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NNLO QCD corrections to $\gamma + \eta_c(\eta_b)$ exclusive production in electron-positron collision

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ABSTRACT: Based on the NRQCD factorization formalism, we calculate the next-to-next-to-leading order QCD corrections to the heavy quarkonium $\eta_c(\eta_b)$ production associated with a photon at electron-positron colliders. By matching the amplitudes calculated in full QCD theory to a series of operators in NRQCD, the short-distance coefficients up to NNLO QCD radiative corrections are determined. It turns out that the full set of master integrals that we obtained could be analytically expressed in terms of Goncharov Polylogarithms, integrals over polylogarithms and complete elliptic integrals, which mostly do not exist in the literature and could be employed in the analyses of other physical processes. In phenomenology, numerical calculations of NNLO K-factors and cross sections of $e^+e^- \rightarrow \gamma + \eta_c(\eta_b)$ processes in BESIII and B-factory experiments are performed, which may stand as a test of the NRQCD higher order calculation while confronting to the data.

KEYWORDS: NLO Computations, QCD Phenomenology

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The study of heavy quarkonium production and decay is very important in the understanding of Quantum Chromodynamics (QCD), and heavy quarkonium as well. In the analysis of hadron, to overcome the obstacle of nonperturbative QCD is inevitable. Fortunately, the advent of Nonrelativistic Quantum Chromodynamics (NRQCD) factorization formalism enables the calculation of quarkonium production and decay in a solid footing [1], rather than models. NRQCD can systematically factorize the nonperturbative sectors out and leave others perturbative QCD(pQCD) calculable. Up to now NRQCD achieves a great success in Born level and first order pQCD calculations,¹ though some unsatisfactory still remains in phenomenological study.

Since the energy scales set by heavy quarks, the charm and bottom quarks, are moderate, the strong interaction in this regime is perturbative expandable, whereas does not converge quickly. The large discrepancy between leading order theoretical calculation [3–5] and experimental data [6, 7], for example in double charmonium production at B-factory, can be indeed amended by the next-to-leading order(NLO) QCD corrections [8, 9]. Higher order calculation itself is critical, challenging and meaningful theoretically, in addition to its phenomenological impact.

The processes of $\eta_c(\eta_b)$ exclusive production in accompany with a photon at electron-positron colliders is right now an interesting topic, because it can be measured accurately and calculated precisely. Furthermore, naive estimation indicates that $e^+e^- \rightarrow \gamma + \eta_c$ cross section overshoots that of $e^+e^- \rightarrow J/\psi + \eta_c$ process by an order of magnitude, and hence $e^+e^- \rightarrow \gamma + \eta_c(\eta_b)$ processes may be well measured in Belle II experiment soon. On theory side, the leading-order calculation and next-to-leading order corrections are ready [10–12], where the radiative corrections are negative.

In calculating high order corrections, the central issue is to calculate the multi-loop Feynman integrals, namely master integrals (MI). The first complete NNLO analytical calculations for quarkonium production and decay, the $J/\psi(\Upsilon) \rightarrow e^+e^-$ and $e^+e^- \rightarrow J/\psi(\Upsilon)$ processes, were achieved in refs. [13] and [14] respectively, and a few years ago even the NNNLO result for $\Upsilon \rightarrow e^+e^-$ process was obtained [15]. In recent years, at the NNLO level many calculations are performed for quarkonium production and decays [16–18]. It is interesting to note that in recently, the NNLO corrections to $\gamma\gamma^* \rightarrow \eta_c(\eta_b)$ transition form factor and $\eta_c(\eta_b) \rightarrow \text{light hardrons}$ were calculated numerically [19, 20]. Numerical calculation is an unique and promising way for higher order radiative corrections, nevertheless right now it experiences the shortage of proper numerical packages, especially for the kinematics in physical region.

