The shortened KK spectrum of IIB supergravity on $Y^{p,q}$

Arash Arabi Ardehali,^a James T. Liu^a and Phillip Szepietowski^b

^a*Michigan Center for Theoretical Physics, Randall Laboratory of Physics, University of Michigan, Ann Arbor, MI 48109–1040, U.S.A.*

^b*Department of Physics, University of Virginia, Box 400714, Charlottesville, VA 22904, U.S.A.*

E-mail: ardehali@umich.edu, jimliu@umich.edu, pgs8b@virginia.edu

ABSTRACT: We examine the shortened KK spectrum of IIB supergravity compactified on $Y^{p,q}$ and conjecture that the spectrum we have obtained is complete. The (untwisted) shortened spectrum on $S^5/\mathbb{Z}_{2p}$ and on $T^{1,1}/\mathbb{Z}_p$ are obtained as special cases when $p = q$ and $q = 0$, respectively. Knowledge of the shortened spectrum allows us to compute the superconformal index of these theories and to find agreement with earlier calculations from the dual field theories. We also employ the shortened spectrum to perform a $1/N^2$ test of AdS/CFT by holographically reproducing the difference of the central charges, $c - a = p/8$, of the dual CFTs.

KEYWORDS: Supergravity Models, AdS-CFT Correspondence, Anomalies in Field and String Theories

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1 Introduction

About a decade ago a new avenue was opened in the exploration of AdS/CFT with reduced supersymmetry by the discovery of an infinite family of Sasaki-Einstein five-manifolds $Y^{p,q}$ [1, 2] and the construction of their dual four-dimensional quiver gauge theories [3, 4]. Various checks of the duality had been performed successfully, the most notable of which being perhaps the matching of the large- N conformal anomalies and the spectrum of baryonic states [4, 5].

However, although many other properties of the family were known from the field theory side, such as the spectrum of the mesonic states [6] and the superconformal index [7], the gravitational computations were obstructed by the difficulty in obtaining the KK spectrum of IIB supergravity on the $Y^{p,q}$ manifolds. In particular, the scalar Laplacian on $Y^{p,q}$ leads to a Heun equation [8–10] whose exact spectrum is not known.

Despite the difficulty in finding the full spectrum of the Heun equation in question, some harmonics were found and identified with their dual mesonic states in [9]. Here we extend the available results by finding the shortened KK spectrum of IIB supergravity compactified on $Y^{p,q}$. We conjecture that this spectrum is complete and perform two checks involving AdS/CFT. This shortened spectrum is the main result of this work and can be found in tables 3 and 4.

The shortened KK spectrum enables us to see the spectrum of all the protected single-trace operators from the gravity side. It also allows us to compute the superconformal index from supergravity and find matching with the earlier field theoretical computation of [7].

Another use of the shortened spectrum is in holographically reproducing the difference of the gauge theory central charges $c - a$ [11]. This is a rather non-trivial test of AdS/CFT beyond large- N . Our success in reproducing the field theory value, $c - a = p/8$, from the shortened spectrum on the gravity side helps to address some of the issues raised in [11].

This paper is organized as follows. Section 2 reviews the KK spectrum of IIB supergravity compactified on a generic Sasaki-Einstein 5-manifold and the possible multiplet shortening patterns. In section 3 we present the shortened KK spectrum of IIB supergravity on $Y^{p,q}$. Section 4 illustrates how the mesonic chiral ring of the dual field theories is mapped to the supergravity states. In section 5 we demonstrate that the untwisted shortened spectrum on $S^5/\mathbb{Z}_{2p}$ and $T^{1,1}/\mathbb{Z}_p$ may be obtained from the shortened spectrum presented here upon setting $p = q$ or $q = 0$, respectively. The superconformal index is computed in section 6, and the holographic $c - a$ is computed in section 7. The closing section includes comments on how the results of the present paper shed light on the issues concerning the $1/N^2$ corrections to the holographic Weyl anomaly raised in [11].

2 The KK spectrum of IIB supergravity on SE_5

A generic compactification of IIB supergravity on $AdS_5 \times SE_5$ yields $\mathcal{N} = 2$ gauged supergravity coupled to a KK tower that can be arranged into $\mathcal{N} = 2$ representations of the $SU(2, 2|1)$ supergroup.

The compactification on S^5 preserves $\mathcal{N} = 8$ supersymmetry, and the KK spectrum was obtained in [12, 13]. The result is particularly simple when given in terms of shortened representations of $SU(2, 2|4)$; at level p ($p \geq 2$), the states transform under the representation $\mathcal{D}(p, 0, 0; 0, p, 0)$, where we have used the notation $\mathcal{D}(E_0, s_1, s_2; l_1, l_2, l_3)$ where (l_1, l_2, l_3) are the Dynkin labels of the $SU(4)_R$ representation.

Subsequently the KK spectroscopy for $T^{1,1}$ was investigated in [14, 15]. The resulting spectrum was given in terms of nine generic KK multiplets — Graviton, Gravitinos I through IV, and Vectors I through IV — along with a Betti vector and Betti hypermultiplet. It was then shown in [16] that this decomposition in terms of nine generic multiplets persists for general $\mathcal{N} = 2$ compactifications. The full spectrum consists of these generic multiplets along with the possible addition of special KK multiplets and Betti multiplets.

In fact, the analysis of [14–16] demonstrates that the generic KK tower can be obtained solely from knowledge of the eigenvalues of the scalar Laplacian on SE_5 . Essentially, the vector and tensor harmonics needed in the decomposition of IIB fields on SE_5 may be related to a combination of scalar harmonics and invariant tensors related to the structure of the manifold. Hence information from the scalar harmonics is sufficient.

It is convenient to define the eigenvalues of the scalar Laplacian on SE_5 according to

$$\square Y = -e_0(e_0 + 4)Y, \quad (2.1)$$

Supermultiplet	Representation	e_0 condition
Graviton	$\mathcal{D}(e_0 + 3, \frac{1}{2}, \frac{1}{2}; r)$	$e_0 \geq 0$
Gravitino I and III	$\mathcal{D}(e_0 + \frac{3}{2}, \frac{1}{2}, 0; r + 1) + \mathcal{D}(e_0 + \frac{3}{2}, 0, \frac{1}{2}; r - 1)$	$e_0 > 0$
Gravitino II and IV	$\mathcal{D}(e_0 + \frac{9}{2}, \frac{1}{2}, 0; r - 1) + \mathcal{D}(e_0 + \frac{9}{2}, 0, \frac{1}{2}; r + 1)$	$e_0 \geq 0$
Vector I	$\mathcal{D}(e_0, 0, 0; r)$	$e_0 > 0$
Vector II	$\mathcal{D}(e_0 + 6, 0, 0; r)$	$e_0 \geq 0$
Vector III and IV	$\mathcal{D}(e_0 + 3, 0, 0; r - 2) + \mathcal{D}(e_0 + 3, 0, 0; r + 2)$	$e_0 \geq 0$

Table 1. The generic $\mathcal{N} = 2$ spectrum of IIB supergravity on SE_5 . The spectrum is given in terms of the eigenvalue e_0 of the scalar Laplacian and the R -charge r .

where we take $e_0 \geq 0$. Note that the eigenvalues e_0 will depend on the R -charge as well as other global quantum numbers on SE_5 . Moreover, it was shown in [16] that e_0 satisfies the bound

$$e_0 \geq \frac{3}{2}r. \quad (2.2)$$

As an example, for $T^{1,1}$, we have

$$e_0(e_0 + 4) = 6[j(j + 1) + \ell(\ell + 1) - r^2/8], \quad (2.3)$$

where (j, ℓ, r) labels the representation under the isometry group $SU(2)_j \times SU(2)_\ell \times U(1)_r$ of $T^{1,1}$, and the R -charge satisfies the bound

$$|r| \leq 2 \min(j, \ell). \quad (2.4)$$

The $e_0 \geq \frac{3}{2}r$ bound is saturated when $j = \ell = |r|/2$.

