Type IIB flux vacua from G-theory II

**Philip Candelas,^a Andrei Constantin,^b Cesar Damian,^c Magdalena Larfors^b
and Jose Francisco Morales^d**

^a*Mathematical Institute, University of Oxford,
Andrew Wiles Building, Radcliffe Observatory Quarter,
Woodstock Road, Oxford, OX2 6GG, U.K.*

^b*Department of Physics and Astronomy, Uppsala University,
SE-751 20, Uppsala, Sweden*

^c*Departamento de Fisica, DCI, Campus Leon, Universidad de Guanajuato,
C.P. 37150, Leon, Guanajuato, Mexico*

^d*INFN — Sezione di Roma “TorVergata”,
Dipartimento di Fisica, Universita di Roma “TorVergata”,
Via della Ricerca Scientifica, 00133 Roma, Italy*

E-mail: candelas@maths.ox.ac.uk, andrei.constantin@physics.uu.se,
cesaredas@fisica.ugto.mx, magdalena.larfors@physics.uu.se,
francisco.morales@roma2.infn.it

ABSTRACT: We find analytic solutions of type IIB supergravity on geometries that locally take the form $\text{Mink} \times M_4 \times \mathbb{C}$ with M_4 a generalised complex manifold. The solutions involve the metric, the dilaton, NSNS and RR flux potentials (oriented along the M_4) parametrised by functions varying only over $\mathbb{C}$. Under this assumption, the supersymmetry equations are solved using the formalism of pure spinors in terms of a finite number of holomorphic functions. Alternatively, the solutions can be viewed as vacua of maximally supersymmetric supergravity in six dimensions with a set of scalar fields varying holomorphically over $\mathbb{C}$. For a class of solutions characterised by up to five holomorphic functions, we outline how the local solutions can be completed to four-dimensional flux vacua of type IIB theory. A detailed study of this global completion for solutions with two holomorphic functions has been carried out in the companion paper [1]. The fluxes of the global solutions are, as in F-theory, entirely codified in the geometry of an auxiliary $K3$ fibration over $\mathbb{CP}^1$. The results provide a geometric construction of fluxes in F-theory.

KEYWORDS: Flux compactifications, Superstring Vacua, String Duality

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1 Introduction

Compactifications of string theories with and without flux is a subject of long history, dating back to the seminal papers [2–4].¹ Without fluxes, supersymmetry requires that the internal manifold in type II string compactification is Calabi-Yau, whereas in the presence of fluxes, it must be of generalised Calabi-Yau type. A generalised Calabi-Yau manifold [10] is characterised by the existence of globally defined spinors. Spinor bilinears define polyforms that behave as pure spinors in the generalised tangent space. Supersymmetry is preserved in the four-dimensional theory if the pure spinors satisfy a system of first order differential

¹For recent reviews (with extensive references) on the subject of supersymmetric and non-supersymmetric flux compactifications of string theory, see [5–9].

equations [11]. If the flux also satisfies the relevant Bianchi identities and the internal manifold is compact, a supersymmetric four-dimensional vacuum is obtained.

In the companion paper [1], we present concrete examples of supersymmetric four-dimensional type IIB vacua where all fields can be explicitly written out in an analytic form, even in the presence of fluxes. The solutions are built by gluing local solutions on $T^4 \times \mathbb{C}$ in a U-duality consistent way. Such local solutions can be found by starting from non-compact Calabi-Yau geometries, and then applying a sequence of U-duality transformations that rotate the metric into fluxes. In this way, different classes of flux solutions characterised by up to three holomorphic functions are generated.

The aim of this paper is to present more general flux solutions that cannot be related to Calabi-Yau geometries by means of U-dualities. We consider geometries that locally take the form $\text{Mink} \times M_4 \times \mathbb{C}$ with M_4 a generalised complex manifold with $SU(2)$ structure. We use an ansatz in which the metric, the dilaton and the type IIB fluxes are parametrised by functions varying over the complex plane, and all form potentials are oriented along M_4 . Under these assumptions, the supersymmetry constraints simplify drastically and can be solved in terms of a finite number of holomorphic functions. We find three classes of solutions with $SU(2)$ structure that we denote A, B and C. The three solutions correspond to different choices of the two angles describing the relative orientations of the two spinors defining the $SU(2)$ structure. The solutions A, B, C in [1] fall into the solution class of that name here for $M_4 = T^4$, and correspond to the case where only three of the holomorphic functions characterising the general solutions are non-constant.

The interest in the solutions under study here lies in the fact that they can be given an auxiliary, completely geometric description, following the approach of [12, 13] (related ideas have been explored in [14–40]). In particular, solutions on T^4 or $K3$ characterised by $n \leq 5$ holomorphic functions can be extended to the whole complex plane (including infinity) away from a finite number of degeneration points. Around these points, the functions undergo non-trivial monodromy transformations in the U-duality group $SO(2, n, \mathbb{Z})$. This group is also the modular group of the space of complex structures of an algebraic $K3$ surface with Picard number $20 - n$. Moreover, the locally holomorphic functions parametrise a coset space that is isomorphic to the complex structure moduli space of this $K3$. The n holomorphic functions characterising the flux solution can thus be identified with the n holomorphic parameters (periods of the holomorphic two-form) characterising the complex structure of the $K3$ surface, and the local charges in the flux solution (e.g. branes, orientifold planes) can be read off from the monodromy transformations of the periods around singular points in the base. The presence of singularities (and thus local sources) allow non-trivial flux solutions even when the base $\mathbb{C}$ is compactified, in agreement with known no-go theorems [41].² Consequently, we can, in this way, construct four-dimensional flux vacua of type IIB string theory in terms of auxiliary geometries that are fibrations of $K3$ surfaces over, for example, a two-sphere. This auxiliary description is an extension of F-theory [42], in that it provides a geometric description of fluxes in F-theory compactifications. The details of this analysis are given in [1] for the case $n = 2, 3$ and will not be repeated here.

²Recall that the only globally defined holomorphic function on a compact space is a constant.

The techniques developed in that paper can also be applied to $K3$ fibrations with $n > 3$ complex parameters, and hence to the local solutions of this paper. Since this computation is very technical, it goes beyond the scope of this paper and is left for future work.

The rest of this note is organised as follows. First, in section 2, we give a very brief review of type IIB supergravity, and present the ansatz we will use for the local solutions. In section 3, we solve the supersymmetry equations and the Bianchi identities (away from local sources) where the internal six-manifold takes the form $M_4 \times \mathbb{C}$. We perform the analysis of the supersymmetry equations using the formalism of pure spinors, briefly reviewed in appendix B. We present three classes of solutions, with different flux and metric content, that can be parametrised in terms of a set of holomorphic functions. We also discuss how these different classes of solutions are related by U-duality transformations. Finally, in section 4 we draw some conclusions. Appendix A summarises our conventions, and appendix C rederives one class of local solutions by the more direct approach of solving the Killing spinor equations.

2 Type IIB supergravity

In this section, we provide a very brief review of type IIB supergravity, in order to clarify our conventions. For more details, we refer the reader to [43] and recent reviews on flux compactifications [5–9]. We also specify the ansatz for the local supersymmetric solutions that will be studied in the next section.

2.1 Action and Bianchi identities

In the low-energy supergravity limit, the bosonic field content of type IIB string theory consists of the Neveu-Schwarz-Neveu-Schwarz (NSNS) fields (a metric g , a scalar field called the dilaton ϕ and a two-form field B) and the Ramond-Ramond (RR) p -form fields C_p , where p is 0, 2, 4. This is complemented by the fermionic fields: two gravitinos Ψ_M^A and two dilatinos λ^A , $A = 1, 2$ of equal chirality.