In this work, we are going to calculate analytically the next-to-next-to-leading order(NNLO) QCD corrections to exclusive $\eta_c(\eta_b)$ production associated with a photon in electron-positron collision, the multi-scale $e^+e^- \rightarrow \gamma + \eta_c(\eta_b)$ processes. Numerical evaluation for BESIII and Belle II experiments will be presented. Note, since the kinematics of the master integrals in our calculation lie obviously in the physical region, due to the lack of

¹For example see ref. [2].

Figure 1. Typical LO, NLO, and NNLO Feynman diagrams of the $e^+e^- \rightarrow \gamma + \eta_c(\eta_b)$ processes.

corresponding numerical calculation package, to our knowledge a full numerical calculation of those integrals is still unrealistic and unreliable.

In our calculation, **FeynArts** [21] is employed to generate the corresponding Feynman diagrams and amplitudes up to NNLO, among them both massive and massless “light by light” diagrams are taken into account. Typical LO, NLO, and NNLO Feynman diagrams of the $e^+e^- \rightarrow \gamma + \eta_c(\eta_b)$ processes are shown in figure 1. We calculate the amplitude and reduce the loop integrals in full QCD, before applying the NRQCD projector to quarkonium. Up to any loop, the fermion chains in the amplitude of concerned process can be generally expressed as

$$\mathcal{M}^{\mu_1\mu_2} = \bar{u}(k_q) \cdot \gamma^{\mu_1} \cdot \gamma^{\mu_2} \cdot \not{k}_\gamma \cdot v(k_q) f_1 + \dots \quad (1)$$

Here, f_1 is a function of kinematics, coupling, renormalization and factorization scales. In above expression, the leptonic current is implied being factorized out, and the ellipses represent terms with the number of γ matrix less than 3. Of the $e^+e^- \rightarrow \gamma^* \rightarrow \gamma + \eta_c(\eta_b)$ processes, only the first term in (1) contributes. In order to get f_1 , we multiply (1) with the projector

$$P_{\mu_1\mu_2} = [\bar{v}(k_q) \cdot \not{k}_\gamma \cdot \gamma_{\mu_2} \cdot \gamma_{\mu_1} \cdot u(k_q) - \bar{v}(k_q) \cdot \gamma_{\mu_1} \cdot \gamma_{\mu_2} \cdot \not{k}_\gamma \cdot u(k_q)] / [(D-3)(D-2)(s-4m_Q^2)^2], \quad (2)$$

and summing over the spinor helicities and trace over the dirac chains. Here, s denotes the center of mass energy squared and m_Q the heavy quark mass. Note that by means of this procedure, one can avoid the ambiguity in the definition of heavy quarkonium projector and γ_5 in the dimensional regularization scheme.

With the reduction of the obtained two-loop amplitudes, we find all related scalar integrals may be attributed to a set of 133 master integrals. Using the method of differential equation [22, 23], and by virtue of the recent development in choosing basis properly [24], it was found that 86 of the 133 master integrals can be expressed as functions of Harmonic and Goncharov polylogarithms [25]. The calculation of massive multi-loop Feynman integrals is a challenging task due to the appearance of new mathematical structures, which cannot be reduced to the well-known multiple polylogarithms yet. Thanks to the very recent developments in differential equation technique in dealing with multiscale Feynman integrals [26–30], we are now able to calculate those massive multi-loop Feynman integrals of our concern beyond multiple polylogarithms, which are found can be classified into

Figure 2. Typical master integrals in elliptic sectors. The internal lines with a dot mean the powers of the propagators are raised to 2.

two categories, the elliptic sectors. One elliptic sector contains two two-loop sunrise integrals with full massive propagators and those integrals containing them as subtopologies, the other sector comprises the two non-planar two-loop three-point integrals. The typical Feynman diagrams of those two elliptic sectors are shown in figure 2.

