the SM+GM EFT onto the SM EFT at the matching scale μ_R given by $M(\mu_R) = \mu_R$, and measuring the change in the Higgs mass parameter.

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[4]There exist extensions of Einstein gravity in which corrections to μ^2 are both calculable and naturally small (see e.g. [46]), although such theories generally also have problems with unitarity.

[5]The bounds we present are therefore bounds on the parameter M (i.e. the $\overline{\mathrm{MS}}$ mass of the GM at the scale M), not the pole mass.
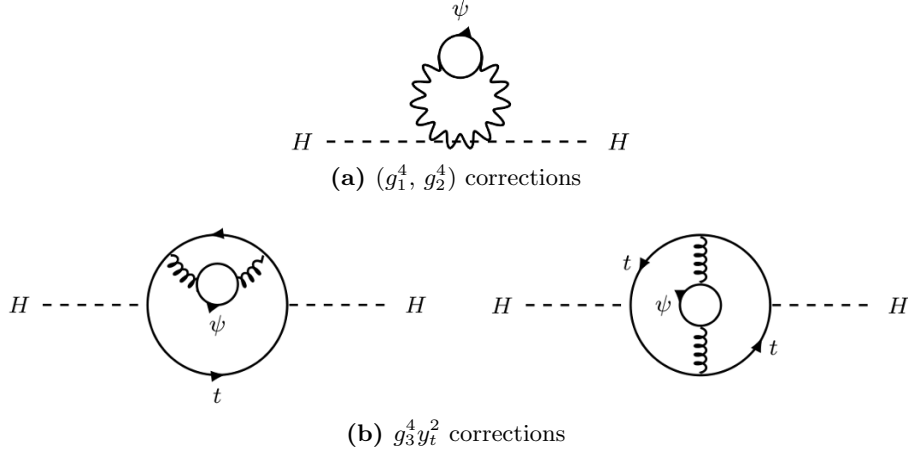
(a) (g_1^4, g_2^4) corrections

(b) $g_3^4 y_t^2$ corrections

Figure 2. Corrections to μ^2 from a heavy vector-like fermion.

3.1 Vector-like fermion

The minimal SM+GM Lagrangian for a vector-like fermion is that of the SM plus

$$\Delta\mathcal{L} = \bar{\psi} D^\nu \gamma_\nu \psi - M\bar{\psi}\psi. \tag{3.1}$$

The high scale input parameters of this model are those of the SM plus the renormalised parameter $M(\Lambda_h)$, i.e. $\mathcal{I}_j = \{\mu^2(\Lambda_h), \lambda(\Lambda_h), g_1(\Lambda_h), g_2(\Lambda_h), g_3(\Lambda_h), y_t(\Lambda_h), y_b(\Lambda_h), y_\tau(\Lambda_h), M(\Lambda_h)\}$. The one-loop matching conditions are trivial,

$$\mu_+^2(\mu_R) = \mu_-^2(\mu_R), \qquad\qquad \lambda_+(\mu_R) = \lambda_-(\mu_R), \tag{3.2}$$

where the $+$ $(-)$ subscript denotes the SM+GM (SM) EFT parameter.

The $\mu^2(\mu_R)$ RGE takes the form of eq. (2.1) with $C_2(\mu_R)$ a function of SM parameters. Recall that it is primarily the $C_2(\mu_R)$ term which leads to a potential naturalness problem, as argued in section 2.1 for constant C_2. In the vector-like fermionic SM+GM EFT it takes the form

$$C_2 = -2Q_3 Q_2 Q_1^2 \frac{g_1^4}{(4\pi)^4} - \frac{1}{2} Q_3 Q_2 (Q_2^2 - 1) \frac{g_2^4}{(4\pi)^4} + 96 Q_2 (N_c^2 - 1) c(r_\psi) \frac{g_3^4 y_t^2}{(4\pi)^6}, \tag{3.3}$$

where $c(r_\psi) = \frac{1}{2}$ (3) for $Q_3 = 3$ (8), and dependence on scale μ_R is implied. Representative diagrams leading to these terms are shown in figure 2. Note that we have added by hand the leading three-loop SU(3) correction, arising from three diagrams [see figure 2b], since otherwise the two-loop RGEs do not capture any SU(3) correction beyond multiplicity factors.[6] This correction turns out to be competitive with the two-loop pure gauge corrections at scales $\mu_R \lesssim 10^5$ GeV due to the relatively large couplings g_3 and y_t below this scale. It is also opposite in sign, thus potentially delaying the growth of $\mu^2(\mu_R)$ (and the corresponding naturalness problem) if it happens to approximately cancel with the other gauge contributions.

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[6]This three-loop correction was calculated with the aid of MATAD [48].

There are no unconstrained high scale dimensionless inputs to minimise over, so the sensitivity measure eq. (2.5) is just

$$\Delta(M, \Lambda_h) = \sqrt{\left(\frac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)}\right)^2 + \left(\frac{\partial \log \mu^2(m_Z)}{\partial \log M(\Lambda_h)}\right)^2}. \tag{3.4}$$

Results are obtained by numerically evaluating $\Delta(M, \Lambda_h)$ at points of interest in (M, Λ_h) space.

3.2 Complex scalar

The minimal SM+GM Lagrangian for a complex scalar is that of the SM plus

$$\Delta\mathcal{L} = D^\nu \Phi^\dagger D_\nu \Phi - \left(M^2 \Phi^\dagger \Phi + \sum \lambda_{\Phi i} \Phi^\dagger \Phi \Phi^\dagger \Phi + \sum \lambda_{Hj} H^\dagger H \Phi^\dagger \Phi\right), \tag{3.5}$$

where H is the SM Higgs field, and the sums are over all possible contractions. Explicitly, we take the following convenient contractions for the portal quartics

$$\Delta\mathcal{L} \supset \lambda_{H1} H^\dagger H \operatorname{Tr}(\Phi^\dagger \Phi) + \lambda_{H2} \left(2 \operatorname{Tr}(H^\dagger \Phi \Phi^\dagger H) - H^\dagger H \operatorname{Tr}(\Phi^\dagger \Phi)\right), \tag{3.6}$$

where the second term is relevant only for $Q_2 \geq 2$. The high scale input parameters of the model are those of the SM plus the extra self and portal quartics and renormalised mass parameter, i.e. $\mathcal{I}_j = \{\mu^2(\Lambda_h), \lambda(\Lambda_h), g_1(\Lambda_h), g_2(\Lambda_h), g_3(\Lambda_h), y_t(\Lambda_h), y_b(\Lambda_h), y_\tau(\Lambda_h), \lambda_{\Phi1}(\Lambda_h), \ldots, \lambda_{H1}(\Lambda_h), \ldots, M^2(\Lambda_h)\}$. The one-loop matching conditions are

$$\mu_+^2(\mu_R) = \mu_-^2(\mu_R) - Q_3 Q_2 \frac{\lambda_{H1}(\mu_R)}{(4\pi)^2} M^2(\mu_R) \left[1 - \log\left(\frac{M^2(\mu_R)}{\mu_R^2}\right)\right], \tag{3.7}$$

$$\lambda_+(\mu_R) = \lambda_-(\mu_R) - \frac{Q_3 Q_2}{2} \frac{\lambda_{H1}^2(\mu_R)}{(4\pi)^2} \log\left(\frac{M^2(\mu_R)}{\mu_R^2}\right), \tag{3.8}$$

where the $+$ $(-)$ subscript denotes the SM+GM (SM) EFT parameter and we have neglected terms suppressed by powers of v^2/M^2. We will always work in the limit where Φ does not obtain a vacuum expectation value. This is a well-motivated simplification, since for masses at the naturalness bounds we will obtain (typically $M > \text{TeV}$), experimental agreement with the canonical Higgs mechanism for electroweak symmetry breaking generically constrains any scalar GM to observe this limit, and of course a coloured scalar multiplet must exactly satisfy it. Evidently the μ^2 term receives a threshold correction when matching is performed at the scale $\mu_R = M(\mu_R)$.

