

Global anomalies, discrete symmetries and hydrodynamic effective actions

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ABSTRACT: We derive effective actions for parity-violating fluids in both (3+1) and (2+1) dimensions, including those with anomalies. As a corollary we confirm the most general constitutive relations for such systems derived previously using other methods. We discuss in detail connections between parity-odd transport and underlying discrete symmetries. In (3+1) dimensions we elucidate connections between anomalous transport coefficients and global anomalies, and clarify a previous puzzle concerning transports and local gravitational anomalies.

KEYWORDS: Anomalies in Field and String Theories, Discrete Symmetries, Effective Field Theories, Space-Time Symmetries

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1 Introduction

Through studies of free field theories [1–4], holographic duality [5–7], phenomenological arguments based on entropy current [8–12], and equilibrium partition functions [13–15], it has been recognized that systems with quantum anomalies exhibit novel transport behavior in the presence of rotation or in a magnetic field (for a recent review see [16]). Effects of anomalies on transport in superfluids, superconductors and topological insulators have also been discussed in [17–20]. (See also [21–28].) These anomalous transports could be relevant in a wide range of physical contexts: from the study of quark-gluon plasma at subnuclear

scales [29–33], to cosmology, where the dynamics of primordial magnetic fields plays an important role in the early stage of the universe [34, 35], and astrophysical phenomena such as pulsar kicks [36, 37]. In addition, there have been various experimental searches for the signatures of anomalies on transports in condensed matter systems, see [38–40].

Given their importance, it is of primary interest to incorporate anomalous transports in an effective field theory framework, which is the goal of this paper. Such a formulation has a number of advantages. Firstly, an effective field theory provides a framework where hydrodynamic fluctuations can be systematically incorporated, thus enabling one to search for new physical effects due to fluctuations in parity-violating systems. Secondly, the effective action approach provides a first-principle derivation of the constitutive relations which automatically incorporates all the phenomenological constraints. Indeed our derivation reproduces fully the constitutive relations of previous approaches. It also highlights some new insights which we will discuss momentarily.

Consider a parity-violating relativistic system in $(3 + 1)$ -dimension with a global $U(1)$ symmetry whose conserved current is \hat{J}^μ . Suppose the symmetry becomes anomalous in the presence of an external source A_μ for \hat{J}^μ ,

$$\nabla_\mu \hat{J}^\mu = \frac{c\hbar}{4} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad (1.1)$$

where F is the field strength for A . Due to (1.1), the Euclidean partition function of the system in the presence of source A_μ is not invariant under *small* gauge transformations of A . We will refer to (1.1) as a local $U(1)$ anomaly, in contrast to a global anomaly in which case the partition function is invariant under small gauge transformations, but not under *large* gauge transformations when the system is put on a topologically nontrivial manifold. To first order in the derivative expansion, the parity-odd part J_o^μ of charge current can be written in the Landau frame as [8, 9, 29, 30]

$$J_o^\mu = \xi_\omega \omega^\mu + \xi_B \mathcal{B}^\mu. \quad (1.2)$$

The first term implies a contribution to the current that is induced by and parallel to, the vorticity $\omega^\mu \equiv \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$ (u^μ is the local velocity field). This is called the chiral vortical effect (CVE). The second term is proportional to the magnetic field strength $\mathcal{B}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$, which is often referred to as the chiral magnetic effect (CME). The transport coefficients ξ_ω and ξ_B receive contributions from local anomaly (1.1) as follows [8, 9, 13]

$$\xi_\omega = -3c\hbar\mu^2 \left(1 - \frac{2}{3}\alpha\right) + 2a_1\mu T(1 - \alpha) + a_2 T^2(1 - 2\alpha) - 2a_3 T^3 \frac{n_0}{\epsilon_0 + p_0}, \quad (1.3)$$

$$\xi_B = -3c\hbar\mu(2 - \alpha) + 2a_1 T(1 - \alpha) - a_2 T^2 \frac{n_0}{\epsilon_0 + p_0}, \quad (1.4)$$

with

$$\alpha \equiv \frac{\mu n_0}{\epsilon_0 + p_0} \quad (1.5)$$

where $a_{1,2,3}$ are constants, and $\mu, T, n_0, \epsilon_0, p_0$ are local chemical potential, temperature, charge density, energy density and pressure respectively.

It is curious that even in the absence of local anomaly (1.1), i.e. with $c = 0$, there can still be chiral vortical and magnetic effects, determined up to three constants. It has been pointed out that for a CTP invariant theory, only a_2 is allowed [13, 18], whose physical origin has generated much recent interest. From holography and free theory examples, a_2 appears to be related to the coefficient λ of the local mixed gravitational anomalies

$$\nabla_\mu \hat{J}^\mu = \lambda \epsilon^{\mu\nu\lambda\rho} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\lambda\rho} \tag{1.6}$$

as [41–47]

$$a_2 = -32\pi^2 \lambda. \tag{1.7}$$

Relation (1.7) is puzzling from the perspective of anomaly matching in a low energy effective theory, as the right hand side of (1.6) contains four derivatives and thus should modify J^μ only at the third derivative order while terms in (1.2) have only one derivative. Furthermore, matching with constitutive relations or partition functions as done in [8, 9, 13, 14] will not lead to any multiplicative factor π as in (1.7). Arguments have been made in [48–52] which show that (1.7) should apply at least to field theory systems smoothly connected to free theories through continuous parameter(s). Alternatively, it has been hinted in [53] and subsequently explicitly worked out in various examples in [54, 55] that the transport coefficient a_2 should be considered as being directly related to global mixed gravitational anomalies when putting the system on a topologically nontrivial manifold. It has also been known that relation like (1.7) is violated for systems with gravitinos [45, 48, 50, 56].

In this paper we work out effective actions for parity-violating fluids in both $(2+1)$ and $(3+1)$ dimensions following the approach developed in [57–59] (see [60–62] for earlier attempts at an effective action for anomalous transports). We assume that at microscopic level the system has an underlying discrete symmetry Θ which includes time reversal. Here Θ can be the time reversal \mathcal{T} itself, or any combinations of \mathcal{C}, \mathcal{P} with \mathcal{T} , such as \mathcal{CPT} . As a corollary we confirm (1.2)–(1.4) as the most general constitutive relation for a parity-violating system in $(3+1)$ -dimensions, and in $(2+1)$ -dimension we confirm the constitutive relations obtained earlier in [13, 14, 63]. In $(2+1)$ -dimension the story is much richer, containing six independent functions of local temperature and chemical potential. The rest of the paper is devoted to detailed derivations of the effective actions. Here we highlight a couple of conceptual points related to (1.3)–(1.4). In particular, we offer an interpretation for (1.7) which reconciles various different perspectives.¹ We find:

1. In both $(3+1)$ and $(2+1)$ dimensions, possible parity-odd transport behavior sensitively depends on the underlying discrete symmetries. Hence hydrodynamic transports can be used to probe microscopic discrete symmetries. For example, given the form (1.2)–(1.4), when \mathcal{PT} is conserved, then $a_{1,2,3} = 0$ and $c = 0$, i.e. no chiral vortical or magnetic effects. If \mathcal{CPT} is conserved, then $a_1 = a_3 = 0$. If only \mathcal{T} is conserved, then all $a_{1,2,3}$ and c are allowed. Thus detection of possible existence of a_1, a_3 can be used to test \mathcal{CPT} violations.

¹While these points follow naturally from our discussion, some aspects could have been realized before using the approaches already discussed in the literature. For example, the connection with global anomalies discussed below could have been read from the results of [13].

While \mathcal{CTP} is preserved for all relativistic local field theories, searching for its possible violations through transports could be interesting. Some condensed matter systems exhibit emergent relativistic symmetries, and transport behavior can then be potentially used to probe whether there is emergent \mathcal{CPT} as well.

2. All three constants $a_{1,2,3}$ in (1.3)–(1.4) are associated with global anomalies, respectively with pure gauge, mixed gauge, and pure gravitational anomalies. More explicitly, consider the partition function of the system on a spatial manifold $S^1 \times S^2$ at a finite temperature, i.e. the full manifold is $S^1_T \times S^1 \times S^2$, with S^1_T denoting the Euclidean time direction along which we put thermal boundary conditions. We also turn on the external metric and source A_μ as

$$ds^2 = g_{00} (d\tau - v_i dx^i)^2 + a_{ij} dx^i dx^j, \quad A_\mu dx^\mu = A_0 (d\tau - v_i dx^i) + b_i dx^i \quad (1.8)$$

with all components to be independent of Euclidean time τ . x^i denotes directions along $S^2 \times S^1$. Let us suppose there is no local gauge anomaly (1.1), i.e. $c = 0$. Then to first derivative order, the partition function should be invariant under the following two U(1) transformations²

$$v_i \rightarrow v_i - \partial_i f, \quad b_i \rightarrow b_i, \quad (1.9)$$

$$v_i \rightarrow v_i, \quad b_i \rightarrow b_i + \partial_i g \quad (1.10)$$

where both f and g are independent of τ . Equation (1.9) arises from time diffeomorphism along the Euclidean time circle³ while (1.10) is the stationary gauge transformation for A_μ . It turns out, however, when $a_{1,2,3}$ are nonzero, the partition function is only invariant under transformations which are smoothly connected to the identity, but not invariant under large gauge transformations.

More explicitly, suppose b_i has a magnetic flux along S^2 , then under a large gauge transformation of b_i and v_i along S^1 we find that the partition function transforms as

$$Z \rightarrow \exp \left[\frac{8\pi^2 m a_1}{q^2} + i \frac{2\pi n a_2}{q} \right] Z, \quad m, n \in \mathbb{Z} \quad (1.11)$$

where q is the minimal U(1) charge of the system. The term proportional to a_2 in (1.11) is fully consistent with the discussion of various examples in [54, 55]. In (1.11) the term in the exponent proportional to a_1 is real; recall that the presence of a_1 breaks \mathcal{CTP} . Similarly when only v_i has a magnetic flux along S^2 , under a large gauge transformation of v_i along S^1 we find that

$$Z \rightarrow e^{-2a_3 r} Z, \quad r \in \mathbb{Z} \quad (1.12)$$

which is again real. The standard lore is that there can be no pure global gravitational anomaly in $d = 4$. But here \mathcal{CPT} is broken and we are at a finite temperature.

²The gravitational anomaly (1.6) does not matter at this derivative order.

³The transformation associated to f is also known in the literature as Kaluza-Klein U(1).

We thus see measuring parity-violating transports can also be used to probe global anomalies of a system. Note that a_2 appears in (1.11) in a phase, so the global anomaly (1.11) only captures the “fractional” part of a_2 , i.e. $a_2 \rightarrow a_2 + kq$ with $k \in \mathbb{Z}$ does not change the phase. In contrast, the factors associated with a_1 and a_3 in (1.11)–(1.12) are real. As a result the global anomalies associated with them are fully equivalent to the corresponding transport coefficients.

The relations between coefficients $a_{1,2,3}$ and global anomalies described above are universal relations which can be deduced solely at the level of low energy effective theory, without any knowledge of UV physics. Now let us come back to the relation (1.7) which from the light of the above discussion may be interpreted as the combination of the following:

- (a) the connection between a_2 -related transports in (1.3)–(1.4) to global gravitational anomaly (1.11) which is a universal low energy relation;
- (b) a relation between local mixed anomaly coefficient λ in (1.6) and the global mixed anomaly (1.11) which has been known to be valid for some class of systems. This relation goes beyond low energy physics.

This resolves the two puzzles mentioned below (1.7): equation (1.7) should not be viewed as a low energy relation. Indeed, from the perspective of low energy effective field theory, neither transport behavior in (1.3)–(1.4) nor the global anomaly in (1.11) has anything to do with (1.6). Nevertheless, when UV physics is taken into consideration, they are controlled by the same number in a large class of systems. In this light the discussion of [48–51] can be considered as establishing (b) for field theory systems smoothly connected to free theories through continuous parameter(s).

The plan of the paper is as follows. In section 2 we briefly review the formalism of [57–59] to set up the notations and the rules for derivations of later sections. In section 3 we obtain the effective action of a parity-violating fluid in $(3+1)$ -dimension. In section 4 we discuss the connection between the effective action and thermal partition function, and connection with global anomalies. In section 5 we discuss the entropy current for $(3+1)$ -systems. In section 6 we repeat the analysis for $(2+1)$ -dimensional parity-violating systems, obtaining the effective action, partition function and the entropy current. We have also included a number of appendices for technical details.

2 Review of hydrodynamical action in physical spacetime

In this section, we review the formulation of the hydrodynamical action introduced in [57–59] to set up the notations and formalism for deriving anomalous transports in later sections.⁴

⁴See also [64–76] for other discussions of action formulation.

2.1 General setup

Consider the closed time path (CTP) generating functional $W[g_1, A_1; g_2, A_2]$ for a system with a U(1) symmetry in some state specified by the density matrix ρ_0

$$e^{W[g_{1\mu\nu}, A_{1\mu}; g_{2\mu\nu}, A_{2\mu}]} \equiv \text{Tr} \left[U(+\infty, -\infty; g_{1\mu\nu}, A_{1\mu}) \rho_0 U^\dagger(+\infty, -\infty; g_{2\mu\nu}, A_{2\mu}) \right] \quad (2.1)$$

where $U(t_2, t_1; g_{1\mu\nu}, A_{1\mu})$ denotes the quantum evolution operator of the system from t_1 to t_2 in the presence of spacetime metric $g_{1\mu\nu}$ and an external vector field $A_{1\mu}$ (sources for the U(1) current). The sources for two legs of the CTP contour are taken to be independent. We introduce the “on-shell” stress tensors and currents for each leg as

$$-i \frac{\delta W}{\delta g_{1\mu\nu}(x)} = \frac{1}{2} \sqrt{-g_1} T_1^{\mu\nu}(x), \quad -i \frac{\delta W}{\delta A_{1\mu}(x)} = \sqrt{-g_1} J_1^\mu(x), \quad (2.2)$$

$$i \frac{\delta W}{\delta g_{2\mu\nu}(x)} = \frac{1}{2} \sqrt{-g_2} T_2^{\mu\nu}(x), \quad i \frac{\delta W}{\delta A_{2\mu}(x)} = \sqrt{-g_2} J_2^\mu(x). \quad (2.3)$$

The expectation values $T^{\mu\nu}, J^\mu$ of the stress tensor and the U(1) current in the state ρ_0 in an external metric $g_{\mu\nu}$ and external background A_μ are obtained by

$$T_{\mu\nu} = T_1^{\mu\nu}|_{g,A} = T_2^{\mu\nu}|_{g,A}, \quad J^\mu = J_1^\mu|_{g,A} = J_2^\mu|_{g,A} \quad (2.4)$$

where $|_{g,A}$ denotes setting $g_{1\mu\nu} = g_{2\mu\nu} = g_{\mu\nu}$ and $A_{1\mu} = A_{2\mu} = A_\mu$.