In the first sector, there are 2 massive two-loop sunrise integrals and 37 integrals containing massive sunrise integrals as subtopologies. The massive sunrise integrals (E_1, E_2) are shown in figure 2. Note, the problem of massive two-loop sunrise integrals was ever solved in ref. [27], where the results are expressed in terms of one-fold integrals over complete elliptic integrals and polylogarithms. With the basis chosen in ref. [27] and the method proposed in ref. [24], we find the differential equations for all integrals containing massive sunrise integrals can be reduced to compact forms, which are different from [24] and will be presented in detail elsewhere [31].

To be more specific, in our calculation the first sector with 39 integrals $E_i (i = 1 \dots 39)$, including 2 massive two-loop sunrise integrals (E_1, E_2) which are already known [27], is treated as subtopologies. With proper basis selection for $A_i (i = 3 \dots 39)$, in linear combination of master integrals $E_i (i = 3 \dots 39)$, differential equations for $A_i (i = 3 \dots 39)$ can be expressed as

$$\frac{d\mathbf{A}}{ds} = \epsilon(\mathbf{W} \cdot \mathbf{A} + \mathbf{Y} \cdot \mathbf{F}) + (\epsilon\mathbf{Q}_1 + \mathbf{Q}_2)\mathbf{A}_1. \quad (3)$$

Here, $\mathbf{A}$ is a 37-dimensional basis vector that contains integrals of $E_i (i = 3 \dots 39)$; $\mathbf{F}$ is a 86-dimensional basis vector that had been calculated in our previous work [25]; $\mathbf{W}$ and $\mathbf{Y}$ are 37×37 and 37×86 matrices, respectively; $\mathbf{Q}_i (i = 1, 2)$ are also 37-dimensional basis vectors; and $\mathbf{A}_1$ is a linear function of the massive sunrise integrals (E_1, E_2), write as

$$\mathbf{A}_1 = \epsilon^2 \frac{12m_Q^2((D-3)((3D-8)E_1 + (s+4m_Q^2)E_2) - (s/2-4m_Q^2)F_1/\epsilon^2)}{(s/2-2m_Q^2)(s/2-10m_Q^2)}, \quad (4)$$

with F_1 being the first integral of vector $\mathbf{F}$ [25]. Note that $(\mathbf{W}, \mathbf{Y}, \mathbf{Q}_1, \mathbf{Q}_2)$ are rational functions of kinematics and hence are free of ϵ . The ϵ^0 term in $\mathbf{Q}_2$ appears only in the inhomogeneous part, and is irremovable. The homogenous parts of the differential equation (3) have been simplified, and the dimensional regularization parameter ϵ in homogeneous parts is factored out from kinematics. With the known inhomogeneous terms containing A_1 [27], the differential equation (3) can be solved recursively in terms of ϵ .

For the second sector, which comprises two non-planar two-loop three-point integrals (B_1, B_2) shown in figure 2, the second order homogenous differential equation for $g = (s - 4m_Q^2)^2 B_1$ can be formulated as

$$\frac{d^2 g}{ds^2} - \frac{s^2 - 8s m_Q^2 - 12m_Q^2}{(s - 4m_Q^2)(s^2 - 8s m_Q^2 + 80m_Q^4)} \frac{dg}{ds} - \frac{64m_Q^2 g}{(s - 4m_Q^2)^2 (s^2 - 8s m_Q^2 + 80m_Q^4)} = 0. \quad (5)$$

To obtain the homogeneous solution for above equation, we make a variable transformation of $v = \frac{-i(s-4m_Q^2)}{8m_Q^2}$. The homogeneous equation for g then turns to

$$\frac{d^2 g}{dv^2} - \frac{1+v^2}{v(1-v^2)} \frac{dg}{dv} + \frac{1}{v^2(1-v^2)} g = 0. \quad (6)$$

Two homogeneous solutions $(g_1(v), g_2(v))$ for (6) are readily obtained and can be expressed as the complete elliptic integrals of the first kind

$$g_1(v) = vK(v^2) \quad \text{and} \quad g_2(v) = vK(1-v^2), \quad (7)$$

with $K(x)$ being defined as

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x t^2)}}. \quad (8)$$

By means of variation of constants, the inhomogeneous solution may be obtained. For details of basis chosen, boundary condition determination, analytical continuation, and solutions for two elliptic sectors, readers are recommended to refer to ref. [31].