The $\mu^2(\mu_R)$ RGE in the SM+GM EFT with a complex scalar takes the form of eq. (2.1) with $C_2(\mu_R)$ a function of both the SM parameters and the extra quartics,

$$\begin{aligned} C_2 = {}& + 2Q_3 Q_2 \frac{\lambda_{H1}}{(4\pi)^2} - 4Q_3 Q_2 \frac{\lambda_{H1}^2}{(4\pi)^4} \\ & + 5Q_3 Q_2 Q_1^2 \frac{g_1^4}{(4\pi)^4} + \frac{5}{4} Q_3 Q_2 (Q_2^2 - 1) \frac{g_2^4}{(4\pi)^4} \\ & + 16 Q_3 Q_2 Q_1^2 \frac{g_1^2 \lambda_{H1}}{(4\pi)^4} + 4 Q_3 Q_2 (Q_2^2 - 1) \frac{g_2^2 \lambda_{H1}}{(4\pi)^4} \\ & - 24 Q_3 \frac{\lambda_{H2}^2}{(4\pi)^4} + 64 Q_2 \frac{g_3^2 \lambda_{H1}}{(4\pi)^4}, \end{aligned} \tag{3.9}$$
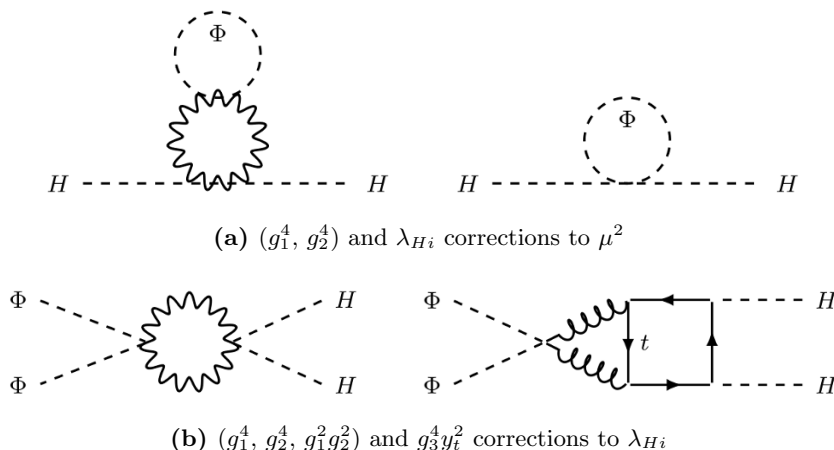
(a) (g_1^4, g_2^4) and λ_{Hi} corrections to μ^2

(b) $(g_1^4, g_2^4, g_1^2 g_2^2)$ and $g_3^4 y_t^2$ corrections to λ_{Hi}

Figure 3. Corrections to μ^2 from a heavy scalar, and related corrections to λ_{Hi}.

where the final line λ_{H2}^2 $(g_3^2 \lambda_{H1})$ term appears only if $Q_2 = 2, 3$ $(Q_3 = 3)$, and dependence on scale μ_R is implied.[7] Representative diagrams which lead to these terms are shown in figure 3a. Recall that the naturalness problem can be ameliorated in the limit $C_2 \to 0$. Indeed, it could be the case that at some scale the λ_{Hi} conspire to give $C_2(\mu_R) = 0$. However, this will not be stable under RG evolution; the portal quartics receive gauge corrections via the diagrams shown in figure 3b:

$$\frac{d\lambda_{H1}}{d\log\mu_R} \supset 3Q_1^2 \frac{g_1^4}{(4\pi)^2} + 3c_2(r_\Phi)\frac{g_2^4}{(4\pi)^2} - 32\frac{y_t^2 g_3^4}{(4\pi)^4} \,, \tag{3.10}$$

$$\frac{d\lambda_{H2}}{d\log\mu_R} \supset 3n_2 Q_1 \frac{g_1^2 g_2^2}{(4\pi)^2} \,. \tag{3.11}$$

Here, $n_2 = 1\,(2)$ and the quadratic Casimir $c_2(r_\Phi) = 3/4\,(2)$ for $Q_2 = 2\,(3)$; and the term proportional to g_3 applies to the case $Q_3 = 3$. Note that there exist corrections with odd power in Q_1; this means that (unlike the fermion case) RG evolution will depend on the sign of the hypercharge, however we do not observe any noticeable consequences from this effect. Also, in this case, we see that the two-loop scalar RGEs *do* capture an SU(3) correction $\sim g_3^4 y_t^2$, through a two-loop correction to the portal quartics. Thus we do not add by hand the three-loop $g_3^4 y_t^2$ term to the μ^2 RGE in our scalar SM+GM analysis.

The portal and self quartics are unknown and unconstrained high scale input parameters which must be projected out to obtain a sensitivity measure which is a function of (M, Λ_h). Our measure eq. (2.6) requires them to take on such values which minimise the Bayes factor, i.e. such values which give a conservative "best case scenario" for Higgs mass sensitivity in the given model. In eq. (2.5), we wrote down an approximation to the full sensitivity measure eq. (2.4), which is valid when the low scale dimensionless SM observables are approximately insensitive to the unconstrained inputs in the vicinity of the minimum.

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[7]We encountered difficulties with PYR@TE when generating the two-loop scalar octet RGEs. Thus, regrettably, they are left out of this study.

In that case we can evaluate the sensitivity measure eq. (2.5),

$$\Delta(M, \Lambda_h) = \min_{\lambda_{Hi}} \left\{ \sqrt{ \begin{array}{c} \left(\dfrac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)}\right)^2 + \left(\dfrac{\partial \log \mu^2(m_Z)}{\partial \log M^2(\Lambda_h)}\right)^2 \\ + \sum \left(\dfrac{\partial \log \mu^2(m_Z)}{\partial \log \lambda_{Hi}(\Lambda_h)}\right)^2 + \sum \left(\dfrac{\partial \log \mu^2(m_Z)}{\partial \log \lambda_{\Phi j}(\Lambda_h)}\right)^2 \end{array} } \Bigg|_{\substack{\lambda_{\Phi j}(M)=0 \\ \lambda_{Hi}}} \right\}, \quad (3.12)$$

where $\lambda_{\Phi j}(M) = 0$ is minimally consistent with our assumption that Φ attains no vacuum expectation value.[8] We will now make an argument for why we indeed expect this approximation to be valid in the scalar SM+GM.

Intuitively, since $C_2 M^2$ is the primary quantity which leads to a naturalness problem, to zeroth approximation we expect the minimum to occur where the average value of $C_2(\mu_R)$ over the RG evolution is zero. Consider the case with only one portal coupling λ_{H1}. Then from eq. (3.9) our expectation requires $\lambda_{H1}(\mu_R)$ to take on values $\mathcal{O}(g^4/(4\pi)^2)$ (in order to cancel the pure gauge contribution), and for $C_2(\mu_R)$ to swap sign along its RG evolution. Indeed, we observe this to be the case in our numerical study, as we will demonstrate in section 5. Now, λ_{H1} enters the one-loop dimensionless SM parameter RGEs only for the Higgs self-quartic λ, as $\sim \lambda_{H1}^2/(4\pi)^2$. Therefore its contribution to the evolution of λ (and all dimensionless SM parameters) is very small at the minimum, and we can say that $\partial \lambda(m_Z)/\partial \lambda_{H1}(\Lambda_h) \simeq 0$. As for the new quartic couplings $\lambda_{\Phi j}$, they do not directly enter any of the dimensionless SM parameter two-loop RGEs, therefore their effect is also very small. Extending to the case with two portal quartics, eq. (3.9) clearly implies that a contour in $(\lambda_{H1}, \lambda_{H2})$ space will satisfy $C_2(\mu_R) = 0$, so our argument for $\mathcal{O}(g^4/(4\pi)^2)$ quartics no longer holds. However the $\partial/\partial \log \lambda_{Hi}(\Lambda_h)$ terms in the sensitivity measure eq. (3.12) are proportional to $\lambda_{Hi}(\Lambda_h)$, so that the minimum will always prefer smaller values for these quartics. The dimensionless SM observables will then be insensitive to variations around $\lambda_{Hi}(\Lambda_h)$ for the reasons already argued.

4 Results

The naturalness bounds for various vector-like fermionic and scalar GMs, derived according to the method detailed in section 3, are presented in tables 1 and 2 for $\Lambda_h = \{M^+, \Lambda_{\text{Pl}}\}$ and $\Delta(\Lambda_h) = \{10, 100, 1000\}$. Contour plots in (M, Δ) and (M, Λ_h) parameter space are also provided in figures 4–7. These constitute the main result of this paper, and we hope that they will serve as a useful reference of naturalness benchmarks for phenomenological model builders.

Before we discuss them in more detail, let us briefly reiterate their meaning. The scale Λ_h corresponds to the input scale of $\overline{\text{MS}}$ parameters in the SM+GM EFT. The quantity

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[8]The full sensitivity measure eq. (3.12) should also involve a minimisation over the $\lambda_{\Phi j}$. However, we found that, after demanding $\lambda_{\Phi j}(M) \geq 0$ to ensure no non-trivial vacuum expectation value, the minimum always occurred for $\lambda_{\Phi j}(M) \simeq 0$. Thus in practice, to improve speed and numerical stability, we set $\lambda_{\Phi j}(M) = 0$ when evaluating the sensitivity measure and note that even varying this up to $\lesssim 0.5$ made little difference to our results. We also note that the $\lambda_{\Phi j}$ always evolve to positive values due to pure gauge contributions to their RGEs at one-loop, and therefore the potential does not become trivially unstable.