The action for the bosonic sector is, in the string frame,

$$S = \frac{1}{2\kappa_{10}} \int d^{10}x \sqrt{-g} \left(e^{-2\phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{2 \cdot 3!} H^2 \right] - \frac{1}{2} F_1^2 - \frac{1}{2 \cdot 3!} F_3^2 - \frac{1}{4 \cdot 5!} F_5^2 \right) - \frac{1}{4\kappa_{10}} \int \left(C_4 \wedge H_3 \wedge F_3 \right), \quad (2.1)$$

where $g = \det g_{MN}$, $H = dB$ and $F_n = dC_{n-1} - H_3 \wedge C_{n-3}$ are the NSNS and RR field strengths, respectively. In what follows, we collectively refer to the RR fluxes using a polyform language

$$F = d_H C = dC - H \wedge C, \text{ where } C = C_0 + C_2 + C_4. \quad (2.2)$$

The fluxes must fulfil the Bianchi identities

$$dH = 0 \quad d_H F = 0. \quad (2.3)$$

If sources (NS 5-branes, D_p -branes and orientifolds) are present, they will modify the right hand side of these equations.

2.2 Killing spinor equations

A purely bosonic supergravity configuration is supersymmetric if and only if the fermionic supersymmetry variations vanish. This leads to the Killing spinor equations (KSE)

$$\delta\Psi_M = \left(\nabla_M + \frac{1}{8} H_{MNO} \Gamma^{NO} \mathcal{P} + \frac{e^\phi}{8} \sum_n \frac{1}{n!} F_{P_1 \dots P_n} \Gamma^{P_1 \dots P_n} \Gamma_M \mathcal{P}_n \right) \epsilon = 0 \quad (2.4)$$

$$\delta\lambda = \left(\Gamma^M \partial_M \phi + \frac{1}{2} H_{MNO} \Gamma^{MNO} \mathcal{P} - \frac{e^\phi}{4} \sum_n \frac{(5-n)}{n!} F_{P_1 \dots P_n} \Gamma^{P_1 \dots P_n} \mathcal{P}_n \right) \epsilon = 0 \quad (2.5)$$

where Ψ_M , ϵ and λ are column vectors containing two Majorana-Weyl spinors of the same chirality, ∇ is the standard covariant derivative, n is odd and Γ are the ten-dimensional Dirac matrices (see appendix A for our spinor conventions). The projection matrices $\mathcal{P}, \mathcal{P}_n$ are given by

$$\mathcal{P} = -\sigma^3 \quad \mathcal{P}_3 = \sigma^1 \quad \mathcal{P}_{1,5} = i\sigma^2. \quad (2.6)$$

where the Pauli matrices σ^i are given in appendix A.

Once the KSE and the Bianchi identities are satisfied in a Minkowski vacuum, it can be shown that all bosonic equations of motion follow [44, 45]; as such, the supersymmetric solutions constructed below will satisfy all the constraints required for local type IIB vacua.

2.3 The ansatz

In the following section we will solve the KSE and the Bianchi identities corresponding to type IIB supergravity on space-times $\mathbb{R}^{1,3} \times M_4 \times \Sigma$, with Σ an open subset of $\mathbb{C}$. The M_4 metric g_{mn} , the dilaton ϕ , the B -field and C_p -fields are assumed to vary over $\Sigma \subset \mathbb{C}$. All the non-trivial fluxes are assumed to be oriented along M_4 .

Let $\{y^1, y^2, y^3, y^4\}$ be real coordinates on M_4 and z a complex coordinate on $\mathbb{C}$. In these coordinates, we write the metric and the fluxes as:

$$ds^2 = ds_4^2 + ds_6^2 = e^{2A} \sum_{\mu=0}^3 dx_\mu dx^\mu + \sum_{m,n=1}^4 g_{mn} dy^m dy^n + e^{2D} |h(z)|^2 dz d\bar{z} \quad (2.7)$$

$$B = \frac{1}{2} b_{mn} dy^m \wedge dy^n, \quad C_2 = \frac{1}{2} c_{mn} dy^m \wedge dy^n, \quad C_4 = c_4 dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4,$$

where $A, D, g_{mn}, b_{mn}, c_{mn}, c_4, C_0$ and ϕ are real z -dependent functions.

To cohere with the ansatz for the bosonic fields, the ten-dimensional Killing spinor $\epsilon = (\epsilon^1, \epsilon^2)^T$ must decompose into four- and six-dimensional spinors, that we denote ζ_i^a and η^i , respectively. The number of four-dimensional spinors is determined by the number of ten-dimensional spinors and the number n of well-defined internal spinor η^i :

$$\epsilon^A = \sum_{i=1}^n (\zeta_{i+}^A \otimes \eta_+^i + \zeta_{i-}^A \otimes \eta_-^i), \quad (2.8)$$

where $\pm$ denotes chiral and anti-chiral components of the spinors, $\zeta_{i-}^A = \zeta_{i+}^{A*}$, and $\eta_-^i = \eta_+^{i*}$. We take $\zeta_{i\pm}^A$ to be constant spinors and assume that η^i vary only along the z -plane, in

accordance with our ansatz for the bosonic fields. On $SU(2)$ structure manifolds there are two globally defined spinors η_+^1 , which can be written in the form (see appendix B for details)

$$\eta_+^1 = e^{\frac{A-i\theta}{2}} \eta_+ \quad \eta_+^2 = e^{\frac{A+i\theta}{2}} (\cos \alpha \eta_+ + \sin \alpha \chi_+) , \quad (2.9)$$

Different choices of the angles α and θ will lead to different kinds of fluxes and brane sources.

3 Local supersymmetric solutions

In this section, we present three classes of local supersymmetric type IIB solutions, that all satisfy the Killing spinor equations and source-free Bianchi identities. As reviewed in appendix B.1, the fact that $M_4 \times \Sigma$ allows two well-defined spinors implies that its structure group is reduced to $SU(2)$. We will use this local $SU(2)$ structure to define pure $O(6,6)$ spinors, and show that these satisfy the supersymmetry equations, once the supergravity fields on M_4 vary holomorphically over Σ . In appendix C, this result is shown without recourse to the pure spinor language for a class of solutions, named type A in the following, with up to five holomorphic functions.

A six-dimensional manifold of the local form $M_4 \times \Sigma$ has a local $SU(2)$ structure if it admits a set of differential forms, (j, Ω_2) on M_4 and K on Σ (cf. appendix B.1). Choosing local holomorphic coordinates (z^1, z^2, z) , we take for the one-form K

$$K = e^D h(z) dz \quad (3.1)$$

and expand the two-forms j, Ω_2 , the NSNS two-form B and the RR potentials C_p in a basis of closed forms on M_4 with coefficients that depend on z . The fluxes of these configurations are given by $H = dB$ and $F_n = dC_{n-1} - H \wedge C_{n-1}$; they satisfy the Bianchi identities automatically, and always have one leg along either dz or its complex conjugate.

We take the $SU(2)$ structure on M_4 to be defined by self-dual two-forms

$$*_4 j = j \quad *_4 \Omega_2 = \Omega_2 . \quad (3.2)$$

We will consider potential forms oriented along M_4 . For a d_4 -closed form χ on M_4 varying only over $\mathbb{C}$, the Hodge dual in six-dimensions can be written as

$$* d\chi = *_4 (*_2 d_2 \chi) = - *_4 d_2^c \chi \quad (3.3)$$

with

$$d = \partial + \bar{\partial} \quad d^c = i(\bar{\partial} - \partial) \quad (3.4)$$

the exterior derivatives on $M_4 \times \mathbb{C}$ and d_2, d_2^c their reductions to $\mathbb{C}$.

The four-dimensional metric will be computed with the help of formula

$$g_{mn} = -j_{mp} I_n^p \quad (3.5)$$

with

$$I_n^p = c' \epsilon^{pm_1 m_2 m_3} (\text{Re} \Omega_2)_{nm_1} (\text{Im} \Omega_2)_{m_2 m_3} \quad (3.6)$$

ϵ the Levi-Civita symbol in four dimensions, and c' fixed such that $I^2 = -1$. These equations follow straightforwardly from the corresponding $SU(3)$ structure identities (B.4) and (B.5), cf. appendix B.