In general, the isometry group may be different. However, the conserved $U(1)_r$ will always be present, as demanded by $\mathcal{N} = 2$ supersymmetry. Thus the KK spectrum can be arranged into representations of $SU(2, 2|1)$ based on the e_0 and r eigenvalues of the scalar Laplacian. The generic KK spectrum is given in table 1. In addition to the generic spectrum, there may be KK towers of special multiplets as well as a finite number of Betti multiplets. The former correspond to two-forms $\omega \in H_{\partial_B}^{1,1}(SE_5)$, and the latter to $\omega \in H^2(SE_5)$ [16].

The scalar Laplacian always admits a constant mode on SE_5 , with corresponding eigenvalues $e_0 = 0$ and $r = 0$. Truncating to $e_0 = 0$ gives the zero mode spectrum shown in table 2. These are the modes that may be retained in the consistent truncation on any squashed Sasaki-Einstein manifold [17–20].

2.1 Multiplet shortening

In general, the spectrum in table 1 fill out long multiplets. However, the multiplets will be shortened whenever some of the unitarity bounds become saturated [21, 22] (see also [23]).

Supermultiplet	Representation	Name given in [18]
Graviton	$\mathcal{D}(3, \frac{1}{2}, \frac{1}{2}; 0)$	supergraviton
Gravitino II and IV	$\mathcal{D}(\frac{9}{2}, \frac{1}{2}, 0; -1) + \mathcal{D}(\frac{9}{2}, 0, \frac{1}{2}, 1)$	LH+RH massive gravitino
Vector II	$\mathcal{D}(6, 0, 0; 0)$	massive vector
Vector III and IV	$\mathcal{D}(3, 0, 0; -2) + \mathcal{D}(3, 0, 0; 2)$	LH+RH chiral

Table 2. The $e_0 = 0$ multiplets that may be retained in a consistent truncation.

There are three multiplet shortening conditions

$$\begin{aligned}
&\text{conserved:} & E_0 = 2 + s_1 + s_2, & \frac{3}{2}r = s_1 - s_2, \\
&\text{chiral (anti-chiral):} & E_0 = \frac{3}{2}r & \left(E_0 = -\frac{3}{2}r \right), \\
&\text{semi-long I (semi-long II):} & E_0 = 2 + 2s_1 - \frac{3}{2}r & \left(E_0 = 2 + 2s_2 + \frac{3}{2}r \right). \quad (2.5)
\end{aligned}$$

Note that the conserved multiplet can be thought of as satisfying the semi-long I and II conditions simultaneously.

We now impose these shortening conditions on the generic KK spectrum in table 1. Making note of the restriction $e_0 \geq \frac{3}{2}|r|$, we find that shortening occurs only under the conditions

$$i) \quad e_0 = \frac{3}{2}|r| \quad \text{or} \quad ii) \quad e_0 = \frac{3}{2}|r| + 2. \quad (2.6)$$

These two possibilities were noted in [14, 15] in the case of $T^{1,1}$. The possible shortenings are given in table 3. Note that $e_0 = \frac{3}{2}r$ ($e_0 = -\frac{3}{2}r$) corresponds to holomorphic (antiholomorphic) functions on the Calabi-Yau cone over SE_5 [16].

3 The shortened spectrum of IIB supergravity on $Y^{p,q}$

The scalar Laplacian on $Y^{p,q}$ and aspects of the spectrum have been investigated in [8–10, 24, 25]. Much of the difficulty in obtaining the full spectrum is due to the fact that, although the Laplacian is separable, one ends up with a second order equation of Heun type. This arises because of the cubic function $q(y) = b - 3y^2 + 2y^3$ that shows up in the metric. It turns out, however, that the Heun equation admits simple solutions once the shortening conditions (2.6) are imposed.

We follow the general analysis of [9] and refer the reader to that reference for additional notation and conventions. The isometry group of $Y^{p,q}$ is $SU(2)_j \times U(1)_\alpha \times U(1)_r$, and there are three commuting Killing vectors, which may be taken to be $\partial/\partial\phi$, $\partial/\partial\psi$ and $\partial/\partial\alpha$. This suggests that we look for solutions to the scalar equation (2.1) of the separable form

$$Y(y, \theta, \phi, \psi, \alpha) = e^{i(N_\phi\phi + N_\psi\psi + N_\alpha\alpha/l)} R(y) \Theta(\theta). \quad (3.1)$$

Note that the R -charge is given by

$$r = 2N_\psi - \frac{N_\alpha}{3l}, \quad (3.2)$$

Multiplet	Representation	Shortening condition		Shortening type
Graviton	$\mathcal{D}(e_0 + 3, \frac{1}{2}, \frac{1}{2}; r)$	$e_0 = 0$ $e_0 = -\frac{3}{2}r$ $e_0 = \frac{3}{2}r$	$r = 0$ $r < 0$ $r > 0$	conserved SLI SLII
Gravitino I	$\mathcal{D}(e_0 + \frac{3}{2}, \frac{1}{2}, 0; r + 1)$	$(e_0 = 1)$ $e_0 = \frac{3}{2}r$ $e_0 = -\frac{3}{2}r$ $e_0 = \frac{3}{2}r + 2$	$(r = -\frac{2}{3})$ $r > 0$ $r < -\frac{2}{3}$ $r > -\frac{2}{3}$	(conserved) chiral SLI SLII
Gravitino II	$\mathcal{D}(e_0 + \frac{9}{2}, \frac{1}{2}, 0; r - 1)$	$e_0 = -\frac{3}{2}r$	$r \leq 0$	SLI
Gravitino III	$\mathcal{D}(e_0 + \frac{3}{2}, 0, \frac{1}{2}; r - 1)$	$(e_0 = 1)$ $e_0 = -\frac{3}{2}r$ $e_0 = \frac{3}{2}r$ $e_0 = -\frac{3}{2}r + 2$	$(r = \frac{2}{3})$ $r < 0$ $r > \frac{2}{3}$ $r < \frac{2}{3}$	(conserved) anti-chiral SLII SLI
Gravitino IV	$\mathcal{D}(e_0 + \frac{9}{2}, 0, \frac{1}{2}; r + 1)$	$e_0 = \frac{3}{2}r$	$r \geq 0$	SLII
Vector I	$\mathcal{D}(e_0, 0, 0; r)$	$e_0 = 2$ $e_0 = \frac{3}{2}r$ $e_0 = -\frac{3}{2}r$ $e_0 = -\frac{3}{2}r + 2$ $e_0 = \frac{3}{2}r + 2$	$r = 0$ $r \geq \frac{2}{3}$ $r \leq -\frac{2}{3}$ $r < 0$ $r > 0$	conserved chiral anti-chiral SLI SLII
Vector II	$\mathcal{D}(e_0 + 6, 0, 0; r)$	—	—	—
Vector III	$\mathcal{D}(e_0 + 3, 0, 0; r - 2)$	$e_0 = -\frac{3}{2}r$ $e_0 = -\frac{3}{2}r + 2$	$r \leq 0$ $r \leq \frac{2}{3}$	anti-chiral SLI
Vector IV	$\mathcal{D}(e_0 + 3, 0, 0; r + 2)$	$e_0 = \frac{3}{2}r$ $e_0 = \frac{3}{2}r + 2$	$r \geq 0$ $r \geq -\frac{2}{3}$	chiral SLII

Table 3. The generic shortening structure. For a given e_0 and r satisfying the shortening condition, there may be an additional degeneracy associated with the global symmetries of SE_5 . The conserved gravitinos are present only if the compactification preserves ≥ 16 real supercharges. Vector Multiplet II is never shortened.

where

$$\frac{1}{l} = \frac{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}{q}. \quad (3.3)$$

Here Θ satisfies the equation

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} (N_\phi + N_\psi \cos \theta)^2 + j(j+1) - N_\psi^2 \right] \Theta = 0, \quad (3.4)$$

and the equation for R can be transformed to the standard form of Heun's equation.