			$\Lambda_h = M^+$			$\Lambda_h = 10^{19}$ GeV		
SU(3)	SU(2)$_L$	U(1)$_Y$	$\Delta = 10$	$\Delta = 100$	$\Delta = 1000$	$\Delta = 10$	$\Delta = 100$	$\Delta = 1000$
1	1	$\pm\frac{1}{6}$	1400	4300	13000	130	420	1300
		$\pm\frac{1}{3}$	690	2200	6800	64	210	670
		$\pm\frac{2}{3}$	350	1100	3400	32	110	340
		± 1	230	730	2300	22	72	230
		± 2	120	370	1200	13	43	140
		± 3	80	250	790	-	-	-
	2	0	70	230	740	11	35	110
		$\pm\frac{1}{2}$	69	220	720	10	34	110
		± 1	65	210	670	9.7	32	100
		± 2	54	170	550	-	-	-
	3	0	35	110	370	6.0	20	64
		± 1	34	110	350	6.1	20	65
		± 2	31	100	330	-	-	-
3	1	0	54	190	700	17	56	190
		$\pm\frac{1}{3}$	54	200	710	17	60	210
		$\pm\frac{2}{3}$	56	210	750	21	77	300
		± 1	59	220	830	72	140	340
		± 2	110	800	1600	-	-	-
	2	0	180	350	850	13	37	110
		± 1	110	250	660	9.0	28	84
		± 2	57	150	440	-	-	-
	3	0	29	86	260	-	-	-
		± 1	27	82	250	-	-	-
		± 2	24	72	220	-	-	-
8	1	0	20	74	270	7.3	25	86
		± 1	21	77	280	-	-	-
		± 2	23	91	360	-	-	-
	2	0	17	67	270	-	-	-
		± 1	18	72	310	-	-	-
		± 2	21	110	780	-	-	-
	3	0	63	110	270	-	-	-
		± 1	50	98	240	-	-	-
		± 2	32	72	190	-	-	-

Table 1. Naturalness bounds on the mass M (in TeV and to 2 significant figures) of various vector-like fermionic gauge multiplets for $\Lambda_h = \{M^+, 10^{19} \text{ GeV}\}$ and $\Delta(\Lambda_h) = \{10, 100, 1000\}$. The dashes indicate that a Landau pole arises below 10^{19} GeV along the $\Delta(\Lambda_h) = 10$ contour.

$\Delta(\Lambda_h)$, defined in eq. (2.6), is a sensitivity measure for the Higgs mass parameter which can be interpreted as a Bayesian evidence on the Jeffreys scale or (more loosely) to a percentage fine-tuning in the Barbieri-Giudice sense. A stringent naturalness constraint is then $\Delta(\Lambda_{\text{Pl}}) < 10$, which (loosely) ensures $< 10\%$ sensitivity for $\mu^2(m_Z) \simeq -(88 \text{ GeV})^2$ when the $\overline{\text{MS}}$ parameters are defined at Λ_{Pl}. If a phenomenological model can satisfy this constraint then we would say it does not induce a Higgs naturalness problem. The bounds

SU(3)	SU(2)$_L$	U(1)$_Y$	$\Lambda_h = M^+$			$\Lambda_h = 10^{19}$ GeV		
			$\Delta = 10$	$\Delta = 100$	$\Delta = 1000$	$\Delta = 10$	$\Delta = 100$	$\Delta = 1000$
1	1	$\pm\frac{1}{6}$	1300	4100	13000	29	96	310
		$\pm\frac{1}{3}$	670	2000	6400	14	47	150
		$\pm\frac{2}{3}$	340	1000	3200	6.8	23	75
		±1	230	690	2200	4.4	15	48
		±2	120	350	1100	2.0	6.5	21
		±3	77	240	740	-	-	-
	2	0	67	210	680	2.3	7.7	25
		$\pm\frac{1}{2}$	65	210	660	2.1	7.2	24
		±1	62	190	620	1.8	6.0	20
		±2	52	160	510	1.1	3.6	12
	3	0	33	100	340	1.1	3.6	12
		±1	32	100	330	0.95	3.2	10
		±2	30	94	300	0.45	1.7	6.7
3	1	0	220	820	2900	12	40	130
		$\pm\frac{1}{3}$	290	1200	5400	12	38	110
		$\pm\frac{2}{3}$	330	880	2500	4.2	14	45
		±1	160	470	1400	2.5	8.4	27
		±2	71	210	660	0.99	3.2	10
	2	±0	40	130	400	1.3	4.3	14
		±1	37	120	370	0.99	3.3	11
		±2	31	96	300	-	-	-
	3	±0	20	62	200	-	-	-
		±1	19	60	190	-	-	-
		±2	18	55	180	-	-	-

Table 2. Naturalness bounds on the mass M (in TeV and to 2 significant figures) of various scalar gauge multiplets for $\Lambda_h = \{M^+, 10^{19}$ GeV$\}$ and $\Delta(\Lambda_h) = \{10, 100, 1000\}$. The dashes indicate that a Landau pole arises below 10^{19} GeV along the $\Delta(\Lambda_h) = 10$ contour.

weaken in the limit $\Lambda_h \to M^+$. Still, rather remarkably, they remain finite in this limit, as we argued in section 2. The $\Delta(M^+) < 10$ bound can therefore be interpreted as a conservative naturalness constraint on M. It is also of interest if $\Delta(\Lambda_{\rm Pl})$ is not applicable, e.g. if new physics arises at a scale above M which markedly affects the $\mu^2(\mu_R)$ evolution, or if the EFT hits a Landau pole below $\Lambda_{\rm Pl}$.

5 Discussion

Some aspects of our results can be understood by scaling relations. At fixed Λ_h, the bounds in tables 1 and 2 scale approximately as $\sqrt{\Delta/C_2^{SM}}$, as one would expect from eq. (2.8) for the simple example discussed in section 2. Where they are violated (particularly for the $\Delta(\Lambda_h)$ bounds) it is due to some cancellation between contributions: the contributions arising from SU(3) charge are opposite in sign to those from SU(2) and U(1)$_Y$ charge. The contour plots in figures 4–7 make these cancellations more obvious, and we will discuss them shortly. Comparing bounds evaluated at disparate Λ_h is more involved. Indeed, this
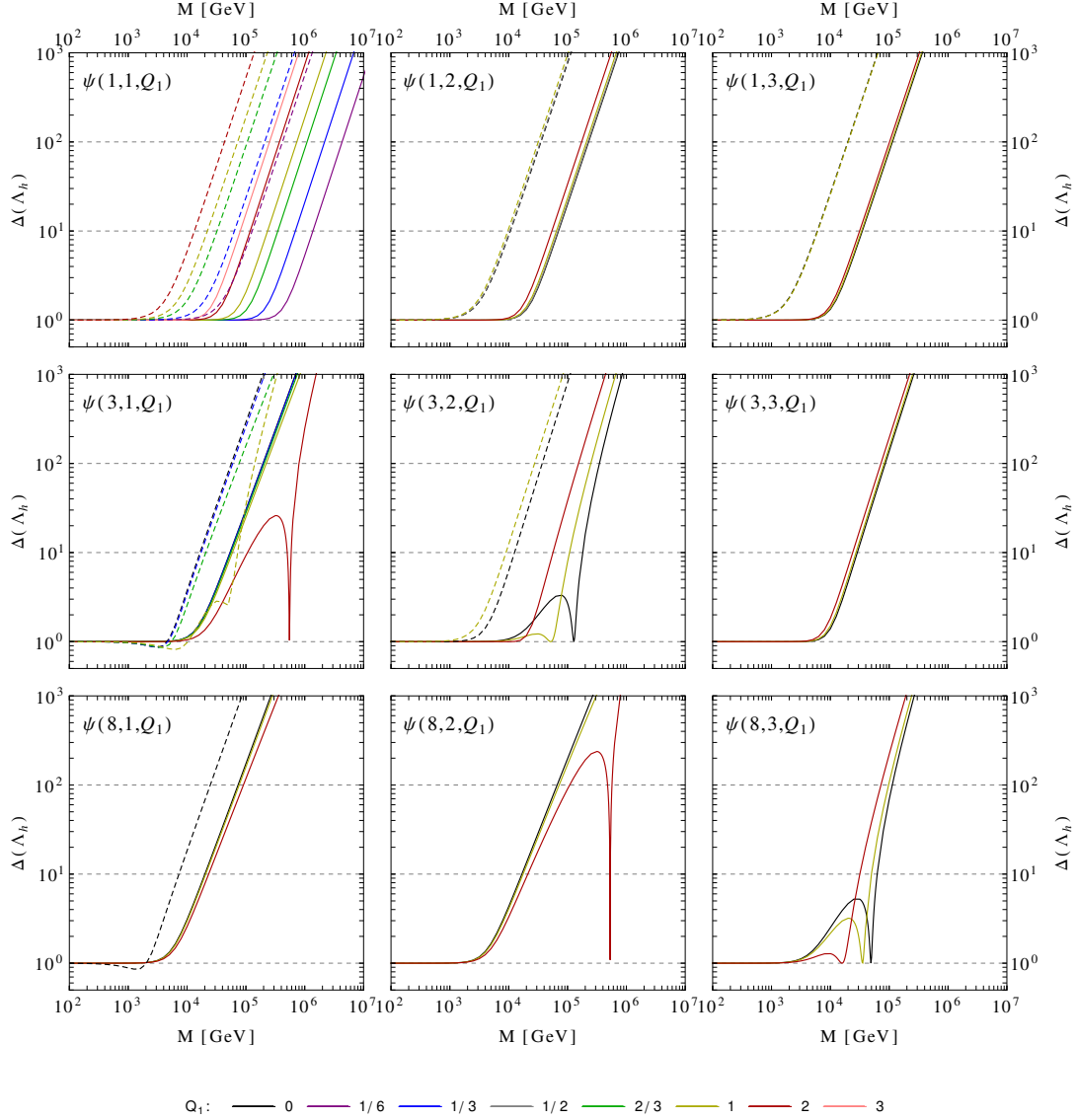
Figure 4. $\Lambda_h = \{M^+, \Lambda_{\mathrm{Pl}}\}$ contours {solid, dashed} in the vector-like fermionic SM+GM EFT. The "throat" features are an artifact of the loop level to which we are working [see section 2.3].

