Published for SISSA by Springer

Received: September 21, 2015
Revised: November 20, 2015
Accepted: January 15, 2016
Published: February 1, 2016

Non-equilibrium steady state in the hydro regime

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Abstract: We study the existence and properties of the non-equilibrium steady state which arises by putting two copies of systems at different temperatures into a thermal contact. We solve the problem for the relativistic systems that are described by the energy-momentum of a perfect hydro with general equation of state (EOS). In particular, we examine several simple examples: a hydro with a linear EOS, a holographic CFT perturbed by a relevant operator and a barotropic fluid, i.e., $P = P(\mathcal{E})$. Our studies suggest that the formation of steady state is a universal result of the hydro regime regardless of the kind of fluid.

Keywords: AdS-CFT Correspondence, Field Theories in Higher Dimensions, Conformal Field Models in String Theory, Bethe Ansatz

ArXiv ePrint: 1509.01162

doi:10.1007/JHEP02(2016)005

JHEP02(2016)005

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1 Introduction

Although the study of equilibrium states has been the focus of most research in many body systems and condensed matter physics, much of the interesting phenomena around us are far from equilibrium. However, thermodynamic study of non-equilibrium states are less advanced since the dynamical equations governing thermodynamic variables are highly non-linear differential equations. Unlike the thermodynamic equilibrium where the processes are reversible and time independent, for a system in a non-equilibrium state processes are generally irreversible and time dependant. That's what makes the study of systems far from equilibrium so difficult. Nevertheless, waiting long enough, most non-equilibrium systems tend to approach to the state of thermodynamic equilibrium unless there is a continuous flow of matter or energy to or from the system.

Yet an interesting subclass of non-equilibrium systems is the *steady state* which is reached by a system subject to a constant driving force. In a non-equilibrium steady state (NESS) there is no time variation, i.e., all thermodynamic properties are time independent, although the entropy production is non-zero and the system is recognised by the presence of fluxes. The fluxes are the flows of currents of the conserved variables driven by the gradient of their corresponding thermodynamic conjugate, e.g., the energy current is driven by the temperature gradient, and so on. Since the dynamic of such a system is dissipative, in order to maintain the steady state one needs the injection of energy at one boundary and subtraction at the other. One simple example of such NESS is a system confined between two heat baths (reservoirs) at different but constant temperature, transporting heat from one to another by the energy current which is generally proportional to the temperature gradient.

However in a recent studies of one dimensional quantum critical systems described by conformal field theory (CFT), Bernard and Doyon [1] found an interesting steady state with a nontrivial energy current choosing a different framework; instead of putting a system in contact with external reservoirs, they picked two copies of some large quantum system at different thermal equilibrium, e.g., different temperatures, and then glued them together at a contact point which result in the energy transfer from one to another. After long enough time, a steady state would be established and parts of each system far from the contact point would effectively behave like heat baths. Another interesting point about their set-up is that although the temperature profile is flat, there exists a non-zero energy current. That is, unlike the usual case for steady states, the energy current is *not* simply a function of temperature gradient. Moreover, the universal character of the resulting steady state is noticeable; the energy transfer only depends on the universal constants and the temperatures of the initial copies they start with.

Motivated by these results, the existence of such a steady state and its universality has been investigated for CFT in higher dimensions [2, 3]. Therefore, one legitimate question might be whether this is a particular feature of CFT, or one may get similar answer starting from initial systems with more general equation of state. In this manuscript we are trying to explore this question relying on the holographic insight given in [3];[1] there it has been shown that the non-equilibrium steady state on the conformal boundary is in fact dual to the Lorentz boosted black brane in the bulk. According to the AdS/CFT correspondence, the stress tensor of the steady state could be obtained from the metric of the dual gravity theory in the bulk. Therefore, the steady state is completely specified if one obtains its temperature and boost velocity, i.e., $\{T, v\}$, in terms of the temperatures of the initial systems at the instant of contact, denoted by T_l and T_r for one system on the left and the other on the right. In fact, at sufficiently large scales two initial systems look like asymptotic reservoirs with the steady state as the intermediate state interpolating between two reservoirs on the left and right. Then in this framework, the problem is similar to a Riemann problem with initial boundary condition T_l for $x < 0$ and T_r for $x > 0$, where a possible solution would be two shock waves emitting at the point of contact and moving to the opposite directions. Note that the development of the steady state in this set-up also requires that two shock waves propagate in the opposite directions without splitting and decomposition, for if this happens the composite waves could lead to cascades and finally thermalizing the system.

