Mapping 6D $\mathcal{N} = 1$ supergravities to F-theory

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ABSTRACT: We develop a systematic framework for realizing general anomaly-free chiral 6D supergravity theories in F-theory. We focus on 6D $(1, 0)$ models with one tensor multiplet whose gauge group is a product of simple factors (modulo a finite abelian group) with matter in arbitrary representations. Such theories can be decomposed into *blocks* associated with the simple factors in the gauge group; each block depends only on the group factor and the matter charged under it. All 6D chiral supergravity models can be constructed by gluing such blocks together in accordance with constraints from anomalies. Associating a geometric structure to each block gives a dictionary for translating a supergravity model into a set of topological data for an F-theory construction. We construct the dictionary of F-theory divisors explicitly for some simple gauge group factors and associated matter representations. Using these building blocks we analyze a variety of models. We identify some 6D supergravity models which do not map to integral F-theory divisors, possibly indicating quantum inconsistency of these 6D theories.

Contents

1 Introduction

String theory appears to provide a framework in which gravity can be consistently coupled to many different low-energy field theories in different dimensions. The problem of understanding precisely which low-energy gravity theories admit a UV completion, and which can be realized in string theory, is a longstanding challenge. Many different string constructions exist, which have been shown to give a variety of low-energy theories through compactifications of perturbative string theory or M/F-theory. In four space-time dimensions, while there are many string constructions, giving a rich variety of field theory models coupled to gravity, there is no general understanding as yet of which gravity theories admit a UV completion and which do not. In six dimensions, however, we may be closer to developing a systematic understanding of the set of allowed low-energy theories and their UV completions through string theory. For chiral (1, 0) supersymmetric theories in six dimensions, cancellation of gravitational, gauge, and mixed anomalies give extremely strong constraints on the set of possible consistent models [1]. In [2], it was shown that (with restrictions to nonabelian gauge group structure and one tensor multiplet) the number of possible distinct combinations of gauge groups and matter representations appearing in such models is finite. In [3], it was conjectured that all consistent models of this type have realizations in string theory. The goal of this paper is to connect the set of allowed chiral 6D supergravity theories to their string realizations by developing a systematic approach to realizing these theories in F-theory.

In a general 6D supergravity theory, the gauge group can be decomposed into a product of simple factors modulo a finite abelian group ($\mathcal{G} = (G_1 \times \cdots \times G_k)/\Gamma$) [In this paper we ignore U(1) factors]. In [2] it was shown that when there is one tensor multiplet, the anomaly cancellation conditions in 6D independently constrain each nonabelian factor G_i in the gauge group, along with the associated matter representations, into a finite number of distinct "building blocks". Each building block makes a contribution to the overall gravitational anomaly $n_h - n_v = 244$, where n_h, n_v respectively are the numbers of hyper and vector multiplets in the theory. An arbitrary model can be constructed by combining these building blocks to saturate the gravitational anomaly (with neutral hypermultiplets added as needed). The basic idea of the approach we take in this paper is to construct a dictionary between these building blocks of anomaly-free 6D theories and geometric structures in F-theory. F-theory [4] is a framework for constructing type IIB string vacua where the axio-dilaton varies over the internal space. The nonperturbative SL(2, $\mathbb{Z}$) symmetry of type IIB is geometrized in F-theory as the modular group of a fictitious T^2 fibered (holomorphically) over the internal space. F-theory on elliptically fibered 3-folds gives rise to a large class of 6D theories with (1, 0) supersymmetry [5–7]. The low-energy theory has one tensor multiplet when the base of the elliptic fibration is a Hirzebruch surface $\mathbb{F}_m$; this is the case we will consider in this paper. We develop a dictionary in which each supergravity building block is associated with a geometric structure in F-theory given by a divisor class on the $\mathbb{F}_m$ base of the elliptic fibration. Then, the construction of an F-theory model associated with a given anomaly-free 6D model proceeds by simply combining the divisors on the F-theory side associated with the building blocks on the supergravity side. The con-

nection between the anomaly cancellation conditions in 6D and the topological constraints on an F-theory construction were analyzed in [8–10]. In those papers, a detailed analysis is given of the F-theory structure associated with specific matter representations in the associated supergravity theory. In this paper we combine the results of that analysis with the block construction of supergravity theories and an explicit map from supergravity blocks to F-theory divisors to give a complete picture of the correspondence between 6D supergravity theories and F-theory models. This correspondence has potential not only to help in understanding the string realization of various supergravity theories in 6D (and perhaps eventually in 4D), but also to assist in understanding the range of geometric singularities possible in F-theory.

In this paper we focus initially on theories with gauge group constructed from products of simple factors $SU(N)$. This provides a clean and fairly simple illustration of the general ideas just described. A similar analysis is also possible for the other classical groups $SO(N)$ and $Sp(N)$, and the exceptional groups E_6, E_7, E_8, F_4, G_2. We give some simple examples of these other groups, leaving a systematic analysis of F-theory geometry associated with arbitrary gauge group and matter representations for future work. We identify some situations in which the map to F-theory violates an integrality condition on divisors in the base of the F-theory construction, so that apparently no F-theory model exists corresponding to these supergravity theories. We speculate on possible associated integrality constraints on the low-energy theories.

In section 2 we review the structure of anomalies in 6D (1, 0) supergravity theories. We summarize the results of [2] showing that the number of consistent theories with one tensor multiplet is finite, and elaborate on the construction of models from building blocks associated with factors in the gauge group. We explicitly describe the allowed factors with gauge group $SU(N)$ and matter in the fundamental and antisymmetric tensor representations, which form a simple example of the general framework presented here. In section 3 we review the relevant basic structures in F-theory. We give an explicit dictionary from $SU(N)$ supergravity building blocks to divisors in F-theory, and find that all product group models built from these blocks in supergravity give rise to topologically allowed combinations of divisors in F-theory. In section 4 we expand the dictionary to include other representations of $SU(N)$ as well as some other simple groups and representations, and describe the corresponding structure in F-theory. In section 5 we discuss the problem of constructing explicit Weierstrass models associated with the topological data given by the dictionary for a given supergravity model. In section 6 we summarize some of the exceptions we have identified to the integrality of the F-theory mapping. We conclude in section 7 with a general discussion and comments on extensions of the results described in this paper. Related work analyzing the interplay of constraints imposed by anomaly cancellation and geometric constraints in F-theory has recently appeared in [11, 12].

2 Anomaly-free (1, 0) supergravity models in 6D

In this section we review the basic anomaly conditions of $(1, 0)$ supersymmetric theories coupled to gravity in six dimensions (subsection 2.1), and the result of [2] showing that

only a finite number of gauge groups and matter content are possible in such theories (subsection 2.2). We then give an example of how a class of such models can be explicitly enumerated by giving a complete classification of all models whose gauge group is a product of SU(N) factors, with matter in fundamental, antisymmetric, and bifundamental representations (subsection 2.3).

2.1 Review of anomaly conditions

In this subsection we give a brief review of the anomaly conditions on 6D (1, 0) supergravity theories [1]. A more complete review of these conditions appears in [2], and we mostly follow the notation and conventions of that paper. We repeat some of the central equations here for convenience.

Throughout the paper we denote traces in the fundamental and adjoint representations by tr, Tr respectively, using tr_R for all other representations R. Traces of second and fourth powers of F in any representation can be expanded as

$$\mathrm{tr}_R F^2 = A_R \mathrm{tr} F^2 \tag{2.1}$$

$$\mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R (\mathrm{tr} F^2)^2 \tag{2.2}$$

We denote the dimension of a general representation R by D_R.

We consider theories with gauge group of the form $\mathcal{G} = (G_1 \times \cdots \times G_k)/\Gamma$ with G_i simple (assuming no U(1) factors) and Γ a finite abelian group. The number of hypermultiplets in representation R of group i is denoted x_R^i, and the number of bifundamental hypermultiplets transforming in (R, S) under G_i, G_j is denoted x_{RS}^{ij}.

We let n_t, n_h and n_v denote the number of tensor multiplets, hypermultiplets, and vector multiplets in our theory. For $n_t = 1$, the anomaly has a term proportional to

$$I_1 = (n_h - n_v - 244) \mathrm{Tr}\, R^4 \,, \tag{2.3}$$

whose vanishing implies

$$n_h - n_v = 244 \tag{2.4}$$

When (2.4) is satisfied, the anomaly polynomial becomes (after rescaling so that the coefficient of $(\mathrm{tr} R^2)^2$ is one)

$$I = (\mathrm{tr} R^2)^2 + \frac{1}{6} \mathrm{tr} R^2 \sum_i \left[\mathrm{Tr} F_i^2 - \sum_R x_R^i \mathrm{tr}_R F_i^2 \right] - \frac{2}{3} \left[\mathrm{Tr} F_i^4 - \sum_R x_R^i \mathrm{tr}_R F_i^4 \right]$$
$$+ 4 \sum_{i,j,R,S} x_{RS}^{ij} (\mathrm{tr}_R F_i^2)(\mathrm{tr}_S F_j^2) \tag{2.5}$$

Anomalies can be cancelled through the Green-Schwarz mechanism [1, 13] when this polynomial can be factorized as

$$I = \left(\mathrm{tr} R^2 - \sum_i \alpha_i \mathrm{tr} F_i^2 \right) \left(\mathrm{tr} R^2 - \sum_i \tilde{\alpha}_i \mathrm{tr} F_i^2 \right) \tag{2.6}$$

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A necessary condition for the anomaly to factorize in this fashion is the absence of any irreducible $\mathrm{tr}F_i^4$ terms. This gives the condition

$$\mathrm{tr}F_i^4: \quad B_{Adj}^i = \sum_R x_R^i B_R^i \tag{2.7}$$

For groups G_i which do not have an irreducible $\mathrm{tr}F_i^4$ term, $B_R^i = 0$ for all representations R and therefore (2.7) is always satisfied. The sum in (2.7) is over all hypermultiplets that transform under any representation R of G_i. For example, a single hypermultiplet that transforms in the representation (R, S, T) of $G_i \times G_j \times G_k$ contributes $\dim(S) \times \dim(T)$ to x_R^i. Note that the anomaly conditions are not sensitive to whether a group transforms in a given representation R or the conjugate representation $\bar{R}$. For example, in a model with gauge group $\mathrm{SU}(N) \times \mathrm{SU}(M)$ with x hypermultiplets in $(N, \bar{M}) + (\bar{N}, M)$ and y hypers in $(N, M) + (\bar{N}, \bar{M})$, anomaly cancellation can only constrain the sum $x + y$. F-theory in its usual formulation generally gives rise only to hypermultiplets in the first category, with $y = 0$. An F-theory realization of models with $y \neq 0$ has not been fully developed, though such supergravity models certainly are possible in six dimensions, as found for example in [14].

For a factorization of the anomaly polynomial (2.5) to exist, in addition to (2.4) and (2.7), the following equations must have a solution for real $\alpha_i, \tilde{\alpha}_i$

$$\alpha_i + \tilde{\alpha}_i = \frac{1}{6}\left(\sum_R x_R^i A_R^i - A_{Adj}^i\right) \tag{2.8}$$

$$\alpha_i \tilde{\alpha}_i = \frac{2}{3}\left(\sum_R x_R^i C_R^i - C_{Adj}^i\right) \tag{2.9}$$

$$\alpha_i \tilde{\alpha}_j + \alpha_j \tilde{\alpha}_i = 4 \sum_{R,S} x_{RS}^{ij} A_R^i A_S^j \tag{2.10}$$

2.2 Finite number of models

In [2], it was proven that there are a finite number of distinct gauge groups and matter representations which satisfy the conditions (2.4), (2.7), (2.8), (2.9), and (2.10) when the additional condition is imposed that all gauge kinetic terms must be positive for some value of the dilaton.

The condition (2.4) plays a key role in this proof of finiteness. The anomaly cancellation conditions constrain the matter transforming under each gauge group so that the quantity $n_h - n_v$ in general receives a positive contribution from each gauge group and associated matter, and the construction of models compatible with (2.4) thus has the flavor of a partition problem. (There are cases where a single gauge group factor and associated matter contribute a negative $n_h - n_v$, but generally only one such factor can appear in any model). Because equations (2.8), (2.7) and (2.9) all depend only upon the numbers of fields transforming in different representations under the gauge group factor G_i, we can consider solutions of these equations as "building blocks", from which complete theories can be constructed by combining building blocks, with the overall constraint (2.4) bounding the

size and complexity of the possible models which can be constructed. In combining blocks in this fashion, it is necessary to keep in mind that some matter transforming under a given gauge group may also have nontrivial transformation properties under another group, so that $n_h - n_v$ is subadditive. The number of such fields transforming under multiple groups, however, is bounded by (2.10), so that the enumeration is still finite.

The parameters $\alpha_i, \tilde{\alpha}_i$ which are fixed through (2.8), (2.9) for each block (up to exchanging the two values) play a key role in the structure of consistent models. These parameters enter the Lagrangian in the kinetic term for the gauge field G_i through

$$\mathcal{L} = -\sum_i (\alpha_i e^{\phi} + \tilde{\alpha}_i e^{-\phi}) \text{tr}(F_i^2) + \dots \tag{2.11}$$

as shown in [15]. Thus, if both $\alpha, \tilde{\alpha}$ are negative for some gauge group, the gauge kinetic term always has the wrong sign; we do not consider theories with this apparent instability.