After taking the above procedures, all 133 master integrals can be analytically expressed in terms of the Goncharov Polylogarithms, integral over polylogarithms and elliptic functions. Note, in elliptic sectors two-fold iterated integrals are employed in expressing the integrals. Goncharov Polylogarithms can be numerically evaluated with the **GINAC** implementation [32, 33], and can also be transformed into functions of $\ln, \text{Li}_n$ and Li_{22} up to weight four, by using the method described in ref. [34]. The numerical evaluation of integrals over polylogarithms and integrals over elliptic functions and polylogarithms can be simply performed by software **MATHEMATICA**. In order to ensure our analytical results being correct, all of them have been checked against the numerical package yields [35, 36], in both physical and non-physical regions. For each integral encountered, we have used at least two different numerical methods to check it up.

About renormalization, for quark wave function and mass, Z_2 and Z_m respectively, we adopt the on-shell scheme [37, 38], while for the strong coupling constant the $\overline{\text{MS}}$ scheme

is taken up to one-loop order. All ultraviolet divergences are then removed after taking the renormalization procedure. The remaining infra divergences are process independent, and can be canceled by the corrections of quarkonium NRQCD matrix elements under $\overline{\text{MS}}$ scheme, with the results depending on the NRQCD factorization scale $\ln \Lambda_\mu$.

Up to NNLO, the cross sections of $e^+e^- \rightarrow \gamma^* \rightarrow \gamma + \eta_c(\eta_b)$ processes can be formulated as

$$\sigma = \sigma^0 \left(1 + \frac{\alpha_s}{\pi} c_1 + \left(\frac{\alpha_s}{\pi} \right)^2 c_2 \right). \quad (9)$$

Here, the LO cross section is well-known and can be reproduced easily, i.e.,

$$\sigma^0 = \frac{32e_Q^4 \pi^2 \alpha^3}{3m_Q s^2} \left(1 - \frac{4m_Q^2}{s} \right) \langle \mathcal{O}_1(^1S_0) \rangle, \quad (10)$$

with $\langle \mathcal{O}_1(^1S_0) \rangle$ being the NRQCD matrix element. The NLO correction coefficient c_1 was obtained in refs. [11, 12], and our calculation agree with them. It writes

$$\begin{aligned} c_1 = & -\frac{y+2}{2y} \text{Li}_2(y+1) - \frac{y+3}{4y} \ln^2 \left(\frac{\sqrt{y}-\sqrt{y+2}}{\sqrt{y}+\sqrt{y+2}} \right) - \frac{(y+2)(3y+2) \ln(-y)}{4(1+y)^2} \\ & - \frac{3\sqrt{y+2} \ln \left(\frac{\sqrt{y}-\sqrt{y+2}}{\sqrt{y}+\sqrt{y+2}} \right)}{2\sqrt{y}} - \frac{\pi^2(y+1)(2y+7) + 3y(9y+8)}{12y(y+1)}, \end{aligned} \quad (11)$$

with $y = \frac{s-4m_Q^2}{2m_Q^2}$. The NNLO correction coefficient c_2 obtained in this work is a function of $(s, m_Q, \mu_r, \mu_\Lambda)$, which can be formulated as

$$c_2 = \frac{\beta_0}{4} \ln \left(\frac{\mu_r^2}{m_Q^2} \right) C_F c_1 - \pi^2 \left(C_F^2 + \frac{C_A C_F}{2} \right) \ln \left(\frac{\mu_\Lambda}{m_Q} \right) + f \left(\frac{s}{m_Q^2} \right), \quad (12)$$

in which the color factor $C_F = \frac{4}{3}$, s represents the center-of-mass-system(CMS) energy squared, m_Q denotes the heavy quark mass, μ_r is the renormalization scale, and μ_Λ for the factorization scale. The function $f(\frac{s}{m_Q^2})$ is quite lengthy, containing multiple polylogarithms, iterative integrals and elliptic integrals, which is not suitable to show in a journal paper, but may be provided in electronic form upon request. To guarantee our result being reliable, we have also reproduced the NNLO correction result for $\eta_c \rightarrow \gamma\gamma$ process [16].