Riemann problem for classical and relativistic hydrodynamics with various equation of states is widely studied in the literature, see for example [4–8]. We follow similar approach of solving Riemann problem to study NESS while considering to have a relativistic perfect fluids for the left and right hydro before bringing them into a thermal contact. We solve the problem assuming a general equation of state, e.g., barotropic fluid. In general, we conclude that the formation of steady state after thermal contact has nothing to do with conformal field theory or even with integrable models; it is a universal result in the hydro regime for just about any kind of fluid.

Therefore, the outline of this paper is as follows: in section 2 we briefly review shock wave solutions to the Riemann problem as well as Rankine-Hugoniot jump condition which

[1]We prefer to follow the method introduced in [3] as opposed to the ansatz given in [2]. The former, due to holographic analysis, has less parameters and is easier to deal with particularly if one is interested in studying a more general equation of state like barotropic case that we investigate in this paper.

are essential in our calculations of specifying the steady state. Then in section 3, starting with the perfect fluid assumption for the initial systems at the instant of contact, we investigate the steady state properties for the systems with more general equation of state (EOS) than the conformal fluid. In particular, we study three examples: systems with linear equation of state, the QFT model described in [9] as a small deviation from CFT, and finally systems with barotropic equation of state where pressure is only a function of energy density however we restrict our calculation to the case where the temperature difference between two systems are very small. We close the paper by some discussion about limitation on the stability and the existence of shock solutions and also give a brief conclusion in section 4.

2 Shock waves and jump condition

Consider a relativistic fluid described by the energy density $\mathcal{E}$ and the pressure P, and then introduce a small perturbations to the pressure and density of the system. It is straightforward to show that these perturbations, the so-called "*sound waves*", propagate through the fluid with the velocity c_s given by

$$c_s^2 \equiv \frac{dP}{d\mathcal{E}}\,, \tag{2.1}$$

known as the "*speed of sound*" in literature. If we assume that the sound waves have infinitesimal amplitude, i.e., the perturbations are very small, then the speed of sound is nearly uniform throughout the fluid. However, in general the speed of sound is a function of density which result in the crest of wave to move faster than the trough. When the crest overtakes the trough a *shock wave* can form due to the steepening of wave. A shock front is a surface that marks a sudden jump in the density and pressure of the fluid. Although this is a way that shocks could form, in general, shock waves are characterized by a rapid, discontinuous change in the density and pressure of the system. The shock is bounded into an infinitesimal region where the fluid properties, such as density and pressure, immediately before and after being shocked are linked by the jump conditions. These conditions, usually referred to as the *Rankine-Hugoniot jump conditions*, are derived from the conservation laws.

Consider a conserved quantity $q(x,t)$ in (1+1)-dimensions[2] satisfying the hyperbolic conservation law [4, 5], i.e.,

$$\partial_t q + \partial_x f(q) = 0\,, \tag{2.2}$$

where f represents the flux of q. Now assume that there exists a solution q which has a discontinuity, i.e., a shock, along the curve $x = \xi(t)$; this is known as the *weak solution* of the PDE (2.2) [10]. Then choose the interval $\xi_- \leq x \leq \xi_+$ such that it intersects with

[2]The hyperbolic conservation laws originally studied in [4, 5] for a theory in (1+1)-dimensions. However, one can follow a similar approach for a theory in higher dimensions by imposing a symmetry so that nothing happens in the remaining transverse directions.

$x = \xi(t)$ at time t. Integrating equation (2.2) over this interval, yields

$$\frac{d}{dt}\left(\int_{\xi_-}^{\xi(t)} q(x,t)dx + \int_{\xi(t)}^{\xi_+} q(x,t)dx\right) + \int_{\xi_-}^{\xi_+} \partial_x f(q(x,t))dx = 0\,. \tag{2.3}$$

If we apply Leibnitz rule to (2.3) using the fact that the conservation equation (2.2) is satisfied on either side of the discontinuity, we obtain

$$\frac{d\xi(t)}{dt}(q_l - q_r) = f_l - f_r\,, \tag{2.4}$$

where q_l and q_r denote the values of q on the left and right sides of $x = \xi(t)$, respectively, and

$$f_l = f(q_l)\,, \qquad f_r = f(q_r)\,.$$

Equation (2.4) is called the *Rankine-Hugoniot jump condition* and is usually written in shorthand notation as

$$u_s[q] = [f]\,, \tag{2.5}$$

where $[q] \equiv q_l - q_r$ and $[f] \equiv f_l - f_r$ are the jumps across the discontinuity (or shock) and $u_s \equiv d\xi(t)/dt$ is referred to as the *shock speed*.