As we will see in section 3, the parameters $\alpha, \tilde{\alpha}$ are the key to the mapping from gauge group building blocks to F-theory. These parameters encode the homology class of the divisor in the base of the F-theory compactification associated with the given gauge group component. From the anomaly cancellation equations, it is clear that there are various constraints on the $\alpha_i, \tilde{\alpha}_i$ parameters for the various gauge group components. For example, we cannot have two gauge group factors which both have $\alpha_i < 0$ and $\tilde{\alpha}_i > 0$, or the number of bifundamental fields charged under these two groups would be negative through (2.10). Similarly, two factors with matter representations giving specific values of the $\alpha, \tilde{\alpha}$'s cannot appear in the same theory unless the product in (2.10) computed using those values of $\alpha, \tilde{\alpha}$ is divisible by 4. These algebraic constraints on supergravity blocks correspond to geometric constraints in F-theory which we will describe in 3.

2.3 Classification of $SU(N)$ models

From the proof of finiteness and the block decomposition structure of a general chiral 6D supergravity theory, in principle it should be possible to systematically classify and enumerate all possible models, at least when restricting to a semisimple gauge group and one tensor multiplet. Each model has a gauge group $\mathcal{G} = (G_1 \times \dots \times G_K)/\Gamma$, and matter multiplets in any representation of $\mathcal{G}$. In classifying all 6D models, a key point is that the values of $\alpha_i, \tilde{\alpha}_i$ for each factor G_i depend on the matter charged under G_i alone, and not on the other factors in the gauge group. Thus, a complete classification of models can proceed heuristically as follows

1. Classify all blocks.

   For each simple group G_i, classify all representations R and matter multiplicities x_R^i, such that $\sum_R x_R^i B_R^i = B_{Adj}^i$ and solutions to (2.8) and (2.9) exist. This gives a set of building blocks, which can be used to build the full gauge group $\mathcal{G}$. We define a *block* as consisting of a gauge group G_i and all the associated charged matter representations. The values for $\alpha_i, \tilde{\alpha}_i$ are determined for a given block (up to exchange).

2. Combine blocks.

   We wish to combine blocks in all possible combinations compatible with the anomaly conditions. The blocks from Step 1 cannot be combined arbitrarily; in a model with

gauge group $\prod_i G_i$, for every pair of indices i, j, the associated blocks can only be combined if there is enough matter which is simultaneously charged under $G_i \times G_j$ to satisfy equation (2.10). This gives a constraint on which blocks can be combined, which becomes quite strong as the number of blocks is increased. Thus, to construct all models we need to classify all possible combinations of the blocks determined in Step 1, subject to the conditions that both (2.10) and the gravitational anomaly condition $n_h - n_v = 244$ are satisfied.

Note that once the blocks have been combined, there are only finitely many choices for the finite abelian group Γ, which is constrained by the matter representation. (For example, an SU(2) block with only adjoint matter could have gauge group $SU(2)/\mathbb{Z}_2 \cong SO(3)$, but if there is fundamental matter it is not possible to take $\Gamma = \mathbb{Z}_2$.)

This general strategy for classifying models is complicated by the fact that even though there are only finitely many models in total, placing a bound on the set of blocks needed in step 1 above is nontrivial. It is the gravitational anomaly condition $n_h - n_v = 244$, which depends on *all* the matter, that ultimately enforces finiteness. Thus, at the level of enumerating the blocks, a block in a given model could contribute more than 244 to $n_h - n_v$, if another block has a negative contribution. Moreover, in the presence of matter charged simultaneously under multiple groups, the contribution to $n_h - n_v$ from a given block is overcounted since many groups "share" the same hypermultiplets.

While the proof of finiteness in [2] demonstrates that a complete enumeration of all models is in principle possible, we do not present here a complete algorithm for efficient enumeration of all models. Instead, we consider a simplified class of models for which we carry out a complete classification of models, as an example of how the bounds from the gravitational anomaly and multiply-charged matter fields can be used to constrain the set of possible models. The approach used in this simplified class could be generalized to include most other gauge groups and matter representations, but we leave a completely general analysis to further work.

Thus, in this paper, we implement an explicit algorithm based on the above strategy to enumerate all models consisting of blocks with an $SU(N)$ gauge group and associated matter in the fundamental and antisymmetric representations. In this section we restrict to $SU(N)$ blocks with $N > 3$, and discuss the special cases $SU(2), SU(3)$ in section 4.[1] We give a simple classification of all blocks of this type (step 1), and find a lower bound for the contribution of each block to $n_h - n_v$ which enables us to systematically classify all models built from these blocks (step 2), under the assumption that the only type of matter charged under more than one gauge group factor is bifundamental matter charged under two simple factors. This gives a fairly simple set of possibilities which provide a clear framework for demonstrating the dictionary for associating blocks and complete models with F-theory constructions, which we describe in section 3. In section 4 we describe other representations for matter charged under $SU(N)$ and other gauge groups, and give some examples of blocks including these structures.

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[1] For SU(2), SU(3), there is no quartic Casimir, and the range of models is slightly larger both on the supergravity side and the F-theory side.

Step 1: SU(N) blocks with fundamental (□) and antisymmetric (⊟) matter

For the fundamental and antisymmetric representations of SU(N) we have

rep. R	A_R	B_R	C_R	D_R
fundamental (f)	1	1	0	N
antisymmetric (a)	$N-2$	$N-8$	3	$N(N-1)/2$

For a gauge group factor $G_i = \mathrm{SU}(N), N > 4$ with matter in only these representations, the F^4 anomaly condition (2.7) can be used to determine a relationship between the number f of fundamental representations and the number a of antisymmetric representations

$$f = 2N - a(N-8).\qquad(2.12)$$

Using this relation, a simple computation shows that the anomaly polynomial automatically factorizes as

$$I = (\mathrm{tr}R^2 - 2\mathrm{tr}F^2)(\mathrm{tr}R^2 - (a-2)\mathrm{tr}F^2)\qquad(2.13)$$

The values of $\alpha, \tilde{\alpha}$ are therefore given by one of the two possibilities

$$\alpha, \tilde{\alpha} = 2, a-2$$
$$\alpha, \tilde{\alpha} = a-2, 2\,.\qquad(2.14)$$

Thus, for this sub-family of SU(N) blocks with fundamental and antisymmetric matter, we have implemented Step 1 of the algorithm above. Each block is specified by integers a, N where $N \geq 4$, and since $f, a \geq 0$, we have the further constraints $2N/(N-8) \geq a \geq 0$.

For example, consider a model where the gauge group has a single simple factor SU(N). With various numbers a of antisymmetric representations, we find solutions which undersaturate the gravitational anomaly condition $n_h - n_v = 244$ up to $a = 10$, with $f = (2-a)N + 8a$, and with N ranging up to $N \leq (15, 15, 16, 18, 16, 13, 9, 6, 5, 4, 4)$ (for a from 0 to 10). The "building blocks" associated with these gauge groups and matter representations are tabulated in table 2.

Note that by plugging 1/2 of (2.12) in to the formula (2.4) for the number of matter fields, we have

$$n_h - n_v = fN + aN(N-1)/2 - N^2 + 1 = N\left(f/2 + 7a/2\right) + 1\,.\qquad(2.15)$$

This shows that the contribution from each block to $n_h - n_v$ is positive, and is greater than $Nf/2$. As we now discuss, the form (2.15) of the contribution to $n_h - n_v$ gives a finite bound on the set of blocks which may enter into complete models and makes it possible to analyze all models composed of these blocks in an efficient fashion.

Step 2: combining SU(N) blocks with fundamental + antisymmetric matter into complete models

We now wish to combine the blocks described above into all possible models with gauge group of the form $\mathcal{G} = (G_1 \times \ldots \times G_K)$ (we do not explicitly carry out the analysis of possible quotients by a discrete group Γ here, but this could be done systematically in a

straightforward fashion). We assume in this analysis that the only kind of multiply-charged matter available is bifundamental matter charged under two groups G_i, G_j. If the number of such bifundamental fields is x_{ij} then the total contribution to $n_h - n_v$ from blocks i, j is decreased by $x_{ij} N_i N_j$. We can subtract half this contribution from the contribution (2.15) from each block to $n_h - n_v$. This removes at most $Nf/2$ from each block. Thus, even with the overcounting from bifundamentals, each block has a contribution of at least

$$(n_h - n_v)_i \geq 7Na/2 + 1 \tag{2.16}$$

to the total gravitational anomaly. This provides an immediate upper bound on the set of individual blocks which can be used. Since we must have $(n_h - n_v)_i \leq 244$, we have $a \leq 17$. For $a > 0$ we have $N \leq 486/(7a)$.

The blocks with $a = 0$ form a special case. For these blocks, $(\alpha, \tilde{\alpha}) = (2, -2)$ or $(-2, 2)$. We cannot have more than one such block. If we chose two blocks with the same $\alpha, \tilde{\alpha}$, there would be a negative number of bifundamentals from (2.10). And if we choose one of each sign, then the gauge kinetic terms (2.11) have opposite signs so one will be negative and unphysical. Thus, we can only have one block with $a = 0$. Without loss of generality we assume it has $(\alpha, \tilde{\alpha}) = (2, -2)$. We cannot have a block of this type and a block with $a = 1$, since we would not have an acceptable number of bifundamentals between these blocks. Thus, if we have a block with $a = 0$, all other blocks must have at least $a = 2$. A similar argument shows that only one block can have $a = 1$. We find therefore that an efficient approach to classifying all models is to begin by classifying all combinations of blocks with $a > 1$, and then for each such combination to check which blocks with $a = 0$ or $a = 1$ can be included. While (2.16) does not provide a bound on N when $a = 0$, if we add the $a = 0$ block last, then $n_h - n_v$ and/or (2.10) provide a strong constraint on the N allowed for the $a = 0$ block.

It is now straightforward to systematically enumerate all models built from SU(N) blocks with a antisymmetric matter fields and f fundamental matter fields, using (2.16) and (2.10). We can do this recursively, starting with 1 block and continuing to K blocks, adding blocks with nonincreasing values of $a > 1$ so that

$$\sum_i \left(\frac{7N_i a_i}{2} + 1 \right) \leq 244, \tag{2.17}$$

where at each step we only add blocks where there are enough fundamentals in each component of the complete model to satisfy (2.10). Given K blocks satisfying (2.17), we can then keep that combination, or add a single block with $a = 1$ or $a = 0$. Given all the blocks, we then check that the total gravitational anomaly is undersaturated

$$n_h - n_v \leq 244 \tag{2.18}$$

and saturate the anomaly with neutral hypermultiplets as needed.

We have carried out this algorithm and enumerated all possible models of this type which are consistent with all anomaly cancellation conditions. The number of models with K blocks is tabulated in table 1. The total number of models with any number of blocks

K	1	2	3	4	5	6	7	8	9
# models	88	1301	4798	5975	3202	882	152	19	1

Table 1. Number of models with K blocks, gauge group product of $SU(N)$ factors ($N > 3$ for each factor) with matter in fundamental and antisymmetric representations.

and this gauge group and matter structure is 16,418. In this enumeration we have restricted to $SU(N)$ blocks with $N > 3$. In section 4.2 we include $SU(3)$ blocks in the enumeration.

As an example of a consistent theory with a product group structure satisfying anomaly cancellation, consider the following two "building blocks" associated with group factors, matter representations, and compatible choices of $\alpha, \tilde{\alpha}$

$$
\begin{aligned}
SU(4): \quad & 2\,\boxed{}\,\boxed{} + 16\,\square, \quad \alpha_1 = 0, \tilde{\alpha}_1 = 2 \\
SU(5): \quad & 4\,\boxed{}\,\boxed{} + 22\,\square \quad \alpha_2 = 2, \tilde{\alpha}_2 = 2\,.
\end{aligned}
\tag{2.19}
$$

Since we have $\alpha_1 \tilde{\alpha}_2 + \alpha_2 \tilde{\alpha}_1 = 4$ there is one bifundamental hypermultiplet transforming under the $(4, 5)$ representation of the gauge group. This uses up 5 of the fundamentals in the $SU(4)$ and 4 of the fundamentals in the $SU(5)$. The total contribution to $n_h - n_v$ from this product group is $n_h - n_v = 167$. This is one of the 1301 two-block models appearing in table 1.

As another example, consider the single model of this type with the most blocks, appearing in the $K = 9$ column in table 1. This model has gauge group

$$
SU(16) \times SU(4)^8
\tag{2.20}
$$

where the $SU(16)$ has $a = 0$ antisymmetric matter fields and each $SU(4)$ has $a = 2$. It follows that the $SU(16)$ block has $\alpha = 2, \tilde{\alpha} = -2$ and the other blocks have $\alpha = 0, \tilde{\alpha} = 2$. There are thus bifundamental fields in the $(16, 4)$ connecting the first component to each other component. This model has a total gravitational anomaly contribution of $n_h - n_v = 233$, so there are 11 neutral hypermultiplets.

We now show how all 16418 of the models classified here and enumerated in table 1 can be embedded in F-theory (at least topologically).

3   F-theory realizations of $SU(N)$ product models

We now describe the mapping from the gauge group block construction of consistent 6D supergravity theories to F-theory, focusing on the simple class of models described in the previous section. We begin with a brief review of some basic aspects of F-theory and then describe the map.