3.1 Pure spinor equations

A particularly elegant reformulation of the supersymmetry constraints is found using $O(6, 6)$ pure spinors (or polyforms) $\Phi_{1,2}$ [11, 46]. In terms of these variables the KSE (2.4)–(2.5) translate into a set of first order differential equations

$$d_H(e^{3A-\phi}\Phi_1) = 0 \quad (3.7)$$

$$d_H(e^{2A-\phi}\text{Re}\Phi_2) = 0 \quad (3.8)$$

$$d_H(e^{4A-\phi}\text{Im}\Phi_2) = \frac{e^{4A}}{8} * \lambda(F), \quad (3.9)$$

with $d_H\chi = d\chi - H \wedge \chi$ for any differential form χ , and $\lambda(F) = F_1 - F_3 + F_5$. In appendix B.1 we review how $\Phi_{1,2}$ are related to the six-dimensional spinors $\eta_{\pm}^i$: the latter define nowhere vanishing differential forms (j, Ω_2, K) which in turn specify two nowhere vanishing polyforms $\Phi_{\pm}$ [47, 48]

$$\begin{aligned} \Phi_1 &= -\frac{1}{8}K \wedge (\sin \alpha e^{-ij} + i \cos \alpha \Omega_2) \\ \Phi_2 &= \frac{e^{-i\theta}}{8}e^{\frac{1}{2}K \wedge \bar{K}} (\cos \alpha e^{-ij} - i \sin \alpha \Omega_2) . \end{aligned} \quad (3.10)$$

By specifying α and θ , we will, in the remainder of this section, find three different types of supersymmetric IIB solutions, that we will label A, B and C. More precisely, we will construct local solutions to the KSE and the Bianchi identities following the ansatz (2.7). In this analysis, we will, from time to time, use the fact that the $SU(2)$ structure defines also an $SU(3)$ structure on the six-dimensional manifold characterised by the forms

$$\Omega_3 = K \wedge \Omega_2 \quad J = j + \frac{i}{2}K \wedge \bar{K} . \quad (3.11)$$

3.2 Solution class A

We start by considering the case $\alpha = 0$ and $\theta = \pi/2$, i.e.

$$\Phi_1 = -\frac{i}{8}\Omega_3 \quad \Phi_2 = -\frac{i}{8}e^{-iJ} \quad (3.12)$$

In this case the two spinors η^A are parallel, but out of phase.³ The constraints (3.7)–(3.9) become

$$d_H(e^{3A-\phi}\Omega_3) = 0, \quad (3.13)$$

$$d_H(e^{2A-\phi}\text{Im}[e^{-iJ}]) = 0, \quad (3.14)$$

$$d_H(e^{4A-\phi}\text{Re}[e^{-iJ}]) = -e^{4A} * \lambda(F), \quad (3.15)$$

The first two supersymmetry equations lead to

$$d(e^{3A-\phi}\Omega_3) = d(e^{2A-\phi}J) = H \wedge J = H \wedge \Omega_3 = 0, \quad (3.16)$$

³This case was studied in [13], where solutions with the local geometry $K3 \times \Sigma$ were found.

where we solve the first two relations by taking

$$\Omega_3 = e^{\phi-3A}\hat{\Omega}_3 \quad J = e^{\phi-2A}\hat{J}, \quad (3.17)$$

with $d\hat{\Omega}_3 = d\hat{J} = 0$. A sufficient condition to solve the last two constraints is to take B anti-self-dual, since this implies that B wedges to zero with both J and Ω_3 . Since B is a closed z -dependent form on M_4 , this implies that $H = dB$ also wedges to zero with these forms.

The third supersymmetry equation (3.15) then reduces to⁴

$$F_5 = dC_4 - H \wedge C_2 = e^{-4A} * d[e^{4A-\phi}] \quad (3.18)$$

$$F_3 = dC_2 - H C_0 = e^{-\phi} * dB \quad (3.19)$$

$$F_1 = dC_0 = -\frac{1}{2}e^{-4A} * d[e^{4A-\phi} J \wedge J]. \quad (3.20)$$

Using (A.7), (B.3) and (3.17), we compute the Hodge duals

$$* \frac{1}{2}(df \wedge J \wedge J) = -d^c f \quad * df = -\frac{1}{2}d^c f \wedge J \wedge J, \quad (3.21)$$

On the other hand, using (3.3) and the anti-self-duality⁵ of B one finds

$$* dB = - *_4 d_2^c B = d^c B \quad (3.22)$$

with d_2 and d_2^c the restrictions of d, d^c to $\mathbb{C}$. Consequently

$$dC_4 - dB \wedge C_2 = \frac{1}{2}d^c[e^{\phi-4A}\hat{J} \wedge \hat{J}] \quad (3.23)$$

$$dC_2 - C_0 dB = e^{-\phi}d^c B \quad (3.24)$$

$$dC_0 = -d^c e^{-\phi} \quad (3.25)$$

where we have used (3.22) and that B is anti-self-dual with respect to $*_4$.

As noticed in [13], these equations can be written in the compact form

$$\bar{\partial}\mathcal{T} = 0 \quad (3.26)$$

with the holomorphic polyform

$$\mathcal{T} = e^{-B}(C + ie^{-\phi}\text{Re}[e^{-ij}]) \quad (3.27)$$

or, in components,

$$\mathcal{T}_0(z) = C_0 + ie^{-\phi} \quad (3.28)$$

$$\mathcal{T}_2(z) = C_2 - \tau B \quad (3.29)$$

$$\mathcal{T}_4(z) = C_4 - B \wedge C_2 + \frac{1}{2}\tau B \wedge B - \frac{i}{2}e^{-\phi}j \wedge j. \quad (3.30)$$

⁴For p -forms on even-dimensional spaces, we have $*^2 = -1$ for odd p , and $*^2 = +1$ for even p .

⁵Notice that for a two-form $*_4 = \hat{*}_4$, i.e. self-duality with respect to the warped and flat metrics associated to j and $\hat{j}$ are equivalent.

Let us take $B = b_a \chi_a^-$, $C_2 = c_a \chi_a^-$, $\mathcal{T}_2 = \beta^{(a)} \chi_a^-$ with χ_a^- a basis of anti-self-dual two-forms on M_4 that satisfy

$$\chi_a^- \wedge \chi_b^- = -2\delta_{ab} dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4. \quad (3.31)$$

A supersymmetric solution is then specified by the set of holomorphic functions

$$\begin{aligned} \tau(z) &= C_0 + i e^{-\phi} \\ \beta^{(a)}(z) &= c_a - \tau b_a \\ \sigma(z) &= -c - 2b_a c_a + \tau b_a^2 + i e^{\phi-4A}, \end{aligned} \quad (3.32)$$

where we have used that $j \wedge j = 2e^{2\phi-4A} dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4$. We conclude that a supersymmetric solution is specified by $(b_2^- + 2)$ -holomorphic functions (where b_2^- is the number of globally defined anti-self-dual two-forms on M_4) characterising the fluxes, and a choice of warped metric for M_4 .