In more physical systems there are more than one conservation law and the dynamical behavior of the fluid is governed by a system of hyperbolic conservation laws as

$$\partial_t Q + \partial_x F = 0\,, \tag{2.6}$$

where $Q = (q_1, \ldots, q_n)$ and F is a function of $q_1, \ldots, q_n$. Therefore, one would derive n jump conditions corresponding to each equation in (2.6) as

$$u_s[Q] = [F]\,. \tag{2.7}$$

However, to obtain a full description of the fluid, we also need to take into account the equation of state (EOS) which relates different thermodynamic variables of the system. Although the conservation laws and the jump conditions are valid for any EOS, the deterministic role of EOS on the nature of the shock waves and their propagations has been widely considered in literature, see for example [11–13].

Therefore, a key concern in studying the fluid dynamics is the existence and uniqueness of the shock solutions in a given fluid, the so-called *Riemann problem*. In general, the Riemann problem is the initial value problem for a system of conservation laws where the initial data are a pair of constant states separated by a jump discontinuity at $x = 0$, i.e.,

$$Q_0(x) = \begin{cases} Q_l & \text{if } x < 0, \\ Q_r & \text{if } x > 0. \end{cases} \tag{2.8}$$

In the next section, we study steady state solutions for the perfect fluid with various EOSs and examine the constraints to have the solution to the Riemann problem.

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3 Perfect fluid and steady state solution

Consider two isolated semi-infinite systems each described by a relativistic perfect fluid, i.e.,

$$T^{\mu\nu} = (P + \mathcal{E})u^\mu u^\nu + P\eta^{\mu\nu}\,, \tag{3.1}$$

and governed by the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{3.2}$$

Let's denote the temperature of one system by T_l and the other by T_r and assume that the energies of two systems are just a function of temperature, i.e., $\mathcal{E}_{l,r} = \mathcal{E}(T_{l,r})$. Now we want to bring two systems into contact at $t = 0$ and study the possible solutions. Indeed, this is a Riemann problem with initial values T_l for $x < 0$ and T_r for $x > 0$. As mentioned in the introduction, this problem has been recently studied for conformal fluid [1–3]. Our goal is to study the problem for an arbitrary equation of state while we follow an approach adopted in [3] to examine a possible development of the steady state through the propagation of shock waves emanating from the contact point.

It is more evident that the conservation law (3.2) is a system of hyperbolic equations when it is written as follows

$$\begin{aligned} \partial_t T^{tt} + \partial_x T^{xt} &= 0\,, \\ \partial_t T^{tx} + \partial_x T^{xx} &= 0\,. \end{aligned} \tag{3.3}$$

Therefore comparing with (2.6) we get

$$Q = \begin{pmatrix} T^{tt} \\ T^{tx} \end{pmatrix}, \qquad F = \begin{pmatrix} T^{tx} \\ T^{xx} \end{pmatrix}. \tag{3.4}$$

Then all we need to solve for the shock solution are the Rankine-Hugoniot jump conditions:

$$u_s(Q_l - Q_r) = F_l - F_r \tag{3.5}$$

where Q and F are given by (3.4) with $Q_l > Q_r$ and $F_l > F_r$.

The existence and uniqueness of the shock solution for an arbitrary EOS is highly influenced by the thermodynamic properties of the fluid as well as the nature of the conservation laws. Nevertheless, for a relativistic perfect fluid satisfying the hyperbolic laws (3.3), the existence and uniqueness of the shock solutions has been shown in [7] provided the difference between quantities on the left and right side of discontinuity is small and the following relations are fulfilled:

$$c_s < 1\,, \qquad \frac{d^2P}{d\mathcal{E}^2} \geqq -2\frac{(1-c_s^2)c_s^2}{P+\mathcal{E}}\,. \tag{3.6}$$

The first inequality ensures the physical requirement that the speed of sound is less than the speed of light, where we have set $c = 1$ throughout this paper.

In the following we study the steady state solutions considering three types of equation of state: a) linear EOS, i.e., $P = \sigma\mathcal{E}$, b) the EOS for a $(d+1)$-dimensional QFT introduced in [9] which is in fact a holographic CFT perturbed by a relevant operator, and c) general barotropic EOS, i.e., $P = P(\mathcal{E})$ where we assume the left and right quantity are slightly different, i.e., $\mathcal{E}_l - \mathcal{E}_r \ll \mathcal{E}_{l,r}$.