3.1   Review of 6D F-theory constructions

Compactifications of F-theory on elliptic Calabi-Yau 3-folds generate a large class of six-dimensional theories. Since we have restricted our attention to models with one tensor multiplet, the base of the elliptic fibration must be a Hirzebruch surface $\mathbb{F}_m$. We briefly review the structure of these compactifications here, for more details see [4, 7].

F-theory provides a geometric understanding of compactifications of type IIB string theory where the axio-dilaton varies over the internal space. F-theory on an elliptically fibered Calabi-Yau M with base B is a type IIB compactification on B, where the axio-dilaton is identified with the complex structure of the elliptic fiber. In our case, M is a 3-fold with base $\mathbb{F}_m$. There is a codimension-one locus in the base where the fiber degenerates; these correspond to 7-branes wrapping a complex curve. Possible degeneration structures are given by the Kodaira classification [16, 17]; for example, a type I singularity along an irreducible curve ξ in the base B is associated with the Dynkin diagram A_{N-1}, and corresponds to N 7-branes wrapping ξ. Such a configuration generally results in an $SU(N)$ gauge group in the low-energy theory. Similarly, all other A-D-E gauge groups can be obtained by engineering the appropriate degeneration on the curve ξ. The set of 7-branes allowed in the compactification is constrained by the condition that the full manifold defined by the elliptic fibration must be Calabi-Yau, and thus have vanishing canonical class. Kodaira's formula expresses this fact as a relationship between the locus of singular fibers and the canonical class K of the base.

$$\sum_\beta a_\beta X_\beta = -12K \,. \tag{3.1}$$

Here the X_β denote the classes of irreducible curves along which the elliptic fibration degenerates. The multiplicities a_β are determined by the singularity type of the elliptic fiber [4]; for an A_{N-1} singularity, the multiplicity is N. Some of the divisors X_β correspond to curves where the singularity in the elliptic fiber results in nonabelian gauge symmetry; we denote the classes of such curves by ξ_i. The remaining curves do not enhance the gauge group in the low-energy theory (singularity type I_1 or II, with multiplicity $a_\beta = 1$ or $a_\beta = 2$, respectively); we denote the sum of the classes of such curves by Y. Given such a decomposition of the singularity locus, the matter content in the theory is found by studying the detailed structure of the singularities and intersections of the divisors ξ_i, Y. The analysis of singularity types associated with matter in various representations of the gauge group is given in [7–10, 19]. The simplest example of a matter field is when two components of the divisor intersect. For example, when an A_{N-1} locus, corresponding to $SU(N)$, intersects an A_0 locus, the singularity type is enhanced to A_N at the intersection. This results in matter hypermultiplets localized at the intersection, which transform in the fundamental of $SU(N)$. The $SU(N)$ transformation properties of these matter hypermultiplets are precisely the same as if they had been obtained from the Higgsing of the adjoint of $SU(N+1)$ [19], and the Higgsing procedure is a good informal guide to the behavior at the intersection point (although, crucially, the gauge group itself is not actually enhanced to $SU(N+1)$). Note that while the way in which the gauge symmetry group is encoded in the singularity structure is completely determined by the Kodaira classification, there is as yet no complete classification of singularity structures associated with matter representations. Indeed, we encounter a number of exotic representations in the classification of 6D supergravity models which should correspond to currently unknown singularity structures on the F-theory side. Some examples of this type are given in section 4.

We are interested in elliptic fibrations over the Hirzebruch surface $\mathbb{F}_m$. These are a

family of surfaces which are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$ indexed by an integer $m \geq 0$. A basis for the set of divisors is given by D_v, D_s, with intersection pairings

$$D_v \cdot D_v = -m, \quad D_v \cdot D_s = 1, \quad D_s \cdot D_s = 0. \tag{3.2}$$

In terms of the fibration, D_v is a section, while D_s corresponds to the class of the fiber. It is sometimes useful to work with $D_u = D_v + mD_s$, which satisfies $D_u \cdot D_u = m$. K, the canonical class of $\mathbb{F}_m$ is given by

$$-K = 2D_v + (2 + m)D_s. \tag{3.3}$$

For $\mathbb{F}_m$, the effective divisors that correspond to irreducible curves are given by

$$D_v, \quad aD_u + bD_s, \ a, b \geq 0. \tag{3.4}$$

We are interested in constructing F-theory compactifications on Calabi-Yau 3-folds elliptically fibered over $\mathbb{F}_m$, for models with gauge groups $\prod_i \mathrm{SU}(N_i)$, $N_i \geq 4$. For this purpose, the singular locus must contain divisors ξ_i corresponding to irreducible curves (3.4), satisfying the Kodaira formula

$$24D_v + (12m + 24)D_s = \sum_i N_i \xi_i + Y \tag{3.5}$$

We now proceed to identify such models by mapping solutions of the anomaly cancellation conditions into F-theory.

3.2 Mapping $\mathrm{SU}(N)$ models into F-theory

We now return to the classification of $\mathrm{SU}(N)$ building blocks for anomaly-cancelling 6D chiral supergravity theories. Associated with each factor $G_i = \mathrm{SU}(N_i)$ in the gauge group we have a set of matter fields in representations satisfying (2.7), (2.8), (2.9); these conditions uniquely determine the coefficients $\alpha_i, \tilde{\alpha}_i$ for each factor. In section 2.3, we showed how these building blocks could be combined to construct complete lists of anomaly-free models with multiple gauge group factors. In this section, we show how the data from anomaly cancellation in the low-energy theory, namely the $\alpha_i, \tilde{\alpha}_i$, determine the structure of the F-theory compactification.

In order to define the dictionary between the low-energy physics and F-theory, we wish to associate with each gauge block a divisor ξ_i on an appropriate Hirzebruch surface, to be used as the base of the elliptic fibration in F-theory. A block specified by $\alpha, \tilde{\alpha}$ is mapped to the divisor

$$(\alpha, \tilde{\alpha}) \ \rightarrow \ \xi = \frac{\alpha}{2}\left(D_v + \frac{m}{2}D_s\right) + \frac{\tilde{\alpha}}{2}D_s. \tag{3.6}$$

For example, an SU(5) gauge group with 3 matter fields in the antisymmetric tensor representation has $\alpha, \tilde{\alpha} = 2, 1$, and can be mapped to the divisor

$$\{\mathrm{SU}(5), 3\,\boxed{} + 19\,\square, \alpha = 2, \tilde{\alpha} = 1\} \ \rightarrow \ \left(D_v + \frac{m}{2}D_s\right) + \frac{1}{2}D_s, = D_v + \frac{m+1}{2}D_s \tag{3.7}$$

a	f	max N	α	$\tilde{\alpha}$	$\mathbb{F}_0$	$\mathbb{F}_1$	$\mathbb{F}_2$
0	$2N$	15	2	-2			D_v
1	$N+8$	15	2	-1		D_v	
2	16	16	2	0	D_v		
			0	2	D_s	D_s	D_s
3	$-N+24$	18	2	1		$D_v + D_s$	
4	$-2N+32$	16	2	2	$D_v + D_s$		$D_v + 2D_s$
5	$-3N+40$	13	2	3		$D_v + 2D_s$	
6	$-4N+48$	9	2	4	$D_v + 2D_s$		$D_v + 3D_s$
			4	2	$2D_v + D_s$	$2(D_v + D_s)$	
7	$-5N+56$	6	2	5		$D_v + 3D_s$	
8	$-6N+64$	5	2	6	$D_v + 3D_s$		$D_v + 4D_s$
			6	2	$3D_v + D_s$		
9	$-7N+72$	4	2	7		$D_v + 4D_s$	
10	$-8N+80$	4	2	8	$D_v + 4D_s$		$D_v + 5D_s$
			8	2	$4D_v + D_s$		

Table 2. Building blocks associated with gauge group factors SU(N) having a 2-index antisymmetric representations, up to $a = 10$. For each block, number of fundamental representations given as function of N. Maximum value of N is indicated such that $n_h - n_v \leq 244$, corresponding to constraint on single block. (Larger N for given a can appear in multi-block models.) Possible values of $\alpha, \tilde{\alpha}$ and associated divisors in $\mathbb{F}_m$ are given for $m = 0, 1, 2$.

This divisor corresponds to the class of an irreducible curve only for $m = 1$. In this case, the block specifies both the divisor ξ and the base $\mathbb{F}_m$.

The correspondence between divisors in F-theory and the coefficients $\alpha, \tilde{\alpha}$ was expressed in related forms in [5, 6, 9, 20]. To check that the map defined by (3.6) is correct, we compare with the results of [9], from which we have

$$\xi_i \cdot (-K) = \frac{1}{6} \left(\sum_R x_R^i A_R^i - A_{Adj}^i \right) = \alpha_i + \tilde{\alpha}_i \tag{3.8}$$

$$2\xi_i \cdot \xi_i = \frac{2}{3} \left(\sum_R x_R^i C_R^i - C_{Adj}^i \right) = \alpha_i \tilde{\alpha}_i \,. \tag{3.9}$$

A short computation using (3.6), (3.3) confirms that these equations are satisfied. Furthermore, we can check that the product of two distinct blocks satisfies

$$4\xi_i \cdot \xi_j = \alpha_i \tilde{\alpha}_j + \alpha_j \tilde{\alpha}_i \,. \tag{3.10}$$

The conditions in equations (3.8) and (3.9) immediately guarantee that α_i and $\tilde{\alpha}_i$ are real. This follows from the discriminant of the corresponding quadratic equation

$$(\xi_i \cdot K)^2 - 4 \cdot 2\xi_i \cdot \xi_i \tag{3.11}$$

being non-negative. But that discriminant is the negative of the determinant of the matrix

$$\begin{bmatrix} \xi_i \cdot \xi_i & \xi_i \cdot K \\ \xi_i \cdot K & K \cdot K \end{bmatrix} \tag{3.12}$$

(since $K \cdot K = 8$ for a Hirzebruch surface $\mathbb{F}_m$), and the Hodge index theorem for algebraic surfaces (cf. [18]) implies that (3.12) has negative or zero determinant.

Equations (3.8) and (3.9) are also the key to generalizing the analysis in this paper to cases with $n_t > 1$, something we plan to pursue in future work.

Note that the map (3.6) can generally take a block to several different choices of $\mathbb{F}_m$. Furthermore, for some blocks, both choices of $\alpha, \tilde{\alpha}$ lead to acceptable divisors. For example, an SU(N) group with 2 antisymmetric representations has either $\alpha = 2, \tilde{\alpha} = 0$ corresponding to the divisor D_v on $\mathbb{F}_0$ or $\alpha = 0, \tilde{\alpha} = 2$ corresponding to the divisor D_s on any $\mathbb{F}_m$. For single-block models of this type, there are distinct realizations on $\mathbb{F}_0, \mathbb{F}_1$, and $\mathbb{F}_2$. In some cases apparently distinct realizations of a given model are actually equivalent by a duality. For example, D_v and D_s on $\mathbb{F}_0$ are related by exchanging the two $\mathbb{P}^1$'s whose product forms $\mathbb{F}_0$, corresponding to S-duality of the supergravity theory. In addition, $\mathbb{F}_2$ can be deformed to $\mathbb{F}_0$ through a complex structure deformation as discussed in [5, 6], so models on these two surfaces may be related by deformations on a single moduli space. It would be good to have a general understanding of when distinct embeddings of a given model are physically equivalent under a duality symmetry and when they are not.

In table 2, we list the SU(N) blocks with fundamental and antisymmetric representations allowed by anomaly cancellation, and the corresponding F-theory divisors on $\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2$. The reason we restrict to $m = 0, 1, 2$ is that at a general point in moduli space, the gauge group is completely broken for blocks listed in table 2. For other values of m, the F-theory compactification has a nonabelian unbroken symmetry of a type other than SU($N \geq 4$) at a general point in moduli space. The table includes all blocks which have $n_h - n_v \leq 244$; larger values of a are possible in models with multiple gauge group factors.

Given the map on building blocks associated with gauge group factors, in principle we can build up an arbitrary model with any product gauge group and matter content from knowledge of the embedding of the blocks. For example, the F-theory construction of the model with gauge group SU(4) × SU(5) with 2 and 4 antisymmetric representations of the factors described in (2.19) is associated with the following set of singular divisors on the F-theory side, using the base $\mathbb{F}_0$

$$X_1 = 4\xi_1, \quad \xi_1 = D_s \tag{3.13}$$

$$X_2 = 5\xi_2, \quad \xi_2 = D_v + D_s \tag{3.14}$$

$$Y = -24K - 4\xi_1 - 5\xi_2 = 19D_v + 15D_s. \tag{3.15}$$

Note that this same model could be constructed in two other ways. The same model can be realized on $\mathbb{F}_2$, where the map (3.6) gives $\xi_2 = D_v + 2D_s$. Alternatively, we could have chosen $\alpha_1 = 2, \tilde{\alpha} = 0$, giving the same gauge group and matter content, but with the F-theory realization having $\xi_1 = D_v$ on $\mathbb{F}_0$. As this example illustrates, some models have several distinct realizations in F-theory. A similar redundancy was noted in [14], where

multiple UV realizations of some specific anomaly-free models were found in the heterotic string, associated with topologically distinct lattice embeddings. It would be nice to have a better understanding of the physical differences between different F-theory realizations of the models considered here.