is why we have gone to the trouble of a two-loop RGE analysis! Still, some qualitative observations will be made presently.

For the fermionic GMs in table 1, the rough scaling relation $\sim \sqrt{1/\sqrt{5}\log(\Lambda_{\mathrm{Pl}}/M)}$ between bounds evaluated at $\Delta(M^+)$ and $\Delta(\Lambda_{\mathrm{Pl}})$, as expected from eq. (2.8), is broken by the RG evolution of C_2. We observe that the naturalness bounds at Λ_{Pl} are more stringent than this relation would suggest for $\psi(1,1,Q_1)$, and less stringent for $\psi(1,Q_2,0)$ and $\psi(Q_3,1,0)$. This is simply because $g_1^4(\mu_R)$ (and therefore C_2) grows at higher energy, whereas the opposite is true for $g_2^4(\mu_R)$ and $g_3^4(\mu_R)y_t^2(\mu_R)$. This effect can be observed in figure 8, where we show the example RG evolution of $\mu^2(\mu_R)$ for gauge multiplets of increasing mass. The naturalness problem which broadly arises from a sensitivity to the high scale input $\mu^2(\Lambda_h)$ is self-evident for large masses.
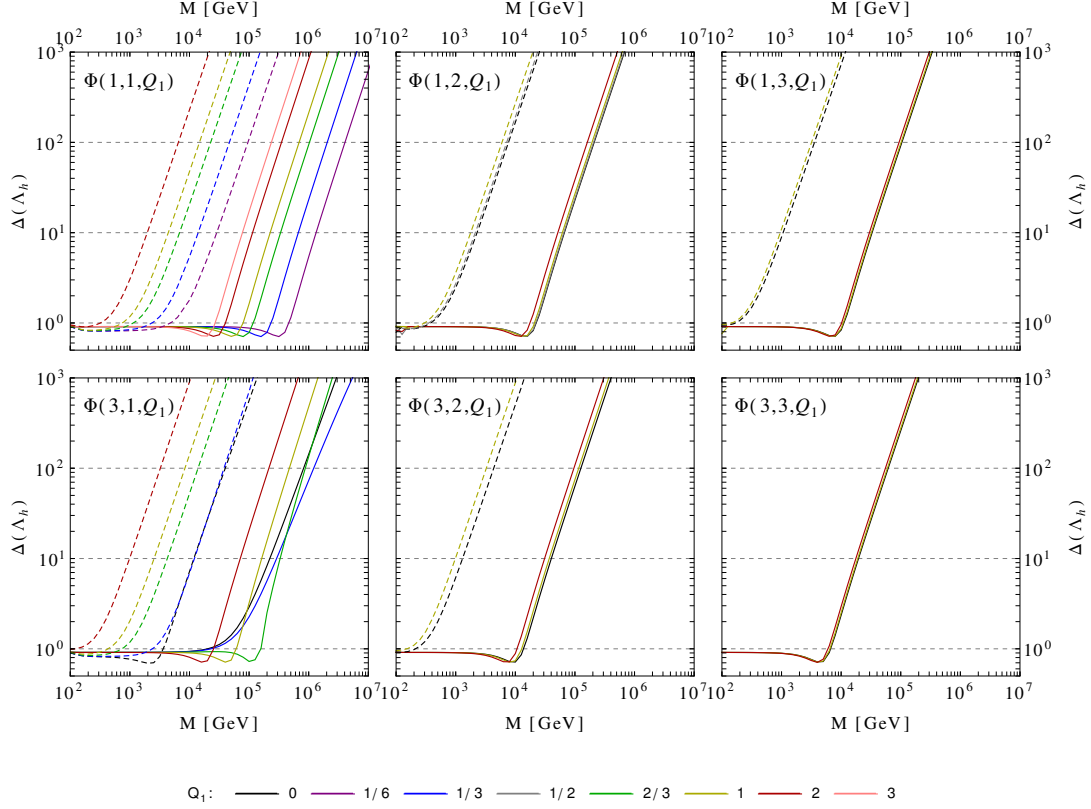
Figure 5. $\Lambda_h = \{M^+, \Lambda_{\rm Pl}\}$ contours {solid, dashed} in the scalar SM+GM EFT.

For the scalar GMs in table 2, there is no obvious scaling relation between naturalness bounds at different scales. One observation is that, although the $\Delta(M^+)$ bounds are similar[9] to those found in the fermionic GM case, the $\Delta(\Lambda_{\rm Pl})$ bounds are much more stringent. This is because the sole one-loop term in the μ^2 RGE involves the portal quartic λ_{H1}, which is *itself* renormalised by pure gauge RG terms at one-loop. Thus the scaling relation is expected to more closely resemble $\sim 1/\log(\Lambda_{\rm Pl}/M)$ [rather than $\sim \sqrt{1/\log(\Lambda_{\rm Pl}/M)}$]. What actually happens is unfortunately quite opaque, since it is hidden by various complexities: a coupled set of RGEs; a non-trivial sensitivity measure eq. (3.12); and a minimisation procedure over the λ_{Hi}. Let us attempt to convey some intuition for what happens by considering the example of a two Higgs doublet model, i.e. the SM+$\Phi(1, 2, 1/2)$. To this end it is useful to define a reduced sensitivity measure

$$\Delta_{red}(M, \Lambda_h) = \left| \frac{\partial \log \mu^2(m_Z)}{\partial \log \mu^2(\Lambda_h)} \right|, \tag{5.1}$$

which is a subcomponent of the full measure eq. (3.12). This reduced measure vanishes in the limit $\mu^2(\Lambda_h) \to 0$. As we already argued in section 3.2, it is always possible to choose the λ_{Hi} such that C_2 swaps sign over its RG evolution and $\mu^2(\Lambda_h) = 0$. Thus one