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3.1 Linear EOS

Consider we start with two sem-infinite fluids described by a linear EOS:

$$P = \sigma \mathcal{E}\,, \tag{3.7}$$

where from the definition (2.1) we infer $\sigma = c_s^2$. Next we bring two systems into instantaneous thermal contact at $x, t = 0$. We show that at long enough time the steady state forms and we obtain the properties of that. In fact what we have, looks like a Riemann problem with initial values

$$\mathcal{E}_0(x) = \begin{cases} \mathcal{E}_l & \text{if } x < 0, \\ \mathcal{E}_r & \text{if } x > 0. \end{cases} \tag{3.8}$$

A consistent solution has two shock waves propagating in opposite direction with respect to each other and therefore all we need to solve are the jump conditions (3.5) for EOS (3.7). At long enough time we have three regions:

1. region left described by an ideal fluid with energy momentum tensor:

$$T_l^{\mu\nu} = \begin{pmatrix} \mathcal{E}_l & 0 & 0 & 0 \\ 0 & \sigma\mathcal{E}_l & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma\mathcal{E}_l \end{pmatrix} \tag{3.9}$$

2. steady state region in the middle described by a boosted fluid with energy $\mathcal{E}$ and the boost velocity v:

$$T_s^{\mu\nu} = \sigma\mathcal{E}\left(\eta^{\mu\nu} + (\sigma^{-1}+1)u^\mu u^\nu\right)\,, \tag{3.10}$$

   where $\eta^{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$ and $u^\mu = \gamma(1, v, 0, \cdots, 0)$ with $\gamma = 1/\sqrt{1-v^2}$.

3. region right described by an ideal fluid with energy momentum tensor:

$$T_r^{\mu\nu} = \begin{pmatrix} \mathcal{E}_r & 0 & 0 & 0 \\ 0 & \sigma\mathcal{E}_r & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma\mathcal{E}_r \end{pmatrix} \tag{3.11}$$

Note that $\mathcal{E}_l > \mathcal{E} > \mathcal{E}_r$. Now we apply the jump relation (3.5) for two shock waves:

I. One shock wave is moving with the speed u_l to the negative x direction from steady state region with energy $\mathcal{E}$ to the left region with energy $\mathcal{E}_l > \mathcal{E}$, therefore we have:

$$u_l(Q_l - Q_s) = F_l - F_s\,, \tag{3.12}$$

where Q and F are given in (3.4).

II. Another shock wave is moving with the speed u_r to the positive x direction from steady state region with energy $\mathcal{E}$ to the right region with energy $\mathcal{E}_r < \mathcal{E}$, therefore we have:

$$u_r(Q_s - Q_r) = F_s - F_r\,. \tag{3.13}$$

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Combining equations (3.12) and (3.13) immediately gives:

$$T_s^{tx} = \frac{T_l^{xx} - T_r^{xx}}{u_l + u_r} = \sigma\left(\frac{\mathcal{E}_l - \mathcal{E}_r}{u_l + u_r}\right), \tag{3.14}$$

where we have used (3.9) and (3.11). Moreover, substituting (3.9)–(3.11) into the equations (3.12)–(3.13), we obtains the four unknowns $\{u_l, u_r, \mathcal{E}, v\}$ in terms of the known boundary conditions, i.e., $\mathcal{E}_r(x > 0)$ and $\mathcal{E}_l(x < 0)$ as:

$$u_l = \sigma\sqrt{\frac{\bar{\chi} + \sigma^{-1}}{\bar{\chi} + \sigma}}, \qquad u_r = \sqrt{\frac{\bar{\chi} + \sigma}{\bar{\chi} + \sigma^{-1}}}, \tag{3.15}$$

$$\mathcal{E} = \sqrt{\mathcal{E}_l\mathcal{E}_r}, \qquad v = \frac{\bar{\chi} - 1}{\sqrt{(\bar{\chi} + \sigma^{-1})(\bar{\chi} + \sigma)}}, \tag{3.16}$$

where $\bar{\chi} \equiv \sqrt{\mathcal{E}_l/\mathcal{E}_r}$. One should note that $\sigma = c_s^2 < 1$ to fulfil relativistic constraint. Accordingly the second expression in (3.6) is satisfied since $0 \leq \sigma < 1$ and therefore the above shock solution is the only solution to this set up for all $\mathcal{E}_l$ and $\mathcal{E}_r$.[3] Moreover, for the special case of $\sigma = 1/d$, where d is the spatial dimensions, the equations (3.14)–(3.16) reduced to the one in [3] for conformal fluid considering the fact that the energy of the conformal fluid is related to the temperature as $\mathcal{E} \propto T^{d+1}$ thus $\bar{\chi}$ here coincide with χ in that reference.

3.2 A QFT model

We start with a $(d + 1)$-dimensional QFT in the regions left and right where the action is given by

$$S_{\mathrm{QFT}} = S_{\mathrm{CFT}} + \lambda\int d^{d+1}x\,\mathcal{O}(x), \tag{3.17}$$

where λ has dimension $d + 1 - \Delta$ and the dimensionless quantity $\lambda/T^{d+1-\Delta} \ll 1$. Note that the unitarity bound for a scalar operator introduces a lower bound on the conformal dimension as $\Delta \geq (d-1)/2$, for most recent study see [14]. Also the action (3.17) describes a holographic CFT perturbed by a *relevant operator* which requires $\Delta < d + 1$ and in the following we investigate the development of the steady state for such a fluid. Recently in [15], a similar study has been done for a (1+1)-dimensional CFT which is perturbed by an *irrelevant operator*.