In the *absence* of any gravitational and U(1) anomalies, $W[g_1, A_1; g_2, A_2]$ should be invariant under independent gauge transformations of A_1, A_2 and independent diffeomorphisms of $g_1 A_1$ and g_2, A_2 , i.e.

$$W[g_1, A_1 + d\lambda_1; g_2, A_2 + d\lambda_2] = W[A_1, A_2] \quad (2.5)$$

$$W[g_1, A_1; g_2, A_2] = W[g_1^{\xi_1}, A_1^{\xi_1}; g_2^{\xi_2}, A_2^{\xi_2}] \quad (2.6)$$

where g^ξ, A^ξ denote diffeomorphisms of g, A generated by a vector field ξ^μ .⁵ Equations (2.5)–(2.6) in turn ensure that

$$\nabla_{s\mu} J_s^\mu = 0, \quad \nabla_{s\nu} T_{s\mu}^\nu = F_{s\mu\nu} J_s^\nu, \quad s = 1, 2 \quad (2.7)$$

where ∇_1 is the covariant derivative associated with $g_{1\mu\nu}$, and $F_{1\mu\nu}$ is the field strength of $A_{1\mu}$. Similarly for quantities with subscript 2.

For slowly varying sources, we can express the generating functional (2.1) in terms of path integrals over *slow* degrees of freedom of the system

$$e^{W[g_1, A_1; g_2, A_2]} = \int D\chi e^{\frac{i}{\hbar} I_{\text{EFT}}[\chi]} \quad (2.8)$$

where χ collectively denotes slow variables of the system which in general also come in two copies. The low energy effective action I_{EFT} depends on ρ_0 and external sources which we have suppressed, and is assumed to be local.

⁵ λ_1, λ_2 and $\xi_{1,2}^\mu$ are all assumed to vanish at spatial and time infinities.

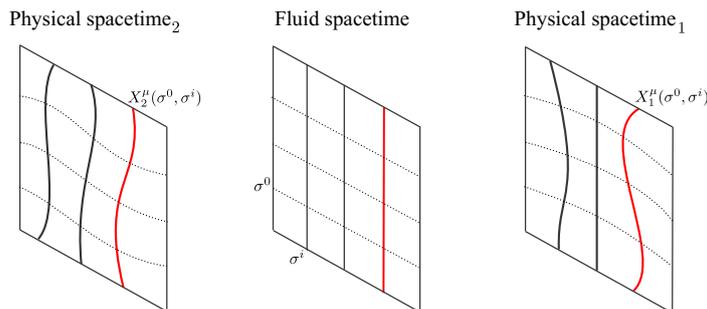


Figure 1. $X_{1,2}^\mu(\sigma^A)$ describe motion of a continuum of fluid elements labelled by σ^A in two copies of physical spacetimes with coordinates $X_{1,2}^\mu$ respectively. $\sigma^A = (\sigma^0, \sigma^i)$ are coordinates for a “fluid spacetime”, where σ^i are interpreted as labels of each fluid element while σ^0 is their “internal” time. The red straight line in the fluid spacetime with constant σ^i is mapped by $X_{1,2}^\mu(\sigma^0, \sigma^i)$ to physical spacetime trajectories (also in red) of the corresponding fluid element.

For ρ_0 describing a medium in local equilibrium, generically the only slow modes are those associated with conserved quantities (2.7), i.e. hydrodynamical modes, with I_{EFT} the corresponding hydrodynamical action I_{hydro} . We will limit ourselves to the generic situation.⁶ The slow variables associated with the stress tensor can be chosen to be $X_{1,2}^\mu(\sigma^A)$ which describe motions of a continuum of fluid elements labelled by σ^A in two copies of physical spacetimes with coordinates $X_{1,2}^\mu$ respectively. See figure 1. The slow variables associated with the U(1) currents are $\varphi_{1,2}(\sigma^A)$ which can be interpreted as U(1) phase rotations associated for each fluid elements. It is also convenient to introduce an additional scalar field $\beta(\sigma^A)$ which gives the local inverse temperature in fluid spacetime.⁷ $X_{1,2}^\mu$ and $\varphi_{1,2}$ are the Stuckelberg fields for diffeomorphisms and gauge transformations (2.5)–(2.6), and we require the hydrodynamical action I_{hydro} to be a local action of pullbacks of $g_{s\mu\nu}$ and $B_{s\mu} = A_{s\mu} + \partial_\mu \varphi_s$, $s = 1, 2$ to the fluid spacetime

$$h_{sAB}(\sigma) = \frac{\partial X_s^\mu}{\partial \sigma^A} g_{s\mu\nu}(X_s(\sigma)) \frac{\partial X_s^\nu}{\partial \sigma^B}, \quad B_{sA}(\sigma) = \frac{\partial X_s^\mu}{\partial \sigma^A} A_{s\mu}(X_s(\sigma)) + \partial_A \varphi_s(\sigma). \quad (2.9)$$

i.e.

$$I_{\text{hydro}} = I_{\text{hydro}}[h_1, B_1; h_2, B_2; \beta]. \quad (2.10)$$

By construction $h_{1,2}$ and $B_{1,2}$ are invariant under independent diffeomorphisms and gauge transformations of the two legs of the CTP contour ($s = 1, 2$):

$$g'_{s\mu\nu}(X'_s) = \frac{\partial X_s^\lambda}{\partial X'^{\lambda\mu}} \frac{\partial X_s^\rho}{\partial X'^{\rho\nu}} g_{s\lambda\rho}(X_s), \quad A'_{s\mu}(X'_s) = \frac{\partial X_s^\lambda}{\partial X'^{\lambda\mu}} A_{s\lambda}(X_s), \quad X'^{\mu}(\sigma) = f'_s{}^\mu(X_s(\sigma)) \quad (2.11)$$

$$A'_{s\mu} = A_{s\mu} - \partial_\mu \lambda_s(X_s), \quad \varphi'_s(\sigma) = \varphi_s(\sigma) + \lambda_s(X_s(\sigma)), \quad (2.12)$$

which along with (2.10) immediately implies (2.5)–(2.6). Furthermore, the form of the action (2.10) implies that the equations of motion of $X_{1,2}^\mu$ and $\varphi_{1,2}$ are equivalent to the

⁶The discussion can be readily generalized to systems such as near a critical point where one should also include the corresponding order parameter(s). See [59, 76].

⁷Note that there is only one temperature field rather than two copies.

conservations of the “off-shell” hydrodynamical stress tensors and currents defined as

$$\frac{\delta I_{\text{hydro}}}{\delta g_{1\mu\nu}(x)} \equiv \frac{1}{2}\sqrt{-g_1}\hat{T}_1^{\mu\nu}(x), \quad \frac{\delta I_{\text{hydro}}}{\delta A_{1\mu}(x)} \equiv \sqrt{-g_1}\hat{J}_1^\mu(x), \quad (2.13)$$

$$\frac{\delta I_{\text{hydro}}}{\delta g_{2\mu\nu}(x)} \equiv -\frac{1}{2}\sqrt{-g_2}\hat{T}_2^{\mu\nu}(x), \quad \frac{\delta I_{\text{hydro}}}{\delta A_{2\mu}(x)} \equiv -\sqrt{-g_2}\hat{J}_2^\mu(x). \quad (2.14)$$

As defined the path integrals (2.8) apply to a general quantum system. At sufficiently high temperatures it is often enough to consider the leading order in a small \hbar expansion. For this purpose we decompose

$$g_{1\mu\nu} = g_{\mu\nu} + \frac{\hbar}{2}g_{a\mu\nu}, \quad g_{2\mu\nu} = g_{\mu\nu} - \frac{\hbar}{2}g_{a\mu\nu}, \quad A_{1\mu} = A_\mu + \frac{\hbar}{2}A_{a\mu}, \quad A_{2\mu} = A_\mu - \frac{\hbar}{2}A_{a\mu} \quad (2.15)$$

$$X_1^\mu = X^\mu + \frac{\hbar}{2}X_a^\mu, \quad X_2^\mu = X^\mu - \frac{\hbar}{2}X_a^\mu, \quad \varphi_1 = \varphi + \frac{\hbar}{2}\varphi_a, \quad \varphi_2 = \varphi - \frac{\hbar}{2}\varphi_a, \quad (2.16)$$

and the action I_{hydro} can be expanded in \hbar as

$$\frac{1}{\hbar}I_{\text{hydro}} = I_{\text{hydro}}^{(0)} + \hbar I_{\text{hydro}}^{(1)} + \dots \quad (2.17)$$

In this limit the path integrals (2.8) survive and describe classical statistical averages. We will refer to variables with subscript a as a -variables and those without as r -variables. r -variables can be considered as describing physical quantities while a -variables correspond to noises. For example, $X^\mu(\sigma^A)$ is interpreted as mapping fluid spacetime into the physical spacetime (now only one copy) with X_a^μ interpreted as the corresponding position noises.

While the hydrodynamical action I_{hydro} is naturally formulated in the fluid spacetime σ^A , one can also formulate it in physical spacetime by inverting $X^\mu(\sigma^A)$, i.e. use $\sigma^A(X)$ as dynamical variables and express all other variables accordingly as functions of X^μ . In the physical spacetime formulation, the dynamical variables are then $\sigma^A(x), \varphi(x), \beta(x)$ and $X_a^\mu(x), \varphi_a(x)$, while the background fields are $g_{\mu\nu}(x), A_\mu(x), g_{a\mu\nu}(x), A_{a\mu}(x)$, where we have replaced X^μ by x^μ to emphasize they are now just coordinates for physical spacetime. The physical spacetime formulation has the advantage of being more physically intuitive and connects more directly with the traditional phenomenological approach.

2.2 Formulation of I_{hydro} in physical spacetime

We now list various symmetries and consistency requirements which I_{hydro} should satisfy when formulated in the physical spacetime to leading order in the \hbar -expansion [57–59]. They can be separated into the following categories:

1. Spacetime diffeomorphisms and gauge transformations. In the absence of any gravitational and charged current anomalies, the action I_{hydro} should be invariant under physical spacetime version of (2.11)–(2.12). Invariance under these transformations implies that a -fields (including both background and dynamical variables) must appear through the combinations

$$G_{a\mu\nu}(x) \equiv g_{a\mu\nu} + \mathcal{L}_{X_a}g_{\mu\nu} = g_{a\mu\nu} + \nabla_\mu X_{a\nu} + \nabla_\nu X_{a\mu}, \quad (2.18)$$

$$C_{a\mu} \equiv A_{a\mu}(x) + \partial_\mu \varphi_a(x) + \mathcal{L}_{X_a}A_\mu = A_{a\mu}(x) + \partial_\mu \varphi_a(x) + X_a^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu X_a^\nu \quad (2.19)$$

while A_μ and φ must appear through

$$B_\mu = A_\mu + \partial_\mu \varphi(x). \quad (2.20)$$

The above variables are the physical spacetime version of (2.9).

2. Spatial and time diffeomorphisms in the fluid spacetime which *define* a fluid. We require the action I_{hydro} be invariant under

$$\sigma^i \rightarrow \sigma'^i(\sigma^i), \quad \sigma^0 \rightarrow \sigma^0 \quad (2.21)$$

$$\sigma^0 \rightarrow \sigma'^0 = f(\sigma^0, \sigma^i), \quad \sigma^i \rightarrow \sigma^i. \quad (2.22)$$

Furthermore we require the action be invariant under the diagonal shift

$$\varphi \rightarrow \varphi - \lambda(\sigma^i(x^\mu)), \quad \varphi_a \rightarrow \varphi_a \quad (2.23)$$

where λ is a function of σ^i only. Invariance under (2.23) defines a normal fluid. For a superfluid where the U(1) symmetry is spontaneously broken this symmetry should be dropped. The symmetries (2.21)–(2.23) involve only dynamical variables, yet they should be viewed as “global gauge symmetries,” i.e. configurations related by such transformations are deemed physically equivalent.

Invariance under (2.21)–(2.23) implies that the only invariant which can be constructed from $K_\mu^A \equiv \partial_\mu \sigma^A$ is the velocity field u^μ defined by

$$u^\mu = \frac{1}{b} (K^{-1})_0^\mu, \quad b^2 = -g_{\mu\nu} (K^{-1})_0^\mu (K^{-1})_0^\nu \quad (2.24)$$

and $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. By definition $u^\mu u_\mu = -1$. B_μ is not invariant under diagonal shift (2.23) of φ , but

$$\mu \equiv u^\mu B_\mu, \quad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.25)$$

are invariant. To summarize, the only combinations of r -variables which can appear are

$$\beta(x), \quad u^\mu, \quad \mu, \quad F_{\mu\nu}, \quad \Delta^{\mu\nu}. \quad (2.26)$$

It is often convenient to combine the first three variables further into

$$\beta^\mu = \beta(x) u^\mu(x), \quad \hat{\mu}(x) = \beta(x) \mu = \beta^\mu(x) B_\mu \quad (2.27)$$

where β^μ is now unconstrained.