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[9]The larger relative difference between the bounds for coloured states may be partly accounted for by the three-loop $g_3^4 y_t^2$ term which is not captured in our pure two-loop scalar analysis.
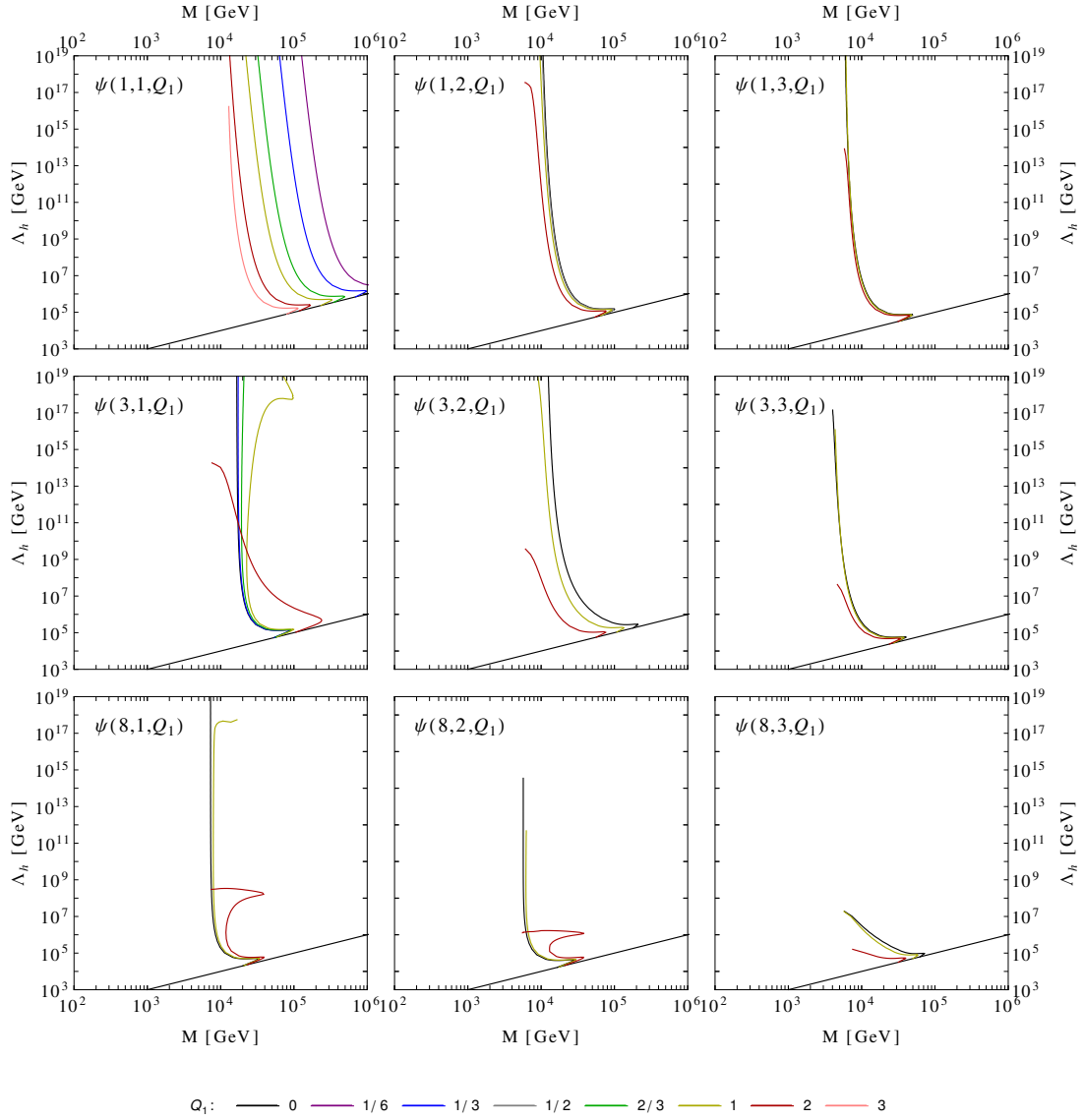
Figure 6. $\Delta(\Lambda_h) = 10$ contours in the vector-like fermionic SM+GM EFT. If a line ends it is because the system hits a Landau pole.

expects a contour in λ_{Hi} space along which $\Delta_{red}(M, \Lambda_h)$ vanishes [49]. In figure 9 we plot $\Delta_{red}(M, \Lambda_h)$ as a function of $(\lambda_{H1}(M), \lambda_{H2}(M))$ for $\Lambda_h = \Lambda_{\rm Pl}$ and $M = 1\,{\rm TeV}$, where such a contour is readily observed. Obviously this contour constitutes a fine-tuning in the λ_{Hi}, and we would hope that our full sensitivity measure captures this tuning and restores a finite naturalness bound. Indeed, it does; also shown in figure 9 is the full sensitivity measure as a function of $(\lambda_{H1}(M), \lambda_{H2}(M))$, with a unique minimum of $\Delta(\Lambda_{\rm Pl}) \simeq 2.7$ nearby the $\mu^2(\Lambda_h) = 0$ contour. In the lower panel of figure 9 we also show the running of $\mu^2(\mu_R)$ and $C_2(\mu_R)M^2(\mu_R)$ at this minimum (and for other example masses). It is seen that C_2 does switch sign, as expected.
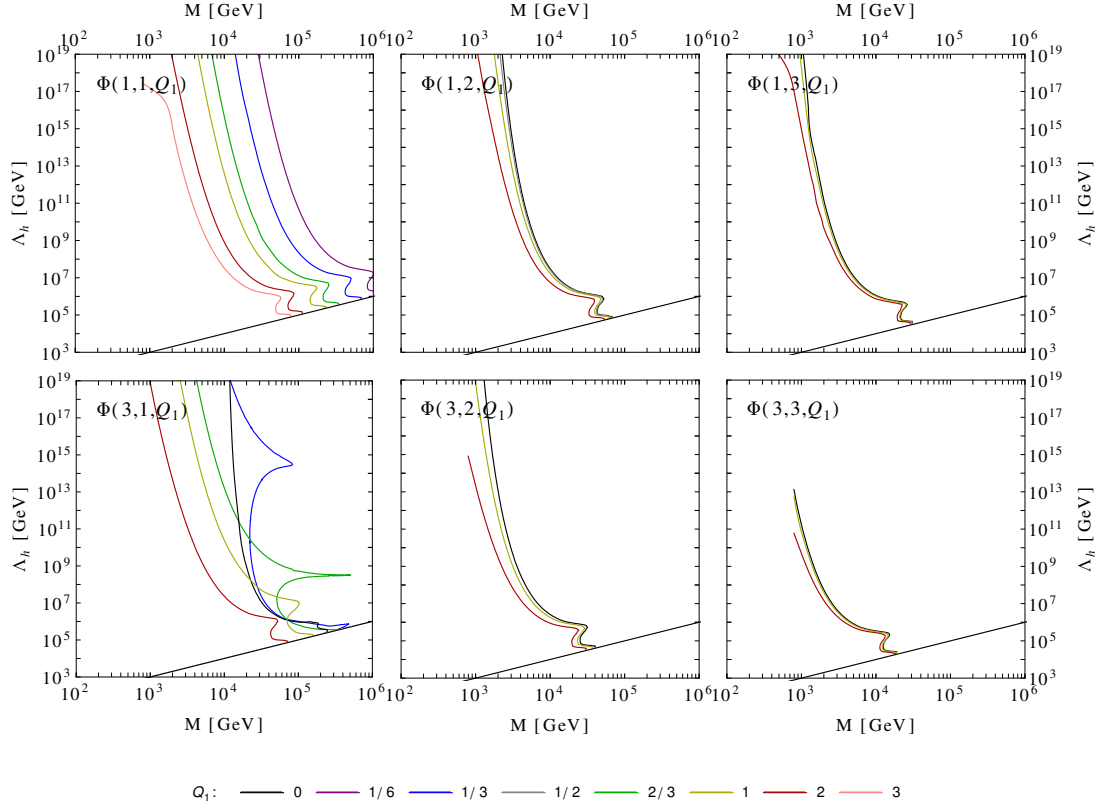
Figure 7. $\Delta(\Lambda_h) = 10$ contours in the scalar SM+GM EFT. If a line ends it is because the system hits a Landau pole.
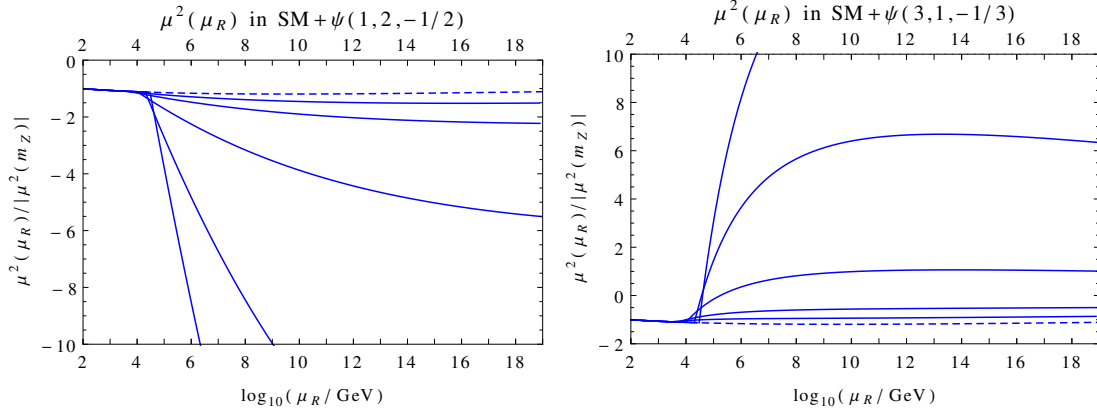

Figure 8. Example running of $\mu^2(\mu_R)$ in the SM+GM EFT for a heavy lepton doublet $\psi(1, 2, -1/2)$ and a heavy down-type quark $\psi(3, 1, -1/3)$ with $M = 3, 5, 10, 20, 30 \, \mathrm{TeV}$. The dashed line is the SM-only case.