The energy density and pressure of a fluid described by (3.17) at finite temperature has been studied perturbatively in [9] and is given by:

$$\begin{aligned}
\mathcal{E}(T) &= \mathcal{A}T^{d+1}\left(1 - \alpha\left(\frac{\lambda}{T^{d+1-\Delta}}\right)^2\right) + \cdots, \\
\mathcal{P}(T) &= \frac{\mathcal{A}}{d}T^{d+1}\left(1 - \left(\frac{\lambda}{T^{d+1-\Delta}}\right)^2\right) + \cdots,
\end{aligned} \tag{3.18}$$

where $\alpha = (2\Delta - d - 2)/d$ and $\mathcal{A}$ is proportional to the central charge of CFT.

[3] It is proved that for a linear EOS the uniqueness constraint (3.6) is valid for all $\mathcal{E}_l$ and $\mathcal{E}_r$ while for general EOS the constraint is valid provided left and right quantities are close. We refer the interested reader to [7] for the proof.

Using (2.1) the speed of sound up to second order in perturbative expansion is given by

$$c_{s,\mathrm{QFT}}^2 = c_{s,\mathrm{CFT}}^2 \left[1 + \frac{2(d+1-2\Delta)(d+1-\Delta)}{d(d+1)} \frac{\lambda^2}{T^{2(d+1-\Delta)}}\right], \tag{3.19}$$

where $c_{s,\mathrm{CFT}}^2 = 1/d$. The second term in the square bracket is negative for $(d+1)/2 < \Delta < d+1$ indicating that $c_{s,\mathrm{QFT}}^2 < c_{s,\mathrm{CFT}}^2$. This is indeed a reasonable expectation: in the study of QCD thermodynamics in four dimensions, it is known that in the hot QCD plasma the speed of sound is approaching the Stefan-Boltzmann limit, equal to conformal limit, i.e., $c_{s,\mathrm{SB}}^2 = 1/3$, from below [16]. For $\Delta < (d+1)/2$ it is evident that $c_{s,\mathrm{QFT}}^2 > c_{s,\mathrm{CFT}}^2$, violating the Stefan-Boltzman limit, yet an allowed solution by unitarity considerations. Also, using the expression (3.19) one can compare the speed of sound in the left and right fluid. For the conformal dimension in the range $\frac{d+1}{2} < \Delta < d+1$ the second term in the bracket is negative, therefore assuming $T_l > T_r$ one easily conclude $c_{s,\mathrm{QFT},l}^2 > c_{s,\mathrm{CFT}}^2 > c_{s,\mathrm{QFT},r}^2$ which supports our intuition that the speed of sound is larger in a fluid with higher temperature. However, for $\frac{d-1}{2} < \Delta < \frac{d+1}{2}$ the second term in the bracket in (3.19) is positive which yields to $c_{s,\mathrm{QFT},l}^2 < c_{s,\mathrm{CFT}}^2 < c_{s,\mathrm{QFT},r}^2$ while $T_l > T_r$. Although this is a mathematically accepted solution, it is not physically relevant case.

Moreover, plugging the expressions (3.18) into equations (3.3)–(3.5) we will get the expression for the temperature of steady state as:

$$T = T_c \left(1 + \frac{\tau\lambda^2}{T_c^{2(d+1-\Delta)}}\right) + \cdots, \tag{3.20}$$

where $T_c = \sqrt{T_l T_r}$ is the temperature obtained for conformal case, and

$$\tau = \frac{(1-\chi^{-\delta})}{2d(d+1)(\chi+1)} \left[d(1-\delta)(1-\chi^{\delta+1}) + (d-\delta)(1-\chi^{\delta-1})\chi\right], \tag{3.21}$$

where we have defined

$$\chi \equiv \left(\frac{T_l}{T_r}\right)^{\frac{d+1}{2}}, \qquad \delta \equiv 2\left(1 - \frac{\Delta}{d+1}\right). \tag{3.22}$$

Note that $\chi > 1$ since we have assumed $T_l > T_r$. Examining the expression (3.20) reveals that depending on the value of conformal dimension Δ the temperature T of the steady state starting with the initial perturbed fluids could be smaller or larger than the temperature T_c of the steady state starting with the initial conformal fluids. Indeed, $\delta < 1$ for $\Delta > (d+1)/2$ which result in both terms in square bracket in (3.21) become negative. That is, in this regime τ is negative and therefore $T < T_c$. However, $\delta > 1$ for $\Delta < (d+1)/2$ which yields positive τ and $T > T_c$. More precisely one gets

$$\begin{aligned} T < T_c \qquad &\text{for} \qquad \frac{d+1}{2} < \Delta < d+1, \\ T > T_c \qquad &\text{for} \qquad \frac{d-1}{2} < \Delta < \frac{d+1}{2}. \end{aligned} \tag{3.23}$$

The above result is valid independent of the value of χ, i.e., relative ratio of right and left temperature, however the temperature T deviates more from T_c when the difference between right and left temperature increases.