3. Classical remnants of constraints from quantum unitarity of (2.1),

$$I_{\text{hydro}}^*[\Lambda_r, \Lambda_a] = -I_{\text{hydro}}[\Lambda_r, -\Lambda_a], \quad (2.28)$$

$$\text{Im } I_{\text{hydro}} \geq 0, \quad (2.29)$$

$$I_{\text{hydro}}[\Lambda_r, \Lambda_a = 0] = 0, \quad (2.30)$$

where $\Lambda_{r,a}$ collectively denote all r - and a -variables including both dynamical and background fields.

4. Discrete spacetime symmetries. If the microscopic system is invariant under charge conjugation \mathcal{C} , parity \mathcal{P} or \mathcal{CP} , such discrete symmetries should be imposed on I_{hydro} and they can be imposed straightforwardly as usual.
5. We assume the microscopic Hamiltonian underlying the macroscopic many-body state ρ_0 is invariant under a discrete symmetry Θ containing *time reversal*. Θ can be time reversal \mathcal{T} itself, or any combinations of \mathcal{C}, \mathcal{P} with \mathcal{T} , such as \mathcal{CPT} . Θ can also be a combination of \mathcal{T} with some other internal discrete operations. Unlike \mathcal{C} or \mathcal{P} , Θ by itself can *not* be imposed directly on I_{hydro} , since Θ does not take the generating functional W to itself, but to a time reversed generating functional W_T .⁸ The fact that the underlying Hamiltonian is invariant under Θ nevertheless leads to important constraints on I_{hydro} as we will discuss in the next item.
6. We require I_{hydro} to be invariant under a Z_2 dynamical KMS symmetry

$$\tilde{I}_{\text{hydro}} \equiv I_{\text{hydro}}[\tilde{\Lambda}_r, \tilde{\Lambda}_a] = I_{\text{hydro}}[\Lambda_r, \Lambda_a] \quad (2.31)$$

where tilde denotes a Z_2 transformation which is a combination of Θ and the Kubo-Martin-Schwinger (KMS) transformation.⁹

Equation (2.31) plays the dual role of imposing microscopic time-reversibility and *local equilibrium*. It should be understood as a mathematical characterization of a state ρ_0 in local equilibrium. The prototype of such a state is the thermal density matrix in slowly varying external sources, but (2.31) is more general, applicable also to pure states. It was found in [57–59] that (2.31) leads to Onsager relations, local first law, local second law, and local fluctuation-dissipation relations.

To leading order in \hbar , the tilde operation in (2.31) can be written schematically as

$$\tilde{\Lambda}_r = \Theta\Lambda_r + O(\hbar), \quad \tilde{\Lambda}_a = \Theta\Lambda_a - i\Theta\Phi_r + O(\hbar) \quad (2.33)$$

where Φ_r denotes certain combination of r -variables with total one derivative. More explicitly, in (2.33) we denoted Θ transformation of a tensor $G(x)$ as

$$\Theta G(x) \equiv \eta_G G(\eta x), \quad (2.34)$$

where we have suppressed tensor indices for G , and η_G should be understood as a collection of phases (± 1) one for each component for G . Similarly for ηx . For example, for $\Theta = \mathcal{T}$ and $G = A_\mu$

$$\eta_A A_\mu = (A_0, -A_i), \quad \eta x^\mu = (-x^0, x^i), \quad (2.35)$$

⁸This is quite intuitive as I_{hydro} contains dissipative terms, thus it cannot be invariant under Θ alone.

⁹As emphasized in [57], when ρ_0 is given by a thermal density matrix, while neither Θ nor the KMS operation takes the generating functional (2.1) to itself, the generating functional W is invariant under the combination of them i.e.

$$W[\tilde{g}_{\mu\nu}, \tilde{A}_\mu; \tilde{g}_{a\mu\nu}, \tilde{A}_{a\mu}] = W[g_{\mu\nu}, A_\mu; g_{a\mu\nu}, A_{a\mu}]. \quad (2.32)$$

Accordingly in I_{hydro} one can *not* impose either Θ or KMS separately, but should impose the combination of them (2.31).

while for $\Theta = \mathcal{CP}\mathcal{T}$

$$\eta_A A_\mu = (-A_0, -A_i), \quad \eta x^\mu = (-x^0, -x^i). \quad (2.36)$$

Since Θ contains \mathcal{T}

$$\Theta i = -i\Theta. \quad (2.37)$$

The second set of equations in (2.33) for a -variables can be written explicitly as

$$\Theta \tilde{\varphi}_a(x) = \varphi_a(x) + i\beta^\mu \partial_\mu \varphi(x), \quad (2.38)$$

$$\Theta \tilde{G}_{a\mu\nu}(x) = G_{a\mu\nu}(x) + i\mathcal{L}_{\beta^\mu} g_{\mu\nu}(x) = G_{a\mu\nu}(x) + i(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu), \quad (2.39)$$

$$\Theta \tilde{C}_{a\mu}(x) = C_{a\mu}(x) + i\mathcal{L}_{\beta^\mu} B_\mu(x) = C_{a\mu}(x) + i(\nabla_\mu \hat{\mu} - \beta^\nu F_{\mu\nu}). \quad (2.40)$$

The explicit transformations for $\Theta = \mathcal{T}, \mathcal{PT}, \mathcal{CP}\mathcal{T}$ for various tensors are given in appendix A.

It is straightforward to write down the most general $I_{\text{hydro}} = \int d^d x \sqrt{-g} \mathcal{L}$ consistent with the above prescriptions. We can expand the corresponding Lagrangian density \mathcal{L} in terms of the number of a -variables and derivatives. The first few terms in the a -field expansion can be written schematically as

$$\mathcal{L} = \frac{1}{2} \hat{T}^{\mu M} G_{a\mu M} + \frac{i}{4} W^{\mu\nu, MN} G_{a\mu M} G_{a\nu N} + \frac{1}{8} Y^{\mu\nu\rho, MNP} G_{a\mu M} G_{a\nu N} G_{a\rho P} + \dots, \quad (2.41)$$

where we have introduced notation

$$G_{a\mu M} = (G_{a\mu\nu}, 2C_{a\mu}), \quad \hat{T}^{\mu M} = (\hat{T}^{\mu\nu}, \hat{J}^\mu), \quad M = (\mu, d), \quad G_{a\mu d} = 2C_{a\mu} \quad (2.42)$$

and $\hat{T}^{\mu M}, W^{\mu\nu, MN}, \dots$ are covariant tensors constructed out of r -variables $\{\beta^\mu, \hat{\mu}, F_{\mu\nu}, \Delta^{\mu\nu}\}$ and covariant derivatives on $G_{a\mu M}$. Given that $G_{a\mu\nu} = g_{a\mu\nu} + \dots$ and $C_{a\mu} = A_{a\mu} + \dots$, we identify $\hat{T}^{\mu\nu}$ and \hat{J}^μ as the ‘‘off-shell’’ hydrodynamic stress tensor and U(1) current, and the equations of motion of X_a^μ, φ_a give the standard hydrodynamic equations.

If we introduce n as the sum of the number of a -fields and the number of derivatives in a term, then since Φ_r in (2.33) contains one derivative, the dynamical KMS transformation (2.31) preserves n , which implies that terms in the action which have the same value of n transform separately among themselves. We can thus write the action as

$$\mathcal{L} = \sum_{n=1}^{\infty} \mathcal{L}_n = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \dots \quad (2.43)$$

where \mathcal{L}_n contains all terms with given n . They are separately invariant under (2.31). \mathcal{L}_1 contains only zeroth derivative term in $\hat{T}^{\mu M}$ while \mathcal{L}_2 contains first derivative terms in $\hat{T}^{\mu M}$ and zeroth derivative terms in $W^{\mu\nu, MN}$. The explicit expressions for (2.41) to order \mathcal{L}_2 for a parity-preserving fluid are given in [58].¹⁰

¹⁰They are given to order \mathcal{L}_3 for conformal fluids.

We now give a brief review of the derivation of the entropy current, whose details are given in [59]. Dynamical KMS invariance (2.31) implies that

$$\tilde{\mathcal{L}} = \mathcal{L} + \nabla_\mu V^\mu, \quad V^\mu = V_0^\mu + V_1^\mu + \dots \quad (2.44)$$

where $\tilde{\mathcal{L}} = \mathcal{L}[\Theta\tilde{\Lambda}_a, \Theta\tilde{\Lambda}_r]$, and V_k^μ contains k factors of a -fields. The entropy current can then be defined as

$$S^\mu = V_0^\mu - \hat{V}_1^\mu - \hat{T}^{\mu\nu}\beta_\nu - \hat{\mu}\hat{J}^\mu, \quad (2.45)$$

where \hat{V}_1^μ is V_1^μ with Λ_a replaced by the corresponding Φ_r as introduced in (2.33). It can be shown upon using equations of motion

$$\nabla_\mu S^\mu = R \geq 0 \quad (2.46)$$

where R is a local non-negative expression.

3 Effective action for parity-violating systems in (3+1)-dimension

In this section we apply the formalism reviewed in the previous section to four-dimensional systems which break parity, including those with a local U(1) anomaly

$$\nabla_\mu \hat{J}^\mu = \frac{c}{4} \hbar \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \quad (3.1)$$

where constant c depends on specific systems. $\epsilon^{\mu\nu\lambda\rho}$ is the fully antisymmetric tensor with $\epsilon^{0123} = \frac{1}{\sqrt{-g}}$. In (3.1) we have made manifest \hbar -dependence so as to be clear about the order in \hbar -expansion at which the corresponding anomalous transports appear in the hydrodynamical action. We assume that the system does *not* have any *local* mixed gravitational anomalies. We will see that the system can nevertheless possess *global* gravitational anomalies which are closely connected to certain novel transports.

3.1 Generating functional

From (3.1), under independent local transformations of $A_{1,2}$, equation (2.5) should be replaced by¹¹

$$-iW[g_1, A_1 - d\lambda_1; g_2, A_2 - d\lambda_2,] = -iW[A_1, g_1; A_2, g_2] + c \int (\lambda_1 F_1 \wedge F_1 - \lambda_2 F_2 \wedge F_2) \quad (3.2)$$

while (2.6) remains. Note that $F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA$, and the second term on the right hand side is independent of metrics. Indeed, from (3.2) the consistent currents introduced in (2.2)–(2.3) now satisfy¹²

$$\nabla_\mu J^\mu = \frac{c\hbar}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \quad (3.3)$$

¹¹We emphasize that here we consider only small gauge transformations and diffeomorphisms, i.e. those vanish at spatial and time infinities and smoothly connected to the identity.

¹²Note that when restoring \hbar , there should be a \hbar factor on the left hand side of various equations in (2.2)–(2.3).

and from diffeomorphism invariance of W we also have

$$\nabla_\nu T^\nu{}_\mu = F_{\mu\nu} J^\nu - A_\mu \nabla_\nu J^\nu = F_{\mu\nu} J^\nu - \frac{c}{4} \hbar A_\mu \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}. \quad (3.4)$$

In (3.3)–(3.4) we have suppressed indices 1, 2. It should be understood there are two copies of them and so are (3.5)–(3.7) below. Defining the covariant current as

$$\mathcal{J}^\mu = J^\mu + c\hbar \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda} \quad (3.5)$$

we can write equations (3.3) and (3.4) as

$$\nabla_\mu \mathcal{J}^\mu = \frac{3c}{4} \hbar \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (3.6)$$

$$\nabla_\nu T^\nu{}_\mu = F_{\mu\nu} \mathcal{J}^\nu. \quad (3.7)$$

Note that the equation for $T^{\mu\nu}$ must be expressible in terms of covariant current \mathcal{J}^μ as $T^{\mu\nu}$ should be gauge invariant (the last term in (3.2) is independent of the metric). To leading order in \hbar -expansion, the anomalous piece in (3.2) becomes (see (2.15)–(2.16) and $\lambda_a = \lambda_1 - \lambda_2$)

$$c \int (\lambda_1 F_1 \wedge F_1 - \lambda_2 F_2 \wedge F_2) = c\hbar \int (\lambda_a F \wedge F + 2\lambda F \wedge F_a) + O(\hbar^2). \quad (3.8)$$

3.2 Parity odd action

We now construct the hydrodynamic action for a parity-violating system with a local U(1) anomaly. We can write the action as

$$I_{\text{hydro}} = I_{\text{even}} + I_{\text{odd}} \quad (3.9)$$

where I_{even} and I_{odd} are parity even and odd parts respectively. I_{odd} can be further decomposed as

$$I_{\text{odd}} = I_{\text{o,inv}} + I_{\text{anom}} \quad (3.10)$$

where I_{anom} is responsible for generating the anomalous term in (3.2), and $I_{\text{o,inv}}$ is invariant under gauge transformations. Given that I_{even} is invariant under gauge transformations we can also write

$$I_{\text{hydro}} = I_{\text{inv}} + I_{\text{anom}}, \quad I_{\text{inv}} = I_{\text{o,inv}} + I_{\text{even}}. \quad (3.11)$$

Note that I_{inv} should depend on $\varphi_{1,2}$ only through $B_{1,2}$ introduced in (2.9), while I_{anom} does not have to.

Since neither the diagonal shift (2.23) nor the dynamical KMS transformations (2.33) mix parity even and odd parts, I_{even} and I_{odd} can be treated independently. I_{even} was discussed in detail in [57, 58]. Here we focus on

$$I_{\text{odd}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{odd}}. \quad (3.12)$$

and will construct \mathcal{L}_{odd} to order \mathcal{L}_2 as defined in (2.43).