As another example of how a complete model is mapped to F-theory using (3.6) consider the model with gauge group $SU(16) \times SU(4)^8$ described below equation (2.20). The $SU(16)$ has $a = 0$ and maps to (16 copies of) D_v on $\mathbb{F}_2$. Each $SU(4)$ has $a = 2$ and maps to $(4 \times) D_s$. The bifundamental in each $SU(16) \times SU(4)$ follows from the intersection number $D_v \cdot D_s = 1$. The total singularity locus for this model is

$$\sum_i X_i = 16D_v + 32D_s\,. \tag{3.16}$$

Note that no more factors of $SU(4)$ can be added in F-theory because then the residual singularity locus $Y = -12K - \sum_i X_i$ could not be expressed as a sum of irreducible components without further singularities on D_v.

Given the map (3.6) we can compare the constraints on models from anomaly cancellation to the geometric constructions on the F-theory side. It is remarkable how neatly specific properties of the anomaly equations are mirrored in the F-theory geometry. For example, on the anomaly side, we know that it is not possible to have more than one gauge group factor with a negative $\tilde{\alpha}$. Thus, we cannot have more than one $SU(N)$ factor with 0 or 1 antisymmetric representations. On the F-theory side, this corresponds to the fact that the divisor D_v on $\mathbb{F}_1$ and $\mathbb{F}_2$ has $D_v \cdot D_v = -m < 0$, associated with the fact that this divisor has no deformations. Thus, all singularities associated with this topological equivalence class are coincident, and only one X_i of this type can appear in the decomposition (3.1).

The genus of an irreducible, non-singular curve in the class ξ_i is determined by the adjunction formula

$$K \cdot \xi_i + \xi_i \cdot \xi_i = 2g_i - 2. \tag{3.17}$$

Note that by equations (3.8) and (3.9), this can also be expressed as

$$g_i = \left(\frac{1}{2}\alpha_i - 1\right)\left(\frac{1}{2}\tilde{\alpha}_i - 1\right). \tag{3.18}$$

On the F-theory side, a genus g curve corresponding to an $SU(N)$ gauge group gives g hypermultiplets in the adjoint representation. In the anomaly analysis of section 2.3 for $SU(N)$ blocks with fundamental and 2-index antisymmetric matter, one of α or $\tilde{\alpha}$ is equal to 2. This fact implies that the genus of a non-singular, irreducible curve in the class ξ corresponding to the map (3.6) is always zero. This is in agreement with the fact that there are no adjoint matter hypermultiplets. We give some examples of blocks with adjoint matter in the following section.

Another property of the anomaly cancellation equations which is mirrored neatly in the F-theory geometry is the fact that blocks with values of a differing in parity cannot appear in the same model, except in special circumstances. In particular, if one $SU(N)$ block has $(\alpha_1, \tilde{\alpha}_1) = (2, a_1 - 2)$ with a_1 odd, the second group cannot have $(\alpha_2, \tilde{\alpha}_2) = (2, \text{even})$, or

the number of bifundamentals would not be integral. In F-theory, this parity constraint arises because for a even/odd with $\alpha = 2$, the map (3.6) gives a divisor on F_m with m even/odd. As a result, a second block of the above form (2, even) would map to a fractional divisor, which is not allowed. Thus, an SU(N) block with $(\alpha_1, \tilde{\alpha}_1) = (2, a_1 - 2)$ where a_1 is odd, can be combined with another block with even a_2, only if $(\alpha_2, \tilde{\alpha}_2) = (a_2 - 2, 2)$ with $a_2 \equiv 2 \pmod 4$. For example, if $(\alpha_1, \tilde{\alpha}_1) = (2, \text{odd})$ and $(\alpha_2, \tilde{\alpha}_2) = (0, 2)$, both blocks can be realized on $\mathbb{F}_1$, with the second block on D_s.

The map (3.6) defines a set of divisors in $\mathbb{F}_m$ for any model with gauge group of the form $\mathcal{G} = \prod_i \text{SU}(N_i)$ and matter in the fundamental and antisymmetric representations. In the next section we discuss the extension of this embedding to other representations and other groups. First, however we discuss the conditions which must be satisfied for the singularity locus defined in this way to give the desired F-theory model.

To show that the models defined in F-theory as described above indeed have the correct structure, we must first check that the matter content of the theory is that desired. Given a gauge group SU(N) with f fundamental and a antisymmetric matter fields, we wish to check that the F-theory model defined through the map (3.6) correctly reproduces these numbers of fields in each representation. As shown in [9], indeed

$$a = \xi \cdot (-K) = \alpha + \tilde{\alpha} \tag{3.19}$$

$$f = -8\xi \cdot K - N\xi \cdot \xi = 8a + N\alpha\tilde{\alpha}/2 = 8a + N(2 - a) \tag{3.20}$$

in agreement with the F^4 relation (2.12).

For a complete model we must also show that $n_h - n_v = 244$. Because the matter content of each gauge group is correctly reproduced by the geometric model produced through (3.6), the only question is whether the number of neutral hypermultiplets associated with the residual discriminant locus $Y = -12K - \sum_i N_i \xi_i$ is the correct number to saturate the gravity anomaly (2.4). Indeed, one of the principal results of [9] was the demonstration that this gravitational anomaly is precisely saturated, based on an explicit calculation of the number of neutral hypermultiplets arising from cusps in Y.

Thus, we have shown that for any model composed of simple blocks of the type considered so far, the map (3.6) gives an appropriate combination of divisor classes in $\mathbb{F}_m$. From the definition of the map, then, it seems plausible that we can construct an F-theory model for any anomaly-free supergravity theory in the class considered so far. To show this conclusively, however, we must check several things.

1. We must show that there are no consistent supergravity models such that the image in F-theory requires a sum of divisors so large that the residual discriminant locus Y cannot be written as a sum of effective irreducible divisors. In such a situation, such as if $\sum_i X_i = aD_v + bD_s$ with $a > 24$, there would not be an F-theory description of the complete model. We have checked that the map (3.6) leads to an acceptable set of divisors for all of the 16,418 SU(N) models explicitly tabulated in table 1. It would be nice to have a more general proof that this always works.

2. Even if all consistent supergravity models lead to configurations with acceptable Y's, we have only described the topological structure of the singularity locus. To guarantee

that the model is well-defined, we need a Weierstrass model explicitly describing the elliptic fibrations (or some other equally explicit description). We believe that such a Weierstrass model should exist for any configuration of divisors satisfying the anomaly cancellation conditions (in particular (2.4)). We return to this question in section 5.

4 More representations and groups

While we defined the map (3.6) from supergravity building blocks to F-theory divisors above in the context of SU(N) blocks with only fundamental and antisymmetric matter, it seems that (up to a constant) this map immediately provides a correct embedding of most 6D chiral supergravity models in F-theory.

In this section we expand the map to include more general SU(N) matter representations as well as other gauge groups. We give examples of various other matter representations and gauge groups, and describe their embedding in F-theory. This works in most cases, but there are some situations in which the image of a block in F-theory does not correspond to an integral divisor. These models may not have F-theory representatives and may suffer from some kind of quantum inconsistency. In other cases we find exotic matter representations for which no corresponding singularity structure has yet been identified in F-theory. We do not attempt a comprehensive analysis here of all possible gauge group and matter blocks, but give examples which display the generality of the supergravity-F-theory map (3.6).

4.1 Other representations of SU(N)

4.1.1 Adjoint representation

As mentioned above, in F-theory the genus g of the divisor determines the number d of adjoint matter representations transforming under the group associated with that divisor. In 6D supergravity, we can include d adjoint matter representations for the group SU(N). The adjoint of SU(N) has $A = B = 2N, C = 6, D = N^2 - 1$. Thus, with the addition of d adjoints the relation (2.12) between the number of fundamentals and N, a becomes

$$f = 2N - 2Nd - a(N - 8).$$ (4.1)

The anomaly equations (2.8) and (2.9) in the presence of adjoint matter are

$$\alpha + \tilde\alpha = a$$ (4.2)
$$\alpha\tilde\alpha = 2a + 4(d - 1).$$ (4.3)

The solutions to these equations are somewhat limited. For example, for one adjoint $(d = 1)$ we have $\alpha\tilde\alpha = 2a$. In this case the solutions for $\alpha, \tilde\alpha$ are only real when $a \geq 8$. For $a = 8$ we have blocks associated with SU(N) with $(\alpha, \tilde\alpha) = (4, 4)$. This maps using (3.6) to $2D_v + 3D_s$ on $\mathbb{F}_1$ which indeed is a genus 1 divisor. Similarly, for $a = 9$ we have $(\alpha, \tilde\alpha) = (6, 3)$ which maps to $3(D_v + D_s)$ which is a genus 1 divisor on $\mathbb{F}_1$.

The story becomes more unusual for $a = 10$, where we have $f = -10(N - 8)$. If we choose $N = 4$, there is a single block model with $n_h - n_v = 220$, where $\alpha, \tilde\alpha = 5 \pm \sqrt{5}$.

Since these α's are not rational, the map (3.6) does not take them to divisors on any $\mathbb{F}_m$. Thus, the one-block model with gauge group SU(4), one adjoint, 10 antisymmetric and 40 fundamental matter hypermultiplets seems to satisfy anomaly cancellation and has gauge kinetic terms with the correct sign but does not seem to have an embedding in F-theory. We comment further on this and other models with irrational $(\alpha, \tilde{\alpha})$ in section 6.

We can perform a similar analysis for $d = 2$. The smallest value of a for which $\alpha, \tilde{\alpha}$ are real is $a = 10$, for which $(\alpha, \tilde{\alpha}) = (4, 6)$ (in either order). This could correspond to various divisors such as $2D_v + 3D_s$ on $\mathbb{F}_0$, all of which have genus $g = 2$.

Because a single adjoint matter hypermultiplet has the same dimension as the vector multiplet, the contribution to $n_h - n_v$ from any block with at least one adjoint matter multiplet is necessarily positive, and is at least $Nf \geq Nf/2$. Thus, the same algorithm as used in section 2 can be used to classify and enumerate all models including those with adjoint matter.

4.1.2   3-index antisymmetric representation

Now, consider including the 3-index antisymmetric representation, which has (these constants, found in [13], can be reproduced by simply considering the action of two orthogonal diagonal SU(N) generators on the states labeled by Young tableaux).

$$A_{3a} = \frac{1}{2}(N^2 - 5N + 6) \qquad\qquad B_{3a} = \frac{1}{2}(N^2 - 17N + 54) \qquad (4.4)$$

$$C_{3a} = (3N - 12) \qquad\qquad D_{3a} = \frac{1}{6}N(N-1)(N-2)\,. \qquad (4.5)$$

Using these relations (2.12) is modified to

$$f = 2N - a(N - 8) - \frac{1}{2}(N^2 - 17N + 54)t\,, \qquad (4.6)$$

where t denotes the number of hypermultiplets in the 3-index antisymmetric representation. The anomaly polynomial again factorizes, in the form

$$I = (\text{tr}R^2 - 2\text{tr}F^2)(\text{tr}R^2 - (a - 2 + (N - 4)t)\text{tr}F^2) \qquad (4.7)$$

so (up to exchange) we have

$$\alpha = 2, \qquad \tilde{\alpha} = a - 2 + (N - 4)t\,. \qquad (4.8)$$

The contribution to the matter bound is

$$n_h - n_v = 1 + N\left(f + a(N - 1)/2 + t(N - 1)(N - 2)/6 - N\right)\,. \qquad (4.9)$$

Restricting to single blocks with $n_h - n_v \leq 244$ there are solutions for $N = 6, 7, 8$ (for $N < 6$ the 3-index antisymmetric representation is equivalent to the fundamental, antisymmetric, or conjugate thereof). For $N = 6$, there can be $t = 1, 2$ or 3 fields in the 3-index antisymmetric representation, with a up to 5, 3, or 1 in these respective cases, for a total of 12 distinct models. For $N = 7$ with $t = 1$ the range of a is up to 3, and there is a model

with $t = 2, a = 0$. For $N = 8$ there are models with $t = 1$ and $a = 0, 1$. Each of these models maps to a corresponding divisor in F-theory. For example, the $N = 8, a = 1$ model has $\tilde{\alpha} = 3$ so maps to $D_v + 2D_s$ in $\mathbb{F}_1$. The singularity structure corresponding to these matter representations for $N = 6, 7, 8$ is described in F-theory in [10]

If we extend to multiple-block models, there may be other possibilities. For example, the block with $N = 9, t = 1, a = 0$ has $f = 27$ fundamentals. By itself, the contribution to $n_h - n_v$ from this block is 247, but it may be possible to combine this with other blocks in a complete model. The singularity type associated with a divisor of this kind is unknown. It would be interesting to either show that this block cannot appear in a complete supergravity theory, or find an F-theory realization of a model containing this block.

4.1.3 Symmetric representation

When we include s symmetric representations the anomaly polynomial no longer has an obvious algebraic factorization in general. The F^4 anomaly condition is then modified from (2.12) to

$$f = 2N - a(N - 8) - s(N + 8). \tag{4.10}$$

Including symmetric representations as well as antisymmetric and fundamental, a systematic analysis finds 44 single-block models with various combinations of f, a, N. One interesting set of cases is when $a = 0, s = 1$. In this case, $f = N - 8$ and the anomaly factorizes with

$$\alpha = 1, \qquad \tilde{\alpha} = -2. \tag{4.11}$$

In this case, the map (3.6) does not take the block to an integral divisor on any $\mathbb{F}_m$. On $\mathbb{F}_4$, the image is $D_v/2$. This is another example of a block which does not have a clear corresponding geometric structure in F-theory. Like the previous example it is characterized by its failure to give an integral divisor under the map (3.6).