Also one can obtain the boost velocity as:

$$v = v_c \left(1 + \frac{\nu \lambda^2}{T_c^{2(d+1-\Delta)}} \right) + \cdots \,, \tag{3.24}$$

where v_c is given in equation (3.16) for the conformal fluid by replacing $\bar{\chi} \to \chi$ and

$$\nu = -\tau \left[\frac{(d+1)^3}{2d} \frac{\chi(\chi+1)(1+\chi^{-\delta})}{(\chi-1)(1-\chi^{-\delta})(\chi+d)(\chi+d^{-1})} \right] . \tag{3.25}$$

The square bracket in the above expression is always positive then the sign of ν is determined by the sign of τ which we already discussed. Therefore from the expression (3.24), one can deduce that the boost velocity is smaller or larger than the conformal case depending on the value of conformal dimension Δ, i.e.,

$$\begin{aligned} v > v_c \qquad &\text{for} \qquad \frac{d+1}{2} < \Delta < d+1 \,, \\ v < v_c \qquad &\text{for} \qquad \frac{d-1}{2} < \Delta < \frac{d+1}{2} \,. \end{aligned} \tag{3.26}$$

One can also obtain the shock speed in the left and right fluid as

$$\begin{aligned} u_l &= u_{c,\,l} \left(1 + \frac{\mathcal{U}_l \lambda^2}{T_c^{2(d+1-\Delta)}} \right) + \cdots \,, \\ u_r &= u_{c,\,r} \left(1 + \frac{\mathcal{U}_r \lambda^2}{T_c^{2(d+1-\Delta)}} \right) + \cdots \,. \end{aligned} \tag{3.27}$$

where $u_{c,\,l}$ and $u_{c,\,r}$ are given in equation (3.15) for conformal fluid by replacing $\bar{\chi} \to \chi$ and

$$\mathcal{U}_l = \frac{\nu(d-1+2\chi)}{1+d} + \frac{(1+d)^2\tau + \alpha(1+d)\chi^{-\delta} - \alpha(d+\chi^{-1}) + \chi^{-1} - 1}{d\chi^{-2}(\chi-1)(d^{-1}+\chi)} \,, \tag{3.28}$$

$$\mathcal{U}_r = \frac{\nu(d-1+2\chi^{-1})}{1+d} - \frac{(1+d)^2\tau + \alpha(1+d)\chi^{\delta} - \alpha(d+\chi) + \chi - 1}{(\chi-1)(d+\chi)} \,. \tag{3.29}$$

It could be shown

$$\begin{aligned} u_{l,r} < u_{c,\,l,r} \qquad &\text{for} \qquad \frac{d+1}{2} < \Delta < d+1 \,, \\ u_{l,r} > u_{c,\,l,r} \qquad &\text{for} \qquad \frac{d-1}{2} < \Delta < \frac{d+1}{2} \,. \end{aligned} \tag{3.30}$$

To check if the above solution is the unique solution of the problem, one needs to check the inequalities (3.6). From equation (3.19) it is evident that for small perturbations, i.e., $\lambda/T^{d+1-\Delta} \ll 1$, the first constraint in (3.6) is always fulfilled. The second constraint is always satisfied for all $\mathcal{E}_l$ and $\mathcal{E}_r$ for the conformal fluid and then it holds as well in our QFT model which is just a small perturbation around CFT. Therefore the shock waves emanating at the contact point are stable and the results obtained for the steady state hold for all $\mathcal{E}_l$ and $\mathcal{E}_r$.

It might be interesting to compare the results of our QFT model in $(1+1)$-dimensions with that of [15]; note that the former is a perturbed CFT by a relevant operator while the latter is a CFT perturbed by an irrelevant operator. Let's start with the left and right fluid which is described by (3.18) with $d = 1$ and $\Delta = 1$, i.e.,

$$\mathcal{E}(T) = \mathcal{A}T^2 \left(1 - \alpha \frac{\lambda^2}{T^2}\right) + \cdots \,, \tag{3.31}$$

$$\mathcal{P}(T) = \mathcal{A}T^2 \left(1 - \frac{\lambda^2}{T^2}\right) + \cdots \,. \tag{3.32}$$