Let us first look at I_{anom} . To match with the anomalous term in (3.2), we take the anomalous action as (written in fluid spacetime)

$$\frac{1}{\hbar} I_{\text{anom}} = c \int [\varphi_1 F_1(X_1) \wedge F_1(X_1) - \varphi_2 F_2(X_2) \wedge F_2(X_2)] \quad (3.13)$$

where $X_{1,2}^\mu$ are functions of σ^A , F_{1AB} is the pull-back of $F_{1\mu\nu}$. Note that under gauge transformations (2.12) we precisely recover (3.2) from (3.13). To see this, for two terms in (3.13) one changes the integration variables to X_1 and X_2 respectively, which then become dummy variables.

Given (3.13) and that I_{inv} depends only on $B_{1,2}$, the equations of motion of φ_s and X_s^μ lead to

$$\nabla_\mu \hat{J}^\mu = \frac{c}{4} \hbar \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (3.14)$$

$$\nabla_\nu \hat{T}^\nu{}_\mu = F_{\mu\nu} \hat{J}^\nu - \frac{c}{4} \hbar A_\mu \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (3.15)$$

where the off-shell stress tensors and consistent currents are defined in (2.13)–(2.14). Again we have suppressed $s = 1, 2$ and each equation should be understood to have two copies. Defining the covariant off-shell currents as

$$\hat{\mathcal{J}}^\mu = \hat{J}^\mu + c \hbar \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda} = \hat{J}_{\text{inv}}^\mu + c \hbar \epsilon^{\mu\nu\rho\lambda} B_\nu F_{\rho\lambda} \quad (3.16)$$

where \hat{J}_{inv}^μ is defined as the off-shell currents corresponding to I_{inv} , we then have

$$\nabla_\mu \hat{\mathcal{J}}^\mu = \frac{3c}{4} \hbar \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (3.17)$$

$$\nabla_\nu \hat{T}^\nu{}_\mu = F_{\mu\nu} \hat{\mathcal{J}}^\nu. \quad (3.18)$$

The last equality of (3.16) makes it manifest that $\hat{\mathcal{J}}^\mu$ is invariant under gauge transformations.

Expanding in small \hbar and rewriting the resulting expressions in physical spacetime we find that (3.13) becomes

$$\frac{1}{\hbar} I_{\text{anom}} = c \hbar \int d^4x (\varphi_a F \wedge F + 2\varphi F \wedge \mathcal{F}_a) \quad (3.19)$$

where

$$\mathcal{F}_a = dC_a = F_a + \mathcal{L}_{X_a} F \quad (3.20)$$

and $C_{a\mu}$ was defined in (2.19). Note that $\frac{1}{\hbar} I_{\text{anom}}$ is of order $O(\hbar)$. Under a diagonal shift (2.23), equation (3.19) transforms as

$$\frac{1}{\hbar} \delta I_{\text{anom}} = 2c \hbar \int \lambda(\sigma^i) F \wedge \mathcal{F}. \quad (3.21)$$

In order for the full odd action (3.10) to be invariant under (2.23), $I_{\text{o,inv}}$ should also not be invariant and its variation should precisely cancel (3.21). At linear order in a -fields (order $O(a)$) we can write

$$\frac{1}{\hbar} \mathcal{L}_{\text{o,inv}} = \frac{1}{2} \hat{T}_o^{\mu\nu} G_{a\mu\nu} + \hat{J}_o^\mu C_{a\mu} \quad (3.22)$$

and the terms on the right hand side may be further expanded in \hbar and derivatives.

Let us first consider $\hat{T}_o^{\mu\nu}$ which as usual can be decomposed as

$$\hat{T}_o^{\mu\nu} = \varepsilon_o u^\mu u^\nu + p_o \Delta^{\mu\nu} + 2u^{(\mu} q_o^{\nu)} + \Sigma_o^{\mu\nu}, \quad (3.23)$$

where q_o^ν and $\Sigma_o^{\mu\nu}$ are transverse to u^μ . Since terms proportional to $G_{a\mu\nu}$ will never generate a term of the form (3.21) under (2.23), $T_o^{\mu\nu}$ should be diagonal shift invariant by itself. At zeroth derivative order there is no such term. At first derivative order the only non-vanishing quantity is q_o^ν which can be written as

$$q_o^\mu = g_1 \omega^\mu + g_2 \mathcal{B}^\mu \quad (3.24)$$

where

$$\omega^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu (\nabla_\lambda u_\rho - \nabla_\rho u_\lambda), \quad \mathcal{B}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu F_{\lambda\rho} \quad (3.25)$$

and g_1, g_2 are some functions of $\beta(x)$ and $\hat{\mu}(x)$.

\hat{J}_o^μ can be written as

$$\hat{J}_o^\mu = -c\hbar \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} B_\rho + \hat{j}_o^\mu \quad (3.26)$$

where the variation of the first term under diagonal shift cancels (3.21) and thus \hat{j}_o^μ should be invariant. From (3.16) we thus find that

$$\hat{\mathcal{J}}_{\text{odd}}^\mu = \hat{j}_o^\mu. \quad (3.27)$$

As discussed above to first derivative order since there is no diagonal shift invariant scalar term \hat{j}_o^μ should then be transverse and can be written as

$$\hat{j}_o^\mu = h_1 \omega^\mu + h_2 \mathcal{B}^\mu \quad (3.28)$$

where h_1, h_2 are some functions of $\beta(x)$ and $\hat{\mu}(x)$.

Now let us consider quadratic terms in a -fields (order $O(a^2)$) to zeroth order in derivative, which should have the form

$$\frac{i}{4} W_o^{\mu\nu, MN} G_{a\mu M} G_{a\nu N} \quad (3.29)$$

where $W_o^{\mu\nu, MN}$ is parity odd and is diagonal shift invariant. Such a term does not exist at zero derivative order so we conclude there are no new parity-odd terms at order $O(a^2)$.

Collecting the above expressions, \mathcal{L}_{odd} can be written as

$$\begin{aligned} \frac{1}{\hbar} \mathcal{L}_{\text{odd}} &= \frac{c}{4} \hbar \epsilon^{\mu\nu\lambda\rho} (\varphi_a F_{\mu\nu} F_{\lambda\rho} + 2\varphi F_{\mu\nu} \mathcal{F}_{a\lambda\rho}) + u^{(\mu} q_o^{\nu)} G_{a\mu\nu} + \left(\hat{j}_o^\mu - c\hbar \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} B_\rho \right) C_{a\mu} \\ &= u^{(\mu} q_o^{\nu)} G_{a\mu\nu} + \left(\hat{j}_o^\mu - c\hbar \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} A_\rho \right) C_{a\mu} + \frac{c}{4} \hbar \epsilon^{\mu\nu\lambda\rho} \varphi_a F_{\mu\nu} F_{\lambda\rho} \end{aligned} \quad (3.30)$$

where $\mathcal{F}_{a\lambda\rho}$ is defined by (3.20), and q_o^μ, \hat{j}_o^μ are given respectively by (3.24), (3.28). Using field redefinitions one can write \mathcal{L}_{odd} in the Landau frame (see section VI of [58] for details)

$$\mathcal{L}_{\text{odd}} = \left(\ell_o^\mu - c\hbar \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} A_\rho \right) C_{a\mu} + \frac{c}{4} \hbar \epsilon^{\mu\nu\lambda\rho} \varphi_a F_{\mu\nu} F_{\lambda\rho} \quad (3.31)$$

with ℓ_o^μ given by

$$\ell_o^\mu \equiv \hat{j}_o^\mu - \frac{n_0}{\epsilon_0 + p_0} q_o^\mu \quad (3.32)$$

where ϵ_0, p_0, n_0 are respectively zeroth order energy, pressure and charge densities.

3.3 Dynamical KMS condition

We now impose the dynamical KMS condition (2.31) on the parity-odd action (3.10). We will consider respectively $\Theta = \mathcal{PT}, \mathcal{T}, \mathcal{CPT}$ and will see that they lead to very different results.

Due to the presence of \hbar on the right hand side of (3.3), $\frac{1}{\hbar}I_{\text{anom}}$ is of order $O(\hbar)$. In $\frac{1}{\hbar}I_{\text{o,inv}}$ the first term in (3.26) is $O(\hbar)$ while g_1, g_2, h_1, h_2 are undetermined at the moment. We will later argue that they should also be $O(\hbar)$. Thus in our discussion below it is enough to consider the leading order terms in dynamical KMS transformations (2.33).¹³

3.3.1 $\Theta = \mathcal{PT}$

We find in this case

$$\frac{1}{\hbar}\tilde{\mathcal{L}}_{\text{odd}} - \frac{1}{\hbar}\mathcal{L}_{\text{odd}} = -2\frac{\mathcal{L}_{\text{odd}}}{\hbar} - u^\mu q_o^\nu \nabla_\mu \beta_\nu \quad (3.33)$$

$$- (\hat{j}_o^\mu - c\hbar\epsilon^{\mu\nu\lambda\rho}F_{\nu\lambda}A_\rho)(\partial_\mu \hat{\mu} + \beta^\alpha F_{\alpha\mu}) - \frac{c\hbar}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\beta^\alpha \partial_\alpha \phi \quad (3.34)$$

KMS invariance at $\mathcal{O}(a)$ then requires

$$g_1 = h_1 = g_2 = h_2 = c = 0. \quad (3.35)$$

3.3.2 $\Theta = \mathcal{T}$

From (3.19) we find that under dynamical KMS transformation, the anomalous action becomes (see appendix B for useful formulae)

$$\begin{aligned} \frac{\tilde{I}_{\text{anom}}}{\hbar} &= c\hbar \int ((\varphi_a + \mathcal{L}_{i\beta}\varphi)F \wedge F + 2\varphi F \wedge (\mathcal{F}_a + \mathcal{L}_{i\beta}F)) \\ &= \frac{I_{\text{anom}}}{\hbar} + 2ic\hbar \int d(\varphi(\beta \cdot F) \wedge F) \end{aligned} \quad (3.36)$$

where $\beta \cdot F \equiv \beta^\mu F_{\mu\nu} dx^\nu$. For dynamical KMS transformation of $I_{\text{o,inv}}$, note that

$$\tilde{q}_o^\mu(-x^0, x^i) = (q_o^0, -q_o^i)(x), \quad \tilde{J}_o^\mu(-x^0, x^i) = (\hat{J}_o^0, -\hat{J}_o^i)(x). \quad (3.37)$$

We then find that

$$\frac{\tilde{I}_{\text{o,inv}}}{\hbar} = \frac{I_{\text{o,inv}}}{\hbar} + i \int d^4x \left(\frac{1}{2}\hat{T}_o^{\mu\nu} \mathcal{L}_\beta g_{\mu\nu} + \hat{J}_o^\mu \mathcal{L}_\beta B_\mu \right). \quad (3.38)$$

For I_{odd} to be invariant, we need the second term of (3.38) to be a total derivative. More explicitly, using (3.24)–(3.28), we find after some algebraic manipulations (see appendix B for useful formulae)

$$i \int d^4x \left(\frac{1}{2}\hat{T}_o^{\mu\nu} \mathcal{L}_\beta g_{\mu\nu} + \hat{J}_o^\mu \mathcal{L}_\beta B_\mu \right)$$

¹³In fact one can check that the structure of $O(\hbar)$ corrections in (2.33) are such that even if g_1, g_2, h_1, h_2 are of order \hbar^0 , at first derivative order for $O(a)$ terms the leading terms in (2.33) are adequate.

$$\begin{aligned}
 &= i \int \left[\left(\frac{\beta h_2}{2} + 3c\hbar\hat{\mu} \right) F \wedge F + (h_2 d\hat{\mu} - g_2 d\beta) \wedge u \wedge F + (h_1 d\hat{\mu} - g_1 d\beta) \wedge u \wedge du \right. \\
 &\quad \left. + h_1(\beta \cdot F) \wedge u \wedge du + g_2(\beta \cdot \omega) \wedge u \wedge F + \frac{g_1 \beta}{2} du \wedge du - 2c\hbar d(\hat{\mu} B \wedge F) \right] \quad (3.39)
 \end{aligned}$$

where $u \equiv u_\mu dx^\mu$. For the above expression to be a total derivative we find that h_1, h_2, g_1, g_2 must arise from derivatives of two functions H_1, H_2 and satisfy the following relations

$$h_2 = \frac{-6c\hbar\hat{\mu} + 2a_1}{\beta}, \quad h_1 = g_2, \quad \partial_{\hat{\mu}} H_2 = h_2, \quad \partial_\beta H_2 = -g_2, \quad \partial_{\hat{\mu}} H_1 = h_1, \quad (3.40)$$

$$\partial_\beta H_1 = -g_1, \quad 2H_1 = g_1 \beta, \quad g_2 \beta = H_2 \quad (3.41)$$

where a_1 is a constant. Note that one could add a constant to the right hand side of equation $2H_1 = g_1 \beta$, but that constant can be absorbed in the definition of H_1 . Similarly with equation $g_2 \beta = H_2$. With (3.40)–(3.41),

$$i \int d^4x \left(\frac{1}{2} \hat{T}_o^{\mu\nu} \mathcal{L}_\beta g_{\mu\nu} + \hat{J}_o^\mu \mathcal{L}_\beta B_\mu \right) = i \int dQ, \quad (3.42)$$

$$Q = -2c\hbar\hat{\mu} B \wedge F + a_1 A \wedge F + H_2 u \wedge F + H_1 u \wedge du. \quad (3.43)$$

Note that Q is defined only up to a closed three-form as such an addition will not change (3.42).

The most general solutions to (3.40)–(3.41) can be written as

$$H_2 = \frac{-3c\hbar\hat{\mu}^2 + 2a_1\hat{\mu} + a_2}{\beta}, \quad H_1 = \frac{-c\hbar\hat{\mu}^3 + a_1\hat{\mu}^2 + a_2\hat{\mu} + a_3}{\beta^2} \quad (3.44)$$

$$h_1 = \frac{-3c\hbar\hat{\mu}^2 + 2a_1\hat{\mu} + a_2}{\beta^2}, \quad h_2 = \frac{-6c\hbar\hat{\mu} + 2a_1}{\beta}, \quad (3.45)$$

$$g_1 = \frac{-2c\hbar\hat{\mu}^3 + 2a_1\hat{\mu}^2 + 2a_2\hat{\mu} + 2a_3}{\beta^3}, \quad g_2 = \frac{-3c\hbar\hat{\mu}^2 + 2a_1\hat{\mu} + a_2}{\beta^2} \quad (3.46)$$

where a_1, a_2, a_3 are constants. Thus to first derivative order I_{odd} is fully determined up to three constants.