Before performing numerical calculation, we need first to fix the input parameters. The number of active quarks n_f is taken to be 4 for η_c production, and 5 for η_b process. The charm quark mass lies in between 1.4 and 1.5 GeV, which agrees with what adopted in [11, 12] for η_c exclusive production. The bottom quark mass first takes the value in $\overline{\text{MS}}$ scheme, i.e. $\overline{m}_b(\overline{m}_b) = 4.18 \text{ GeV}$ in PDG [39], and then converts to the 2-loop pole mass $m_b = 4.78 \text{ GeV}$ [19, 20]. In order to evaluate the bottom-quark-mass dependence of the final result, in our numerical calculation the bottom quark mass varies from 4.7 to 4.8 GeV. Of the strong coupling, we first choose the well-determined value at Z-pole, that is $\alpha_s(M_Z) = 0.1184$ [39], and then evolve it to the heavy quark mass scales by employing the program **RunDec** [40] in four-loop accuracy. The NRQCD matrix elements for heavy quarkonium are taken from fittings in refs. [41, 42], i.e.,

$$\langle \mathcal{O}_1(^1S_0) \rangle_{\eta_c} = 0.470 \text{ GeV}^3, \quad \langle \mathcal{O}_1(^1S_0) \rangle_{\eta_b} = 3.069 \text{ GeV}^3. \quad (13)$$

Figure 3. The cross sections of exclusive $\gamma + \eta_c$ production up to NNLO in electron-positron collision with center-of-mass energy from 4.0 to 5.5 GeV. The subscript denotes the magnitude of charm quark mass.

For $\gamma + \eta_c(1S)$ exclusive production in BESIII experiment, the renormalization scale μ_r is set to be at $2m_Q$. It was found in refs. [13, 19] that the choice of factorization scale $\mu_\Lambda = 1$ GeV results in a better convergence in perturbative expansion rather than the conventional choice of $\mu_\Lambda = m_c$ for charmonium. We find this conclusion still holds for $\gamma + \eta_c$ exclusive production, and hence $\mu_\Lambda = 1$ GeV is adopted in our numerical evaluation. While for bottomonium, the NRQCD factorization appears to work reasonably well, and the conventional choice of $\mu_\Lambda = m_Q$ is adequate to get a good convergence in perturbative expansion.

In figure 3 we show respectively the LO, NLO, and NNLO cross sections versus the CMS energy, which varies in between 4.0 and 5.5 GeV. From the figure we can read that NLO and NNLO corrections are both negative. With the NNLO correction, the production rate in this region is greatly suppressed, about 3 times reduced from the LO result, and the NNLO cross section varies more smoothly than LO and NLO results with the change of CMS energy, as expected. Recently, BESIII collaboration performed a measurement on $e^+e^- \rightarrow \gamma + \eta_c$ process with CMS energy between 4.01 and 4.60 GeV in data set corresponding to a total integrated luminosity of 4.6 fb^{-1} [43], and found no evidence of direct $\gamma + \eta_c(1S)$ production, which is in accord with the higher order QCD predictions. That is to say, the NNLO result also prefers the enhancement in $e^+e^- \rightarrow \gamma + \eta_c$ process between 4.23 and 4.36 GeV being from exotic particle $Y(4260)$ decay.