The temperature and boost velocity of the steady state which develops in the intermediate region would be specified by (3.20) and (3.24), respectively, while the speed of shocks moving to the left and right are given by (3.27). However, one would immediately recover that the subleading terms in all the expansions vanish for $d = 1$ and $\Delta = 1$ and obtain

$$T = T_c \,, \qquad v = v_c \,, \qquad u_l = u_{l,c} \,, \qquad u_r = u_{r,c} \,, \tag{3.33}$$

for the steady state which is the same with that of conformal fluids. That is, even though the initial fluids that we start with on the left and right are perturbed CFTs given by EOS (3.32), the steady state developing in between left and right regions shows the same properties as one starts with conformal fluids as the initial states to start with. This is different from the results in [15] for a steady state interpolating between left and right CFTs perturbed by an irrelevant operator where they observe deviation from the conformal case for the temperature, shock speeds and etc.

We emphasis that expression (3.33) is only true for steady state if the conformal dimension of the relevant operator is set to one in $(1+1)$ dimensions. For any other allowed values of Δ one should use expansions (3.20), (3.24), (3.27) to express the properties of steady state and it definitely deviates from the conformal case. However, similar story happens in an arbitrary dimensions if one chooses an operator where $\Delta = (d+1)/2$.

3.3 Barotropic fluid

Consider a barotropic fluid for which pressure is a function of energy density, i.e.,

$$P = P(\mathcal{E}) \,, \tag{3.34}$$

where the energy density is only a function of temperature, $\mathcal{E} = \mathcal{E}(T)$. We also assume that the difference between left and right temperature is very small compared to either left or right temperature which implies

$$\mathcal{E}_l - \mathcal{E}_r \ll \mathcal{E}_{l,r} \,, \tag{3.35}$$

with $\mathcal{E}_l > \mathcal{E}_r$. Now one can use the Taylor expansion for pressure around some $\mathcal{E}_0$ as

$$P(\mathcal{E}) = P_0 + c_s^2(\mathcal{E} - \mathcal{E}_0) + \frac{1}{2}\kappa(\mathcal{E} - \mathcal{E}_0)^2 + \cdots \tag{3.36}$$

where we have defined $P(\mathcal{E}_0) \equiv P_0$ and

$$\kappa \equiv \frac{d^2 P}{d\mathcal{E}^2}\,, \tag{3.37}$$

is a characteristic parameter of the fluid and should satisfy the second inequality in (3.6), in order to have a unique shock solution to the Riemann problem with initial values $\mathcal{E}_l$ for $x < 0$ and $\mathcal{E}_r$ for $x > 0$. Now solving equations (3.3)–(3.5) perturbatively we will get the following expression for the energy density of the steady state fluid

$$\mathcal{E} = \mathcal{E}_0 + \frac{1}{16c_s^2}\left(\kappa - \frac{2(1+c_s^2)c_s^2}{H_0}\right)\Delta\mathcal{E}^2 + \cdots\,, \tag{3.38}$$

where we have defined $H_0 \equiv P_0 + \mathcal{E}_0$ which is the enthalpy of the equilibrium state where $\mathcal{E}_l = \mathcal{E}_r$. Also, we have chosen $\mathcal{E}_0$ to be the mean energy of the left and right fluid, i.e.,

$$\mathcal{E}_0 = \frac{\mathcal{E}_l + \mathcal{E}_r}{2}\,, \tag{3.39}$$

and

$$\Delta\mathcal{E} = \mathcal{E}_l - \mathcal{E}_r\,. \tag{3.40}$$

Note that the first correction which appears in energy density is of second order. Indeed, this is not surprising as one would expect only even powers of $\Delta\mathcal{E}$ appears in the energy expansion simply because the energy of the steady state should not change if we start by the initial condition where we have replaced $\mathcal{E}_l \leftrightarrow \mathcal{E}_r$. Furthermore, the boost velocity is given by

$$v = \frac{c_s}{2\,H_0}\,\Delta\mathcal{E} + \cdots\,. \tag{3.41}$$

Here, we expect only odd powers of $\Delta\mathcal{E}$ appears in the boost expansion since again the magnitude of the boost velocity should not change by replacing $\mathcal{E}_l \leftrightarrow \mathcal{E}_r$, however one need to boost in the opposite direction.

The speed of sound in the left and right fluid to the first order in $\Delta\mathcal{E}$ is given by

$$\begin{aligned} c_{s,l} &= c_s\left(1 + \frac{\kappa}{4c_s^2}\Delta\mathcal{E} + \cdots\right)\,, \\ c_{s,r} &= c_s\left(1 - \frac{\kappa}{4c_s^2}\Delta\mathcal{E} + \cdots\right)\,. \end{aligned} \tag{3.42}$$

We did not include the second order term in the above expansion in order to avoid introducing a new characteristic parameter for the fluid. For $\kappa > 0$ it is clear that $c_{s,l} > c_s > c_{s,r}$ which is consistent with our assumption for the temperature that $T_l > T_r$. On the other hand for $\kappa < 0$ we get $c_{s,r} > c_s > c_{s,l}$ while we have assumed $T_l > T_r$. This is not what we usually expect in a normal physical system. Note that the speed of sound on left and right would be replaced under $\mathcal{E}_l \leftrightarrow \mathcal{E}_r$.