There are other configurations with symmetric representations which are better behaved. If we have $s \geq 1$ with $a > 8$, there are a variety of solutions. For example, for $N = 4$ there are one-block solutions with $s = 1, a = 9, \ldots, 12$, as well as with $(s, a) = (2, 13), (3, 15), (4, 17), (6, 20)$. Other similar solutions exist for N up to 8. As an example of a block of this type we have

$$\mathrm{SU}(4), s = 5, a = 18, f = 20, \alpha = 6, \tilde{\alpha} = 7 \tag{4.12}$$

For these solutions the associated divisors are generally integral.

In F-theory, symmetric matter does not arise from a local enhancement of the singularity and cannot be determined just from the topological class of the singularity locus. When the curve of A_{N-1} singularities is itself singular with s double points, we have s symmetric hypermultiplets [8]. Since the map (3.6) only determines the topological class of the discriminant locus, more information is needed to encode models with this type of matter in F-theory. This additional information about the number of double points, must be included to explicitly construct a Weierstrass model for a theory with matter transforming under the symmetric representation of $\mathrm{SU}(N)$.

4.1.4   4-index antisymmetric representations

We can consider still larger representations. For example, it is natural to consider the 4-index antisymmetric representation of $SU(N)$. There are a couple of exotic blocks with $SU(8)$ gauge group and matter content

$$\square + 3\,\square\square + 2\,\squarebox + \squarebox \quad (n_h - n_v = 243) \tag{4.13}$$

$$2\,\squarebox + 3\,\square\square + 2\,\squarebox \quad (n_h - n_v = 241)\,. \tag{4.14}$$

Both these blocks have $(\alpha, \tilde{\alpha}) = (6, 5)$. We are not aware of a singularity structure in F-theory which would produce the 4-index antisymmetric tensor representation, but it is possible that such an exotic singularity structure could exist.

4.1.5   Larger representations

As the matter representations become larger, the contribution to $n_h - n_v$ from these hypermultiplets increases. As a consequence, for more complicated representations than those considered above there are very few values of N which do not immediately oversaturate the $n_h - n_v = 244$ bound. We have not attempted to completely classify the supergravity blocks which may include these larger representations. We leave the investigation of these more exotic models to future work.

4.2 $SU(2)$ and $SU(3)$

Blocks with gauge group $SU(2)$, $SU(3)$ are special in the $SU(N)$ series, as they do not have an irreducible fourth-order invariant. In addition, since $\pi_6(SU(2)) = \mathbb{Z}_{12}$ and $\pi_6(SU(3)) = \mathbb{Z}_6$, we have to consider possible global anomalies [21]. We consider blocks with $SU(2)$ or $SU(3)$ gauge group and f hypermultiplets in the fundamental representation.[2] These groups were analyzed in [20], and we simply state the results of applying the map (3.6) in these cases. From the anomaly polynomial, the values of $\alpha, \tilde{\alpha}$ are

$$SU(2): \quad (\alpha, \tilde{\alpha}) = \left(2, \frac{f - 16}{6}\right) \tag{4.15}$$

$$SU(3): \quad (\alpha, \tilde{\alpha}) = \left(2, \frac{f - 18}{6}\right) \tag{4.16}$$

Fractional values of $\alpha, \tilde{\alpha}$ map to non-integral divisor classes under the map (3.6). This appears at first to give another class of non-integral exceptional cases for the map to F-theory, but in this case global anomalies constrain the number of fundamental hypermultiplets modulo 6 through

$$SU(2): \quad f \equiv 4 \,(\mathrm{mod}\ 6)$$
$$SU(3): \quad f \equiv 0 \,(\mathrm{mod}\ 6) \tag{4.17}$$

---

[2]The 2-index antisymmetric of $SU(2)$ is trivial, and of $SU(3)$ is just the anti-fundamental.

Thus, the absence of global anomalies implies the integrality of $\alpha, \tilde{\alpha}$.

The constraints from global anomalies in equation (4.17), first derived in [20], can be understood from Higgsing. Consider a model with gauge group $SU(N)$ with f fundamental and a antisymmetric hypermultiplets. We can Higgs the gauge group down to $SU(N-1)$ by turning on a VEV for the fundamental hypermultiplets. Thus, we end up with a model with gauge group $SU(N-1)$ and f' fundamentals and a' antisymmetrics. The Higgsing can be worked out in the more familiar 4D, $\mathcal{N} = 2$ language, and it turns out that $f' = f - 2 + a$, $a' = a$. Note that $f' = 2(N-1) - a'((N-1) - 8)$, which implies that Higgsing preserves the form of the trF^4 condition for $N \geq 4$. However, if we Higgs from $SU(4) \rightarrow SU(3)$, there is no trF^4 condition. Moreover, the antisymmetric representation is equivalent to the (anti) fundamental. Therefore, for $SU(3)$, $f' = f - 2 + 2a \Rightarrow f' = 6(a+1)$. This is in agreement with (4.17). When the $SU(3)$ is then Higgsed down to $SU(2)$, we must have $f'' = f' - 2 = 4 + 6a$, which again agrees with (4.17).

The gravitational anomaly requires that $f \leq 118$ for $SU(2)$ and $f \leq 84$ for $SU(3)$. We now check the validity of the divisor map (3.6) for the $SU(2)$ model with 118 fundamental hypermultiplets.

$$(\alpha, \tilde{\alpha}) = (2, 17) \longrightarrow D_v + 9D_s \text{ on } \mathbb{F}_1 \tag{4.18}$$

This does not oversaturate the Kodaira formula (3.5). For the $SU(3)$ model with 84 fundamentals,

$$(\alpha, \tilde{\alpha}) = (2, 11) \longrightarrow D_v + 6D_s \text{ on } \mathbb{F}_1 \tag{4.19}$$

The divisor map (3.6) thus works without exception for this class of $SU(2)$ and $SU(3)$ blocks.

We have incorporated all possible blocks with $SU(3)$ gauge groups into the systematic analysis described in 2.3. Including $SU(3)$ blocks increases the total number of possible models to 68,997, with the number of models for a fixed number of factors maximized at 20,639 models with 4 factors. The largest number of factors possible including $SU(3)$ blocks is 13, which occurs for a single model with gauge group

$$G = SU(18) \times SU(3)^{12}, \tag{4.20}$$

where there is a single bifundamental representation $(18, 3)$ for each factor $SU(3)$, and no other matter fields transforming under any of the gauge group components. The F-theory map takes the $SU(18)$ to D_v on $\mathbb{F}_2$, and each $SU(3)$ factor to D_s. We discuss this case in more detail in section 5.2.3. Note that the total of 68,997 models including $SU(3)$ blocks includes 46 models containing an $SU(3)$ with no fundamental matter. Such a block has $(\alpha, \tilde{\alpha}) = (2, -3)$, and is associated with the divisor D_v on $\mathbb{F}_3$; the 46 models containing this block can only be realized on $\mathbb{F}_3$.

4.3 Tri-fundamental representation of $SU(M) \times SU(N) \times SU(P)$

It is possible to have matter charged simultaneously under three factors of the gauge group. The anomaly conditions constrain the number of hypermultiplets that are simultaneously charged under two factors of the gauge group. A tri-fundamental representation can occur only if the anomaly conditions allow for sufficiently many bifundamentals between

every pair of groups. Through a complete enumeration of three block models with gauge groups $SU(M) \times SU(N) \times SU(P)$, we find 848 models with one hypermultiplet in the tri-fundamental representation.

In F-theory, a tri-fundamental of $SU(2) \times SU(2) \times SU(N)$ can be realized if the three singular loci corresponding to the three factors intersect at a point, and at that point the singularity type is enhanced to D_{N+2} [19]. Similarly, for $SU(2) \times SU(3) \times SU(5)$, we would require the locus of A_1, A_2 and A_4 singularities to intersect at a point, with enhancement to E_8. In an analogous manner, we can realize tri-fundamentals of $SU(2) \times SU(3) \times SU(4)$ and $SU(2) \times SU(3) \times SU(3)$ through enhancements to E_7 and E_6 respectively. In our exhaustive enumeration of three-stack models, we find that there are two models with gauge group $SU(2) \times SU(3) \times SU(6)$ with matter content

$$40(\square, 1, 1) + 36(1, \square, 1) + 8(1, 1, \square) + 1(1, 1, \boxminus) + 1(\square, \square, \square)$$
$$43(\square, 1, 1) + 40(1, \square, 1) + 1(\square, \square, 1) + 1(\square, \square, \square).$$

The other models can all be realized using the singularity types discussed above. We are not aware of the singularity structure in F-theory that can realize the $SU(2) \times SU(3) \times SU(6)$ models. We postpone the analysis of these cases to future work. One possible realization of tri-fundamental matter fields might be through string junctions (see, e.g., [22]) which end on three 7-brane stacks and hence carry charge under three groups.

4.4   SO(N)

So far, we have used the map (3.6) to take blocks with $SU(N)$ gauge group to F-theory divisors. In fact, essentially the same map works for all simple groups, up to an overall constant which depends upon the group. In this section, we consider the case of $SO(N)$. If we only have fundamental representations (or bifundamental), then the F^4 condition gives

$$f = N - 8 \tag{4.21}$$

and the anomaly polynomial factorizes as

$$I = (\rho - \phi)(\rho + 2\phi). \tag{4.22}$$

Thus, $\alpha, \tilde{\alpha} = 1, -2$ (in either order), and the gauge group can only have one such factor. For a single SO(N) block we have $fN = N(N-8) \le 244 + N(N-1)/2$ so $N \le 30$.

For the gauge group $SO(N)$, we use the map

$$\alpha(D_v + m/2 D_s) + \tilde{\alpha} D_s. \tag{4.23}$$

Note that the normalization factor is different from the divisor map (3.6) for the $SU(N)$ case by a factor of 2. This normalization factor depends on the choice of trace convention in the fundamental representation. We choose the normalization factor here to give an integral divisor. The divisor is irreducible and effective only for $m = 4$. F-theory on a CY 3-fold with base F_4 is dual to the SO(32) heterotic string. At a general point in its moduli space, there is an unbroken SO(8) gauge group. This corresponds to the model

with 0 fundamentals. The maximal gauge group SO(30) can be realized with the SO(32) heterotic string, by choosing a U(1) gauge bundle of instanton number 24 [1, 14].

In [3], we found a model with gauge group SU(24) × SO(8) with 3 hypermultiplets in the ($\square$, 1) representation. The values of $(\alpha, \tilde{\alpha})$ are $(1, 2)$ for the SU(24) and $(1, -2)$ for the SO(8). From the divisor map (4.23), the SO(8) is realized on D_v in the base F_4. The SU(24) singularity, however, is mapped to a fractional divisor $\frac{1}{2}D_u$. This gives another example of an apparently anomaly-free supergravity model with a block which maps to a non-integral divisor in F-theory.

4.5 Exceptional groups

For the E_n groups, we can again compute the map in the same way. From [13] we have the following anomaly coefficients for the fundamental and adjoint representations of E_6, E_7, E_8

Group	Representation	A_R	B_R	C_R
E_6	fundamental	1	0	$\frac{1}{12}$
	adjoint	4	0	$\frac{1}{2}$
E_7	fundamental	1	0	$\frac{1}{24}$
	adjoint	3	0	$\frac{1}{6}$
E_8	fundamental	1	0	$\frac{1}{100}$
	adjoint	1	0	$\frac{1}{100}$

(Note that the adjoint of E_8 is equivalent to the fundamental, up to a constant.)