Quarkonium study in B-factory has a unique importance, where some critical measurement had ever been achieved. With the SuperKEKB/BelleII run in near future, it is reasonable to expect more from it on quarkonium physics, including the measurement on $\gamma + \eta_c(\eta_b)$ exclusive production. In table 1 the cross sections of $\gamma + \eta_c(\eta_b)$ production at B-factory energy $\sqrt{s} = 10.6$ GeV are presented up to NNLO corrections. With the target luminosity of $8 \times 10^{35} \text{ cm}^{-2}\text{s}^{-1}$ of SuperKEKB, the exclusive $e^+e^- \rightarrow \gamma + \eta_c$ and $e^+e^- \rightarrow \gamma + \eta_b$ processes should be observed, at least the former, otherwise there will

$\sigma(\text{fb})$	LO	NLO	NNLO
$\eta_c(1.4)$	89.7	75.2	44.6
$\eta_c(1.5)$	82.8	68.5	45.2
$\eta_b(4.7)$	2.50	1.77	1.75
$\eta_b(4.8)$	2.07	1.47	1.46

Table 1. Cross sections of $e^+e^- \rightarrow \gamma + \eta_c(\eta_b)$ processes up to NNLO accuracy at the B-factory, i.e., $\sqrt{s} = 10.6 \text{ GeV}$. The renormalization scale here is set to be $\mu_r = \sqrt{s}/2$.

Figure 4. The renormalization scale dependence of NLO and NNLO K-factors for $\gamma + \eta_c$ exclusive production at the B-factory. Here, the factorization scale $\mu_\Lambda = 1 \text{ GeV}$, subscripts 1.4 and 1.5 mean the charm quark mass in GeV.

be a surprise. The NLO and NNLO corrections at B-factory are still negative, and the NNLO cross sections are some half of the LO predictions. Note that though NNLO corrections for $e^+e^- \rightarrow \gamma + \eta_c$ is highly suppressed, it still surpasses the NLO cross section of $e^+e^- \rightarrow J/\psi + \eta_c$ process. Therefore, the measurement of exclusive processes $e^+e^- \rightarrow \gamma + \eta_c$ and $e^+e^- \rightarrow \eta_c + l^+ + l^-$, l denoting leptons, are highly desired to further examine the quarkonium production mechanism and high-order QCD calculation reliability.

In figures 4 and 5 we show the renormalization scale dependence of NLO and NNLO K-factors in $\gamma + \eta_c$ and $\gamma + \eta_b$ production at B-factory respectively. It is noticeable that though the NLO and NNLO K-factors are all negative, for $\gamma + \eta_c$ production the NNLO corrections are large and give no help to diminish the final result renormalization scale dependence, even with the choice of factorization scale at $\mu_\Lambda = 1 \text{ GeV}$. Whereas for $\gamma + \eta_b$ production, the NNLO corrections are moderate, the NNLO corrections reduce the renormalization scale dependence evidently. These observations imply that the perturbative expansion converges well for $\gamma + \eta_b$ production, but not for its counterpart in charm sector, which casts a shadow over the higher order corrections and further suggests that NRQCD factorization works more soundly for bottomonium system.

Figure 5. The renormalization scale dependence of NLO and NNLO K-factors for $\gamma + \eta_b$ exclusive production at the B-factory. Here, the factorization scale $\mu_\Lambda = m_b$, subscripts 4.7 and 4.8 mean the bottom quark mass in GeV.

In summary, we calculate the NNLO QCD corrections for $\gamma + \eta_c(\eta_b)$ exclusive production in electron-positron collision within the NRQCD formalism. The result tells that the NRQCD factorization still works well at the two-loop level. Thanks to the recently development in differential equation method, we find that all the master integrals can be analytically expressed in terms of Goncharov Polylogarithms, integrals over polylogarithms and complete elliptic integrals. The calculation details of all the master integrals can be found in refs. [25, 31]. To the best of our knowledge, this is the first time that a NNLO calculation for multiscale process about heavy quarkonium has been achieved analytically. Numerical results indicate that the NLO and NNLO cross sections are both negative in charmonium and bottomonium production. In BESIII regime, the cross section is greatly suppressed with NNLO QCD correction, which agrees with the experimental measurement and implies the $Y(4260) \rightarrow \gamma + \eta_c(1S)$ decay mode within data. Hopefully, in the forthcoming SuperKEKB run and Belle II measurement on $\gamma + \eta_c$ and $\gamma + \eta_b$ production, quarkonium production mechanism might be further elucidated.