3.2.1 Example A: 5 holomorphic functions

As an example we can consider $M_4 = T^4$ with trivial $SU(2)$ structure and $K = e^D h dz$, which give the $SU(3)$ structure forms

$$\begin{aligned} \Omega_3 &= e^{\phi-3A} h dz \wedge (dy^1 + i dy^4) \wedge (dy^2 + i dy^3) \\ J &= e^{\phi-2A} [dy^1 \wedge dy^4 + dy^2 \wedge dy^3] + \frac{i}{2} e^{2D} |h|^2 dz \wedge d\bar{z}. \end{aligned} \quad (3.33)$$

For this choice $\frac{1}{2}\Omega_2 \wedge \bar{\Omega}_2 = j \wedge j$ implies $D = -A$ and a basis of anti-selfdual two-forms can be taken to be

$$\chi_a^- = \{dy^1 \wedge dy^2 - dy^3 \wedge dy^4, dy^1 \wedge dy^3 + dy^2 \wedge dy^4, dy^1 \wedge dy^4 - dy^2 \wedge dy^3\}. \quad (3.34)$$

The solution is then parametrised by five holomorphic functions

$$\tau = \tau_1 + i\tau_2, \quad \sigma = \sigma_1 + i\sigma_2, \quad \beta^{(a)} = \beta_1^{(a)} + i\beta_2^{(a)}, \quad a = 1, 2, 3,$$

and can be written as

$$\begin{aligned} ds^2 &= e^{2A} \sum_{\mu=0}^3 dx^\mu dx_\mu + e^{\phi-2A} \sum_{m,n=1}^4 \delta_{mn} dy^m dy^n + e^{-2A} |h(z)|^2 dz d\bar{z} \\ e^{-\phi} &= \tau_2, \quad C_0 = \tau_1, \\ B &= -\frac{1}{\tau_2} \beta_2^{(a)} \chi_a^-, \quad C_2 = \left(\beta_1^{(a)} - \frac{\tau_1}{\tau_2} \beta_2^{(a)} \right) \chi_a^-, \\ C_4 &= \left(-\sigma_1 + \frac{2}{\tau_2} \vec{\beta}_1 \cdot \vec{\beta}_2 - \frac{\tau_1}{\tau_2^2} \vec{\beta}_2 \cdot \vec{\beta}_2 \right) dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4, \end{aligned} \quad (3.35)$$

with $\vec{\beta}_i \cdot \vec{\beta}_j = \sum_a \beta_i^{(a)} \beta_j^{(a)}$ and

$$e^{-2A} = \sqrt{\sigma_2 \tau_2 - \vec{\beta}_2 \cdot \vec{\beta}_2}. \quad (3.36)$$

The metric has been computed by inserting J and Ω_3 into (B.4) and (B.5). In appendix C, we rederive this solution by directly solving the equations (2.4)–(2.5) for the ten-dimensional Killing spinors $\epsilon^{1,2}$.

3.3 Solution class B

In the case $\alpha = \pi/2$, the two spinors η^A are orthogonal. The pure spinors are

$$\Phi_1 = -\frac{1}{8}K e^{-ij} \quad \Phi_2 = -\frac{i}{8}\Omega_2 e^{\frac{1}{2}K \wedge \bar{K}} \quad (3.37)$$

where the $SU(2)$ structure forms K, j and Ω_2 are defined in (B.6) and we set $\theta = 0$ since this phase can be trivially reabsorbed in the definition of Ω_2 .

The supersymmetry equations (3.7)–(3.9) then require

$$d_H(e^{3A-\phi} K e^{-ij}) = 0, \quad (3.38)$$

$$d_H(e^{2A-\phi} \text{Im}\Omega_2) = 0, \quad (3.39)$$

$$d_H(e^{4A-\phi} \text{Re}\Omega_2) = e^{4A} * \lambda(F), \quad (3.40)$$

where we use that $K \wedge \bar{K}$ is closed and $d\chi \wedge K \wedge \bar{K} = 0$ for any form χ that is closed on M_4 . The first equation implies

$$d(e^{3A-\phi} K) = K \wedge d(B + ij) = 0 \quad (3.41)$$

that is solved by taking

$$K = e^{\phi-3A} h(z) dz \quad d_4 j = 0 \quad B + ij = \gamma(z) \quad (3.42)$$

with $h(z)$ and $\gamma(z)$ holomorphic zero and two-forms respectively. Equation (3.39) implies

$$d(e^{2A-\phi} \text{Im}\Omega_2) = H \wedge \text{Im}\Omega_2 = 0. \quad (3.43)$$

Since B is parallel to j in order to satisfy (3.42), the second constraint is automatic. The first may be solved by

$$\Omega_2 = e^{\phi-2A} \hat{\Omega}_2 \quad (3.44)$$

with $d\hat{\Omega}_2 = 0$. The third supersymmetry equation (3.40) decomposes to

$$F_1 = dC_0 = 0 \quad (3.45)$$

$$F_3 = d(C_2 - BC_0) = e^{-4A} * d(e^{4A-\phi} \text{Re}\Omega_2) = d^c(e^{-2A} *_4 \text{Re}\hat{\Omega}_2) \quad (3.46)$$

$$F_5 = dC_4 - H \wedge C_2 = 0. \quad (3.47)$$

Using that Ω_2 is self-dual, $*_4\Omega_2 = \Omega_2$, this is solved by

$$-C_2 + C_0 B + i e^{-2A} \text{Re}\hat{\Omega}_2 = \rho(z) \quad (3.48)$$

$$C_4 = \frac{1}{2} C_0 B \wedge B \quad (3.49)$$

with C_0 a constant and $\rho(z)$ a holomorphic two-form. Writing $B = b_a \chi_a$, $j = j_a \chi_a$, $\gamma = \gamma_a \chi_a$ and $C_2 - C_0 B = -c \text{Re}\hat{\Omega}_2$, the solution is specified by the holomorphic functions

$$\rho(z) = c + i e^{-2A} \quad \gamma_a(z) = b_a + i j_a \quad (3.50)$$

In the case $M_4 = T^4$, after fixing a complex two-form $\hat{\Omega}_2$, the j_a span a four-dimensional space orthogonal to $\hat{\Omega}_2$. The dilaton is fixed by the $SU(2)$ condition $j \wedge j = \frac{1}{2} \Omega_2 \wedge \bar{\Omega}_2$. The solution is then specified by five holomorphic functions, one from ρ and four from the γ_a 's.

3.3.1 Example B: 4 holomorphic functions

As an example of solution in the B-class we can take $M_4 = T^4$, $\rho(z) = i$, i.e. $c = A = 0$, and

$$\begin{aligned}\Omega_2 &= e^\phi(dy^1 + i dy^4) \wedge (dy^2 + i dy^3) \\ j &= \tau_2 dy^1 \wedge dy^4 + \sigma_2 dy^2 \wedge dy^3 - \beta_2^{(1)}(dy^1 \wedge dy^3 + dy^2 \wedge dy^4) \\ &\quad + \beta_2^{(2)}(dy^1 \wedge dy^2 - dy^3 \wedge dy^4) .\end{aligned}\tag{3.51}$$

The condition $j \wedge j = \frac{1}{2}\Omega_2 \wedge \bar{\Omega}_2$ implies

$$e^{2\phi} = \sigma_2 \tau_2 - \vec{\beta}_2 \cdot \vec{\beta}_2\tag{3.52}$$

Plugging (3.51) into (3.5)–(3.6) (or the corresponding J and Ω_3 into (B.4)–(B.5)) one finds for the metric on T^4

$$g_{mn} = \begin{pmatrix} \tau_2 & -\beta_2^{(1)} & -\beta_2^{(2)} & 0 \\ -\beta_2^{(1)} & \sigma_2 & 0 & \beta_2^{(2)} \\ -\beta_2^{(2)} & 0 & \sigma_2 & -\beta_2^{(1)} \\ 0 & \beta_2^{(2)} & -\beta_2^{(1)} & \tau_2 \end{pmatrix}\tag{3.53}$$