3.3.3 $\Theta = \mathcal{CPT}$

The analysis for $\Theta = \mathcal{CPT}$ is very similar. Note that

$$\tilde{F}_{\mu\nu}(-x) = F_{\mu\nu}, \quad \tilde{\mathcal{F}}_{a\mu\nu}(-x) = \mathcal{F}_{a\mu\nu}(x) + i\mathcal{L}_\beta F_{\mu\nu}(x) \quad (3.47)$$

and equation (3.36) again applies. For $I_{\text{o,inv}}$, we now have

$$\tilde{q}_o^\mu(-x) = -g_1(-\hat{\mu}, \beta)\omega^\mu(x) + g_2(-\hat{\mu}, \beta)\mathcal{B}^\mu(x) \quad (3.48)$$

$$\tilde{J}_o^\mu(-x) = c\hbar\epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} B_\rho - h_1(-\hat{\mu}, \beta)\omega^\mu(x) + h_2(-\hat{\mu}, \beta)\mathcal{B}^\mu(x). \quad (3.49)$$

and the dynamical KMS condition at $O(a)$ level requires

$$g_1(-\hat{\mu}) = -g_1(\hat{\mu}), \quad g_2(-\hat{\mu}) = g_2(\hat{\mu}), \quad h_1(-\hat{\mu}) = h_1(\hat{\mu}), \quad h_2(-\hat{\mu}) = -h_2(\hat{\mu}). \quad (3.50)$$

The analysis for $O(a^0)$ terms is the same as before and (3.40)–(3.43) apply. Imposing (3.50) on the solutions (3.44)–(3.46) we find that $a_1 = a_3 = 0$, and thus

$$h_1 = \frac{-3c\hbar\hat{\mu}^2 + a_2}{\beta^2}, \quad h_2 = -\frac{6c\hbar\hat{\mu}}{\beta}, \quad g_1 = \frac{-2c\hbar\hat{\mu}^3 + 2a_2\hat{\mu}}{\beta^3}, \quad g_2 = \frac{-3c\hbar\hat{\mu}^2 + a_2}{\beta^2}, \quad (3.51)$$

$$H_2 = \frac{-3c\hbar\hat{\mu}^2 + a_2}{\beta}, \quad H_1 = \frac{-c\hbar\hat{\mu}^3 + a_2\hat{\mu}}{\beta^2}. \quad (3.52)$$

Thus for a macroscopic system whose underlying Hamiltonian is invariant under \mathcal{CPT} to first derivative order I_{odd} is fully determined up to a single constant.

3.4 Explicit expressions for q_o^μ and \hat{j}_o^μ

We can now write down the explicit expressions for q_o^μ and \hat{j}_o^μ to be used in (3.30) or (3.31). It is enough to do it for $\Theta = \mathcal{T}$. The expressions for $\Theta = \mathcal{CPT}$ can be obtained by setting $a_1 = a_3 = 0$, while those for $\Theta = \mathcal{PT}$ can be obtained by setting $a_1 = a_2 = a_3$ to zero.

From (3.45)–(3.46) we find that

$$q_o^\mu = -c\hbar\frac{\hat{\mu}^2}{\beta^2} \left(\frac{2\hat{\mu}}{\beta}\omega^\mu + 3\mathcal{B}^\mu \right) + \frac{2a_1\hat{\mu}}{\beta^2} \left(\frac{\hat{\mu}}{\beta}\omega^\mu + \mathcal{B}^\mu \right) + \frac{a_2}{\beta^2} \left(\frac{2\hat{\mu}}{\beta}\omega^\mu + \mathcal{B}^\mu \right) + \frac{2a_3}{\beta^3}\omega^\mu \quad (3.53)$$

$$\hat{j}_o^\mu = -3c\hbar\frac{\hat{\mu}}{\beta} \left(\frac{\hat{\mu}}{\beta}\omega^\mu + 2\mathcal{B}^\mu \right) + \frac{2a_1}{\beta} \left(\frac{\hat{\mu}}{\beta}\omega^\mu + \mathcal{B}^\mu \right) + \frac{a_2}{\beta^2}\omega^\mu. \quad (3.54)$$

The frame independent quantity ℓ_o^μ (3.32) is then given by

$$\begin{aligned} \ell_o^\mu &= -3c\hbar\frac{\hat{\mu}}{\beta} \left[\left(1 - \frac{2\alpha}{3} \right) \frac{\hat{\mu}\omega^\mu}{\beta} + (2 - \alpha)\mathcal{B}^\mu \right] + \frac{2a_1}{\beta}(1 - \alpha) \left(\frac{\hat{\mu}}{\beta}\omega^\mu + \mathcal{B}^\mu \right) \\ &\quad + \frac{a_2}{\beta^2} \left[(1 - 2\alpha)\omega^\mu - \frac{n_0}{\epsilon_0 + p_0}\mathcal{B}^\mu \right] - \frac{2a_3}{\beta^3} \frac{n_0}{\epsilon_0 + p_0}\omega^\mu \\ &= \xi_\omega\omega^\mu + \xi_B\mathcal{B}^\mu \end{aligned} \quad (3.55)$$

where we have introduced

$$\alpha \equiv \frac{n_0\hat{\mu}}{\beta(\epsilon_0 + p_0)} \quad (3.56)$$

and

$$\begin{aligned} \xi_\omega &= -3c\hbar\frac{\hat{\mu}^2}{\beta^2} \left(1 - \frac{2\alpha}{3} \right) + \frac{2a_1\hat{\mu}}{\beta^2}(1 - \alpha) + \frac{a_2}{\beta^2}(1 - 2\alpha) - \frac{2a_3}{\beta^3} \frac{n_0}{\epsilon_0 + p_0}, \\ \xi_B &= -3c\hbar\frac{\hat{\mu}}{\beta}(2 - \alpha) + \frac{2a_1}{\beta}(1 - \alpha) - \frac{a_2}{\beta^2} \frac{n_0}{\epsilon_0 + p_0}. \end{aligned} \quad (3.57)$$

Equations (3.55)–(3.57) reproduce previous results in the literature obtained from entropy current [8, 9] and equilibrium partition function [13], confirming that these methods indeed give the complete answer for the current problem. However, those methods did not pinpoint the exact discrete symmetry a system should have for (3.55)–(3.57). Ref. [13] did point out for \mathcal{CPT} invariant theories one should set $a_1 = a_3 = 0$.

We presented our results in terms of $\omega^\mu, \mathcal{B}^\mu$ which were defined in (3.25) from respective “field strengths” of u_μ and B_μ . But note that $u^\mu B_\mu \neq 0$. We now present (3.53)–(3.55) in a slightly different basis which makes their expressions a bit more transparent. Introduce

$$b_\mu = \Delta_\mu^\nu B_\nu = B_\mu + \hat{\mu}v_\mu, \quad v_\mu = \frac{u_\mu}{\beta}, \quad b_\mu v^\mu = 0 \quad (3.58)$$

and

$$\mathfrak{w}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}v_\nu(\nabla_\lambda v_\rho - \nabla_\rho v_\lambda), \quad \mathfrak{B}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}v_\nu(\nabla_\lambda b_\rho - \nabla_\rho b_\lambda). \quad (3.59)$$

Note that $v^\mu b_\mu = 0$, and

$$\omega^\mu = \beta^2 \mathfrak{w}^\mu, \quad \mathcal{B}^\mu = \beta(\mathfrak{B}^\mu - \hat{\mu}\mathfrak{w}^\mu). \quad (3.60)$$

Then equations (3.53)–(3.54) and (3.55) can be rewritten as

$$q_o^\mu = \frac{c\hbar\hat{\mu}^2}{\beta}(\hat{\mu}\mathfrak{w}^\mu - 3\mathfrak{B}^\mu) + \frac{2a_1\hat{\mu}}{\beta}\mathfrak{B}^\mu + \frac{a_2}{\beta}(\hat{\mu}\mathfrak{w}^\mu + \mathfrak{B}^\mu) + \frac{2a_3}{\beta}\mathfrak{w}^\mu, \quad (3.61)$$

$$\hat{j}_o^\mu = 3c\hbar\hat{\mu}(\hat{\mu}\mathfrak{w}^\mu - 2\mathfrak{B}^\mu) + 2a_1\mathfrak{B}^\mu + a_2\mathfrak{w}^\mu, \quad (3.62)$$

and

$$\begin{aligned} \ell_o^\mu &= 3c\hbar\hat{\mu} \left[\left(1 - \frac{\alpha}{3}\right)\hat{\mu}\mathfrak{w}^\mu - (2 - \alpha)\mathfrak{B}^\mu \right] \\ &\quad + 2a_1(1 - \alpha)\mathfrak{B}^\mu + a_2 \left[(1 - \alpha)\mathfrak{w}^\mu - \frac{\alpha}{\hat{\mu}}\mathfrak{B}^\mu \right] - \frac{2a_3\alpha}{\hat{\mu}}\mathfrak{w}^\mu \\ &= \xi_{\mathfrak{w}}\mathfrak{w}^\mu + \xi_{\mathfrak{B}}\mathfrak{B}^\mu \end{aligned} \quad (3.63)$$

with

$$\xi_{\mathfrak{w}} = 3c\hbar\hat{\mu}^2 \left(1 - \frac{\alpha}{3}\right) + a_2(1 - \alpha) - \frac{2a_3\alpha}{\hat{\mu}}, \quad \xi_{\mathfrak{B}} = -3c\hbar\hat{\mu}(2 - \alpha) + 2a_1(1 - \alpha) - \frac{a_2\alpha}{\hat{\mu}}. \quad (3.64)$$

Similarly Q of (3.43) can be written more transparently in the basis of (3.58) as

$$Q = -c\hbar(2\hat{\mu}b \wedge db - \hat{\mu}^2 b \wedge dv) + a_1 b \wedge db + a_2 v \wedge db + a_3 v \wedge dv \quad (3.65)$$

where we have dropped an exact three-form as mentioned earlier Q is defined only up to a closed three-form.

4 Equilibrium partition function and global gravitational anomalies

In this section we first explain how to obtain the equilibrium partition function from the hydrodynamical effective action. We discuss two different ways of doing it. We then apply the procedures to I_{odd} found in the last section to obtain the parity-odd part of the equilibrium partition function. We will see that in the absence local anomalies, i.e. $c = 0$, all the parity-odd transport terms are connected to *global* anomalies. When the underlying theory is only invariant under \mathcal{T} , terms proportional to a_1, a_2, a_3 in (3.53)–(3.57) are respectively associated with global U(1), mixed gravitational, and gravitational anomalies. With \mathcal{CPT} invariance, only a global mixed gravitational anomaly is present. This connection also implies that $a_{1,2,3}$ should be proportional to \hbar .

4.1 Equilibrium partition function from effective action

We will now describe two methods of obtaining the equilibrium partition function from the effective action when ρ_0 in (2.1) is given by the thermal density matrix with an inverse temperature β_0 . By definition the generating functional W of (2.1) becomes identically zero when we set the external fields for the two legs to be the same. Nevertheless, as already indicated in [57–59], the equilibrium partition function can be extracted from the effective action with the help of the dynamical KMS condition. We will again work to leading order in small \hbar expansion.

For notational simplicity we will now denote the sources collectively by ϕ_i and their corresponding operators \mathcal{O}_i with index i labelling different operators/components. In [57] it was shown that a generating functional W satisfying the combined Θ and KMS transformation (2.32) can be “factorized” in the stationary limit. That is, when the sources ϕ_{1i}, ϕ_{2i} are time independent, to leading order in the a -field expansion we can write W as

$$W[\phi_{1i}, \phi_{2i}] = i\tilde{W}[\phi_{1i}] - i\tilde{W}[\phi_{2i}] + \dots \quad (4.1)$$

where \dots denotes terms of order $O(a^2)$, $\tilde{W}[\phi_i(\vec{x})]$ is a functional defined on the spatial manifold of the spacetime, and satisfies

$$\tilde{W}[\phi_i(\vec{x})] = \tilde{W}[\Theta\phi_i(\vec{x})], \quad (4.2)$$

where Θ here should be understood as the extension of (2.34) to time-independent field configurations. Equation (4.1) implies that

$$\langle \mathcal{O}_i(\omega = 0, \vec{x}) \rangle_\phi = \frac{\delta \tilde{W}[\phi_i(\vec{x})]}{\delta \phi_i(\vec{x})}. \quad (4.3)$$

Writing the equilibrium partition function Z as

$$Z = e^{-\beta_0 F}, \quad (4.4)$$

where F is the free energy, and doing analytic continuation of \tilde{W} to Euclidean signature,¹⁴ from (4.3) we can identify $-\tilde{W}$ with $\beta_0 F$.

The free energy F (and thus \tilde{W}) should have a local expansion in terms of external sources, as the equilibrium partition function can be computed by putting the system on a Euclidean manifold with a periodic time circle, which generates a finite gap. As discussed in [57] we can obtain \tilde{W} from the contact terms in I_{hydro} as follows. One first obtains the source action I_s by setting the dynamical fields in I_{hydro} to the following equilibrium values

$$\varphi = \varphi_a = X_a^\mu = 0, \quad u^\mu = \frac{1}{b}(1, \vec{0}), \quad \beta = \beta_0 b, \quad b = \sqrt{-g_{00}} \quad (4.5)$$

which give

$$G_{a\mu\nu} = g_{a\mu\nu}, \quad C_{a\mu} = A_{a\mu}, \quad B_\mu = A_\mu, \quad \hat{\mu} = \beta_0 A_0, \quad u_0 = -b, \quad u_i = \frac{g_{0i}}{b}. \quad (4.6)$$

¹⁴See section 4.2 for an explicit example of the continuation.