We first consider the Θ equation. This may be solved in terms of the Jacobi polynomials $P_n^{(\alpha, \beta)}$

$$\Theta = \left(\sin \frac{\theta}{2}\right)^{|N_\phi + N_\psi|} \left(\cos \frac{\theta}{2}\right)^{|N_\phi - N_\psi|} P_{j - |N_\phi + N_\psi|/2 - |N_\phi - N_\psi|/2}^{(|N_\phi + N_\psi|, |N_\phi - N_\psi|)}(\cos \theta), \quad (3.5)$$

where

$$j \geq \max(|N_\phi|, |N_\psi|), \quad (3.6)$$

and either

$$\{j, N_\phi, N_\psi\} \in \mathbb{Z} \quad \text{or} \quad \{j, N_\phi, N_\psi\} \in \mathbb{Z} + \frac{1}{2}. \quad (3.7)$$

These conditions ensure regularity at $\theta = 0$ and π .

For the $R(y)$ equation, it may be converted to a standard Heun form by taking [9]

$$x = \frac{y - y_1}{y_2 - y_1}, \quad (3.8)$$

where y_1, y_2 and y_3 are the roots of the cubic $q(y)$ ordered from smallest to largest. (The physical range of y is $y_1 \leq y \leq y_2$.) Then

$$R = x^{\alpha_1} (1 - x)^{\alpha_2} (a - x)^{\alpha_3} h(x), \quad (3.9)$$

where $h(x)$ satisfies Heun's equation

$$h''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) h'(x) + \frac{\alpha\beta x - k}{x(x-1)(x-a)} h(x) = 0. \quad (3.10)$$

The exponents α_i are given by

$$\begin{aligned} \alpha_1 &= \frac{1}{4} \left| r - N_\alpha \left(p + q - \frac{1}{3l} \right) \right|, \\ \alpha_2 &= \frac{1}{4} \left| r - N_\alpha \left(-p + q - \frac{1}{3l} \right) \right|, \\ \alpha_3 &= \frac{1}{4} \left| r - N_\alpha \left(-2q + \frac{2}{3l} \right) \right|. \end{aligned} \quad (3.11)$$

The parameters in Heun's equation are

$$\begin{aligned} \alpha &= -\frac{1}{2}e_0 + \alpha_1 + \alpha_2 + \alpha_3, & \beta &= 2 + \frac{1}{2}e_0 + \alpha_1 + \alpha_2 + \alpha_3, \\ \gamma &= 1 + 2\alpha_1, & \delta &= 1 + 2\alpha_2, & \epsilon &= 1 + 2\alpha_3, \end{aligned} \quad (3.12)$$

along with

$$\begin{aligned} a &= \frac{q + \sqrt{4p^2 - 3q^2}}{2q}, \\ k &= (\alpha_1 + \alpha_3)(1 + \alpha_1 + \alpha_3) - \alpha_2^2 + a \left[(\alpha_1 + \alpha_2)(1 + \alpha_1 + \alpha_2) - \alpha_3^2 \right] \\ &\quad - \frac{p}{q} \left[\frac{1}{12} \left(1 + \frac{q}{p}(1 + a) \right) e_0(e_0 + 4) - j(j + 1) + \frac{1}{16} \left(\frac{2}{3} \frac{N_\alpha}{l} - r \right)^2 \right]. \end{aligned} \quad (3.13)$$

Before we proceed to examine the shortening conditions, note that constant solutions $h(x)$ require $\alpha\beta = 0$ and $k = 0$. For $e_0 \geq 0$, this reduces to

$$h(x) = 1 \quad \Leftrightarrow \quad \alpha = 0, \quad k = 0. \quad (3.14)$$

We also note that a linear solution requires $(\alpha+1)(\beta+1) = 0$, along with a more complicated condition on k that may be written as

$$(k + \gamma + \epsilon)(k + a(\gamma + \delta)) - a\delta\epsilon = 0. \quad (3.15)$$

3.1 The shortening condition $e_0 = \frac{3}{2}|r|$

We first consider the case $e_0 = \frac{3}{2}r$. Examining α , we find

$$\begin{aligned} 2\alpha &= -\frac{3}{2}r + 2(\alpha_1 + \alpha_2 + \alpha_3) \\ &= -\left(3N_\psi - N_\alpha \frac{1}{2l}\right) + \left|N_\psi - N_\alpha \frac{p+q}{2}\right| + \left|N_\psi - N_\alpha \frac{-p+q}{2}\right| + \left|N_\psi - N_\alpha \left(-q + \frac{1}{2l}\right)\right|. \end{aligned} \quad (3.16)$$

If we were to ignore the absolute values, then this expression simply yields $\alpha = 0$. This suggests that $e_0 = \frac{3}{2}r$ shortening corresponds to constant solutions to Heun's equation. Following through on this conjecture, we see that the absolute values are such that α vanishes whenever

$$N_\psi \geq \max\left(N_\alpha \frac{p+q}{2}, -N_\alpha \frac{p-q}{2}, N_\alpha \left(\frac{1}{2l} - q\right)\right). \quad (3.17)$$

Since the third quantity lies between the first two, it does not provide any further restriction in the inequality. Which of the first two quantities is greater depends on the sign of N_α , and we find

$$\begin{aligned} N_\psi &\geq N_\alpha \frac{p+q}{2} && \text{for } N_\alpha \geq 0; \\ N_\psi &\geq (-N_\alpha) \frac{p-q}{2} && \text{for } N_\alpha \leq 0, \end{aligned} \quad (3.18)$$

as a necessary condition for obtaining a constant solution.

For a constant solution to exist, we must also demand $k = 0$. Assuming that N_ψ satisfies the condition (3.18), which corresponds to simply dropping the absolute values in (3.11), we find

$$k = \frac{p}{q}(j - N_\psi)(j + 1 + N_\psi). \quad (3.19)$$

In this case, $k = 0$ corresponds to either $j = N_\psi$ or $j = -(N_\psi + 1)$. Since both j and N_ψ are non-negative, we conclude that $j = N_\psi$ is required. Putting everything together then gives

$$\text{For } e_0 = \frac{3}{2}r : \quad j = N_\psi \geq 0 \quad \text{and} \quad -N_\psi \frac{2}{p-q} \leq N_\alpha \leq N_\psi \frac{2}{p+q}. \quad (3.20)$$

Note that, for a given j , we must also include the $(2j+1)$ -fold degeneracy of the $SU(2)$ harmonics, corresponding to $-j \leq N_\phi \leq j$.