The solution becomes

$$\begin{aligned}ds^2 &= \sum_{\mu=0}^3 dx^\mu dx_\mu + \sum_{m,n=1}^4 g_{mn} \delta y^m \delta y^n + e^{2\phi} |h(z)|^2 dz d\bar{z} , \\ B &= \tau_1 dy^1 \wedge dy^4 + \sigma_1 dy^2 \wedge dy^3 - \beta_1^{(1)}(dy^1 \wedge dy^3 + dy^2 \wedge dy^4) \\ &\quad + \beta_1^{(2)}(dy^1 \wedge dy^2 - dy^3 \wedge dy^4) , \\ C_0 &= 0 , \quad C_2 = 0 , \quad C_4 = 0 .\end{aligned}\tag{3.54}$$

3.4 Solution class C

Finally, we consider the case $\alpha = 0$ and $\theta = \pi$. Like in case A, the two spinors η^A are parallel but now the relative phase is simply a sign. The pure spinors are

$$\Phi_1 = \frac{1}{8}\Omega_3 \quad , \quad \Phi_2 = -\frac{1}{8}e^{-iJ} .\tag{3.55}$$

The supersymmetry equations (3.7)–(3.9) thus require

$$d_H(e^{3A-\phi}\Omega_3) = 0 ,\tag{3.56}$$

$$d_H(e^{2A-\phi}\text{Re}[e^{-iJ}]) = 0 ,\tag{3.57}$$

$$d_H(e^{4A-\phi}\text{Im}[e^{-iJ}]) = -e^{4A} * \lambda(F) .\tag{3.58}$$

The first two equations imply

$$d(e^{3A-\phi}\Omega_3) = d(e^{2A-\phi}) = H = dJ \wedge J = 0\tag{3.59}$$

of which the first three constraints can be solved by taking

$$\phi = 2A \quad H = 0 \quad \Omega_3 = e^{-A} \hat{\Omega}_3\tag{3.60}$$

with $\delta\hat{\Omega}_3 = 0$. The six-dimensional manifold is then warped complex but need not be Kähler. Using K from (3.1) in (3.11), we conclude that $\Omega_2 = e^{-A-D}\hat{\Omega}_2$ with $\hat{\Omega}_2$ a closed two-form varying holomorphically along the $\mathbb{C}$ -plane. On the other hand the last equation in (3.59) implies

$$0 = dJ \wedge J = dj \wedge j - \frac{i}{2}dj \wedge K \wedge \bar{K} \quad \Leftrightarrow \quad dj \wedge j = 0, \quad d_4j = 0, \quad (3.61)$$

so j is a closed form on M_4 that varies with z in such a way to keep $j \wedge j$ constant. The equation $dj \wedge j = 0$, or equivalently $d^c j \wedge j = 0$, can be solved⁶ by taking $d^c j$ anti-self-dual with respect to $*_4$

$$*_4 d^c j = -d^c j. \quad (3.62)$$

On the other hand, from $j \wedge j = \frac{1}{2}\Omega_2 \wedge \bar{\Omega}_2$ we conclude that

$$D = -A \quad \Rightarrow \quad \Omega_2 = \hat{\Omega}_2 \quad (3.63)$$

Finally the third supersymmetry equation in (3.58) decomposes into (recall that $H = 0$)

$$F_1 = dC_0 = 0 \quad (3.64)$$

$$F_3 = dC_2 = -e^{-4A} * d(e^{2A}j) = -d^c(e^{-2A}) \wedge j - e^{-2A} * dj = -d^c(e^{-2A}j) \quad (3.65)$$

$$F_5 = dC_4 = 0. \quad (3.66)$$

where we used (3.3) and (3.62). Eqs. (3.64)–(3.66) can then be solved by taking C_0, C_4 constant and

$$\gamma = C_2 + i e^{-2A} j \quad (3.67)$$

holomorphic. We recall that j is a two-form orthogonal to Ω_2 that satisfies $d(j \wedge j) = 0$. Just as discussed above for solutions of type B, when $M_4 = T^4$ we can expand j in a four-dimensional basis of two-forms orthogonal to Ω_2 . Writing $C_2 = c_a \chi_a$ we build four holomorphic functions

$$\gamma_a = c_a + i e^{-2A} j_a. \quad (3.68)$$

The flux content of general solutions in this class is then characterised by four holomorphic functions. Additionally, there may be holomorphic functions that parametrise Ω_2 .

3.4.1 Example C: 4 holomorphic functions

As an example of solution in the C-class we choose $M_4 = T^4$ and

$$\begin{aligned} \Omega_2 &= (dy^1 + i dy^4) \wedge (dy^2 + i dy^3) \\ j &= e^{2A} [\tau_2 dy^1 \wedge dy^4 + \sigma_2 dy^2 \wedge dy^3 - \beta_2^{(1)} (dy^1 \wedge dy^3 + dy^2 \wedge dy^4) \\ &\quad + \beta_2^{(2)} (dy^1 \wedge dy^2 - dy^3 \wedge dy^4)] \end{aligned} \quad (3.69)$$

The condition $j \wedge j = \frac{1}{2}\Omega_2 \wedge \bar{\Omega}_2$ implies

$$e^{-2A} = \sqrt{\sigma_2 \tau_2 - \vec{\beta}_2 \cdot \vec{\beta}_2}. \quad (3.70)$$

We recall that $\phi = 2A = -2D$.

⁶For $M_4 = T^4$ or $M^4 = K3$, which have three-dimensional bases of self-dual two-forms, this is the most general solution, since $d^c j \wedge j = d^c j \wedge \Omega_2 = 0$ implies that $d^c j$ is anti-selfdual with respect to $*_4$.

The T^4 metric computed from (3.5) and (3.6) is

$$g_{mn} = e^{2A} \begin{pmatrix} \tau_2 & -\beta_2^{(1)} & -\beta_2^{(2)} & 0 \\ -\beta_2^{(1)} & \sigma_2 & 0 & \beta_2^{(2)} \\ -\beta_2^{(2)} & 0 & \sigma_2 & -\beta_2^{(1)} \\ 0 & \beta_2^{(2)} & -\beta_2^{(1)} & \tau_2 \end{pmatrix} \quad (3.71)$$

and the solution becomes

$$\begin{aligned} ds^2 &= e^{2A} \sum_{\mu=0}^3 dx^\mu dx_\mu + \sum_{m,n=1}^4 g_{mn} \delta y^m \delta y^n + e^{-2A} |h(z)|^2 dz d\bar{z}, \\ C_2 &= \tau_1 dy^1 \wedge dy^4 + \sigma_1 dy^2 \wedge dy^3 - \beta_1^{(1)} (dy^1 \wedge dy^3 + dy^2 \wedge dy^4) \\ &\quad + \beta_1^{(2)} (dy^1 \wedge dy^2 - dy^3 \wedge dy^4), \\ C_0 &= 0, \quad C_2 = 0, \quad C_4 = 0. \end{aligned} \quad (3.72)$$

3.5 Relations between local solutions

In the preceding sections, we presented three types of supersymmetric local solutions to type IIB supergravity. For each class, we displayed an example of solutions on T^4 characterised by 4 holomorphic functions. These three solutions can be related to each other acting with T- and S-dualities.