Again, defining the divisor map for each gauge group requires a choice of normalization constant. Choosing constant factors $3, 6, 30$ for E_6, E_7, E_8 gives the only possible map from these groups without matter to acceptable F-theory divisors

$$E_6: \quad (\alpha, \tilde{\alpha}) = \left(\frac{1}{3}, -1\right) \rightarrow D_v \text{ on } \mathbb{F}_6 \tag{4.24}$$

$$E_7: \quad (\alpha, \tilde{\alpha}) = \left(\frac{1}{6}, -\frac{2}{3}\right) \rightarrow D_v \text{ on } \mathbb{F}_8 \tag{4.25}$$

$$E_8: \quad (\alpha, \tilde{\alpha}) = \left(\frac{1}{30}, -\frac{1}{5}\right) \rightarrow D_v \text{ on } \mathbb{F}_{12} \tag{4.26}$$

Note that essentially the same choice of constant was made in the analysis of [9] to relate geometric structure to anomaly conditions. In general, then, the map is defined as

$$E_6: \quad (\alpha, \tilde{\alpha}) \longrightarrow 3\left[\alpha(D_v + \frac{m}{2}D_s) + \tilde{\alpha}D_s\right] \tag{4.27}$$

$$E_7: \quad (\alpha, \tilde{\alpha}) \longrightarrow 6\left[\alpha(D_v + \frac{m}{2}D_s) + \tilde{\alpha}D_s\right] \tag{4.28}$$

$$E_8: \quad (\alpha, \tilde{\alpha}) \longrightarrow 30\left[\alpha(D_v + \frac{m}{2}D_s) + \tilde{\alpha}D_s\right] \tag{4.29}$$

For an E_n block with f fundamental matter fields, we thus have

$$E_6: (\alpha, \tilde{\alpha}) = \left(\frac{1}{3}, \frac{f-6}{6}\right) \longrightarrow D_v + \frac{m+f-6}{2}D_s \text{ on } \mathbb{F}_m \tag{4.30}$$

$$E_7: (\alpha, \tilde{\alpha}) = \left(\frac{1}{6}, \frac{f-4}{6}\right) \longrightarrow D_v + \frac{m+2f-8}{2}D_s \text{ on } \mathbb{F}_m \tag{4.31}$$

We can confirm that this map works by considering the heterotic string on a K3 surface at the point with gauge symmetry $E_7 \times E_8$. This is obtained by having all 24 instantons in a single $SU(2) \subset E_8$, which breaks E_8 down to the maximal subgroup $SU(2) \times E_7$. From the index theorem, the matter content can be worked out to be 10 hypermultiplets (or 20 half-hypermultiplets) in the fundamental of E_7. Since this model corresponds to a point on the branch of the heterotic string with instanton numbers $(24, 0)$, the dual F-theory construction has base $\mathbb{F}_{12}$. This is in agreement with the divisor map — the E_8 block with no charged matter is realized on D_v and the E_7 block with 10 $\mathbf{56}$ hypermultiplets is realized on D_u. More generally, we could have instanton numbers $(12 - k, 12 + k)$ in $E_8 \times E_8$, and put all the instantons in a single $SU(2)$ subgroup of each E_8 factor, resulting in the gauge group $E_7 \times E_7$. The matter content computed by the index theorem gives $(8 - k)/2$ $(56, 1)$ and $(k + 8)/2$ $(1, 56)$. To obtain fermions of the right chirality to form hypermultiplets, we need $k \leq 8$. When k is odd, we end up with a half-hypermultiplet, which is allowed as the 56 of E_7 is pseudoreal. From the divisor map (4.31), the first E_7 is realized on $D_v + \frac{m-k}{2} D_s$, which is irreducible only for $m = k$, thus fixing m. The second E_7 is realized on $D_v + m D_s = D_u$. There is no bifundamental matter, in agreement with $D_u \cdot D_v = 0$. This verifies the consistency of this map with known heterotic constructions through the F-theory-heterotic duality [4–6].

4.6 Non-simply laced groups

A similar analysis to the previous cases gives the map for the non-simply laced groups. For F_4 and G_2, which have no quartic invariant, including f matter fields in the fundamental representation, we have

$$F_4 : (\alpha, \tilde{\alpha}) = \left(\frac{1}{3}, \frac{f - 5}{6} \right) \longrightarrow D_v + \frac{m + f - 5}{2} D_s \text{ on } \mathbb{F}_m \tag{4.32}$$

$$G_2 : (\alpha, \tilde{\alpha}) = \left(1, \frac{f - 10}{6} \right) \longrightarrow D_v + \frac{3m + f - 10}{6} D_s \text{ on } \mathbb{F}_m \tag{4.33}$$

In the case of G_2, we see that $f \equiv 1 \pmod{3}$ is needed for an integer divisor on some $\mathbb{F}_m$, in agreement with global anomaly cancellation conditions.

For $Sp(N)$ with f fundamentals, the story is similar to $SO(N)$. Cancellation of the F^4 anomaly gives

$$f = 2N + 8, \tag{4.34}$$

and the values of $\alpha, \tilde{\alpha}$ are 2, -1, associated with D_v on $\mathbb{F}_4$.

5 Realizations in F-theory

As discussed in section 3, the map from supergravity models to F-theory gives the topological data of the discriminant locus and singularity structure needed for the corresponding F-theory construction, but this does not immediately lead to an explicit construction of these elliptic fibrations through something like a Weierstrass model.

Conjecture: *every combination of effective divisors X_i and residual divisor Y associated through (3.6) with a 6D supergravity theory satisfying the anomaly conditions, including the gravity bound (2.4) associated with the Euler character of the total space of the elliptic fibration as described in [9], gives rise to an explicit elliptic fibration through a Weierstrass model.*

We do not have a proof of this conjecture in general. In a number of cases we have considered explicitly, however, the contribution of $n_h - n_v$ to the total gravitational anomaly for a supergravity block can be identified directly with the number of degrees of freedom in the Weierstrass model which are fixed in imposing the desired singularity structure on the associated divisor. This suggests that there is a generic sense in which this conjecture should hold, since in any model the number of unfixed degrees of freedom in the Weierstrass model should correspond to the number of neutral hypermultiplets in the corresponding supergravity theory. We give a concrete example of how this works for a specific class of Weierstrass models below.

Extending the map defined in this paper to all possible building blocks with arbitrary simple gauge groups and matter content, along with a proof of this conjecture, would suffice to prove the "string universality" conjecture [3] for chiral 6D supergravity theories, to the extent that all configurations of gauge groups and matter fields allowed in consistent models could be embedded in F-theory. Note that for general models including arbitrary matter types and non-simply laced groups, the construction of appropriate Weierstrass models must include all appropriate singularity types and monodromies to realize the supergravity matter content and gauge group.

To demonstrate the plausibility of the above conjecture, we now give some explicit examples of elliptic fibrations over $\mathbb{F}_m$ for single-block models with gauge group $SU(N)$. We also consider some cases with gauge group E_6 and E_7 with fundamental matter. We show that anomaly-free supergravities in these classes can be realized as explicit F-theory compactifications through Weierstrass models.

5.1 Weierstrass models on Hirzebruch surfaces

We first review the basics of Hirzebruch surfaces as presented in [5–7]. The surface $\mathbb{F}_m$ is defined as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ as follows

$$\mathbb{F}_m := \{(u, v, s, t) \in \mathbb{C}^4 \backslash Z : (u, v, s, t) \sim (\mu \lambda^m u, \mu v, \lambda s, \lambda t), \ \lambda, \mu \in \mathbb{C}^*\} \qquad (5.1)$$

Z is the set of fixed points of the $\mathbb{C}^*$-action specified by λ, μ. The divisors D_u, D_v and D_s as discussed in section 2.1, correspond to the curves $u = 0$, $v = 0$ and $s = 0$ respectively. An elliptically fibered Calabi-Yau 3-fold on the base $\mathbb{F}_m$ can be specified by the Weierstrass equation

$$y^2 = x^3 + f(s, t, u, v)xz^4 + g(s, t, u, v)z^6 \qquad (5.2)$$

in the weighted projective space $\mathbb{P}^{2,3,1}$. The functions f, g are sections of the line bundles $-4K$ and $-6K$ respectively, where K is the canonical bundle. In this section, we consider fibrations over $\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2$, where the fiber suffers an A_{N-1} (type I_{N-1}) degeneration on the

m	0	1	2	3	4	5	6	7	8	9	10	11	12
DOF	244	243	242	251	268	294	318	348	376	404	433	453	482

Table 3. Degrees of freedom (DOF) in terms of coefficients of polynomials that appear in the Weierstrass equation describing an elliptic fibration over $\mathbb{F}_m$.

locus $v = 0$. In the coordinate patch $w = v/u$, $z = s/t$, the defining polynomials $f(w, z)$ and $g(w, z)$ take the form

$$f(w, z) = \sum_{i=0}^{8} w^i f_{8-4m+mi}(z) \tag{5.3}$$

$$g(w, z) = \sum_{j=0}^{12} w^j g_{12-6m+mi}(z) \tag{5.4}$$

The limits in the summations above need to be adjusted to ensure that all polynomials have non-negative degree.

The degeneration locus of the elliptic fibration is given by the vanishing of the discriminant of the defining equation (5.2).

$$\Delta(w, z) = 4f(w, z)^3 + 27g(w, z)^2 \tag{5.5}$$

For the total space of the elliptic fibration to be Calabi-Yau, we need $m \leq 12$. The number of degrees of freedom in f, g associated with the coefficients of the polynomials is shown in table 3. We have subtracted the deformations that correspond to symmetries of $\mathbb{F}_m$, and the overall scale in the discriminant. The dimension of the automorphism group of $\mathbb{F}_m$ can be computed to be $m + 5$ using the formula in [23]. In the specific case of $\mathbb{F}_2$, we show by example how the neutral hypermultiplets from the supergravity theory exactly match with the degrees of freedom available in the Weierstrass model.

5.2 SU(N)

In order to have an A_{N-1} degeneration on the locus $w = 0$, we require that $\mathrm{ord}_{w=0}(\Delta) = N$ and $\mathrm{ord}_{w=0}(f) = \mathrm{ord}_{w=0}(g) = 0$. In addition, Tate's algorithm [7] requires an auxiliary polynomial to factorize, corresponding to the I_N split condition; we discuss this condition in the following section. If the discriminant is of the form $\Delta = w^N(p(z) + wq(z, w))$, the locus $w = 0$ is intersected by the other component $p(z) + wq(z, w) = 0$ at the zeroes of the polynomial $p(z)$. At these points $z = \zeta$, the singularity type of the fiber is enhanced to A_N. In terms of the low-energy theory, this implies that a matter hypermultiplet in the fundamental representation of SU(N) is localized at every zero ζ. For 2-index antisymmetric matter, we require that the singularity type be enhanced to D_N at special points on the locus $w = 0$. (At these points, f and g will vanish, whereas they do not vanish when the fiber is enhanced to A_N.) In this section, we construct Weierstrass models on bases $\mathbb{F}_1$ and $\mathbb{F}_2$ with A_{N-1} locus $w = 0$, which correspond to models with gauge group SU(N) and matter hypermultiplets in the fundamental and 2-index anti-symmetric representation (see table 2). At a general point in moduli space for these models, i.e. with a general choice of polynomials $f(w, z), g(w, z)$, the gauge group is completely broken.

5.2.1 $\mathbb{F}_2$

On $\mathbb{F}_2$, as shown in table 3, the coefficients in the polynomials f, g encode 242 independent degrees of freedom. With an A_{N-1} singularity along the locus D_v ($w = 0$) of $\mathbb{F}_2$, we can realize models with gauge group SU(N) and $N_f = 2N$ hypermultiplets in the fundamental representation. The gravitational anomaly condition requires that $n_h - n_v = 244$ (including neutral hypermultiplets), and this implies that $N \leq 15$. The matter content requires that the discriminant take the form

$$\Delta(w, z) = w^N (p_{2N}(z) + wq(z) + \ldots), \quad N \leq 15 \tag{5.6}$$

where $p_{2N}(z)$ is a polynomial with $2N$ distinct zeroes ζ_i and $q(\zeta_i) \neq 0$. This requirement in fact, uniquely picks out the base $\mathbb{F}_2$.

The functions $f(w, z), g(w, z)$ for the base $\mathbb{F}_2$ can be written as

$$f(w, z) = \sum_{i=0}^{8} w^i f_{2i}(z) \tag{5.7}$$

$$g(w, z) = \sum_{j=0}^{12} w^j g_{2j}(z) \tag{5.8}$$

The discriminant is

$$\Delta(w, z) = 4f^3 + 27g^2 \tag{5.9}$$

$$= 4f_0^3 + 27g_0^2 + w(12f_0^2 f_2 + 54g_0 g_2) + \ldots \equiv \sum_{k=0}^{24} C_{2k}(z) w^k \tag{5.10}$$

The coefficients $C_{2k}(z)$ in the expansion of Δ are polynomials of degree $2k$ in z. In order to have an A_{N-1} singularity along $w = 0$, we need to tune the polynomials f_{2i} and g_{2j} so that $C_0 = C_2 = \ldots = C_{2N-2} = 0$. With the first N coefficients set to zero, the discriminant is of the form

$$\Delta = w^N (C_{2N}(z) + wC_{2N+2}(z) + \ldots) \tag{5.11}$$

At the zeroes of $C_{2N}(z)$, the singularity type is enhanced from A_{N-1} to A_N, and as discussed earlier, this leads to $2N$ matter hypermultiplets in the fundamental representation. (Note that since $-K \cdot D_v = 0$, neither f nor g will vanish along D_v, so there is no anti-symmetric matter.)

We can see how the discriminant can be made to vanish with $\mathrm{ord}_{w=0}(\Delta) = N$ order by order as follows. Since the overall scale in the discriminant polynomial does not matter, we can set $f_0 = -3$ without using up any degrees of freedom. We can now fix $g_0 = 2$, without using any degrees of freedom as this just fixes the location of singularity at $w = 0$. By choosing $g_2(z) \equiv -f_2(z)$, we use up 3 degrees of freedom and the discriminant is of the form

$$\Delta(w, z) = -9(f_2^2 - 12(f_4 + g_4))w^2 + \mathcal{O}(w^3) \tag{5.12}$$

Next, by choosing

$$g_4(z) \equiv \frac{1}{12} \left(f_2^2 - 12f_4 \right) \tag{5.13}$$

the discriminant can be made to vanish to order three, and we have used up another 5 degrees of freedom. In this manner, by an appropriate choice of polynomial $g_{2k}(z)$, the coefficient $C_{2k}(z)$ in the discriminant can be made to vanish for $k \leq 12$. Thus, in order to obtain a gauge symmetry SU(N), $N \leq 13$, we need to fix the polynomials $g_2, g_4, \ldots g_{2N-2}$ and therefore use up $N^2 - 1$ degrees of freedom. The number of residual degrees of freedom works out to $242 - N^2 + 1 = 243 - N^2$, and these should correspond to neutral hypermultiplets. This agrees beautifully with a similar calculation from the anomaly: $n_h - n_v = N^2 + 1$, and we need to add $243 - N^2$ neutral hypermultiplets to satisfy the gravitational anomaly condition. Thus, we see that at each value of N, the number of neutral hypermultiplets on the supergravity side precisely corresponds to the number of unfixed degrees of freedom in the F-theory polynomials. We expect that this will be the case quite generally, so that a correspondence can be made between the contribution of any supergravity block to $n_h - n_v$ and the additional coefficients which must be fixed in the Weierstrass polynomials to encode the corresponding singularity. We leave a general proof of this assertion as a challenge for the future.