We now consider how the U(1) quantum numbers $\{N_\phi, N_\psi, N_\alpha\}$ are quantized. Based on the periodicities of the U(1) circles, all three quantities are integer spaced. However, there may be a shift imposed by regularity at the poles of the spheres in the $Y^{p,q}$ metric. For $\theta = 0$ and π , the regularity condition for the SU(2) harmonics was given in (3.7). However, we must also consider regularity at $y = y_1$ and y_2 . To examine this, we note that the α circle is given by

$$d\left(\frac{\alpha}{l}\right) + \frac{b - 2y + y^2}{6(b - y^2)}(d\psi - \cos\theta d\phi). \quad (3.21)$$

At $y = y_1$, this becomes

$$d\left(\frac{\alpha}{l}\right) + \frac{p + q}{2}(d\psi - \cos\theta d\phi), \quad (3.22)$$

while at $y = y_2$, this becomes

$$d\left(\frac{\alpha}{l}\right) - \frac{p - q}{2}(d\psi - \cos\theta d\phi). \quad (3.23)$$

Thus at $y = y_1$, the natural U(1) coordinate is $\alpha/l + (p + q)\psi/2$, and at $y = y_2$, the natural coordinate is $\alpha/l - (p - q)\psi/2$. While N_α remains integral, this shifts the quantization of N_ψ , depending on whether $p + q$ is even or odd. For $p + q$ even, we have

$$N_\alpha \in \mathbb{Z} \quad \text{and} \quad N_\psi \in \mathbb{Z}, \quad (3.24)$$

and for $p + q$ odd, we have

$$\begin{aligned} N_\alpha \in 2\mathbb{Z} \quad \text{and} \quad N_\psi \in \mathbb{Z}, \\ \text{or} \quad N_\alpha \in 2\mathbb{Z} + 1 \quad \text{and} \quad N_\psi \in \mathbb{Z} + \frac{1}{2}. \end{aligned} \quad (3.25)$$

Note that these quantization conditions apply not only to the solutions with the shortening condition $e_0 = \frac{3}{2}|r|$, but also to the ones with $e_0 = \frac{3}{2}|r| + 2$ discussed below.

As noted in [8], the constant solutions to Heun's equation correspond to holomorphic functions on the Calabi-Yau cone. Since these are the only functions that saturate the bound $e_0 \geq \frac{3}{2}r$ [16], we conclude that the identification of constant solutions with the shortening condition $e_0 = \frac{3}{2}r$ is complete.

In the above, we have considered positive R -charge. The negative r modes may be obtained by taking the complex conjugate. The result is

$$\text{For } e_0 = -\frac{3}{2}r : \quad j = -N_\psi \geq 0 \quad \text{and} \quad N_\psi \frac{2}{p + q} \leq N_\alpha \leq -N_\psi \frac{2}{p - q}. \quad (3.26)$$

Before proceeding to the other type of shortening, it is worth noting that from (3.20), (3.24) and (3.25), it is not difficult to convince oneself that, except for $Y^{1,0} = T^{1,1}$, there are no shortened multiplets in the spectrum of $Y^{p,q}$ manifolds that have $e_0 < 2$. Thus, $T^{1,1}$ is the only member of the family of $Y^{p,q}$ manifolds that has shortened multiplets with scalar fields that need to be quantized with Neumann boundary conditions in AdS₅.

3.2 The shortening condition $e_0 = \frac{3}{2}|r| + 2$

We now consider the case $e_0 = \frac{3}{2}r + 2$. Again, we start with α . Since e_0 is increased by 2, the expression (3.16) becomes

$$2\alpha = -\left(2 + 3N_\psi - N_\alpha \frac{1}{2l}\right) + \left|N_\psi - N_\alpha \frac{p+q}{2}\right| + \left|N_\psi - N_\alpha \frac{-p+q}{2}\right| + \left|N_\psi - N_\alpha \left(-q + \frac{1}{2l}\right)\right|. \quad (3.27)$$

If we impose the conditions (3.18), we would now obtain $\alpha = -1$, which suggests a linear solution to Heun's equation. However, there is a second possibility that $\alpha = 0$, which gives a constant solution under the right conditions.

We first consider $\alpha = -1$. Based on (3.18), the Heun parameters are

$$\begin{aligned} \alpha &= -1, & \beta &= 3 + 3N_\psi - \frac{N_\alpha}{2l}, \\ \gamma &= 1 + N_\psi - N_\alpha \frac{p+q}{2}, & \delta &= 1 + N_\psi + N_\alpha \frac{p-q}{2}, & \epsilon &= 1 + N_\psi - N_\alpha \left(-q + \frac{1}{2l}\right), \end{aligned} \quad (3.28)$$

Substituting this into (3.15) then gives

$$\frac{p^2}{q^2}(j - N_\psi)(j - N_\psi - 1)(j + 1 + N_\psi)(j + 2 + N_\psi) = 0. \quad (3.29)$$

Since $N_\psi \geq 0$, this is solved by either $j = N_\psi$ or $j = N_\psi + 1$.

The second possibility, $\alpha = 0$, may be obtained by relaxing the conditions (3.18), so that one of the arguments inside the absolute values in (3.27) becomes negative. There are two possibilities, depending on the sign of N_α

$$\begin{aligned} N_\alpha &> 0, & N_\psi &= N_\alpha \frac{p+q}{2} - 1, \\ \text{or} & & N_\alpha &< 0, & N_\psi &= -N_\alpha \frac{p-q}{2} - 1. \end{aligned} \quad (3.30)$$

Note that these cases must be restricted to $N_\psi \geq 0$. In particular, this indicates that $|N_\alpha| \geq 2$ for $Y^{1,0}$. Furthermore, the negative N_α case must be restricted to $N_\alpha \leq -2$ for $Y^{p,p-1}$, and does not exist for $Y^{p,p}$. In all cases, we find that $J = N_\psi$ is required to give $k = 0$, and hence a constant solution to Heun's equation.

Combining the $\alpha = 0$ and $\alpha = -1$ possibilities, we find that the $e_0 = \frac{3}{2}r + 2$ states are given by

$$\begin{aligned} \text{For } e_0 = \frac{3}{2}r + 2: & \quad i) \ j = N_\psi \geq 0 \quad \text{and} \quad -(N_\psi + 1) \frac{2}{p-q} \leq N_\alpha \leq (N_\psi + 1) \frac{2}{p+q}, \\ & \quad ii) \ j = N_\psi + 1 \geq 1 \quad \text{and} \quad -N_\psi \frac{2}{p-q} \leq N_\alpha \leq N_\psi \frac{2}{p+q}. \end{aligned} \quad (3.31)$$

The $e_0 = -\frac{3}{2}r + 2$ states can be obtained by complex conjugation.

We have found that the shortening conditions $e_0 = \frac{3}{2}|r|$ and $e_0 = \frac{3}{2}|r| + 2$ are satisfied by a combination of constant and linear solutions to Heun's equation. For $e_0 = \frac{3}{2}|r|$, the solutions are complete, while for $e_0 = \frac{3}{2}|r| + 2$ we have been unable to prove that these

Shortening	Condition for j	Condition for N_α	Condition for N_ψ
$e_0 = \frac{3}{2}r$	$j = N_\psi$	$-N_\psi \frac{2}{p-q} \leq N_\alpha \leq N_\psi \frac{2}{p+q}$	$N_\psi \geq 0$
$e_0 = -\frac{3}{2}r$	$j = -N_\psi$	$N_\psi \frac{2}{p+q} \leq N_\alpha \leq -N_\psi \frac{2}{p-q}$	$N_\psi \leq 0$
$e_0 = \frac{3}{2}r + 2$	$j = N_\psi$	$-(N_\psi + 1) \frac{2}{p-q} \leq N_\alpha \leq (N_\psi + 1) \frac{2}{p+q}$	$N_\psi \geq 0$
	$j = N_\psi + 1$	$-N_\psi \frac{2}{p-q} \leq N_\alpha \leq N_\psi \frac{2}{p+q}$	$N_\psi \geq 0$
$e_0 = -\frac{3}{2}r + 2$	$j = -N_\psi$	$-(-N_\psi + 1) \frac{2}{p+q} \leq N_\alpha \leq (-N_\psi + 1) \frac{2}{p-q}$	$N_\psi \leq 0$
	$j = -N_\psi + 1$	$N_\psi \frac{2}{p+q} \leq N_\alpha \leq -N_\psi \frac{2}{p-q}$	$N_\psi \leq 0$

Table 4. The scalar eigenstates leading to the shortened spectrum on $Y^{p,q}$. The quantum numbers N_ψ and N_α are restricted to satisfy the conditions (3.24) and (3.25). r is given by (3.2). Each state is $(2j+1)$ -fold degenerate, corresponding to the allowed values of $|N_\phi| \leq j$.

are the only possible solutions. Nevertheless, the evidence that we present below for the superconformal index and for $c-a$ strongly suggests that these solutions are complete. We have also checked this numerically for the low-lying spectrum with $j \leq 5$ and $e_0 \leq 15$ for $Y^{p,q}$ with $0 < q < p < 10$.