We can further obtain the speed of shock waves on the left and right fluid as

$$\begin{aligned} u_l &= c_{s,\,l}\left[1 - \frac{1}{8c_{s,\,l}^2}\left(\kappa + 2\frac{(1-c_{s,l}^2)c_{s,l}^2}{H_0}\right)\Delta\mathcal{E} + \cdots\right]\,, \\ u_r &= c_{s,\,r}\left[1 + \frac{1}{8c_{s,\,r}^2}\left(\kappa + 2\frac{(1-c_{s,\,r}^2)c_{s,\,r}^2}{H_0}\right)\Delta\mathcal{E} + \cdots\right]\,. \end{aligned} \tag{3.43}$$

The bracket in the second term is always positive due to the stability constraint (3.6), therefore the left shock is subsonic compared to the speed of sound on the left while the right shock is supersonic with respect to the speed of sound on the right.

If the characteristic parameter $\kappa > 0$ then the second constraint in (3.6) is always satisfied and therefore the above solution is a unique solution. This is indeed a well-known fact, even for a classical fluid, that $\kappa > 0$ is a sufficient condition for the existence and uniqueness of shock waves to the Riemann problem, known as Bethe-Weyl theorem [11, 17]. However for an arbitrary equation of state with $\kappa < 0$, it is still possible to have a unique shock solution for the Riemann problem as long as the inequality in (3.6) fulfilled.

4 Discussion

We argued the possibility of performing the NESS when we bring two copies of systems at different temperatures into a thermal contact in the framework of Bernard and Doyon [1], however, our systems enjoy more general EOS than the conformal fluid. The key feature in the BD set-up is that the steady state is *not* driven by the temperature gradient. There are no external reservoirs, rather the initial systems play the role of heat baths at sufficiently large scales. Furthermore, we used the holographic insight of [3] to describe the steady state as an intermediate Lorentz boosted state of the initial systems, interpolating between two asymptotic heat baths after long enough time. In this approach the formation of the steady state relies on the two single shock waves emanating at the point of contact moving in the opposite directions. Since both systems are assumed to be semi-infinite then the claim is that there is no chance for the shocks to reflect back, forming a cascade and thermalizing the system. Although this seems to be sufficient condition to construct the steady state in the case of CFT, it is not enough for a system with a general EOS, e.g., barotropic fluid that we studied in this paper. In general, in order to perform the steady state, shock waves should not split otherwise thermalization may happen. This is due to the fact that if they split, they may move in the opposite directions and form composite waves leading to cascades and finally thermalizing the system. For a relativistic perfect fluid with general EOS the condition (3.6) is sufficient in order to avoid splitting and to have stable shock solutions. Therefore the development of a steady state is guaranteed and one can obtain the properties of this NESS in terms of the characteristics of the fluid and the initial values, see equations (3.38) and (3.41).

Nonetheless, in our studies we only considered perfect fluids, for which viscosity is zero. It is interesting to see how the construction of the steady states would be affected in this set-up, if we move away from ideal hydro and take into account the viscosity. Since performing of the steady state depends on the propagation of shock waves as mentioned before, then we can turn around and ask how viscosity will affect the formation of shock waves at the point of contact. More precisely, one should investigate the possibility of shock solutions for the initial condition Riemann problem in viscous fluid. In fact, this question has been already studied in literature, e.g., see [18] where the solutions to the relativistic Riemann problem for viscous fluid has been investigated numerically. As a result, by varying the ratio of shear viscosity to entropy density, i.e., η/s, from zero to infinity, a transition from ideal shock waves to viscous one has been shown; starting from ideal fluid with zero viscosity one obtains shock waves with zero width. By increasing η/s the solutions with non-zero width will appear, the so-called viscous shocks. An upper limit for the η/s has been estimated for which shocks can still be observed experimentally on the proper time scale. However, as one continues to increase the ratio η/s above this upper limit, the free-streaming will occur, i.e., shock solution is completely washed out.

Acknowledgments

I would like to express my gratitude to Rob Myers for the initial motivation and for his continuous guidance throughout the progress of this paper. I would also like to thank Perimeter Institute for hospitality at an early stage of this work. The research of R.P. is supported by Icelandic Research Fund grant 130131-053.

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JHEP02(2016)005