All external fields are taken to be time independent. Then to leading order in the a -field expansion

$$I_s = \tilde{W}[\phi_1] - \tilde{W}[\phi_2] + \dots \quad (4.7)$$

where \dots denotes terms of order $O(a^2)$. That I_s is factorizable at this order is warranted by the dynamical KMS condition.¹⁵

There is also an alternative way to obtain the equilibrium free energy as follows. The dynamical KMS condition (2.31) implies that

$$\tilde{\mathcal{L}} = \mathcal{L} + \partial_\mu V^\mu \quad (4.8)$$

where $\tilde{\mathcal{L}}$ is defined as $\tilde{\mathcal{L}} = \mathcal{L}[\Theta\tilde{\Lambda}_a, \Theta\tilde{\Lambda}_r]$ (see (2.33)). V^μ can be further expanded in terms of a -fields as

$$V^\mu = iV_0^\mu + \dots \quad (4.9)$$

where V_0^μ contains r -fields only. From the discussion of the entropy current in [59], we can then identify¹⁶

$$\int d^{d-1}x \sqrt{-g} V_0^0|_{\text{eq}} = \log Z = -\beta_0 F \quad (4.10)$$

where $V_0^0|_{\text{eq}}$ denotes the expression obtained by setting dynamical fields in V_0^0 to equilibrium values (4.5).

The equivalence of the two methods can be considered as a consequence of equivalence of local KMS condition of [57] and the dynamical KMS condition (2.31) as shown in [58]. One can readily check that applied to the parity even part of the effective action I_{even} the two methods indeed give the same answers and are equivalent to the results discussed in [13, 14].

4.2 Parity-odd equilibrium partition function and global anomalies

We now obtain the parity-odd partition function from I_{odd} following the procedures discussed in the previous subsection. It can be readily checked that the two approaches give the same answers. The second approach is significantly simpler technically, which we will describe here. Recall that from our analysis for $\Theta = \mathcal{PT}$ there is no parity-odd contribution to the partition function at first derivative order. The results below are for $\Theta = \mathcal{T}$; to obtain $\Theta = \mathcal{CPT}$ one needs to take $a_{1,3} = 0$ together with (3.50).

From (3.36), (3.43), and (4.10) we immediately obtain that

$$\log Z = \int [(-2c\hbar\hat{\mu} + a_1) A \wedge dA + H_2 u \wedge dA + H_1 u \wedge du] \quad (4.11)$$

where the integration is over the *spatial* manifold with $A \equiv A_i dx^i, u \equiv u_i dx^i$. Using the basis of (3.65), equation (4.11) can be written more transparently as

$$\log Z = \int \left[-c\hbar\beta_0 (2A_0 b \wedge db - A_0^2 b \wedge dv) + a_1 b \wedge db + \frac{a_2}{\beta_0} v \wedge db + \frac{a_3}{\beta_0^2} v \wedge dv \right] \quad (4.12)$$

¹⁵In [57] the KMS condition on I_{hydro} was imposed by requiring I_s to satisfy the combination of Θ and KMS, dubbed the local KMS condition there. In [58] it was shown the dynamical KMS (2.31) and local KMS conditions are equivalent.

¹⁶See equation (3.14) there. The second term \hat{V}_1^0 vanishes in the stationary limit.

where

$$v = v_i dx^i, \quad b = b_i dx^i, \quad v_i = -\frac{g_{0i}}{g_{00}}, \quad b_i = A_i + v_i A_0. \quad (4.13)$$

Equation (4.12) precisely agrees with that given in [13].

Let us now explore a bit further physical implications of (4.12). The background fields in (4.12) are those for a stationary Lorentzian manifold with

$$ds^2 = g_{00} (dt - v_i dx^i)^2 + a_{ij} dx^i dx^j, \quad A_\mu dx^\mu = A_0 dt + A_i dx^i \quad (4.14)$$

and $g_{00} < 0$. Note that (4.14) is preserved by time reparameterizations $t \rightarrow t + f(\vec{x})$, under which

$$v_i \rightarrow v_i - \partial_i f, \quad A_i \rightarrow A_i + A_0 \partial_i f, \quad b_i \rightarrow b_i, \quad (4.15)$$

and time-independent U(1) transformations $A_i \rightarrow A_i + \partial_i \lambda(\vec{x})$ under which

$$v_i \rightarrow v_i, \quad b_i \rightarrow b_i + \partial_i \lambda. \quad (4.16)$$

Below we will refer to (4.15) as time U(1) and (4.16) as flavor U(1).

The thermal partition function is usually calculated by analytically continuing to Euclidean signature with $t \rightarrow -i\tau$ (with τ on a circle with period β_0), and the background fields are taken so that they are real in Euclidean signature. We take the Euclidean metric and gauge field to be of form

$$ds^2 = g_{00} (d\tau - v_i dx^i)^2 + a_{ij} dx^i dx^j \quad (4.17)$$

$$A_\mu dx^\mu = A_0 d\tau + A_i dx^i \quad (4.18)$$

Here, $g_{00} > 0$. Thus, under the analytic continuation $t \rightarrow -i\tau$, we get the replacements

$$v_i \rightarrow -iv_i, \quad A_0 \rightarrow iA_0 \quad (4.19)$$

after which (4.12) becomes

$$\log Z = \int \left[-i c \hbar \beta_0 (2A_0 b \wedge db - A_0^2 b \wedge dv) + a_1 b \wedge db - i \frac{a_2}{\beta_0} v \wedge db - \frac{a_3}{\beta_0^2} v \wedge dv \right]. \quad (4.20)$$

Note that \mathcal{CPT} invariant terms become pure imaginary while the terms proportional to a_1 and a_3 remain real.

Now let us consider a system with *no* local anomalies, i.e. $c = 0$. Then in (4.20) we have three Chern-Simons terms, respectively, for flavor U(1), mixed time and flavor U(1), and time U(1). A defining feature of Chern-Simons terms is that they are not invariant under “large” gauge transformations i.e. those are not connected to the identity. Consider for example the flavor U(1) Chern-Simons term

$$a_1 \int b \wedge db. \quad (4.21)$$

Let us take the spatial manifold to have the topology of $S^1 \times S^2$, where S^1 has size L . We can choose b to have a monopole configuration on S^2 , i.e.

$$\int_{S^2} db = \frac{2\pi n}{q}, \quad n \in \mathbb{Z} \quad (4.22)$$

where q is the minimal charge under $U(1)$.

A large gauge transformation of b_x (x is the circle direction) is

$$b_x \rightarrow b_x + \frac{2\pi m}{qL}, \quad m \in \mathbb{Z} \tag{4.23}$$

we then have [77, 78]

$$Z \rightarrow e^{\frac{8\pi^2 m n a_1}{q^2}} Z. \tag{4.24}$$

Under Kaluza-Klein reduction, v couples to matter as a $U(1)$ gauge field with minimal “charge” $\frac{2\pi}{\beta_0}$, thus a large gauge transformation of v_x is

$$v_x \rightarrow v_x + \frac{k\beta_0}{L}, \quad k \in \mathbb{Z} \tag{4.25}$$

we have

$$Z \rightarrow e^{-i n k \frac{2\pi}{q} a_2} Z. \tag{4.26}$$

For the last term in (4.20) we need to consider a monopole configuration for v_i on S^2 ,

$$\int_{S^2} dv = l\beta_0, \quad l \in \mathbb{Z} \tag{4.27}$$

and then under just a large gauge transformation (4.25) we have

$$Z \rightarrow e^{-2a_3 k l} Z \tag{4.28}$$

Note in (4.24) and (4.28) the partition function transforms by a real number rather than a phase. As mentioned earlier non-vanishing a_1 or a_3 breaks \mathcal{CPT} .

Thus we find in the absence of local anomaly, all the anomalous transports are associated with global gauge or gravitational anomalies for putting the system on a Euclidean four-manifold with a thermal time circle.

In the presence of a local anomaly, i.e. $c \neq 0$, then the transport coefficients in (3.61)–(3.63) are then mixed among local and global anomalies. The same thing happens to the partition function. But note that a_1, a_3 are terms, being real, are not mixed with local anomalies.

Possible connections of the term proportional to a_2 with mixed global gravitational anomaly was first hinted in [53] and shown explicitly in [54, 55] in some free theory models.

5 Entropy current

In this section we obtain the entropy current for a $(3+1)$ -dimensional parity-violating fluid by applying (2.45). One thing to notice is that the anomalous action (3.19) does not have the same structure of the rest of the action. At $O(a)$ the latter has the form (now also including the parity-even part, see (3.11))

$$\frac{1}{\hbar} \mathcal{L}_{\text{inv}} = \frac{1}{2} \hat{T}^{\mu\nu} G_{a\mu\nu} + \hat{J}_0^\mu C_{a\mu} + \dots, \tag{5.1}$$

which is the form assumed in [59]. The fact that I_{anom} has a different structure does not cause a problem, as I_{anom} is KMS invariant by itself. We can then simply apply the procedure of (2.45) to I_{inv} which will generate an entropy current with non-negative divergence.

Now applying (2.45) we find that

$$S^\mu = V_0^\mu - \hat{T}^{\mu\nu} \beta_\nu - \hat{\mu} \hat{\mathcal{J}}^\mu + c \hat{\hbar} \hat{\mu} \epsilon^{\mu\nu\alpha\beta} B_\nu F_{\alpha\beta} \quad (5.2)$$

and

$$\partial_\mu S^\mu = R_{\text{even}} \geq 0 \quad (5.3)$$

with R_{even} to be divergence of the entropy current of the parity-even part.¹⁷ Equation (5.3) means that parity-odd part does not contribute to entropy dissipation.

From (3.42)–(3.43), for the parity-odd part, V_0^μ is simply the dual of Q , giving the following odd-parity contribution to the entropy

$$\begin{aligned} S_o^\mu &= -2u^{(\mu} q_o^{\nu)} \beta_\nu - \hat{\mu} \hat{j}_o^\mu + \left(\frac{a_1}{2} \epsilon^{\mu\nu\alpha\beta} A_\nu F_{\alpha\beta} + H_2 \mathcal{B}^\mu + H_1 \omega^\mu \right) \\ &= \frac{a_1 \hat{\mu}^2 + 2a_2 \hat{\mu} + 3a_3}{\beta^2} \omega^\mu + \frac{2a_1 \hat{\mu} + 2a_2}{\beta} \mathcal{B}^\mu + \frac{a_1}{2} \epsilon^{\mu\nu\alpha\beta} A_\nu F_{\alpha\beta} \\ &= a_1 \epsilon^{\mu\nu\lambda\rho} b_\nu \partial_\lambda b_\rho + 3a_3 \mathfrak{w}^\mu + 2a_2 \mathfrak{B}^\mu \end{aligned} \quad (5.4)$$

where we have dropped a term which is dual to an exact 3-form. Note that this expression is independent of c . The entropy current in the Landau frame is then given by

$$S_o^\mu = -\hat{\mu} \ell_o^\mu + \left(\frac{a_1}{2} \epsilon^{\mu\nu\alpha\beta} A_\nu F_{\alpha\beta} + H_2 \mathcal{B}^\mu + H_1 \omega^\mu \right) \quad (5.5)$$

which gives

$$\begin{aligned} S_o^\mu &= \left(\frac{2c\hat{\hbar}\hat{\mu}^3}{\beta^2} (1-\alpha) + \frac{a_1\hat{\mu}^2}{\beta^2} (2\alpha-1) + \frac{2\alpha a_2}{\beta^2} \hat{\mu} + \frac{a_3(1+2\alpha)}{\beta^2} \right) \omega^\mu \\ &+ \left(\frac{3c\hat{\hbar}\hat{\mu}^2}{\beta} (1-\alpha) + \frac{2a_1\hat{\mu}}{\beta} \alpha + \frac{a_2}{\beta} (1+\alpha) \right) \mathcal{B}^\mu + \frac{1}{2} a_1 \epsilon^{\mu\nu\alpha\beta} A_\nu F_{\alpha\beta}. \end{aligned} \quad (5.6)$$

The parts of the expression which involve the anomaly coefficient agree with the Landau frame entropy current given in [8] when $a_1 = a_2 = a_3 = 0$. Furthermore, there is also agreement with [9] when $a_1 = 0$. After dropping duals of exact three forms, the vector above can be written in the new basis introduced here as

$$\begin{aligned} S_o^\mu &= a_1 \epsilon^{\mu\nu\lambda\rho} b_\nu \partial_\lambda b_\rho + c\hat{\hbar}\hat{\mu}^2 (1-\alpha) (3\mathfrak{B}^\mu - \hat{\mu} \mathfrak{w}^\mu) + a_2 ((1+\alpha)\mathfrak{B}^\mu - (1-\alpha)\hat{\mu} \mathfrak{w}^\mu) \\ &+ a_3 (1+2\alpha) \mathfrak{w}^\mu + 2a_1 \hat{\mu} (\alpha-1) \mathfrak{B}^\mu. \end{aligned} \quad (5.7)$$

6 Parity-violating action in 2 + 1-dimension

Let us now consider the action for parity-violating terms in 2+1-dimension. The procedures are exactly parallel to those of the 3 + 1-dimensional story. So we will be brief, only giving

¹⁷ R_{even} is given explicitly in equation (5.40) of section V C of [58]. It was denoted as Q_2 there.

the main results. We will again work to the level of \mathcal{L}_2 as defined in (2.43). The results below are fully consistent with the constitutive relations presented in [63] from entropy current analysis and those presented in [13, 14] using stationary partition function.