These shortening conditions are summarized in table 4. The generic shortened spectrum of IIB supergravity on $Y^{p,q}$ (with $p > q > 0$) is then obtained by combining these scalar eigenvalues with the shortening structure given in table 3. The complete shortened KK spectrum is given by these generic towers along with Betti multiplets that are in one to one correspondence to the ones present in the case of $T^{1,1}$ [14, 15]. This is because $Y^{p,q}$ and $T^{1,1}$ share the same topology $S^2 \times S^3$. In principle, the spectrum on a generic Sasaki-Einstein manifold may include special multiplets. However, these do not arise in the case of $Y^{p,q}$. Finally, note that, other than the Betti vector multiplet, there are four conserved vector multiplets (one with $j = 0$ and three with $j = 1$) in the spectrum related to the global isometry group $SU(2) \times U(1)$.

4 AdS/CFT state-operator correspondence in the mesonic chiral ring

The full mesonic chiral ring of the dual field theories were constructed in [6]. The gauge theory dual to $Y^{p,q}$ was realized to have three types of mesonic blocks denoted by $\mathcal{S}$, $\mathcal{L}_+$, and $\mathcal{L}_-$. The product of $\mathcal{L}_+$ and $\mathcal{L}_-$ results in an operator of the form $\mathcal{S}^p$. Thus a general mesonic chiral BPS operator $\mathcal{O}_{s,N_\alpha}$ can be written as

$$\mathcal{O}_{s,N_\alpha} = \mathcal{S}^s \mathcal{L}^{N_\alpha}, \quad (4.1)$$

where $\mathcal{L} = \mathcal{L}_+$ and $\mathcal{L}^{-1} = \mathcal{L}_-$. According to [6] the operator constructed in this way has an R-charge

$$Q_R[\mathcal{O}_{s,N_\alpha}] = 2s + p|N_\alpha| + N_\alpha \left(q - \frac{1}{3l} \right), \quad (4.2)$$

and transforms in an irreducible $SU(2)$ representation with spin j

$$j[\mathcal{O}_{s,N_\alpha}] = s + |N_\alpha|\frac{p}{2} + N_\alpha\frac{q}{2}. \quad (4.3)$$

From (3.2) and (4.2) it is clear that one has to identify

$$N_\psi = s + |N_\alpha|\frac{p}{2} + N_\alpha\frac{q}{2}, \quad (4.4)$$

and therefore $N_\psi = j \geq 0$. Then from positivity of s the range of N_α in (3.20) follows. The quantization conditions (3.24) and (3.25) result by demanding s to be an integer.

This establishes that the full mesonic chiral ring of the quiver gauge theory is dual to the supergravity KK states with $e_0 = \frac{3}{2}r$ in the Vector I tower. Extending the analysis to all the protected operators and shortened multiplets is straightforward following [16].

5 Extension to $p = q$ and $q = 0$

The family of $Y^{p,q}$ manifolds, $p > q > 0$, is often formally extended to include the cases $p = q$ where $Y^{p,p} \equiv S^5/\mathbb{Z}_{2p}$ and $q = 0$ where $Y^{p,0} \equiv T^{1,1}/\mathbb{Z}_p$. Although the $Y^{p,q}$ metric is not well-defined for these values of p and q , such assignments are natural from the point of view of the toric diagrams [3]. Here we find that the shortened spectrum presented above gives in fact the untwisted shortened spectrum on $S^5/\mathbb{Z}_{2p}$ and $T^{1,1}/\mathbb{Z}_p$ as one specializes to $p = q$ and $q = 0$, respectively.¹ The reason for indicating untwisted here is because $S^5/\mathbb{Z}_{2p}$ also has a twisted sector that we will discuss in the next section.

The untwisted shortened spectrums on $S^5/\mathbb{Z}_{2p}$ and $T^{1,1}/\mathbb{Z}_p$ were discussed in [11]. To see that the shortened KK spectrum presented above is consistent with [11] when $p = q$ or $q = 0$, only a comparison of the conventions is required.

For $p = q$, we get $l = 1/2p$ and $r = 2N_\psi - \frac{2p}{3}N_\alpha$. Both N_ψ and N_α have integer quantization in this case since $p + q = 2p$ is even. To make contact with the spectrum presented in [11], one has to identify N_ψ and pN_α with $k/2$ and $-Q/2$, respectively, in table 3 of that work.²

For $q = 0$, we have $1/l = 0$ and $r = 2N_\psi$. To make contact with the spectrum, as discussed in [11], recall that $T^{1,1}$ has a $SU(2)_j \times SU(2)_l \times U(1)_r$ isometry. The $SU(2)_j$ is broken to $U(1)_j$ by the orbifold group $\mathbb{Z}_p$. After we identify $N_\psi (= j)$ with the $SU(2)_l$ quantum number and N_α with the $U(1)_j$ quantum number, the spectrum given here matches the one discussed in [11]. The fact that the quantization conditions of N_ψ and N_α are complicated according to (3.24) or (3.25), depending on p , is related to the complicated expressions that were obtained for the $\gamma_j^{(p)}$ in that work.

¹That this must be the case is obvious if one combines the AdS/CFT state-operator correspondence with the observation that the protected operators of the dual field theories are obtainable in the limiting cases $p = q$ and $q = 0$.

²Unlike in [11], here we are using Q for the q -charge of the orbifold multiplets to avoid confusion with the q in $Y^{p,q}$. For a similar reason the KK level will be denoted in the present paper by L , instead of p which was used in that work.

Shortening	Representation	$\mathcal{I}^R$	$\mathcal{I}^L$
conserved	$\mathcal{D}(E_0, s_1, s_2, r)$	$(-1)^{2(s_1+s_2)+1} t^{3r+6s_2+6} \chi_{s_1}(y)$	$(-1)^{2(s_1+s_2)+1} t^{-3r+6s_1+6} \chi_{s_2}(y)$
chiral	$\mathcal{D}(E_0, s_1, 0, r)$	$(-1)^{2s_1} t^{3r} \chi_{s_1}(y)$	0
anti-chiral	$\mathcal{D}(E_0, 0, s_2, r)$	0	$(-1)^{2s_2} t^{-3r} \chi_{s_2}(y)$
SLI	$\mathcal{D}(E_0, s_1, s_2, r)$	0	$(-1)^{2(s_1+s_2)+1} t^{-3r+6s_1+6} \chi_{s_2}(y)$
SLII	$\mathcal{D}(E_0, s_1, s_2, r)$	$(-1)^{2(s_1+s_2)+1} t^{3r+6s_2+6} \chi_{s_1}(y)$	0

Table 5. Contributions to the superconformal index from the various shortened multiplets, where $\chi_j(y)$ is the spin- j SU(2) character as defined in (6.3). While we focus on the right-handed index $\mathcal{I}^R$ in the text, we have included the contributions to the left-handed index $\mathcal{I}^L$ for completeness.