The gravitational anomaly condition imposes $N \leq 15$. In the analysis above, we showed that SU(13) gauge symmetry could be obtained by just using the g_{2k} polynomials. In order to go further, we need to use the degrees of freedom in the f_{2k} polynomials. The discriminant for SU(13) gauge symmetry is of the form

$$\Delta(w, z) = C_{26} w^{13} + C_{28} w^{14} + C_{30} w^{15} + \ldots \tag{5.14}$$

We have $243 - 13^2 = 74$ actual degrees of freedom in the polynomials f_{2k}. It is easy to see that with an appropriate choice of coefficients in these polynomials, C_{26} and C_{28} could generically be made to vanish, but not C_{30}. This agrees nicely with the computation from anomaly cancellation in the low-energy theory. We have an explicit solution for the SU(14) case, which we present below.

$$f(w, z) = -3 + 12h_8 w^4 + 12h_{10} w^5 + 6h_{10} h_2 w^6 + 2h_{14} w^7 + (-12h_8^2 - h_{14} h_2) w^8 \tag{5.15}$$

$$\begin{aligned} g(w, z) = {}& 2 - 12h_8 w^4 - 12h_{10} w^5 - 6h_{10} h_2 w^6 - 2h_{14} w^7 + (24h_8^2 + h_{14} h_2) w^8 \\ & + 24h_{10} h_8 w^9 + 12h_{10}(h_{10} + h_8 h_2) w^{10} + 4(h_8 h_{14} + 3h_{10}^2 h_2) w^{11} \\ & + (-16h_8^3 - 2h_8 h_{14} h_2 + h_{10}(4h_{14} + 3h_{10} h_2^2)) w^{12} \end{aligned} \tag{5.16}$$

The h_i are general polynomials in z. The discriminant is

$$\Delta(w, z) = -36 w^{14}(h_{14}^2 - 3h_{10} h_{14} h_2^2 + 3h_{10}^2(24h_8 + f_6 h_2) + \mathcal{O}(w)) \tag{5.17}$$

We have not found an explicit Weierstrass model for the SU(15) case. While this problem seems difficult algebraically, counting degrees of freedom seems to indicate that this should be possible. Furthermore, while similar algebraic difficulties appear in an analogous construction for SO(N) theories on $\mathbb{F}_4$ at $N = 30$, we know that the SO(30) model has an explicit string construction, as mentioned in section 4.4. We leave the explicit construction of the SU(15) model in this family as a challenge for the future.

All the supergravity models in this family are related to the (hypothetical) SU(15) model by Higgsing; turning on a VEV for a fundamental hypermultiplet in SU(N), breaks

the gauge group to $SU(N-1)$ and a mass term is generated for two fundamental hyper-multiplets. This shows that the number of $SU(N-1)$ hypermultiplets in the low-energy theory is $f - 2 = 2(N-1)$, in agreement with (2.12) for $a = 0$.

5.2.2 $\mathbb{F}_1$

In this subsection, we construct $SU(N)$ models with $N + 8$ fundamental hypermultiplets and one 2-index antisymmetric hypermultiplet. To accomplish this, we engineer an A_{N-1} singularity along the $w = 0$ locus of $\mathbb{F}_1$, which corresponds to the divisor D_v (see table 2).

The polynomials f, g on $\mathbb{F}_1$ take the form –

$$f(w, z) = \sum_{i=0}^{8} w^i f_{i+4}(z) \tag{5.18}$$

$$g(w, z) = \sum_{j=0}^{12} w^j g_{j+6}(z) \tag{5.19}$$

The discriminant locus is of the form

$$\Delta(w, z) = 4f_4^3 + 27g_6^2 + w(12f_4^2 f_5 + 54g_6 g_7) + \ldots \equiv \sum_{k=0}^{24} C_{k+12}(z) w^k \tag{5.20}$$

An $SU(N)$ singularity requires the discriminant to vanish at order N on the locus $w = 0$. We will see that once this singularity is engineered, the matter content works out very nicely in accordance with anomaly cancellation. To obtain $SU(4)$ gauge symmetry, we can choose

$$f_4(z) = -3q_2(z)^2, \qquad f_5(z) = p_3(z)q_2(z),$$

$$g_6(z) = 2q_2(z)^3, \qquad g_7(z) = -f_5(z)q_2(z), \, g_8(z) = \left(-f_6(z) + \frac{1}{12}p_3(z)^2\right)q_2(z), \quad (5.21)$$

$$g_9(z) = \frac{1}{216}\left(36f_6(z)p_3(z) + p_3(z)^3 - 216f_7(z)q_2(z)\right)$$

Here $q_2(z)$ and $p_3(z)$ are arbitrary polynomials of degree 2 and 3 respectively. For the singularity to produce $SU(4)$ gauge symmetry, the polynomial $q_2(z)$ must be perfect square, so $q_2(z) = \lambda(z - z_0)^2$. This corresponds to the split I_4 singularity in Tate's algorithm discussed in [7]. With this choice, the discriminant takes the form

$$\Delta(w, z) = w^4 \left[(z - z_0)^4 C_{12}(z) + \mathcal{O}(w)\right] \tag{5.22}$$

Here $C_{12}(z)$ is a general polynomial of degree 12 with distinct roots. The locus $w = 0$ is intersected by the residual locus at the point $z = z_0$. The singularity type is enhanced to D_4, and thus, we obtain one antisymmetric tensor of $SU(4)$.

In the general $SU(N)$ case, when $N = 2k$ the structure of the singular locus of the fibration is similar to the $SU(4)$ case. The discriminant is of the form

$$\Delta(w, z) = w^N \left[(z - z_0)^4 C_{N+8}(z) + \mathcal{O}(w)\right] \tag{5.23}$$

The fact that the polynomial $C_{N+8}(z)$ has $N+8$ distinct roots results in $N+8$ fundamental hypermultiplets, in agreement with the anomaly calculation. At $z = z_0$, the singularity is enhanced to D_N (since f and g vanish there), and we have antisymmetric matter localized at this point. When $N = 2k + 1$, however, the singularity structure is slightly different, and we discuss this in the SU(5) case. For an SU(5) singularity, in addition to the choices made in (5.21), we need

$$f_6 = -\frac{1}{12}p_3^2 + p_4 q_2, \qquad g_{10}(z) = \frac{1}{12}\left(2f_7 p_3 - 12 f_8 q_2 + p_4^2 q_2\right) \tag{5.24}$$

where $q_2(z) = \lambda(z - z_0)^2$, and p_3, p_4 are general polynomials in z of degree 3 and 4 respectively. The discriminant is

$$\Delta(w, z) = w^5 \left[(z - z_0)^6 C_{11}(z) + \mathcal{O}(w)\right] \tag{5.25}$$

At first sight, this appears to be at odds with the anomaly conditions, since we would have only 11 fundamental hypermultiplets at the roots of $C_{11}(z)$. It turns out that at the point $w = 0, z = z_0$, the singularity type is enhanced all the way to D_6 (split I_2^* according to Tate's algorithm). This enhancement results in an antisymmetric tensor of SU(5) and 2 fundamental hypermultiplets, so all together we still have 13 fundamental hypermultiplets as required by the anomaly conditions. For the general SU(N) case, when N is odd, the discriminant takes the form

$$\Delta(w, z) = w^N \left[(z - z_0)^6 C_{N+6}(z) + \mathcal{O}(w)\right] \tag{5.26}$$

At $z = z_0$ the singularity type is enhanced all the way to D_{N+1} which provides the additional 2 hypermultiplets in the fundamental.

The gravitational anomaly restricts $N \leq 15$, and as in the previous case, this agrees with the counting of degrees of freedom in the Weierstrass model. Again, we have explicitly constructed a Weierstrass model only for the SU(14) case, though a count of the degrees of freedom suggest that an SU(15) Weierstrass model should exist.

The basic method of construction followed here for singularities on the divisor D_v of $\mathbb{F}_{1,2}$ is easily adapted for A_{N-1} singularities on the base $\mathbb{F}_m$, $m = 0$ with divisor D_v or on $\mathbb{F}_m$, $m = 0, 1, 2$ with divisor D_u or D_s. In each case, we can find Weierstrass models compatible with the topological data provided by the map 3.6 for most values of N. Although in both the $\mathbb{F}_2$ and the $\mathbb{F}_1$ cases we encountered algebraic difficulties in extending the construction to the maximum value $N = 15$, in both cases a degree of freedom counting argument suggests that solutions should exist. Furthermore, as mentioned above the existence of a string construction for the analogous SO(30) model gives us additional confidence that the Weierstrass models for SU(15) blocks can be realized on $\mathbb{F}_1, \mathbb{F}_2$ despite the apparent complexity of the algebra in these cases.

5.2.3 SU(18) × SU(3)^{12}

In the systematic enumeration of models, including SU(3) blocks, described in section 4.2, the model with the greatest number of blocks (13) has gauge group SU(18) × SU(3)^{12}. The

matter content consists of bifundamental hypermultiplets charged under the SU(18)×SU(3) for each SU(3) factor. This model was first constructed in a different context in [24]. The SU(18) block contains a total of 36 fundamental hypermultiplets, all in bifundamental representations, and thus belongs to the family in table 2 with SU(N) gauge group and $2N$ fundamentals. In the case of single block models, the gravitational anomaly $n_h - n_v \leq 244$ restricted the gauge group to SU(15) in this family. In this case, however, the other factors contribute negatively to $n_h - n_v$ because the matter hypermultiplets are all "shared" between the various gauge factors.

From the map (3.6), as stated in 4.2, we know that this model can be realized on $\mathbb{F}_2$, with the SU(18) factor on D_v and the various SU(3) factors on D_s. It is possible to construct a Weierstrass model for this combination of singularities. The polynomials f and g are given by

$$f(w,z) = -3(h_0 + h_2 w + h_4 w^2)(9(h_0 + h_2 w + h_4 w^2)^3 - 2h_{12} w^6) \tag{5.27}$$

$$g(w,z) = 54(h_0 + h_2 w + h_4 w^2)^6 + h_{12}^2 w^{12} - 18 h_{12} w^6 (h_0 + h_2 w + h_4 w^2)^3 \tag{5.28}$$

where h_i are polynomials of degree i in z. The discriminant is

$$\Delta(w,z) = -27 h_{12}^3 w^{18} (4(h_0 + h_2 w + h_4 w^2)^3 - h_{12} w^6). \tag{5.29}$$

It is clear that the $w = 0$ locus gives an SU(18) gauge symmetry. In addition, at each zero of $h_{12}(z)$, we have an SU(3) gauge symmetry since the discriminant vanishes at third order. Each SU(3) locus of the form $z = z_\alpha$, where z_α is a root of $h_{12}(z)$, intersects the $w = 0$ locus with the SU(18) gauge symmetry at one point. This gives one bifundamental between SU(18) and each SU(3) factor. The fact that even for the largest multi-block model, the map (3.6) gives an acceptable set of F-theory divisors which admit an explicit Weierstrass model provides contributing evidence for the conjecture that all models with topologically acceptable divisors can be explicitly realized in F-theory.

5.3 E_6

For E_6 gauge symmetry on the locus $w = 0$, we need $\text{ord}(f) \geq 3$, $\text{ord}(g) = 4$ and $\text{ord}(\Delta) = 8$. The locus $w = 0$ is intersected by other components of the discriminant locus, and at these points the singularity type is enhanced to E_7. This implies that a fundamental hypermultiplet is localized at every such intersection [7, 19]. In this section, we give explicit Weierstrass models of E_6 gauge symmetry with fundamental matter. The divisor map (4.30) determines the divisor on $\mathbb{F}_m$, given the number f of fundamentals.

$$f \rightarrow D_v + \frac{m + f - 6}{2} D_s \tag{5.30}$$

We focus on the case where the E_6 symmetry is realized on D_v for simplicity. This is the case when $f = 6 - m$, $m = 0, 1, 2, 3, 4$. In the neighborhood around $w = 0$,

$$f(w,z) = w^3 f_{8-m}(z) + \ldots + w^8 f_{8+4m}(z) \tag{5.31}$$

$$g(w,z) = w^4 g_{12-2m}(z) + w^5 g_{12-m}(z) + \ldots + w^{12} g_{12+6m}(z) \tag{5.32}$$

The discriminant locus is of the form

$$\Delta(w,z) = w^8[27g_{12-2m}(z)^2 + w(54g_{12-2m}(z)g_{12-m}(z) + 4f_{8-m}(z)^3) + \ldots)] \qquad (5.33)$$

As explained in [7], the polynomial $g_{12-2m}(z) = g_{6-m}(z)^2$ in order to obtain an E_6 singularity. The fundamentals of E_6 are localized at the zeroes of $g_{6-m}(z)$.