We will now briefly comment on some features in the (M, Δ) and (M, Λ_h) contour plots of figures 4–7. In figure 4 there is a sharp "Veltman throat" in the $\Delta(M^+)$ contours for coloured fermions. This occurs when the three-loop colour contribution cancels with the electroweak contributions such that $C_2(M) = 0$. It was already noted in section 2.3 that
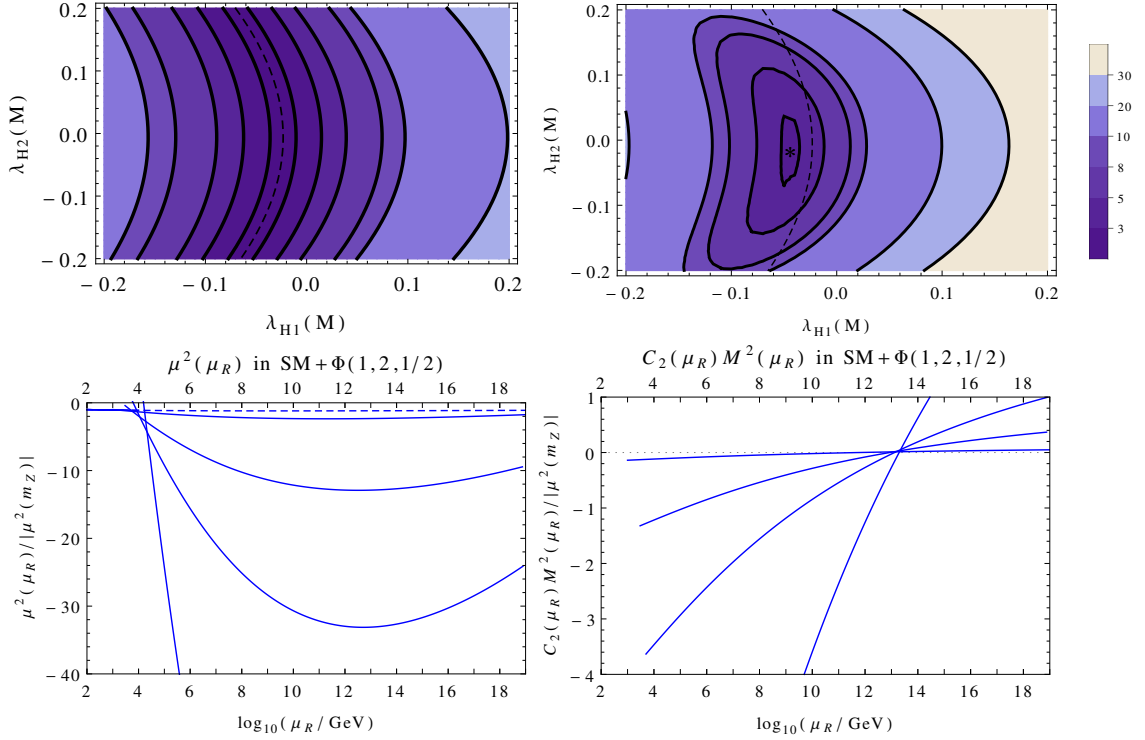
Figure 9. Sensitivity measure and RG evolution in the SM+$\Phi(1, 2, 1/2)$ (i.e. the 2HDM). Upper panel: the not-yet-minimised sensitivity measure for $M = 1\,\mathrm{TeV}$ as a function of $(\lambda_{H1}(M), \lambda_{H2}(M))$ for: (left) $\Delta_{red}(\Lambda_{\mathrm{Pl}})$ of eq. (5.1), and (right) $\Delta(\Lambda_{\mathrm{Pl}})$ of eq. (3.12). The dashed line shows the $\mu^2(\Lambda_{\mathrm{Pl}}) = 0$ contour and the star denotes the global minimum. Lower panel: RG evolution of $\mu^2(\mu_R)$ and $C_2(\mu_R)M^2(\mu_R)$ for $M = 1, 3, 5, 10\,\mathrm{TeV}$ evaluated at the $(\lambda_{H1}, \lambda_{H2})$ points which minimise $\Delta(\Lambda_{\mathrm{Pl}})$.

this is only an artifact of the loop level to which we are working. The $\Delta(\Lambda_{\mathrm{Pl}})$ contour for the $\psi(3, 1, 1)$ GM demonstrates how this feature is effectively removed when $\Lambda_h > M$. In figure 5 the qualitative form of the contours in the $\Phi(3, 1, Q_1)$ scalar case is seen to change as Q_1 is increased from 0 to 2. This is due to a transition in dominance between colour and hypercharge effects, which are opposite in sign.

In figure 6 a cusp feature is observed when Λ_h is just above M. This can be understood from the toy model eq. (2.8): it is the point where $2\log(\Lambda_h/M) \simeq 1$ and the $\partial/\partial \log M^2$ sensitivity measure is minimised. Also, the "turn-around" features in the $\psi(3, 1, Q_1)$, $\psi(8, 1, Q_1)$, and $\psi(8, 2, Q_1)$ plots can again be understood as a balance between the colour and electroweak contributions. In figure 7 a number of cusp features are observed, mostly occurring at $\Lambda_h \sim 20M$. These features all have the same origin: they occur for solutions where $\mu^2(\Lambda_h) \approx 0$. For example, in the $\Phi(1, 2, 1/2)$ case at $\Lambda_h = \Lambda_{\mathrm{Pl}}$ we saw that the λ_{Hi} took on values such that $\mu^2(\Lambda_{\mathrm{Pl}}) < 0$ (see figure 9). It turns out that, for $\Lambda_h \lesssim 20M$, the sensitivity measure is minimised for values such that $\mu^2(\Lambda_h) > 0$. At the transition point the reduced sensitivity measure eq. (5.1) vanishes, and hence the full sensitivity measure is somewhat reduced. Note that in the cases where Φ is coloured the transition occurs later due to the competing contributions between gauge contributions.

Before concluding we would like to make a few comments about the applicability of these bounds in the context of extended models. Firstly, one might contend that our bounds (especially the $\Delta(\Lambda_{\mathrm{Pl}})$ bounds), which are only derived in the context of minimal SM+GM extensions, are not applicable in a realistic model with additional high scale states. This is true in a quantitative sense: the bounds are sure to change. Nonetheless, this does not imply that they are not qualitatively useful. It would take very special physics to ameliorate these bounds by a significant amount. For example, one could try to introduce new states with particular properties at $\Lambda_h \sim M$ such that loop contributions approximately cancel at this scale [38]. In the absence of a symmetry which introduces the appropriate correlations between parameters at this scale, and a symmetry which ensures the cancellation remains satisfied under RG evolution, naturalness bounds similar to those we have derived will be quickly reintroduced at $\Lambda_h > M$. Actually, such symmetry requirements are just those provided by supersymmetric theories, and herein lies the connection between our RG description and the usual naturalness arguments in the context of supersymmetry. In any case, the framework we have outlined in appendix A is fully generalisable to perturbative models with more states. Naturalness of the low scale Higgs mass parameter can be quantified by the Bayesian sensitivity measure eq. (A.7), as long as one is prepared to calculate and solve RGEs at least at two-loop order with one-loop matching between intermediate physical scales.