6 The superconformal index for $Y^{p,q}$

Knowledge of the shortened spectrum allows us to compute the superconformal index of IIB supergravity on $\text{AdS}_5 \times Y^{p,q}$. Recall that the superconformal index for an $\mathcal{N} = 1$ SCFT is defined as [26, 27]³

$$\mathcal{I}^R = \text{Tr}(-1)^F e^{-\beta\delta} t^{2(E+s_2)} y^{2s_1}, \quad (6.1)$$

where $\delta = E - \frac{3}{2}r - 2s_2$. Only states with $\delta = 0$ contribute to the index. This condition means that only states which lie within shortened representations of the superconformal algebra will contribute to the index. The index is therefore a protected quantity and is independent of the coupling. One may also refine the index to include chemical potentials for any global symmetries of the theory by defining

$$\mathcal{I}^R = \text{Tr}(-1)^F e^{-\beta\delta} t^{2(E+s_2)} y^{2s_1} \prod a_i^{2q_i}, \quad (6.2)$$

where the a_i are exponentiated chemical potentials and the q_i are charges under the relevant symmetries.

For holographic theories, the Kaluza-Klein tower provides access to the spectrum of single trace operators of the SCFT. We can thus use knowledge of the KK tower to compute the contribution to the index from these single trace operators. The contribution to the index from the different types of shortened supergravity multiplets is given in table 5.

In [7], the contribution to the index from single trace operators was computed in the large N limit of the quiver gauge theories dual to $\text{AdS}_5 \times Y^{p,q}$ using matrix model techniques. Lacking knowledge of the shortened supergravity spectrum for generic $Y^{p,q}$ a precise check with the supergravity result was done only for the case of $T^{1,1}$, for which the results of [7] were shown to be in agreement with the supergravity calculation of [7, 28].

Following this, [16] performed a general analysis for quiver gauge theories dual to $\text{AdS}_5 \times \text{SE}_5$ for arbitrary smooth Sasaki-Einstein manifolds. These authors understood the index of a quiver gauge theory as the Euler characteristic of the cyclic homology of

³As defined below, this is the “right-handed” index. One can also define a “left-handed” index $\mathcal{I}^L$ in which one replaces r with $-r$ and swaps s_1 and s_2 in both (6.1) and the definition of δ . Also, we focus on the single-particle index throughout this paper.

Ginzburg's dg algebra associated to the quiver, and related it to the Kohn-Rossi Cohomology of the dual internal geometry. This established the gauge/gravity matching of the index for arbitrary smooth Sasaki-Einstein manifolds. Specializing to the case where the Calabi-Yau cone over the Sasaki-Einstein manifold is toric (as is the case for the $Y^{p,q}$ geometries) they arrived at relations that required only combinatorial computations to obtain the index.⁴ However, an explicit expression for the index of the $Y^{p,q}$ theories computed purely from supergravity was not presented. In the following we will use our results for the shortened spectrum on $\text{AdS}_5 \times Y^{p,q}$ to compute the supergravity result for these geometries and compare to the explicit result of [7].

For the $Y^{p,q}$ theories we include chemical potentials a_1 and a_2 corresponding to the $\text{SU}(2)_j \times \text{U}(1)_\alpha$ global symmetries of $Y^{p,q}$. The accompanying charges are given by $q_1 = j$ and $q_2 = pN_\alpha/2$. The chemical potential for the $\text{SU}(2)_j$ has the effect of introducing a factor of $\chi_j(a_2)$, which is the spin- j $\text{SU}(2)$ character given by

$$\chi_j(x) = \sum_{k=0}^{2j} x^{2j-2k} = \frac{x^{2j+1} - x^{-(2j+1)}}{x - x^{-1}}. \quad (6.3)$$

We now move to evaluate the index. Referring to table 5, we see that only conserved, chiral and SLII multiplets contribute. As happens for the holographic $c - a$, the conserved multiplets contribute the same as evaluating the contribution from a semi-long multiplet at the appropriate level. Also note that the contribution from the Betti multiplets vanishes as noted for $T^{1,1}$ in [28]. To compute the index, we sum the contributions from all of the relevant multiplets in table 3 over the allowed values in table 4. Finally, we multiply each contribution by

$$\frac{1}{(1 - y^{-1}t^3)(1 - yt^3)}, \quad (6.4)$$

which takes into account the geometric sum arising from the contribution of the infinite set of operators constructed by acting with space-time derivatives on the bare operators.

After performing all of the sums we arrive, for $p > q$, at the following result for the evaluated index

$$\begin{aligned} \mathcal{I}^R = & \frac{a_1^p a_2^{p+q} t^{3(q+p)-1/l}}{1 - a_1^p a_2^{p+q} t^{3(q+p)-1/l}} + \frac{a_1^p a_2^{-p-q} t^{3(q+p)-1/l}}{1 - a_1^p a_2^{-p-q} t^{3(q+p)-1/l}} \\ & + \frac{a_1^{-p} a_2^{p-q} t^{-3(q-p)+1/l}}{1 - a_1^{-p} a_2^{p-q} t^{-3(q-p)+1/l}} + \frac{a_1^{-p} a_2^{-p+q} t^{-3(q-p)+1/l}}{1 - a_1^{-p} a_2^{-p+q} t^{-3(q-p)+1/l}}, \end{aligned} \quad (6.5)$$

which, upon replacing $1/l$ with (3.3), precisely agrees with the result of [7] for $\mathcal{I}^R$. This provides evidence that the shortened spectrum we have obtained above is complete. Although this result was strictly derived for $p > q$, eq. (6.5) also gives the correct expression for the index when $p = q > 1$, which corresponds to $S^5/\mathbb{Z}_{2p}$.⁵

The derivation of (6.5) for the case of $p = q$ requires extra care for two reasons. First, since these geometries are singular, there are twisted sector modes that must be taken into

⁴We would like to thank R. Eager for correspondence on this point.

⁵The case $p = 1$, corresponding to the $\mathbb{Z}_2$ orbifold, is a special case as it preserves $\mathcal{N} = 2$ supersymmetry and the index receives additional contributions.

KK level	Representation	Shortening type
$L = 0$	$\mathcal{D}(2, 0, 0; 0) \mathbf{1}_0$	conserved
	$\mathcal{D}(2, 0, 0; \frac{4}{3}) \mathbf{1}_{-2}$	chiral
	$\mathcal{D}(2, 0, 0; -\frac{4}{3}) \mathbf{1}_2$	anti-chiral
$L \geq 1$	$\mathcal{D}(L+1, 0, 0; \frac{2}{3}(L+1)) \mathbf{1}_{2L+2}$	chiral
	$\mathcal{D}(L+\frac{3}{2}, \frac{1}{2}, 0; \frac{2}{3}(L+\frac{3}{2})) \mathbf{1}_{2L}$	
	$\mathcal{D}(L+2, 0, 0; \frac{2}{3}(L+2)) \mathbf{1}_{2L-2}$	
	$\mathcal{D}(L+1, 0, 0; -\frac{2}{3}(L+1)) \mathbf{1}_{-2L-2}$	anti-chiral
	$\mathcal{D}(L+\frac{3}{2}, 0, \frac{1}{2}; -\frac{2}{3}(L+\frac{3}{2})) \mathbf{1}_{-2L}$	
	$\mathcal{D}(L+2, 0, 0; -\frac{2}{3}(L+2)) \mathbf{1}_{-2L+2}$	

Table 6. The twisted sector states for the orbifold $S^5/\mathbb{Z}_2$ written in an $\mathcal{N} = 2$ language. We use the same $SU(2) \times U(1)_Q \times U(1)_r$ decomposition as in [11]. The restriction to states with $U(1)_Q = 0 \bmod 2p$ yields the twisted states of $Y^{p,p}$.

account. Second, unlike for other $Y^{p,q}$, here the SLII multiplets of the Gravitino I, Vector I and Vector IV towers with $N_\alpha < 0$ have their N_ψ bounded from below by zero, instead of the generic expression $-\frac{p-q}{2}N_\alpha - 1$ which would give the unacceptable value of -1 . A similar comment applies to the CP conjugate SLI multiplets for which the generic higher bound of $+1$ would be unacceptable for N_ψ .