At $O(a)$ the hydro Lagrangian has terms

$$\frac{1}{2}\hat{T}_o^{\mu\nu}G_{a\mu\nu} + \hat{J}_o^\mu C_{a\mu} \quad (6.1)$$

and as usual we can decompose $\hat{T}_o^{\mu\nu}$ and \hat{J}_o^μ as

$$\hat{T}_o^{\mu\nu} = \varepsilon_o u^\mu u^\nu + p_o \Delta^{\mu\nu} + 2u^{(\mu} q_o^{\nu)} + \Sigma_o^{\mu\nu}, \quad (6.2)$$

$$\hat{J}_o^\mu = n_o u^\mu + j_o^\mu \quad (6.3)$$

where q_o^ν, j_o^μ and $\Sigma_o^{\mu\nu}$ are transverse to u^μ . For this purpose let us list all the parity-odd scalars, vectors, and tensors which are diagonal shift invariant at first derivative order¹⁸

$$\text{scalars: } s_1 = \epsilon^{\mu\nu\lambda} u_\mu \nabla_\nu u_\lambda, \quad s_2 = \frac{1}{2} \epsilon^{\mu\nu\lambda} u_\mu F_{\nu\lambda} \quad (6.4)$$

$$\text{vectors: } t_1^\mu = \epsilon^{\mu\nu\lambda} u_\nu v_{1\lambda}, \quad t_2^\mu = \epsilon^{\mu\nu\lambda} u_\nu v_{2\lambda}, \quad t_3^\mu = \epsilon^{\mu\nu\lambda} u_\nu \partial_\lambda \beta, \quad t_4^\mu = \epsilon^{\mu\nu\lambda} u_\nu \partial_\lambda \hat{\mu} \quad (6.5)$$

$$\text{tensors: } \sigma_o^{\mu\nu} = \sigma_\lambda^{(\nu} \epsilon^{\mu)\rho\lambda} u_\rho \quad (6.6)$$

where we have introduced

$$v_{1\mu} = \partial u_\mu - \beta^{-1} \Delta_\mu^\nu \partial_\nu \beta, \quad v_{2\mu} = \beta^{-1} \Delta_\mu^\nu \nabla_\nu \hat{\mu} - u^\nu F_{\mu\nu}, \quad (6.7)$$

$$\partial \equiv u^\mu \nabla_\mu, \quad \sigma^{\mu\nu} \equiv \Delta^{\mu\lambda} \Delta^{\nu\rho} (\nabla_\lambda u_\rho + \nabla_\rho u_\lambda - g_{\lambda\rho} \nabla_\alpha u^\alpha) \quad (6.8)$$

We can then expand various quantities in (6.2)–(6.3) as

$$\varepsilon_o = g_1 s_1 + g_2 s_2, \quad p_o = h_1 s_1 + h_2 s_2, \quad n_o = f_1 s_1 + f_2 s_2 \quad (6.9)$$

$$q_o^\mu = \sum_{i=1}^4 k_i t_i^\mu, \quad j_o^\mu = \sum_{i=1}^4 l_i t_i^\mu, \quad \Sigma_o^{\mu\nu} = \eta_o \sigma_o^{\mu\nu} \quad (6.10)$$

where all coefficients $g_{1,2}, h_{1,2}, f_{1,2}, k_{1,2,3,4}, l_{1,2,3,4}, \eta_o$ are functions of $\beta, \hat{\mu}$.

At $O(a^2)$ the complete action at zero derivative order is

$$-i\mathcal{L}^{(2)} = \frac{1}{4} s_{11} (u^\mu u^\nu G_{a\mu\nu})^2 + \frac{1}{4} s_{22} (\Delta^{\mu\nu} G_{a\mu\nu})^2 - \frac{1}{2} s_{12} u^\mu u^\nu G_{a\mu\nu} \Delta^{\mu\nu} G_{a\mu\nu} \quad (6.11)$$

$$+ \frac{1}{4} s_{33} (u^\mu C_{a\mu})^2 + s_{23} u^\mu C_{a\mu} \Delta^{\alpha\beta} G_{a\alpha\beta} - s_{13} u^\mu u^\nu G_{a\mu\nu} u^\alpha C_{a\alpha} + t(G_{a<\mu\nu>})^2 \quad (6.12)$$

$$+ r_{11} \Delta^{\alpha\beta} u^\nu u^\mu G_{a\mu\alpha} G_{a\nu\beta} + r_{22} \Delta^{\mu\nu} C_{a\mu} C_{a\nu} + 2r_{12} \Delta^{\mu\nu} u^\alpha G_{a\mu\alpha} C_{a\nu} \quad (6.13)$$

$$+ r \epsilon^{\mu\nu\lambda} u_\mu C_{a\nu} u^\rho G_{a\lambda\rho}. \quad (6.14)$$

Note that only the last term is parity-odd. Here, angular brackets deonte the transverse symmetric traceless part of the corresponding tensor, e.g.,

$$C_{\langle\mu\nu\rangle} = \Delta_\mu^\alpha \Delta_\nu^\beta \left(C_{(\alpha\beta)} - \frac{1}{d-1} \Delta_{\alpha\beta} C^{\gamma\delta} \Delta_{\gamma\delta} \right). \quad (6.15)$$

¹⁸Note the identities $\frac{1}{2} \Delta^\mu_\nu \epsilon^{\nu\lambda\rho} F_{\lambda\rho} = \epsilon^{\mu\nu\lambda} u_\nu F_{\lambda\rho} u^\rho$ and $\Delta^\mu_\nu \epsilon^{\nu\lambda\rho} \nabla_\lambda u_\rho = -\epsilon^{\mu\nu\lambda} u_\nu \partial u_\lambda$.

Non-negativity of the imaginary part of the action, eq. (2.29), leads to various constraints among the coefficients of $\mathcal{L}^{(2)}$. The constraints on the parity even part (6.11)–(6.13) were analyzed in detail in [57]. Among other constraints we have

$$r_{11}, r_{22} > 0, \quad r_{11}r_{22} - r_{12}^2 \geq 0. \quad (6.16)$$

When the parity-odd coefficient r is nonzero, the second inequality of the above becomes

$$r_{11}r_{22} - r_{12}^2 \geq \frac{r^2}{4}. \quad (6.17)$$

To summarize, to level \mathcal{L}_2 the parity-odd action can be written as

$$\mathcal{L}_{\text{odd}} = \frac{1}{2}(\varepsilon_o u^\mu u^\nu + p_o \Delta^{\mu\nu} + 2u^{(\mu} q_o^{\nu)}) + \Sigma_o^{\mu\nu})G_{a\mu\nu} + (n_o u^\mu + j_o^\mu)C_{a\mu} + i r \epsilon^{\mu\nu\lambda} u_\mu C_{a\nu} G_{a\lambda\rho} u^\rho. \quad (6.18)$$

Using field redefinitions one can write \mathcal{L}_{odd} as (see section VI of [58] for details)

$$\mathcal{L}_{\text{odd}} = \frac{1}{2}\theta_o \Delta^{\mu\nu} G_{a\mu\nu} + \ell_o^\mu \Delta_\mu^\nu C_{a\nu} \quad (6.19)$$

with frame independent quantities θ_o, ℓ_o^μ defined by

$$\theta_o = p_o - \varepsilon_o \partial_\varepsilon p_o - n_o \partial_n \varepsilon_o, \quad \ell_o^\mu = j_o^\mu - \frac{n_o}{\varepsilon_o + p_o} q_o^\mu \quad (6.20)$$

where ε_o, p_o, n_o are respectively zeroth order energy, pressure and charge densities. Note that the coefficient r can be defined away using field redefinitions, so (6.17) does not lead to new constraints on transport coefficients.

The outcome of the dynamical KMS condition (2.31) again depends very much on the choice Θ , which we will discuss separately.

6.1 $\Theta = \mathcal{T}$

In this case, we find all coefficients in (6.9)–(6.10) are zero, except for k_2 and l_1 which satisfy the relation

$$-k_2 = l_1 = \frac{1}{2}\beta r. \quad (6.21)$$

The full parity-odd action to level \mathcal{L}_2 can then be written as

$$\mathcal{L}_{\text{odd}} = -\frac{\beta r}{4}(u^\mu t_2^\nu + u^\nu t_2^\mu)G_{a\mu\nu} + \frac{\beta r}{2}t_1^\mu C_{a\mu} + i r \epsilon^{\mu\nu\lambda} u_\mu C_{a\nu} G_{a\lambda\rho} u^\rho. \quad (6.22)$$

The above Lagrangian satisfies

$$\tilde{\mathcal{L}} = \mathcal{L} \quad (6.23)$$

which can be seen by noting the relation

$$\hat{T}_o^{\mu\nu} \nabla_\mu \beta_\nu + \hat{J}_o^\mu (\partial_\mu \hat{\mu} - \beta^\alpha F_{\mu\alpha}) = -r \beta^2 \epsilon^{\mu\nu\rho} u_\mu v_{2\nu} v_{1\rho}. \quad (6.24)$$

Due to (6.23), there is no parity-odd contribution to the thermal partition function to first derivative order. The entropy current is given by

$$S^\mu = p\beta^\mu - T^{\mu\nu} \beta_\nu - \hat{\mu} J^\mu \quad (6.25)$$

where $p, T^{\mu\nu}, J^\mu$ also include the parity-even part, and

$$\nabla_\mu S^\mu = R_{\text{even}} + r\beta^2 \epsilon^{\mu\nu\rho} u_\mu v_{2\nu} v_{1\rho} \quad (6.26)$$

where R_{even} is the parity-even expression. Note that the second term in the right hand side of (6.26) vanishes by ideal fluid equation of motion

$$v_{1\mu} = -\frac{n_0}{\epsilon_0 + p_0} v_{2\mu}. \quad (6.27)$$

6.2 $\Theta = \mathcal{TP}$

The dynamical KMS condition implies that the coefficients in (6.9)–(6.10) should satisfy

$$h_1 = h_2 = r = 0, \quad k_2 = l_1 \quad (6.28)$$

$$g_1 = \beta k_3, \quad -f_1 = \beta k_4, \quad g_2 = \beta l_3, \quad -f_2 = \beta l_4 \quad (6.29)$$

$$\partial_\beta(\beta l_4) = \partial_{\hat{\mu}}(\beta l_3), \quad l_3 + k_4 = \partial_{\hat{\mu}}(\beta k_3) - \partial_\beta(\beta k_4). \quad (6.30)$$

The first equation of (6.30) implies that there exists a function Y such that

$$\beta l_3 = \frac{\partial Y}{\partial \beta}, \quad \beta l_4 = \frac{\partial Y}{\partial \hat{\mu}} \quad (6.31)$$

while the second equation of (6.30) can be further written as

$$\partial_\beta(\beta^2 k_4) + \beta l_3 = \partial_{\hat{\mu}}(\beta^2 k_3) \quad (6.32)$$

which upon using (6.31) implies that there exists a function X such that

$$\beta^2 k_3 = \partial_\beta X, \quad \beta^2 k_4 + Y = \partial_{\hat{\mu}} X. \quad (6.33)$$

k_1, l_2, η_o are unconstrained. Thus there are altogether six independent functions of $\hat{\mu}$ and β : $X, Y, k_1, l_1, l_2, \eta_o$.

Applying the above relations to (6.9)–(6.10) we then have

$$\hat{T}_o^{\mu\nu} = \left(\frac{1}{\beta} \partial_\beta X s_1 + \partial_\beta Y s_2 \right) u^\mu u^\nu + 2u^{(\mu} q_o^{\nu)} + \eta_o \sigma_o^{\mu\nu}, \quad (6.34)$$

$$q_o^\mu = k_1 t_1^\mu + l_1 t_2^\mu + \frac{1}{\beta^2} \partial_\beta X t_3^\mu + \frac{1}{\beta^2} (\partial_{\hat{\mu}} X - Y) t_4^\mu, \quad (6.35)$$

$$\hat{J}_o^\mu = \left(\frac{1}{\beta} (Y - \partial_{\hat{\mu}} X) s_1 - \partial_{\hat{\mu}} Y s_2 \right) u^\mu + l_1 t_1^\mu + l_2 t_2^\mu + \frac{1}{\beta} \partial_\beta Y t_3^\mu + \frac{1}{\beta} \partial_{\hat{\mu}} Y t_4^\mu. \quad (6.36)$$

It can be checked that the above expressions satisfy

$$\hat{T}_o^{\mu\nu} \nabla_\mu \beta_\nu + \hat{J}_o^\mu (\partial_{\hat{\mu}} \hat{\mu} + \beta^\alpha F_{\alpha\mu}) = \nabla_\mu V_0^\mu \quad (6.37)$$

with

$$V_0^\mu = \frac{1}{\beta} \partial_\beta X t_3^\mu + \frac{1}{\beta} (\partial_{\hat{\mu}} X - Y) t_4^\mu + \frac{Y}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} \quad (6.38)$$

which gives

$$\tilde{\mathcal{L}}_{\text{odd}} - \mathcal{L}_{\text{odd}} = i \nabla_\mu V_0^\mu. \quad (6.39)$$

The entropy current can then be obtained as

$$S^\mu = p\beta^\mu + \frac{1}{\beta}(\partial_\beta X)t_3^\mu + \frac{1}{\beta}(\partial_{\hat{\mu}}X - Y)t_4^\mu + \frac{Y}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} - T^{\mu\nu}\beta_\nu - \hat{\mu}J^\mu \quad (6.40)$$

with

$$\nabla_\mu S^\mu = R_{\text{even}}. \quad (6.41)$$

To compare with [63], note that we need to first add the total derivative with zero divergence $-\nabla_\nu(\epsilon^{\mu\nu\rho}u_\rho\tilde{v}_5)$ to their expression of the entropy current. This has the consequence of redefining

$$\tilde{v}_1 \rightarrow \tilde{v}_1 + \partial_T\tilde{v}_5 \quad (6.42)$$

$$\tilde{v}_3 \rightarrow \tilde{v}_3 + \partial_{\hat{\mu}}\tilde{v}_5 \quad (6.43)$$

in their expressions. Further comparing corresponding terms, we find eq. (3.22), eq. (3.23) and eq. (3.24) in [63] are reproduced if we make the identifications

$$X = \frac{\mathcal{M}_\Omega}{T^2} + \int^T \frac{f_\Omega(x)}{x^3} dx \quad (6.44)$$

$$Y = \tilde{v}_4 = \beta\mathcal{M}_B \quad (6.45)$$

For stationary sources (4.14), the thermal partition function is obtained from the zeroth component of V_0^μ with dynamical fields set to their equilibrium values. We find that

$$\log Z = \int d^2x \sqrt{-g} \left(\frac{1}{\beta_0} \epsilon^{ij} \partial_i v_j \left(X - \beta_0 A_0 Y \right) + Y \epsilon^{ij} \partial_i b_j \right) \quad (6.46)$$

where b_i is as defined in (4.13). The above expression of the partition function agrees with [13, 14].