Under T-duality along a direction y , the metric in the string frame and the NSNS/RR fields transform as [49–51]:⁷

$$\begin{aligned} g'_{yy} &= \frac{1}{g_{yy}}, & e^{2\phi'} &= \frac{e^{2\phi}}{g_{yy}}, & g'_{ym} &= \frac{B_{ym}}{g_{yy}}, & B'_{ym} &= \frac{g_{ym}}{g_{yy}} \\ g'_{mn} &= g_{mn} - \frac{g_{my}g_{ny} - B_{my}B_{ny}}{g_{yy}}, & B'_{mn} &= B_{mn} - \frac{B_{my}g_{ny} - g_{my}B_{ny}}{g_{yy}} \\ C'_{m\dots n\alpha y} &= C_{m\dots n\alpha} - (n-1) \frac{C_{[m\dots n|y}g_{y|\alpha]}}{g_{yy}} \\ C'_{m\dots n\alpha\beta} &= C_{m\dots n\alpha\beta y} - nC_{[m\dots n\alpha}B_{\beta]y} - n(n-1) \frac{C_{[m\dots n|y}B_{|\alpha|y}g_{|\beta]y}}{g_{yy}} \end{aligned} \quad (3.73)$$

On the other hand, under S-duality for backgrounds with $C_0 = 0$ is

$$\phi' = -\phi \quad g' = e^{-\phi} g \quad C'_2 = -B \quad B' = C_2. \quad (3.74)$$

Using these formulas one can check that solutions in sections 3.2.1, 3.3.1 and 3.4.1 are related by the duality maps

$$A \xleftrightarrow{T_{14}} C \xleftrightarrow{S} B \quad (3.75)$$

if $\beta^{(3)}$ is set to zero in section 3.2.1. It is important to notice that unlike in the case of three holomorphic solutions studied in the companion paper [1], solutions with four holomorphic functions cannot be mapped to purely metric backgrounds using dualities. Indeed, a simple inspection of (3.54) shows that B have legs long all 6 two-cycles of the T^4 so there is no way so translate it into metric via T-dualities.

⁷Our conventions are such that $B \rightarrow -B$, $B' \rightarrow -B'$ with respect to [51].

4 Conclusions and outlook

In this paper, we presented explicit solutions where the ten-dimensional spacetime takes the local form $\mathbb{R}^{1,3} \times M_4 \times \Sigma$, with M_4 a generalised complex manifold with $SU(2)$ structure and Σ an open subset of $\mathbb{C}$. The metric, dilaton, NSNS and RR potentials are oriented along M_4 and assumed to vary only along $\mathbb{C}$. We display explicit examples for $M_4 = T^4$ specified by up to four holomorphic functions. These solutions can be viewed as supersymmetric solutions of $\mathcal{N} = (2, 2)$ maximal supergravity in six dimensions with a set of scalar fields varying over the z -plane.

This theory is parametrised by a scalar manifold

$$\mathcal{M}_{IIB \text{ on } T^4} = SO(5, 5, \mathbb{Z}) \backslash \frac{SO(5, 5, \mathbb{R})}{SO(5, \mathbb{R}) \times SO(5, \mathbb{R})} \quad (4.1)$$

of dimension 25: 9 fields parametrise the symmetric and traceless metric on T^4 , 2×6 fields correspond to NSNS and RR two-forms and 4 fields are related to the dilaton, the T^4 -volume, the RR zero- and four-forms. The holomorphic functions $\varphi_I(z)$ characterising the local solutions span a complex sub-manifold of (4.1). For example, for solutions of class A with conformally flat metric, the holomorphic functions span the $n = 1, \dots, 5$ -complex dimensional submanifold

$$\mathcal{M}_{BPS} = SO(2, n, \mathbb{Z}) \backslash \frac{SO(2, n, \mathbb{R})}{SO(2, \mathbb{R}) \times SO(n, \mathbb{R})} \subset \mathcal{M}_{IIB \text{ on } T^4} . \quad (4.2)$$

Explicit solutions from class B and C with $n = 1, \dots, 4$ were constructed in 3.3.1, 3.4.1. They are U-dual versions of solutions in class A, and thus share the same moduli space (4.2); in this case, the three solution classes correspond to different orientations of $\mathcal{M}_{BPS}$ inside $\mathcal{M}_{IIB \text{ on } T^4}$.

The moduli space (4.2) is isomorphic to the moduli space $\mathcal{M}_{K3, n}$ of complex structures for an algebraic $K3$ surface with Picard number $20 - n$. The holomorphic functions characterising the flux solutions can then be viewed as the complex structure of an auxiliary $K3$ surface varying holomorphically over a plane. A consistent fibration of the $K3$ surface then defines a fully consistent, non-perturbative flux solution of type IIB supergravity using results in [1, 12, 13]. In particular, a compact Calabi-Yau threefold composed of a $K3$ surface fibered over $\mathbb{CP}^1$ can be used to construct a four-dimensional vacuum of type IIB supergravity with non-trivial fluxes. In the global solution, the local solutions are glued together using U-dualities to cover the whole complex plane $\mathbb{C}$. Branes are associated to singular points in $\mathbb{CP}^1$ where the complex structure of the $K3$ fiber degenerates and around which the holomorphic functions defining the local solutions have non-trivial U-duality monodromies. For a critical number of branes, the two-dimensional metric can be chosen to be regular at infinity, thus compactifying the $\mathbb{C}$ plane into $\mathbb{CP}^1$. In this procedure, fluxes translate into geometry and we can exploit the well developed techniques of algebraic geometry to find new supersymmetric flux vacua. The reader is referred to the accompanying paper [1], where the details of the flux/geometry dictionary are studied and discussed in great detail.

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A Conventions

We use $(M, N, P, Q \dots)$ to index ten-dimensional quantities, $(m, n, p, q \dots)$ in the internal six dimensions and $(\mu, \nu, \rho, \dots)$ for the four space-time dimensions. Flat tangent space indices will sometimes be used, and we denote them with a hat: $\hat{M}, \hat{m}$ etc.

Gamma matrices. Ten-dimensional Dirac matrices are denoted by Γ^M , and six-dimensional ones by γ^m . We will choose the latter to be hermitian, $\gamma_{\hat{m}}^\dagger = \gamma_{\hat{m}}$, imaginary and antisymmetric. All Dirac matrices satisfy the Clifford algebra (e.g. in six dimensions $\{\gamma_m, \gamma_n\} = 2g_{mn}$) and totally antisymmetric products of gamma matrices are denoted $\gamma_{m_1 m_2 \dots m_k}$, where e.g.

$$\gamma_{mn} = \frac{1}{2}[\gamma_m, \gamma_n]. \quad (\text{A.1})$$

The chirality operator in d dimensions is given by

$$\gamma_{d+1} = \text{i}^{-d/2} \gamma_{\hat{m}_1 \dots \hat{m}_d} = \text{i}^{-d/2} \frac{1}{\sqrt{|g|}} \gamma_{m_1 \dots m_d}, \quad (\text{A.2})$$

where we use hatted letters for flat tangent space indices. The eigenvalues of γ_{d+1} are ± 1 (-1) for chiral (antichiral) spinors. Thus, a six-dimensional spinor η can be decomposed into chiral and anti-chiral components $\eta_\pm$, where $\gamma_7 \eta_\pm = \pm \eta_\pm$ ($\eta_- = \eta_+^*$). Without loss of generality, we will take $\eta_\pm$ to have unit norm, $\eta_+^\dagger \eta_+ = \eta_-^\dagger \eta_- = 1$.

The two-dimensional Pauli matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & \text{i} \\ -\text{i} & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.3})$$

Differential forms and Hodge duals. We define the components of a differential p -form by

$$A = \frac{1}{p!} A_{m_1 \dots m_p} dx^{m_1} \wedge \dots \wedge dx^{m_p}, \quad (\text{A.4})$$

The contraction of a q -form with a p -form ($p > q$) is

$$B \lrcorner A = \frac{1}{(p-q)!} B^{m_1 \dots m_q} A_{m_1 \dots m_p} dx^{m_{q+1}} \wedge \dots \wedge dx^{m_p}. \quad (\text{A.5})$$

Our convention for the Hodge star operation $*$, when acting on a p -form, is

$$*A = \frac{\sqrt{|g|}}{p!(d-p)!} \epsilon_{m_1 \dots m_{d-p}}^{n_1 \dots n_p} A_{n_1 \dots n_p} dx^{m_1} \wedge \dots \wedge dx^{m_{d-p}}. \quad (\text{A.6})$$

with $\epsilon_{1\dots d} = 1$, cf. [48, 52]. Another very useful relation is the combined identity

$$*(B \wedge A) = \frac{(-1)^q}{q!} B_{\lrcorner}(*A), \quad (\text{A.7})$$

where A is a p -form, B a q -form, and $(d-p) > q$. Further relevant identities can be found in [53].