These cases were partially investigated in [29], where $Y^{1,1} \equiv S^5/\mathbb{Z}_2$ was completely dealt with. For $Y^{p,p}$ with $p > 1$, the contribution to the index from the untwisted sector was compared with the field theory computation and the difference was conjectured to come from the twisted modes. Here we confirm the prediction made in [29] by an explicit computation of the twisted sector contribution. However, in the case of $S^5/\mathbb{Z}_2$, we find an answer that disagrees with [29] but agrees with [30].⁶ We refer the reader to [30] for additional discussion of the $Y^{1,1}$ case and focus on $Y^{p,p}$ with $p > 1$ in the following.

The twisted sector states of these theories were discussed in an $\mathcal{N} = 2$ language suitable for the $SU(2,2|1)$ index computation in [11]. These states arise from the KK reduction on the S^1 of the $(2,0)$ theory on $AdS_5 \times S^1$. The twisted states of $Y^{1,1}$ are shown in table 6. To obtain the twisted states of $Y^{p,p}$ one keeps only the states with $U(1)_Q = 0 \bmod 2p$.

The states with $U(1)_Q = 0$ can be regarded as Betti multiplets and give canceling contributions to the index. The chiral multiplets give contributions according to table 5. Adding everything up and multiplying by the prefactor (6.4) yields for $p > 1$

$$\mathcal{I}_{Y^{p,p} \text{ twisted}}^R = \frac{t^{2p}}{1 - t^{2p}}. \quad (6.6)$$

⁶The index computed in [30] is the $\mathcal{N} = 2$ index appropriate for the $\mathbb{Z}_2$ orbifold theory which preserves $\mathcal{N} = 2$ supersymmetry. Using our results, we find agreement with the twisted sector index of [30] upon setting $v = 1$ in that paper to reduce their result to the $\mathcal{N} = 1$ index.

This coincides with the conjecture made in [29] for $S^5/\mathbb{Z}_{2p}$ with $\mathbb{Z}_{2p} \subset \text{SU}(3)$ generated by

$$\Omega = \begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega^{-2} \end{pmatrix}, \quad (6.7)$$

with $\omega^{2p} = 1$. In particular, adding the result (6.6) from the twisted sector to the result of [29] for the untwisted sector one recovers (6.5) evaluated at $p = q > 1$ with $a_1 = a_2 = 1$.

7 Holographic $c - a$ for $Y^{p,q}$ and a $1/N^2$ test of AdS/CFT

The quiver CFTs dual to IIB supergravity on $\text{AdS}_5 \times Y^{p,q}$ have $2p$ nodes and a number of chiral fields in bifundamentals [4]. A simple computation demonstrates that each node in the quiver contributes a factor of $1/16$ to $c - a$ that is related to the presence of a decoupled $\text{U}(1)$. Our aim is to reproduce this result $c - a = 2p/16 = p/8$ from the gravitational dual using the shortened KK spectrum.

As discussed in [11], the holographic formula for $c - a$ is [31–33]

$$c - a = -\frac{1}{360} \sum (-1)^F (E_0 - 2) d(s_1, s_2) (1 + f(s_1) + f(s_2)), \quad (7.1)$$

where the sum is in principle over all fields in the spectrum, but can be restricted to only the shortened representations in the KK tower, as the contribution from long multiplets automatically vanish. Here, E_0 , s_1 and s_2 are the quantum numbers of the maximal compact subgroup of $\text{SO}(4,2)$ labeling the bulk fields, $d(s_1, s_2) = d(s_1)d(s_2) = (2s_1 + 1)(2s_2 + 1)$ is the dimension of the $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)$ representation and $f(X) = X(X + 1)(6X(X + 1) - 7)$.

The sum in (7.1) is divergent and requires regularization. Following [11, 32–34], we regularize the sum by multiplying each term by z^L , where L is the ‘KK level’ of the bulk field, and then keep the finite part of the resulting function of z as $z \rightarrow 1$. Since all the fields in the same multiplet have the same KK level this regularization manifestly preserves supersymmetry. As explained in [11], for a generic Sasaki-Einstein manifold there is no well-defined notion of a KK level. Nevertheless one can assign the bulk KK multiplets such levels L according to what they would have been had the multiplets come from compactification on S^5 . The level assignment is as follows. We choose

$$L = E_0 - s_1 - s_2, \quad (7.2)$$

for the Graviton, Gravitino I, Gravitino III and Vector I towers, and

$$L = E_0 - s_1 - s_2 - 1, \quad (7.3)$$

for the Gravitino II, Gravitino IV, Vector III and Vector IV towers. E_0 is again the lowest AdS energy eigenvalue (corresponding to the conformal dimension Δ in the CFT dual) in the multiplet. These level assignments for the shortened multiplets along with their contribution to the holographic $c - a$ are listed in table 7.

Multiplet	Representation	Shortening	Level	$c - a$ for one
Graviton	$\mathcal{D}(e_0 + 3, \frac{1}{2}, \frac{1}{2}; r)$	conserved SLI SLII	$L = e_0 + 2$	$-\frac{5}{8}$ $-\frac{5}{48}(e_0 + 3)$ $-\frac{5}{48}(e_0 + 3)$
Gravitino I	$\mathcal{D}(e_0 + \frac{3}{2}, \frac{1}{2}, 0; r + 1)$	(conserved) chiral SLI SLII	$L = e_0 + 1$	$(\frac{35}{192})$ $-\frac{5}{48}e_0$ $-\frac{1}{96}(e_0 + \frac{3}{2})$ $\frac{5}{48}(e_0 + 1)$
Gravitino II	$\mathcal{D}(e_0 + \frac{9}{2}, \frac{1}{2}, 0; r - 1)$	SLI	$L = e_0 + 3$	$-\frac{1}{96}(e_0 + \frac{9}{2})$
Gravitino III	$\mathcal{D}(e_0 + \frac{3}{2}, 0, \frac{1}{2}; r - 1)$	(conserved) anti-chiral SLII SLI	$L = e_0 + 1$	$(\frac{35}{192})$ $-\frac{5}{48}e_0$ $-\frac{1}{96}(e_0 + \frac{3}{2})$ $\frac{5}{48}(e_0 + 1)$
Gravitino IV	$\mathcal{D}(e_0 + \frac{9}{2}, 0, \frac{1}{2}; r + 1)$	SLII	$L = e_0 + 3$	$-\frac{1}{96}(e_0 + \frac{9}{2})$
Vector I	$\mathcal{D}(e_0, 0, 0; r)$	conserved chiral anti-chiral SLI SLII	$L = e_0$	$\frac{1}{32}$ $-\frac{1}{96}(e_0 - \frac{3}{2})$ $-\frac{1}{96}(e_0 - \frac{3}{2})$ $\frac{1}{96}(e_0 - \frac{1}{2})$ $\frac{1}{96}(e_0 - \frac{1}{2})$
Vector II	$\mathcal{D}(e_0 + 6, 0, 0; r)$	—	—	
Vector III	$\mathcal{D}(e_0 + 3, 0, 0; r - 2)$	anti-chiral SLI	$L = e_0 + 2$	$-\frac{1}{96}(e_0 + \frac{3}{2})$ $\frac{1}{96}(e_0 + \frac{5}{2})$
Vector IV	$\mathcal{D}(e_0 + 3, 0, 0; r + 2)$	chiral SLII	$L = e_0 + 2$	$-\frac{1}{96}(e_0 + \frac{3}{2})$ $\frac{1}{96}(e_0 + \frac{5}{2})$

Table 7. Level assignment and the contribution of a single shortened multiplet to $c - a$ for the shortened spectrum of table 3. The latter must be multiplied by the degeneracy of the multiplet, which depends on the compactification manifold. The conserved gravitinos are present only if the compactification preserves ≥ 16 real supercharges, hence absent from all $Y^{p,q}$ except $Y^{1,1}$.