Secondly, in deriving our sensitivity measure we have made the assumption of flat priors on the logarithms of $\overline{\mathrm{MS}}$ input parameters at scale Λ_h. This particular set of priors is determined by demanding insensitivity to units or parameter rescalings, and makes logical sense in a bottom-up approach where one would like to remain maximally agnostic to the higher scale UV theory. However, it is true that if one were to derive these priors as posteriors arising from a flat set of priors in the UV theory, they would almost certainly not be flat. Furthermore, one would generally expect correlations between the parameters. Hence one might expect that our results are only broadly applicable if those derived priors are approximately flat.[10]

Nevertheless, it is still possible to argue that the naturalness bounds derived here are not expected to significantly change even if the priors are peaked. Consider for example altering the $\log \mu^2(\Lambda_h)$ prior such that it is locally scaled by a factor κ within some window; i.e. the prior is instead flat in a function $f(\log \mu^2)$ with $\partial f(\log \mu^2)/\partial \log \mu^2 = \kappa$ (1) within (outside of) the window. In such a case the sensitivity measure, e.g. eq. (3.12), is of the same form except with $\partial \log \mu^2(\Lambda_h)$ replaced by $\partial f(\log \mu^2(\Lambda_h))$. Now, consider the case where the prior is locally increased ($\kappa > 1$) within a window centered on the realised value of $\log \mu^2(\Lambda_h)$. Then the contribution of the $|\partial \log \mu^2(m_Z)/\partial \log \mu^2(\Lambda_h)|$ term to the sensitivity measure will be scaled down by a factor $1/\kappa$. However, the contribution from the other terms does not change. Generally, as can be understood from our toy examples (see eq. (2.8)), the contribution from other terms (and at very least the $|\partial \log \mu^2(m_Z)/\partial \log M|$ term) is of similar order. Thus, in this case, one does not expect the bound to significantly

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[10]In particular, some might argue that this is unlikely for $\log \mu^2(\Lambda_h)$ in the presence of gravity, but then some might choose to remain agnostic.

change unless the the prior in $\log M$ (and other parameters) is also locally peaked. This is not surprising, since we have increased the probability of a specific initial high scale boundary value for $\mu^2(\Lambda_h)$, but we have not altered the prior on the slope, which is controlled by the size of M and other parameters entering the RGEs. Thus if the bound is to be significantly affected by a set of peaked priors at high scale, they need to be sharply peaked at very particular values in more than one parameter. Unless one has a plausible explanation for such priors then this has only shifted the naturalness issue.

6 Conclusion

The aim of this paper was to confront the question, *at what mass does a heavy gauge multiplet introduce a physical Higgs naturalness problem?* In section 2 we described a physical way to understand the Higgs naturalness problem which might be introduced when perturbative heavy new physics is added to the SM. The description is of particular interest in bottom-up extensions of the SM. The premise is essentially as follows. In any perturbative EFT, the low scale Higgs mass parameter $\mu^2(m_Z) \simeq -(88 \text{ GeV})^2$ can be connected by renormalisation group equations to $\overline{\text{MS}}$ "input" parameters defined at some high scale Λ_h. If $\mu^2(m_Z)$ is especially sensitive to these input parameters, then this signifies a Higgs naturalness problem. In particular, this can occur if a heavy particle of mass M is added to the SM.

In order to sensibly quantify this potential problem, we derived a sensitivity measure using Bayesian probabilistic arguments. The measure can be interpreted as a Bayesian model comparison [see eq. (2.4)] which captures the "naturalness price" paid for promoting the Higgs mass parameter to a high scale input parameter of the model as opposed to a purely phenomenological input parameter at low scale. It is fully generalisable to any perturbative QFT, with the details provided in appendix A. The measure reduces in a certain (relevant) limit to an intuitively motivated Barbieri-Giudice-like fine-tuning measure [see eq. (2.5)]. The resulting sensitivity measure is generally a function of unknown high scale inputs. We conservatively projected these out by minimising over them, thereby obtaining the sensitivity measure eq. (2.6), which is a function of Λ_h and the mass M of a heavy new particle.

This sensitivity measure was used to set naturalness bounds on the masses of various gauge multiplets, using a full two-loop RGE analysis with one-loop matching. An interesting outcome is that, once RG effects are taken into account and finite threshold corrections are captured, a naturalness bound on M remains even in the limit $\Lambda_h \to M^+$. The resulting bounds are presented in tables 1 and 2, and as contours in figures 4–7. They form the main result of this paper, and we hope they are of interest to model builders. For $\Lambda_h = \Lambda_{\text{Pl}}$ we find "10% fine-tuning" bounds of $M < \mathcal{O}(1\text{–}10)\,\text{TeV}$ on the masses of various gauge multiplets, with the bounds on fermionic gauge multiplets significantly weaker than for scalars. In the limit $\Lambda_h \to M^+$ the bounds weaken to $M < \mathcal{O}(10\text{–}100)\,\text{TeV}$; these can be considered as conservative naturalness bounds, of interest if new physics is expected to substantially alter the RG evolution of $\mu^2(\mu_R)$ above the scale M. We also found that the bounds on coloured multiplets are no more severe than on electroweak multiplets, since they correct the Higgs mass directly at three-loop order.

Acknowledgments

A Sensitivity measure as a Bayesian model comparison

In this appendix we show how a Barbieri-Giudice-like fine-tuning measure for $\mu^2(m_Z)$ arises in a certain limit of our Bayesian model comparison. Similar connections have been made in earlier works, e.g. refs. [5, 9].

Bayesian probability allows one to assign a degree of belief to some hypothesis, in our case a particle physics model. The model $\mathcal{M}$ consists of a set of input parameters $\mathcal{I}$ and a rule for connecting these to a set of observables $\mathcal{O}$. Let us assume that there are n fundamental input parameters $\mathcal{I} = \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ and $m \leq n$ independent observables $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_m\}$. The rule is just a map $\mathcal{R} : \mathcal{I} \to \mathcal{O}$ from input space to observable space with $(\mathcal{I}_1, \ldots, \mathcal{I}_n) \mapsto \mathcal{R}(\mathcal{I}_1, \ldots, \mathcal{I}_n) = (\mathcal{O}_1, \ldots, \mathcal{O}_m)$. In this paper the models consist of the SM plus a new gauge multiplet of mass M, with inputs as the logarithms of $\overline{\text{MS}}$ parameters of the full Lagrangian defined at scale Λ_h, observables as the logarithms of $\overline{\text{MS}}$ SM Lagrangian parameters at scale m_Z, and $\mathcal{R}$ given by the RGEs. The logarithms are taken to avoid dependence on units or rescalings of the Lagrangian.[11]

The Bayesian evidence B for $\mathcal{M}$ is the probability that the observables $\mathcal{O}$ attain their experimentally observed values $\mathcal{O}_{ex}$, assuming $\mathcal{M}$ is true:

$$B(\mathcal{M}) := p(\mathcal{O} = \mathcal{O}_{ex}|\mathcal{M}) = \int p(\mathcal{O} = \mathcal{O}_{ex}|\mathcal{I}) \, p(\mathcal{I}) \, d\mathcal{I}, \qquad (A.1)$$

where $p(\mathcal{O} = \mathcal{O}_{ex}|\mathcal{I})$ is also called the likelihood function $\mathcal{L}(\mathcal{I})$, and $p(\mathcal{I})$ is the prior density for the model parameters. The prior density represents the degree of belief in the values of the input parameters before any observations are made. In the absence of any knowledge about the complete UV theory, we should assume priors which are maximally agnostic. This corresponds to a flat prior in the n-dimensional input space $\mathcal{I}$. The mapping $\mathcal{R}$ can be used to express some point in input space $(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ in terms of a new set of coordinates $(\mathcal{O}, \mathcal{I}') \equiv (\mathcal{O}_1, \ldots, \mathcal{O}_m, \mathcal{I}_{m+1}, \ldots, \mathcal{I}_n)$ simply by $\mathcal{I} \mapsto \mathcal{R}'(\mathcal{I}) \equiv (\mathcal{R}(\mathcal{I}), \mathcal{I}')$. We assume that this is a one-to-one mapping (indeed, it is for RGEs in the perturbative regime). If we assume perfectly measured observables, then eq. (A.1) becomes

$$B(\mathcal{M}) \propto \int \delta(\mathcal{O} - \mathcal{O}_{ex}) \, p \circ \mathcal{R}'^{-1}(\mathcal{O}, \mathcal{I}') \left| \begin{pmatrix} \frac{\partial \mathcal{O}_1}{\partial \mathcal{I}_1} & \cdots & \frac{\partial \mathcal{O}_1}{\partial \mathcal{I}_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{O}_m}{\partial \mathcal{I}_1} & \cdots & \frac{\partial \mathcal{O}_m}{\partial \mathcal{I}_m} \end{pmatrix} \right|^{-1} d\mathcal{O}_1 \cdots d\mathcal{O}_m d\mathcal{I}_{m+1} \cdots d\mathcal{I}_n,$$

$$(A.2)$$

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[11]Absolute values inside the logarithms are implied. The signs of the parameters can be considered as separate inputs. Explicitly including them with a flat prior probability mass function does not change the final result, and we ignore them henceforth for clarity.