5.4 E_7

For E_7 gauge symmetry on the locus $w = 0$, we need $\mathrm{ord}(f) = 3$, $\mathrm{ord}(g) \geq 5$ and $\mathrm{ord}(\Delta) = 9$. For f fundamental hypermultiplets, we need the E_7 singularity to enhance to E_8 at f distinct points. From the divisor map (4.31), an E_7 singularity on D_v in $\mathbb{F}_m$ realizes models with $\frac{8-m}{2}$ fundamentals.

$$f(w,z) = w^3 f_{8-m}(z) + \ldots + w^8 f_{8+4m}(z) \qquad (5.34)$$

$$g(w,z) = w^5 g_{12-m}(z) + \ldots + w^{12} g_{12+6m}(z) \qquad (5.35)$$

The discriminant locus is of the form

$$\Delta(w,z) = w^9[f_{8-m}(z)^3 + w(3f_{8-m}(z)^2 f_{16}(z) + g_{12-m}(z)^2) + \ldots)] \qquad (5.36)$$

The adjoint representation of E_8 branches under the maximal subgroup $E_7 \times \mathrm{SU}(2)$ as

$$\mathbf{248} = (\mathbf{133}, 1) + (1, \mathbf{3}) + (\mathbf{2}, \mathbf{56}) \qquad (5.37)$$

The $\mathbf{56}$ is pseudoreal, and so we have a half-hypermultiplet localized at the $8 - m$ zeroes of $f_{8-m}(z)$, or equivalently $\frac{8-m}{2}$ hypermultiplets.

6 Some exceptional cases

In this paper we have found an explicit map from six-dimensional chiral supergravity theories to topological data for F-theory constructions. This map seems to give a realization of a significant fraction of the finite number of anomaly-free chiral 6D supergravity theories with one tensor multiplet in terms of the F-theory limit of string theory.

We have, however, encountered a number of exceptional cases in which the map does not give a well-defined geometry in F-theory. In this section we briefly summarize some of the types of cases encountered. This list is presumably not comprehensive, as we have only explored some groups and representations. It seems likely that there are a number of other types of gauge groups and matter blocks which share the features of these exceptional cases. There may even be more unusual classes of exceptions which we have not encountered.

Understanding whether these exceptional cases represent situations in which there are quantum inconsistencies in apparently reasonable classical low-energy models, or as-yet undiscovered types of string compactifications, will hopefully be a productive way of extending our understanding of the correspondence between string theory and low-energy supergravity theories in six dimensions.

Some of the cases we have found in which the map from supergravity blocks to topological F-theory data does not give well-defined integral divisor classes are the following:

a) For SU(N) with $(N-8)\,\square + 1\,\square\square$ matter hypermultiplets, the image divisor seems to have a component $\frac{1}{2}D_v$.

b) Similarly, the SU(24) block in the anomaly-free SU(24) $\times$ SO(8) model with $3(\square\square,1)$ hypermultiplets encountered in [14] gives a divisor which is $1/2$ of D_u.

c) For SU(N) with one adjoint and $10\,\square\square+10(8-N)\,\square$, we get irrational values $\alpha = 5\pm\sqrt{5}$ for the α's, which do not map to a divisor with integer coefficients.

The common thread in these exceptional cases is that the image of the block through the map (3.6) is not an integral divisor in the $\mathbb{F}_m$ base of the F-theory compactification. We encountered one class of cases in which such potential exceptions are already ruled out by a known mechanism: for SU(2) and SU(3) with f fundamentals, the image of the map is only an integral divisor if f is congruent to 4 or 0 modulo 6. In these cases, the blocks whose images would correspond to non-integral divisors are ruled out by global anomaly cancellation requirements.

It seems possible that other quantum consistency conditions may rule out the low-energy theories associated with the other exceptional cases listed above. This may arise from some other kind of global anomaly or related mechanism. Or, since the terms in the action proportional to α have the flavor of Chern-Simons terms, it is possible that some mechanism analogous to the quantization of Chern-Simons level may enforce an integrality condition on the coefficients $\alpha, \tilde{\alpha}$. Such an argument certainly seems plausible in ruling out the type of exceptional case exemplified in case c, with irrational values of $\alpha, \tilde{\alpha}$. On the other hand, there may be some other topological class of string theory compactifications, for example in another discrete part of the moduli space (as considered in [25]), perhaps corresponding to a compactification on a space with some discrete quotient structure, which gives rise to the models which appear to have a half-integral divisor in the image of the map from the supergravity blocks. Understanding these exceptional cases better should be a fruitful direction for future research.

In addition to these cases in which the image of the supergravity block is not an integral divisor, we have encountered a number of exotic representations whose F-theory geometry is not yet understood. For example, we found configurations with 4-index antisymmetric representations of SU(8) and others with trifundamental representations of groups like SU(2) $\times$ SU(3) $\times$ SU(6), which do not correspond to any known geometric structure in F-theory. These also are interesting cases for future study.

7 Conclusions

In this paper we have described an explicit mapping from the set of low-energy chiral six-dimensionful supergravity theories (with one tensor multiplet and nonabelian gauge group) to F-theory. This gives a global picture of how low-energy theory and string theory are connected in a reasonably tractable component of the string landscape. Further study of this correspondence promises to shed light both on the set of allowed string

theory compactifications and on constraints satisfied by low-energy supergravity theories with UV completions.

Following the proof [2] that there are only a finite number of possible gauge groups and matter content for such theories, the results presented here represent a further step towards proving the conjecture stated in [3] that all UV-consistent 6D chiral supergravity theories can be realized in string theory. There are a number of issues which must be clarified to make further progress in this direction. First, we have not systematically enumerated all the possible 6D supergravity theories, and the gauge group and matter types which can appear in such theories. This can in principle be done. The enumeration of the finite set of possible models on the supergravity side seems quite tractable computationally. Second, given such an enumeration it would be necessary to identify the structures in F-theory corresponding to all matter representations appearing in the list. We have identified in this paper a number of matter representations whose corresponding geometry is not yet known; the finite number of such exotic representations appearing in acceptable 6D supergravity models should provide a good guide to understanding the corresponding allowed singularity structures in F-theory. Third, we have found a number of situations where the image of the map is not an integral divisor in F-theory. Some of these are summarized in the previous section. Showing that these exceptional cases are associated with quantum inconsistencies, or new string vacua, would be necessary to complete the global picture of the map described here. Fourth, we have not shown that explicit Weierstrass models are possible for all topological F-theory constructions, although we have shown this to be possible for certain families. In certain cases, we have a dimension-counting argument which supports the conjecture stated in section 5 that all topologically allowed models in the image of the map can be realized explicitly through Weierstrass models. It would be nice to have a more general argument along these lines. Finally, as mentioned above, we have restricted attention so far to nonabelian models with one tensor multiplet; it is clearly of interest to expand the analysis to include multiple tensor multiplets and U(1) factors in the gauge group.

There is an enormous literature on how different approaches to string compactifications can give rise to different low-energy field theories coupled to gravity in various dimensions, including the six-dimensional case considered here and the four-dimensional case of physical interest. In particular, in [5–7, 9], a detailed analysis was made of the different singularity structures in F-theory and the associated gauge structure and matter content in the associated low-energy theory. In most of this work the emphasis has been on going from string theory to the low-energy theory. In this paper, we have approached the problem from the other direction, by formulating a map from the space of low-energy theories to the space of string theories. Both approaches lead to valuable lessons about the connection between low-energy theory and string theory. It seems likely, however, that further progress in understanding the map from low-energy theories to string theory may be of particular value both in explicit model-building efforts and in understanding the general structure of the landscape.

The map we have described in this paper from supergravity theories to topological F-theory data is not unique in all cases. For some combinations of gauge group and matter content, there are different ways of mapping the theory to F-theory, either by choosing

distinct base spaces $\mathbb{F}_m$, or by switching the values of α and $\tilde{\alpha}$ in the gauge group factors. Thus, there may be multiple F-theory models with given gauge group and matter content. In some cases these F-theory models are related through a known duality symmetry, but in other cases they are not. In general, the number of discrete choices for a given supergravity theory is fairly small. A similar phenomenon was found in [14], where for many models the heterotic realization was uniquely determined by a lattice embedding satisfying certain criteria, but in some cases multiple distinct lattice embeddings give rise to distinct string theory realizations of a specific gauge group and matter content. We have not explored in detail how these models or the distinct F-theory realizations found here would differ, or when in general such models are related by a duality symmetry; we leave exploration of these questions for future work. Note that in principle it is possible to imagine many distinct low-energy Lagrangians for theories with the same gauge group and matter content, but more detailed considerations may place constraints on which Lagrangians lead to consistent theories. We have not explored this issue here either, having focused essentially only on the topological data of the models studied here.

In this paper we have focused on chiral six-dimensional supergravity theories, which are strongly constrained by anomaly cancellation. In developing a dictionary connecting the low-energy supergravity theories to string theory, we find explicit relationships between the constraints imposed by the framework of string compactifications and the anomaly cancellation constraints in 6D. In other situations, such as non-chiral six-dimensional supergravity theories, or general supergravity theories in four dimensions, there are no gravitational anomalies, and the constraints we know of on low-energy theories are weaker. Nonetheless, in these cases there are similar constraints on the space of string compactifications. By understanding the dictionary between low-energy theories and string theory more clearly in the chiral six-dimensional case, it may be possible to generalize this dictionary to other cases in which the low-energy constraints are less well understood. In particular, for $\mathcal{N} = 2$ non-chiral supergravity theories in 6D, and for chiral or non-chiral supergravity theories in four dimensions with extended supersymmetry, there should be similar constraints on the set of F-theory constructions, which may be a useful guide in discovering new constraints on which low-energy field theories can consistently be coupled to quantum gravity in four or six dimensions. We hope that the work presented here will play a useful role in leading to developments in this direction.

Acknowledgments

References

[1] M.B. Green, J.H. Schwarz and P.C. West, *Anomaly Free Chiral Theories in Six-Dimensions*, *Nucl. Phys.* **B 254** (1985) 327 [SPIRES].

[2] V. Kumar and W. Taylor, *A bound on 6D N = 1 supergravities*, *JHEP* **12** (2009) 050 [arXiv:0910.1586] [SPIRES].

[3] V. Kumar and W. Taylor, *String Universality in Six Dimensions*, arXiv:0906.0987 [SPIRES].

[4] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022] [SPIRES].

[5] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi–Yau Threefolds – I*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114] [SPIRES].

[6] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi–Yau Threefolds – II*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161] [SPIRES].

[7] M. Bershadsky et al., *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys.* **B 481** (1996) 215 [hep-th/9605200] [SPIRES].

[8] V. Sadov, *Generalized Green-Schwarz mechanism in F-theory*, *Phys. Lett.* **B 388** (1996) 45 [hep-th/9606008] [SPIRES].

[9] A. Grassi and D.R. Morrison, *Group representations and the Euler characteristic of elliptically fibered Calabi-Yau threefolds*, math/0005196. MATH/0005196;

[10] A. Grassi, D.R. Morrison, *Anomalies and the Euler characteristic of elliptically fibered Calabi-Yau threefolds*, to appear.

[11] K.-S. Choi, *Extended Gauge Symmetries in F-theory*, *JHEP* **02** (2010) 004 [arXiv:0910.2571] [SPIRES].

[12] R.A. Laamara, A. Belhaj, L.J. Boya, L. Medari and A. Segui, *On F-theory Quiver Models and Kac-Moody Algebras*, arXiv:0910.4852 [SPIRES].

[13] J. Erler, *Anomaly cancellation in six-dimensions*, *J. Math. Phys.* **35** (1994) 1819 [hep-th/9304104] [SPIRES].

[14] V. Kumar and W. Taylor, *Freedom and Constraints in the K3 Landscape*, *JHEP* **05** (2009) 066 [arXiv:0903.0386] [SPIRES].

[15] A. Sagnotti, *A Note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett.* **B 294** (1992) 196 [hep-th/9210127] [SPIRES].

[16] K. Kodaira, *On compact analytic surfaces II*, *Ann. Math.* **77** (1963) 563.

[17] K. Kodaira, *On compact analytic surfaces III*, *Ann. Math.* **78** (1963) 1.

[18] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin, Germany (2004).

[19] S.H. Katz and C. Vafa, *Matter from geometry*, *Nucl. Phys.* **B 497** (1997) 146 [hep-th/9606086] [SPIRES].

[20] M. Bershadsky and C. Vafa, *Global anomalies and geometric engineering of critical theories in six dimensions*, `hep-th/9703167` [SPIRES].

[21] E. Witten, *An SU(2) anomaly*, *Phys. Lett.* **B 117** (1982) 324 [SPIRES].

[22] O. DeWolfe and B. Zwiebach, *String junctions for arbitrary Lie algebra representations*, *Nucl. Phys.* **B 541** (1999) 509 [`hep-th/9804210`] [SPIRES].

[23] D.A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, American Mathematical Society, Providence, U.S.A. (2000) [SPIRES].

[24] O.J. Ganor, D.R. Morrison and N. Seiberg, *Branes, Calabi-Yau spaces and toroidal compactification of the $N = 1$ six-dimensional E_8 theory*, *Nucl. Phys.* **B 487** (1997) 93 [`hep-th/9610251`] [SPIRES].

[25] J. de Boer et al., *Triples, fluxes and strings*, *Adv. Theor. Math. Phys.* **4** (2002) 995 [`hep-th/0103170`] [SPIRES].