The above regularization has proven successful for $S^5/\mathbb{Z}_n$ and $T^{1,1}/\mathbb{Z}_n$ with n an arbitrary natural number [11]. Here we find that it continues to be successful for all $Y^{p,q}$ manifolds with $p > q > 0$. The details of the computation are quite similar to the cases discussed in [11]. As an example, we present the computation of $c - a$ for the graviton tower. We have

$$\begin{aligned}
 c - a \Big|_{\text{graviton}} &= -\frac{5}{8} + 2 \times \sum_{N_\alpha, N_\psi, N_\phi} z^{e_0+2} \left(-\frac{5}{48} \right) (e_0 + 3) \\
 &= -\frac{5}{8} + 2 \times \sum_{N_\alpha, N_\psi} z^{\frac{3}{2}(2N_\psi - N_\alpha/3l)+2} \left(-\frac{5}{48} \right) \left(\frac{3}{2} \left(2N_\psi - \frac{N_\alpha}{3l} \right) + 3 \right) (2N_\psi + 1).
 \end{aligned} \tag{7.4}$$

We first sum over N_ψ and then over N_α . This is facilitated by taking the three cases $N_\alpha = 0$, $N_\alpha > 0$ and $N_\alpha < 0$ separately. For $N_\alpha = 0$, the sum over N_ψ should go from 1 to infinity, whereas for $N_\alpha > 0$ the sum over N_ψ should go from $\frac{p+q}{2}N_\alpha$ to infinity, and finally for $N_\alpha < 0$ the sum over N_ψ should go from $-\frac{p-q}{2}N_\alpha$ to infinity. Note that (as in the computation of the index) while N_α is always an integer, N_ψ can be integer or half integer depending on $N_\alpha/(p+q)$, but it always increases in unit steps for any fixed N_α . Evaluating the sum and expanding around $z = 1$ gives

$$c - a \Big|_{\text{graviton}} = \frac{\alpha_4}{(z-1)^4} + \frac{\alpha_3}{(z-1)^3} + \frac{\alpha_2}{(z-1)^2} + \alpha_0 + \dots, \quad (7.5)$$

where $\alpha_{0,2,3,4}$ are complicated functions of p and q and the ellipsis denotes terms vanishing as $z \rightarrow 1$.

The contributions from the other towers can be worked out in a similar manner. In addition, the contribution from the Betti vector $(1/32)$ cancels against that from the Betti hyper $(-1/32)$. Putting everything together, one arrives at

$$c - a \Big|_{Y^{p,q}} = \frac{\alpha(Y^{p,q})}{(z-1)^2} + \frac{\alpha(Y^{p,q})}{(z-1)} + \frac{p}{8} + \dots, \quad (7.6)$$

where

$$\alpha(Y^{p,q}) = \frac{q^2 \left(4p - \sqrt{4p^2 - 3q^2} \right)}{4p^2 \left(2p^2 - 3q^2 - p\sqrt{4p^2 - 3q^2} \right)}. \quad (7.7)$$

This computation is valid for $p > q$.

While the fourth and third order poles cancel at $z = 1$, the second and first order poles do not. This is similar to what happened for $Y^{p,p}$ and $Y^{p,0}$ that were considered in [11]. Following [11, 32, 34], we drop the pole terms, so we are left with the finite result $c - a = p/8$ for $Y^{p,q}$, in perfect agreement with the field theory result.

8 Discussion

We conjecture that the shortened spectrum of the scalar Laplacian presented in section 3 is complete. While we have no proof that table 4 gives the complete shortened spectrum of IIB supergravity on $Y^{p,q}$, the successful computation of the superconformal index in section 6 gives us confidence that the shortened spectrum is in fact fully obtained.

Using the shortened spectrum, we holographically computed $c - a$ of the corresponding dual field theories and found agreement. The computation involves a regularization that we advocated in [11]. It is not entirely clear to us why this is the correct procedure to use, but it had been successful for orbifolds of S^5 and $T^{1,1}$. The fact that this regularization succeeds for $Y^{p,q}$ makes us believe that the method is correct and the matchings found previously [11, 33, 34] were not coincidental. Thus we are able to restate some of the conclusions made in [11] more confidently:

- At least for the models we have considered (four-dimensional CFTs dual to IIB string theory on $\text{AdS}_5 \times X^5$, with X^5 either an orbifold of S^5 , an orbifold of $T^{1,1}$ or

$Y^{p,q}$), holographic $c - a$ can be obtained purely from ten dimensional supergravity, and as such, does not necessitate a stringy origin. Massive string loop (or α') corrections to holographic $c - a$ must vanish, as opposed to claims made to the contrary in [35] and [36].

- The shift in holographic $c - a$ due to alternative boundary conditions (or the shift in $c - a$ due to a relevant double-trace deformation) must vanish (at least in the case of $T^{1,1}$), as opposed to the claim made to the contrary in [37].

Another observation one can make is that holographic $c - a$ as a function of z can be thought of as an index, since it is computed only from the shortened spectrum. It is of course most interesting when it is expanded around $z = 1$, where the constant term gives the actual field theoretical $c - a$. But one can also pay attention to the pole terms near $z = 1$. Eq. (7.6) shows that there are first and second order poles in the expansion of this ‘index’, with equal coefficients. This also happens for the even orbifolds of S^5 , where the pole coefficients can interestingly⁷ be obtained from (7.7) by setting $p = q$. However, there are no pole terms for the odd orbifolds of S^5 [11]. Since the pole terms are related to the asymptotic growth of the terms in the sum (7.1), guided by the analysis of [38] and the above special cases, we conjecture that the general expression for the coefficient of the pole terms is

$$\alpha(\text{SE}_5) = -\frac{5}{32\pi^3} \left(\frac{\int_{\text{SE}_5} \text{Riem}^2}{40} - \text{vol}(\text{SE}_5) \right). \quad (8.1)$$

More interesting than (8.1) would be a geometrical expression for the constant piece in the expansion of the $c - a$ index, i.e. the field theoretical $c - a$. This remains a challenge, perhaps, until a better understanding of the required regularization method is gained. Possible relations between holographic $c - a$ and the superconformal index are also currently under investigation.

Finally, note that in field theory $c - a$ is proportional to $\text{tr} R$ which arises in the $U(1)_R$ ’t Hooft anomaly from the $U(1)$ -gravitational-gravitational triangle diagram. The expression (7.1) for the holographic $c - a$ seems to provide an alternative explicit field theoretical formula for this strictly in terms of the quantum numbers of protected single trace operators. It would be interesting to explore this relation further.

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⁷Note that no twisted states have entered the derivation of equation (7.6). It seems that for $p > 1$ the twisted states of $S^5/\mathbb{Z}_{2p}$ precisely make up for the states lost by setting $p = q$ in the spectrum of $Y^{p,q}$.

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