where the likelihood has become a delta function multiplied by a constant term, and $|\,(\,\cdot\,)\,| \equiv |\det[(\,\cdot\,)]|$ is the determinant of the Jacobian associated with the coordinate transformation. Performing the integration over the observables,

$$B(\mathcal{M}) \propto \int p'(\mathcal{I}') \left| \left(\begin{array}{ccc} \frac{\partial \mathcal{O}_1}{\partial \mathcal{I}_1} & \cdots & \frac{\partial \mathcal{O}_1}{\partial \mathcal{I}_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{O}_m}{\partial \mathcal{I}_1} & \cdots & \frac{\partial \mathcal{O}_m}{\partial \mathcal{I}_m} \end{array} \right)^{-1} d\mathcal{I}_{m+1} \cdots d\mathcal{I}_n \right|_{\mathcal{O}=\mathcal{O}_{ex}}, \tag{A.3}$$

where $p'(\mathcal{I}') \equiv p \circ \mathcal{R}^{-1}(\mathcal{O}_{ex}, \mathcal{I}')$. The requirement $\mathcal{O} = \mathcal{O}_{ex}$ has carved out an experimentally allowed $(n-m)$ dimensional submanifold within the original n dimensional input space. We know that, since the original prior was flat in n dimensions, the prior on the submanifold must be flat with respect to the induced volume element (as opposed to the volume element $d\mathcal{I}_{m+1} \cdots d\mathcal{I}_n$). We can rescale the existing volume element to write, equivalently,

$$B(\mathcal{M}) \propto \int p'(\mathcal{I}') \frac{\left| \left(\begin{array}{ccc} \frac{\partial \mathcal{O}_1}{\partial \mathcal{I}_1} & \cdots & \frac{\partial \mathcal{O}_1}{\partial \mathcal{I}_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{O}_m}{\partial \mathcal{I}_1} & \cdots & \frac{\partial \mathcal{O}_m}{\partial \mathcal{I}_m} \end{array} \right)^{-1} \right|}{\sqrt{\left| \left(\begin{array}{ccc} \frac{\partial \mathcal{I}_1}{\partial \mathcal{I}_{m+1}} & \cdots & \frac{\partial \mathcal{I}_1}{\partial \mathcal{I}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{I}_n}{\partial \mathcal{I}_{m+1}} & \cdots & \frac{\partial \mathcal{I}_n}{\partial \mathcal{I}_n} \end{array} \right)^T \left(\begin{array}{ccc} \frac{\partial \mathcal{I}_1}{\partial \mathcal{I}_{m+1}} & \cdots & \frac{\partial \mathcal{I}_1}{\partial \mathcal{I}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{I}_n}{\partial \mathcal{I}_{m+1}} & \cdots & \frac{\partial \mathcal{I}_n}{\partial \mathcal{I}_n} \end{array} \right) \right|}} d\Sigma \Bigg|_{\mathcal{O}=\mathcal{O}_{ex}}, \tag{A.4}$$

where the quantity under the square root is the determinant of the induced metric, $d\Sigma$ is the induced volume element, and the prior $p'(\mathcal{I}')$ is constant with respect to this volume element. This reduces to[12]

$$B(\mathcal{M}) \propto \int \frac{p'(\mathcal{I}')}{\sqrt{|JJ^T|}} \, d\Sigma \Bigg|_{\mathcal{O}=\mathcal{O}_{ex}}, \tag{A.5}$$

where J is the $m \times n$ matrix defined by $J_{ij} = \partial \mathcal{O}_i / \partial \mathcal{I}_j$ [9]. Additionally, by taking a delta function prior on $\mathcal{I}'$ we can evaluate (and compare) Bayesian evidence for the model $\mathcal{M}$ with unconstrained input parameters $(\mathcal{I}_{m+1}, \ldots, \mathcal{I}_n)$ taking on specific values:

$$B(\mathcal{M}; \mathcal{I}') \propto \frac{1}{\sqrt{|JJ^T|}} \Bigg|_{\substack{\mathcal{O}_{ex} \\ \mathcal{I}'}}. \tag{A.6}$$

Let us now put this in the context of minimal extensions of the SM by a gauge multiplet of mass M. We take $\mathcal{I}_1 = \log \mu^2(\Lambda_h)$ and $\mathcal{O}_1 = \log \mu^2(m_Z)$. The remaining inputs and observables are logarithms of the $\overline{\text{MS}}$ Lagrangian parameters. The Bayesian evidence

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[12]To show this requires the use of: Sylvester's identity $\det(\mathbb{I}_m + AB) = \det(\mathbb{I}_n + BA)$ for $m \times n$ and $n \times m$ matrices A and B; and the matrix identity $(A|B)(A|B)^T = AA^T + BB^T$ for A and B matrices with equal number of rows.

eq. (A.6) is not enough by itself; it can only be interpreted with respect to some reference model. We will, after all, be interested in the sensitivity of $\mu^2(m_Z)$ to the input parameters, and we have not so far treated the μ^2 parameter in any special way. The reference model we choose to compare to is the model $\mathcal{M}_0$ in which the Higgs mass parameter is instead taken as a "phenomenological" input parameter at scale m_Z, i.e. $\mathcal{I}_1 = \mathcal{O}_1 = \log \mu^2(m_Z)$. In $\mathcal{M}_0$ we have that $J_{11} = 1$ and $J_{1j} = 0$ for $j > 1$. The Bayes factor between these two models is then

$$K(\mathcal{M};\mathcal{I}') := \left.\frac{B(\mathcal{M}_0;\mathcal{I}')}{B(\mathcal{M};\mathcal{I}')}\right|_{\substack{\mathcal{O}_{ex}\\\mathcal{I}'}}. \tag{A.7}$$

Since the dimensionful parameter $\mu^2(\mu_R)$ does not enter the (mass independent) RGEs of the remaining dimensionless observables, we have that $J_{i1} = \partial\mathcal{O}_i/\partial\mathcal{I}_1 = 0$ for $i > 1$. Additionally, in the special case that the dimensionless observables are approximately insensitive to the unconstrained inputs, i.e. $J_{ij} \simeq 0$ for $i > 1$ and $j \geq m + 1$, eq. (A.6) becomes

$$B(\mathcal{M};\mathcal{I}') \propto \left.\frac{\left|\begin{pmatrix}\frac{\partial\mathcal{O}_2}{\partial\mathcal{I}_2} & \cdots & \frac{\partial\mathcal{O}_2}{\partial\mathcal{I}_m}\\ \vdots & \ddots & \vdots\\ \frac{\partial\mathcal{O}_m}{\partial\mathcal{I}_2} & \cdots & \frac{\partial\mathcal{O}_m}{\partial\mathcal{I}_m}\end{pmatrix}\right|^{-1}}{\sqrt{\left(\frac{\partial\log\mu^2(m_Z)}{\partial\mathcal{I}_1}\right)^2 + \sum_{j\geq m+1}\left(\frac{\partial\log\mu^2(m_Z)}{\partial\mathcal{I}_j}\right)^2}}\right|_{\substack{\mathcal{O}_{ex}\\\mathcal{I}'}}. \tag{A.8}$$

We can see that a Barbieri-Giudice-like fine-tuning measure has appeared in the denominator. In this case the quantity $B(\mathcal{M}_0;\mathcal{I}')$ becomes independent of $\mathcal{I}'$, and the Bayes factor eq. (A.7) is

$$K(\mathcal{M};\mathcal{I}') = \left.\sqrt{\left(\frac{\partial\log\mu^2(m_Z)}{\partial\log\mu^2(\Lambda_h)}\right)^2 + \sum_{j\geq m+1}\left(\frac{\partial\log\mu^2(m_Z)}{\partial\mathcal{I}_j}\right)^2}\right|_{\substack{\mathcal{O}_{ex}\\\mathcal{I}'}}. \tag{A.9}$$

This is reminiscent of the Barbieri-Giudice fine-tuning measure. We observe the interesting emergence of additional terms quantifying Higgs mass sensitivity only to the unconstrained parameters of the model. Conceptually, K is a comparison between a flat prior in $\log \mu^2(m_Z)$ and the RG devolved (to m_Z) flat prior in $\log \mu^2(\Lambda_h)$, in the vicinity $\mu^2(m_Z) \simeq -(88 \text{ GeV})^2$. A Bayes factor of $K > 10$ corresponds to the onset of strong evidence (on the Jeffreys scale) for $\mathcal{M}_0$ over $\mathcal{M}$.

Lastly, note that the Bayes factor K is still a function of the unknown parameters $\mathcal{I}'$. In order to write down a sensitivity measure for the model $\mathcal{M}$ as a function of a subset of these unknown parameters (e.g. the gauge multiplet mass M), we might want some way of projecting out the nuisance unknowns. One way is to integrate over some region of $\mathcal{I}'$, i.e. evaluate eq. (A.5). However, in this paper we instead choose the following conservative projection:

$$\Delta(\mathcal{M}) = \min_{\mathcal{I}'}\left\{K(\mathcal{M};\mathcal{I}')\right\}. \tag{A.10}$$

This identifies the best case scenario for Higgs mass naturalness in $\mathcal{M}$ by finding the point in $\mathcal{I}'$ with the lowest Bayes factor.

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