The six-dimensional exterior derivative can be decomposed into holomorphic and antiholomorphic parts, and with local holomorphic coordinates z^a , $a = 1, 2, 3$, we have

$$d = \partial + \bar{\partial} \quad d^c = i(\bar{\partial} - \partial) \quad (\text{A.8})$$

with

$$\partial = dz^a \frac{\partial}{\partial z^a} \quad \bar{\partial} = d\bar{z}^a \frac{\partial}{\partial \bar{z}^a}. \quad (\text{A.9})$$

The 2d Hodge dual satisfies

$$*_2 dz = i dz. \quad *_2 1 = d\text{vol}_2 = \sqrt{|g_2|} dz \wedge d\bar{z}. \quad (\text{A.10})$$

B Spinors, structure groups and pure spinors

B.1 SU(3) and SU(2) structures

If a six-dimensional manifold admits a nowhere vanishing spinor η its structure group is reduced to SU(3). Another way to express this constraint is in terms of differential forms. The spinor can also be used to define a set of pure O(6,6) spinors. In this appendix, we briefly review these different formalisms, in order to pave the way for the analysis of the local supersymmetry conditions of type IIB compactifications.

In string compactifications to four dimensions, supersymmetry requires the existence on M_6 of at least one globally defined and nowhere vanishing spinor η . The six-dimensional spinors $\eta_{\pm}$ can be used to build a nowhere vanishing real two-form J and a complex decomposable three-form Ω_3 on the six-dimensional manifold⁸

$$J_{mn} = -i\eta_+^\dagger \gamma_{mn} \eta_+ \quad \Omega_{mnp} = -i\eta_-^\dagger \gamma_{mnp} \eta_+ \quad (\text{B.1})$$

For manifolds of strict SU(3) structure, i.e. those for which η is unique, these are the only nowhere vanishing forms can be defined on M_6 .⁹ J and Ω_3 are subject to the constraints

$$J \wedge \Omega_3 = 0 \quad \frac{1}{6} J \wedge J \wedge J = \frac{i}{8} \Omega_3 \wedge \bar{\Omega}_3 = d\text{vol}_6. \quad (\text{B.2})$$

⁸Gamma matrices are in our conventions imaginary and complex.

⁹In particular, there are no globally defined one-forms on M_6 ; for spinors $\eta_{1,2}$ of the same chirality, bilinears $\eta_1^\dagger \gamma^{m_1 \dots m_k} \eta_2$ vanish for odd k , and $\eta_{\pm}^\dagger \gamma^m \eta_{\pm} = \eta_{\pm}^T \gamma^m \eta_{\pm} = 0$ follows from the antisymmetry of γ^m .

The Hodge duals of J and Ω_3 are

$$*J = \frac{1}{2} J \wedge J \quad , \quad *\Omega_3 = -i\Omega_3 . \quad (\text{B.3})$$

One can show that

$$I^p{}_n = c \epsilon^{pm_1\dots m_5} (\text{Re } \Omega_3)_{nm_1m_2} (\text{Re } \Omega_3)_{m_3m_4m_5} , \quad (\text{B.4})$$

satisfies $I_m{}^p I_p{}^n = -\delta_m^n$ for a given normalisation constant c . The matrix I thus defines an almost complex structure [54] (see also section 3.1 in [55]). Moreover, the contraction of I with J gives a metric

$$g_{mn} = -J_{mp} I^p{}_n . \quad (\text{B.5})$$

On a Calabi-Yau manifold, J and Ω_3 are closed and I is an integrable complex structure. The Ricci-flat Calabi-Yau metric is given by (B.5) once the Kähler form and holomorphic top-form have been correctly identified in the cohomology classes of J and Ω_3 .

If a six-dimensional manifold allows two orthogonal nowhere-vanishing spinors, η and χ , its structure group is further reduced to $\text{SU}(2)$. Again, without loss of generality, we take the chiral and antichiral parts of the spinors to have unit norm. The $\text{SU}(2)$ structure is characterised by the existence of a nowhere vanishing complex one-form K , a real two-form j , and a complex two-form Ω_2 given by:

$$K_m = \eta_-^\dagger \gamma_m \chi_+ \quad j_{mn} = -i\eta_+^\dagger \gamma_{mn} \eta_+ + i\chi_+^\dagger \gamma_{mn} \chi_+ \quad \Omega_{2mn} = \eta_-^\dagger \gamma_{mn} \chi_- \quad (\text{B.6})$$

The $\text{SU}(2)$ structure can be embedded into the $\text{SU}(3)$ via the relations

$$J = j + \frac{i}{2} K \wedge \bar{K} \quad , \quad \Omega_3 = K \wedge \Omega_2 . \quad (\text{B.7})$$

Using these relations and (B.2), it is straightforward to show the $\text{SU}(2)$ structure relations

$$j \wedge \Omega_2 = 0 \quad , \quad j \wedge j = \frac{1}{2} \Omega_2 \wedge \bar{\Omega}_2 \quad , \quad K \lrcorner j = K \lrcorner \Omega_2 = 0 . \quad (\text{B.8})$$

and vice versa these conditions imply that J and Ω_3 given by (B.7) is an $\text{SU}(3)$ structure (using the fact that $|K|^2 = 2$, which follows from the unit norm of χ_+).

B.2 Pure spinors

For manifolds with $\text{SU}(2)$ structure, one can write

$$\eta_+^1 = a\eta_+ \quad \eta_+^2 = b(\cos \alpha \eta_+ + \sin \alpha \chi_+) , \quad (\text{B.9})$$

In presence of D-branes the modulus of the two spinor should match,¹⁰ and supersymmetry requires $|a|^2 = |b|^2 = e^A$. We write $a = |a|e^{i\theta_1}$, $b = |b|e^{i\theta_2}$. The parameter α interpolates

¹⁰This follows from the fact that D-brane boundary conditions relate left and right moving spinors. In particular for a Dp-brane one finds $\epsilon_1 = \hat{\Gamma}^{i_0\dots i_p} \epsilon_2$.

between strict SU(3) ($\alpha = 0$, parallel spinors) and SU(2) ($\alpha = \pi/2$, orthogonal spinors) structures. The O(3) spinors $\eta_{1,2}$ can be used to define two pure O(6,6) spinors

$$\Phi_{\pm} = \frac{1}{|a|^2} \eta_{\pm}^1 \eta_{\pm}^{2\dagger} = \frac{1}{8|a|^2} \sum_{k=0}^6 \frac{1}{k!} \eta_{\pm}^{2\dagger} \gamma_{m_k \dots m_1} \eta_{\pm}^1 \gamma^{m_1 \dots m_k} \quad (\text{B.10})$$

The right hand side of (B.10) can be thought as a polyform via the Clifford map

$$\gamma^{m_1 m_2 \dots m_k} \longleftrightarrow dx^{m_1} \wedge dx^{m_2} \wedge \dots \wedge dx^{m_k} \quad (\text{B.11})$$

In particular bilinears made out of spinors of the same (different) chirality lead to odd (even) forms.

The various contributions can be written as

$$\begin{aligned} \chi_+ &= \frac{1}{2} K_m \gamma^m \eta_- \\ \eta_+ \eta_+^\dagger &= \frac{1}{8} e^{-ij + \frac{1}{2} K \wedge \bar{K}} \\ \chi_- \eta_-^\dagger &= \frac{1}{8} \Omega_2 e^{\frac{1}{2} K \wedge \bar{K}}. \end{aligned} \quad (\text{B.12})$$

Equations (B.12) are equivalent to (B.6) and can be used as an alternative definition of an SU(2) structure. The equivalence between the two can be shown by multiplying the last two relations in (B.12) by $(\mathbb{I}, \gamma_m, \gamma_{mn}, \dots, \gamma_{mnpqrs})$ and tracing over spinor indices; for the first relation one should multiply χ_+ in (B.12) with $\eta_-^\dagger \gamma_n$ from the left.