Finally let us note that the frame independent quantities (6.20) can be written as

$$\theta_o = \chi_\Omega s_1 + \chi_B s_2, \quad (6.47)$$

$$\chi_\Omega = -\partial_{\epsilon_0} p_0 \left(\frac{1}{\beta} \partial_\beta X \right) + \partial_{n_0} p_0 \frac{1}{\beta} (\partial_{\hat{\mu}} X - Y), \quad (6.48)$$

$$\chi_B = -\partial_{\epsilon_0} p_0 \partial_\beta Y + \partial_{n_0} p_0 \partial_{\hat{\mu}} Y \quad (6.49)$$

and

$$\ell_o^\mu = -\tilde{\sigma} t_2^\mu + \tilde{\chi}_E \tilde{E}^\mu - \frac{\tilde{\chi}_T}{\beta^2} t_3^\mu, \quad \tilde{E}^\mu \equiv \beta^{-1} t_4^\mu - t_2^\mu = \epsilon^{\mu\nu\lambda} u_\nu F_{\lambda\rho} u^\rho \quad (6.50)$$

$$\tilde{\chi}_E = \partial_{\hat{\mu}} Y - \frac{n_0}{\beta(\epsilon_0 + p_0)} (\partial_{\hat{\mu}} X - Y), \quad (6.51)$$

$$\tilde{\chi}_T = -\beta \partial_\beta Y + \frac{n_0}{\epsilon_0 + p_0} \partial_\beta X, \quad (6.52)$$

$$-\tilde{\sigma} = l_2 - \frac{2n_0}{\epsilon_0 + p_0} l_1 + \left(\frac{n_0}{\epsilon_0 + p_0} \right)^2 k_1 + \tilde{\chi}_E. \quad (6.53)$$

Note that in the above expressions we have used ideal fluid equation (6.27) which makes t_1^μ and t_2^μ equivalent. As a result the number of independent functions reduce to four: $X, Y, \tilde{\sigma}, \eta_o$.

6.3 $\Theta = \mathcal{CPT}$

Dynamical KMS invariance requires that $k_1, k_3, l_2, l_4, g_1, f_2, r, \eta_o$ be even functions of $\hat{\mu}$, while k_4, l_3, g_2, f_1 be odd functions of $\hat{\mu}$,

$$h_1 = h_2 = 0, \quad k_2 - l_1 = -\beta r, \quad (6.54)$$

$$-k_2(-\hat{\mu}, \beta) = k_2(\hat{\mu}, \beta) + \beta r(\hat{\mu}, \beta), \quad l_1(-\hat{\mu}, \beta) = -l_1(\hat{\mu}, \beta) + \beta r(\hat{\mu}, \beta), \quad (6.55)$$

and (6.29)–(6.33), except that now X should be an even function of $\hat{\mu}$ and Y should be odd. Thus equations (6.34) and (6.36) are unchanged while (6.35) should be modified to

$$q_o^\mu = k_1 t_1^\mu + (l_1 - \beta r) t_2^\mu + \frac{1}{\beta^2} \partial_\beta X t_3^\mu + \frac{1}{\beta^2} (\partial_{\hat{\mu}} X - Y) t_4^\mu. \quad (6.56)$$

Equations (6.39)–(6.46) still apply except that for (6.41) the covariant derivative of S^μ now yields

$$\nabla_\mu S^\mu = R_{\text{even}} + r \beta^2 \epsilon^{\mu\nu\rho} u_\mu v_{2\rho} v_{1\nu} \quad (6.57)$$

with the second term on right hand side again vanishing from ideal fluid equation of motion (6.27).

Eqs. (6.47)–(6.52) are unmodified, while

$$-\tilde{\sigma} = l_2 - \frac{2n_0}{\epsilon_0 + p_0} l_1 + \left(\frac{n_0}{\epsilon_0 + p_0} \right)^2 k_1 + \frac{\beta n_0}{\epsilon_0 + p_0} r + \tilde{\chi}_E. \quad (6.58)$$

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A Explicit expressions for various discrete transformations

In this appendix we list transformations of various tensors under various discrete symmetries. They are important for obtaining the explicit forms of dynamical KMS transformations (2.33) and (2.38)–(2.40) of various tensors. For notational simplicity we have suppressed the transformations of the arguments of all the functions, which are given in the first line of each table.

Discrete transformations in 3+1-dimension			
	\mathcal{T}	\mathcal{PT}	\mathcal{CPT}
x^μ	$(-x^0, x^i)$	$-(x^0, x^i)$	$-(x^0, x^i)$
u^μ	$(u^0, -u^i)$	(u^0, u^i)	(u^0, u^i)
ω^μ	$(\omega^0, -\omega^i)$	$-(\omega^0, \omega^i)$	$-(\omega^0, \omega^i)$
A_μ	$(A_0, -A_i)$	(A_0, A_i)	$-(A_0, A_i)$
\mathcal{B}^μ	$(\mathcal{B}_0, -\mathcal{B}_i)$	$-(\mathcal{B}_0, \mathcal{B}_i)$	$(\mathcal{B}_0, \mathcal{B}_i)$
∂_μ	$(-\partial_0, \partial_i)$	$-(\partial_0, \partial_i)$	$-(\partial_0, \partial_i)$
$\partial = u^\mu \partial_\mu$	$-\partial$	$-\partial$	$-\partial$
$\hat{\mu}$	$\hat{\mu}$	$\hat{\mu}$	$-\hat{\mu}$
$g_{\mu\nu}$	$(g_{00}, -g_{0i}, g_{ij})$	$g_{\mu\nu}$	$g_{\mu\nu}$
φ	$-\varphi$	$-\varphi$	φ

Discrete transformations in 2+1-dimension			
	\mathcal{T}	\mathcal{PT}	\mathcal{CPT}
x^μ	$(-x^0, x^i)$	$(-x^0, -x^1, x^2)$	$(-x^0, -x^1, x^2)$
u^μ	$(u^0, -u^i)$	$(u^0, u^1, -u^2)$	$(u^0, u^1, -u^2)$
A_μ	$(A_0, -A_i)$	$(A_0, A_1, -A_2)$	$(-A_0, -A_1, A_2)$
∂_μ	$(-\partial_0, \partial_i)$	$(-\partial_0, -\partial_1, \partial_2)$	$(-\partial_0, -\partial_1, \partial_2)$
$\partial = u^\mu \partial_\mu$	$-\partial$	$-\partial$	$-\partial$
$\hat{\mu}$	$\hat{\mu}$	$\hat{\mu}$	$-\hat{\mu}$
v_1^μ	$(-v_1^0, v_1^i)$	$(-v_1^0, -v_1^1, v_1^2)$	$(-v_1^0, -v_1^1, v_1^2)$
v_2^μ	$(-v_2^0, v_2^i)$	$(-v_2^0, -v_2^1, v_2^2)$	$(v_2^0, v_2^1, -v_2^2)$
$g_{\mu\nu}$	$(g_{00}, -g_{0i}, g_{ij})$	$(g_{00}, g_{01}, -g_{02}, -g_{12}, g_{11}, g_{22})$	$(g_{00}, g_{01}, -g_{02}, -g_{12}, g_{11}, g_{22})$
s_1	$-s_1$	s_1	s_1
s_2	$-s_2$	s_2	$-s_2$
$t_\alpha^\mu, \alpha = 1, 3$	$(-t_\alpha^0, t_\alpha^i)$	$(t_\alpha^0, t_\alpha^1, -t_\alpha^2)$	$(t_\alpha^0, t_\alpha^1, -t_\alpha^2)$
$t_\alpha^\mu, \alpha = 2, 4$	$(-t_\alpha^0, t_\alpha^i)$	$(t_\alpha^0, t_\alpha^1, -t_\alpha^2)$	$(-t_\alpha^0, -t_\alpha^1, t_\alpha^2)$
$\sigma_o^{\mu\nu}$	$(-\sigma_o^{00}, \sigma_o^{0i}, -\sigma_o^{ij})$	$(\sigma_o^{00}, \sigma_o^{01}, -\sigma_o^{02}, -\sigma_o^{12}, \sigma_o^{11}, \sigma_o^{22})$	$(\sigma_o^{00}, \sigma_o^{01}, -\sigma_o^{02}, -\sigma_o^{12}, \sigma_o^{11}, \sigma_o^{22})$

Note that in all cases $C_{a\mu}, J^\mu$ transform as A_μ , while $T^{\mu\nu}, g_{a\mu\nu}$ have the same transformations as $g_{\mu\nu}$, and $\sigma^{\mu\nu}$ transforms the same as $g_{\mu\nu}$ but with an overall minus sign. ω^μ and \mathcal{B}^μ are defined below (1.2).

B Some useful formulae

In this appendix we give some useful formulae used in deriving equations such as (3.36) and (3.39).

We first note an identity in (3 + 1)-dimension

$$V_\mu \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} G_{\gamma\delta} = -2\epsilon^{\alpha\beta\gamma\delta} F_{\mu\alpha} V_\beta G_{\gamma\delta} - 2\epsilon^{\alpha\beta\gamma\delta} G_{\mu\alpha} V_\beta F_{\gamma\delta} \quad (\text{B.1})$$

which can be written in differential forms as

$$\xi \cdot VF \wedge G = -(\xi \cdot F) \wedge V \wedge G - (\xi \cdot G) \wedge V \wedge F. \quad (\text{B.2})$$

where ξ is a vector field, F, G are two-forms, and V is a one-form. As an example, given $u \equiv u_\mu dx^\mu$, $w = du$, and $\beta^\mu = \beta u^\mu$, we then have

$$\beta F \wedge w = (\beta \cdot F) \wedge u \wedge w + (\beta \cdot w) \wedge u \wedge F. \quad (\text{B.3})$$

It is also useful to recall that for a differential form λ and a vector field ξ

$$d(\xi \cdot \lambda) = \mathcal{L}_\xi \lambda - \xi \cdot d\lambda. \quad (\text{B.4})$$

It then follows that for some vector v^μ

$$\int \mathcal{L}_v \varphi F \wedge F = -2 \int \varphi F \wedge \mathcal{L}_v F + 2 \int d(\varphi F \wedge (v \cdot F)) \quad (\text{B.5})$$

which can be used to derive (3.36).

To see (3.39), we note that:

$$\begin{aligned} -c\hbar \int d^4x \sqrt{-g} \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} B_\rho \mathcal{L}_\beta B_\mu &= -2c\hbar \int \mathcal{L}_\beta B \wedge B \wedge F \\ &= -2c\hbar \int (d\hat{\mu} + \beta \cdot F) \wedge B \wedge F \\ &= 3c\hbar \int \hat{\mu} F \wedge F - 2c\hbar \int d(\hat{\mu} B \wedge F) \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \int d^4x \sqrt{-g} \hat{j}_o^\mu \mathcal{L}_\beta B_\mu &= \int \mathcal{L}_\beta B \wedge u \wedge (h_1 du + h_2 F) \\ &= \int (d\hat{\mu} + \beta \cdot F) \wedge u \wedge (h_1 du + h_2 F) \\ &= \int \left[d\hat{\mu} \wedge u \wedge (h_1 du + h_2 F) + h_1 (\beta \cdot F) \wedge u \wedge du + \frac{\beta h_2}{2} F \wedge F \right] \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \frac{1}{2} \int d^4x \sqrt{-g} T_o^{\mu\nu} \mathcal{L}_\beta g_{\mu\nu} &= \int d^4x \sqrt{-g} ((\beta \cdot du)_\mu - \nabla_\mu \beta) q_o^\mu \\ &= \int \left[\frac{g_1 \beta}{2} du \wedge du + g_2 (\beta \cdot du) \wedge u \wedge F - d\beta \wedge u \wedge (g_1 du + g_2 F) \right]. \end{aligned} \quad (\text{B.8})$$

For (2 + 1)-dimension we have

$$V_\mu \epsilon^{\alpha\beta\gamma} W_\alpha G_{\beta\gamma} = W_\mu \epsilon^{\alpha\beta\gamma} V_\alpha G_{\beta\gamma} - 2\epsilon^{\alpha\beta\gamma} G_{\mu\alpha} V_\beta W_\gamma \quad (\text{B.9})$$

or in differential form

$$(\xi \cdot V)W \wedge G = (\xi \cdot W)V \wedge G - (\xi \cdot G) \wedge V \wedge W \quad (\text{B.10})$$

where W is a one-form and G a two-form. Here are two examples:

$$\epsilon^{\mu\nu\lambda} u_\nu F_{\lambda\rho} u^\rho = \frac{1}{2} \Delta^\mu{}_\nu \epsilon^{\nu\lambda\rho} F_{\lambda\rho} \quad (\text{B.11})$$

$$-\epsilon^{\mu\nu\lambda} u_\nu \partial u_\lambda = \epsilon^{\mu\nu\lambda} u_\nu w_{\lambda\rho} u^\rho = \frac{1}{2} \Delta^\mu{}_\nu \epsilon^{\nu\lambda\rho} w_{\lambda\rho} \quad (\text{B.12})$$

where $w = du$.

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