Plugging (B.12) into (B.10) one finds [47, 48]

$$\begin{aligned} \Phi_- &= -\frac{e^{i\theta_+}}{8} K \wedge (\sin \alpha e^{-ij} + i \cos \alpha \Omega_2) \\ \Phi_+ &= \frac{e^{i\theta_-}}{8} e^{\frac{1}{2} K \wedge \bar{K}} (\cos \alpha e^{-ij} - i \sin \alpha \Omega_2), \end{aligned} \quad (\text{B.13})$$

with $\theta_{\pm} = \theta_1 \pm \theta_2$. The phase θ_+ can be reabsorbed into the definition of K so we discard this phase and rename $\theta_- = -\theta$ in the main text. By specifying α and θ , we find different supersymmetric solutions (see section 3).

C Explicit solution for the Killing spinor

In this appendix, we present explicit local type A solutions of the KSE (2.4)–(2.5) that are parametrised by up to five holomorphic functions. It can be checked that this solution satisfies the equations of motion of type IIB supergravity, and we have done so using Mathematica.

We start from the ansatz

$$\begin{aligned} ds^2 &= e^{2A} \sum_{\mu=0}^3 dx^\mu dx_\mu + e^{\phi-2A} \sum_{m,n=1}^4 \delta_{mn} dy^m dy^n + e^{2D} |h(z)|^2 dz d\bar{z} \\ B &= b_a \chi_a^- \quad C_2 = c_a \chi_a^- \quad C_4 = c dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 \end{aligned} \quad (\text{C.1})$$

where we choose a basis of anti-self forms on T^4 as in (3.34)

$$\chi_a^- = \{dy^1 \wedge dy^2 - dy^3 \wedge dy^4, dy^1 \wedge dy^3 + dy^2 \wedge dy^4, dy^1 \wedge dy^4 - dy^2 \wedge dy^3\} \quad (\text{C.2})$$

The KSE are then solved by Killing spinors ϵ^i , that satisfy

$$\epsilon^2 = i\epsilon^1 \quad \Gamma^z \epsilon^1 = 0 \quad \Gamma^{12} \epsilon^1 = \Gamma^{34} \epsilon^1, \quad (\text{C.3})$$

with ϵ_0 a constant spinor. The condition $\Gamma^{12} \epsilon^1 = \Gamma^{34} \epsilon^1$, together with the anti-self-duality of B and C_2 implies that

$$B_{mn} \Gamma^{mn} \epsilon = C_{mn} \Gamma^{mn} \epsilon = 0, \quad (\text{C.4})$$

where ϵ is the two-component vector $(\epsilon^1, \epsilon^2)^T$. The dilatino equation then reduces to the holomorphicity condition on the axio-dilaton field,

$$\bar{\partial} \tau = \bar{\partial}(C_0 + i e^{-\phi}) = 0 \quad (\text{C.5})$$

in agreement with the pure spinor analysis in section 3.2.

The vanishing of gravitino variations split into two conditions

$$\left(-H_{mn\bar{z}} \Gamma^{n\bar{z}} + \frac{i}{2} e^{\phi} F_{\bar{z}np} \Gamma^{\bar{z}np} \Gamma_m \right) \epsilon^1 = 0 \quad (\text{C.6})$$

$$\left(\nabla_m - i \frac{e^{\phi}}{8} \sum_{n=1,5} \frac{1}{(n)!} F_{P_1 \dots P_n} \Gamma^{P_1 \dots P_n} \Gamma_m \right) \epsilon^1 = 0 \quad (\text{C.7})$$

Equation (C.6) for $m = y^i$ give us

$$\left(-\bar{\partial} B_{y^i n} \Gamma^{n\bar{z}} + \frac{i}{2} e^{\phi} (\bar{\partial} C_{np} - C_0 \bar{\partial} B_{np}) \Gamma^{\bar{z}np} \Gamma_{y^i} \right) \epsilon^1 = 0 \quad (\text{C.8})$$

Writing

$$\Gamma^{\bar{z}np} \Gamma_{y^i} = \{ \Gamma^{\bar{z}np}, \Gamma_{y^i} \} - \Gamma_{y^i} \Gamma^{\bar{z}np} = 2 \Gamma^{\bar{z}[n} \delta_i^{p]} - \Gamma_{y^i} \Gamma^{\bar{z}np} \quad (\text{C.9})$$

and using (C.4) to discard the contribution of the last term in (C.9) one finds

$$i e^{\phi} \bar{\partial} (C_{ny^i} - \tau B_{ny^i}) \Gamma^{\bar{z}n} \epsilon^1 = 0 \quad (\text{C.10})$$

that implies

$$\bar{\partial} (C_{mn} - \tau B_{mn}) = 0 \quad (\text{C.11})$$

Let us consider now (C.7),

$$\left(\frac{1}{4} \omega_{mnp} \Gamma^{np} - i \frac{e^{\phi}}{8} \left(F_n \Gamma^n + \frac{1}{5!} F_{mnopq} \Gamma^{mnopq} \right) \Gamma_m \right) \epsilon^1 = 0 \quad (\text{C.12})$$

The non-trivial components of the spin connection are

$$\begin{aligned} \omega_{y^i y^i \bar{z}} &= \bar{\partial}(e^{\phi-2A}) \\ \omega_{zz\bar{z}} &= w_{\bar{z}\bar{z}z}^* = \frac{1}{2} \bar{\partial} \ln \frac{(\sigma_2 \tau_2 - \vec{\beta}_2^2)}{|h|^2} \end{aligned} \quad (\text{C.13})$$

Plugging this into (C.12), for $m = y^i$ one finds

$$\left(2i\bar{\partial}(e^{\phi-2A}) + e^{2\phi-4A}(\bar{\partial}C_0 - e^{4A-2\phi}F_{\bar{z}1234})\right)\Gamma^{y^i\bar{z}}\epsilon^1 = 0 \quad (\text{C.14})$$

where we used $\Gamma^{1234}\epsilon^1 = e^{4A-2\phi}\epsilon^1$ and $\Gamma_{y^i} = e^{\phi-2A}\Gamma^{y^i}$. Writing $\bar{\partial}C_0 = -i\bar{\partial}e^{-\phi}$ one finds

$$i\bar{\partial}e^{\phi-4A} - F_{1234\bar{z}} = 0 \quad (\text{C.15})$$

or equivalently

$$\bar{\partial}(C_4 - B \wedge C_2 + \frac{1}{2}\tau B \wedge B - ie^{\phi-4A}d^4y) = 0. \quad (\text{C.16})$$

Thus, all three conditions in (3.26) are reproduced.

Finally taking $m = z$ in (C.7) one finds the differential equation

$$\begin{aligned} \left(\partial_{\bar{z}} - \frac{1}{8}\partial_{\bar{z}}\ln\frac{\sigma_2\tau_2 - \vec{\beta}_2^2}{|h|^2}\right)\epsilon^1 &= 0, \\ \left(\partial_z + \frac{1}{8}\partial_z\ln\frac{\sigma_2\tau_2 - \vec{\beta}_2^2}{|h|^2}\right)\epsilon^1 &= 0, \end{aligned} \quad (\text{C.17})$$

where we use the compact notation $\vec{\beta}_2^2 = \sum_a(\beta_2^{(a)})^2$. These equations are solved by

$$\epsilon^1 = \left(\frac{\bar{h}(\bar{z})}{h(z)(\sigma_2\tau_2 - \vec{\beta}_2^2)^{1/2}}\right)^{1/4} \epsilon_0. \quad (\text{C.18})$$

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