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References

- [1] G.T. Bodwin, E. Braaten and G.P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, *Phys. Rev. D* **51** (1995) 1125 [*Erratum ibid.* **D 55** (1997) 5853] [[hep-ph/9407339](#)] [[INSPIRE](#)].
- [2] N. Brambilla et al., *Heavy quarkonium: progress, puzzles and opportunities*, *Eur. Phys. J. C* **71** (2011) 1534 [[arXiv:1010.5827](#)] [[INSPIRE](#)].
- [3] E. Braaten and J. Lee, *Exclusive double charmonium production from e^+e^- annihilation into a virtual photon*, *Phys. Rev. D* **67** (2003) 054007 [*Erratum ibid.* **D 72** (2005) 099901] [[hep-ph/0211085](#)] [[INSPIRE](#)].
- [4] K.-Y. Liu, Z.-G. He and K.-T. Chao, *Problems of double charm production in e^+e^- annihilation at $\sqrt{s} = 10.6$ GeV*, *Phys. Lett. B* **557** (2003) 45 [[hep-ph/0211181](#)] [[INSPIRE](#)].
- [5] K. Hagiwara, E. Kou and C.-F. Qiao, *Exclusive J/ψ productions at e^+e^- colliders*, *Phys. Lett. B* **570** (2003) 39 [[hep-ph/0305102](#)] [[INSPIRE](#)].
- [6] BELLE collaboration, K. Abe et al., *Observation of double $c\bar{c}$ production in e^+e^- annihilation at $\sqrt{s} \simeq 10.6$ GeV*, *Phys. Rev. Lett.* **89** (2002) 142001 [[hep-ex/0205104](#)] [[INSPIRE](#)].
- [7] BABAR collaboration, B. Aubert et al., *Measurement of double charmonium production in e^+e^- annihilations at $\sqrt{s} = 10.6$ GeV*, *Phys. Rev. D* **72** (2005) 031101 [[hep-ex/0506062](#)] [[INSPIRE](#)].
- [8] Y.-J. Zhang, Y.-J. Gao and K.-T. Chao, *Next-to-leading order QCD correction to $e^+e^- \rightarrow J/\psi + \eta_c$ at $\sqrt{s} = 10.6$ GeV*, *Phys. Rev. Lett.* **96** (2006) 092001 [[hep-ph/0506076](#)] [[INSPIRE](#)].
- [9] Y.-J. Zhang and K.-T. Chao, *Double charm production $e^+e^- \rightarrow J/\psi + c\bar{b}arc$ at B factories with next-to-leading order QCD correction*, *Phys. Rev. Lett.* **98** (2007) 092003 [[hep-ph/0611086](#)] [[INSPIRE](#)].
- [10] M.A. Shifman and M.I. Vysotsky, *Form-factors of heavy mesons in QCD*, *Nucl. Phys. B* **186** (1981) 475 [[INSPIRE](#)].
- [11] W.-L. Sang and Y.-Q. Chen, *Higher order corrections to the cross section of $e^+e^- \rightarrow$ quarkonium + γ* , *Phys. Rev. D* **81** (2010) 034028 [[arXiv:0910.4071](#)] [[INSPIRE](#)].
- [12] D. Li, Z.-G. He and K.-T. Chao, *Search for $C = +$ charmonium and bottomonium states in $e^+e^- \rightarrow \gamma + X$ at B factories*, *Phys. Rev. D* **80** (2009) 114014 [[arXiv:0910.4155](#)] [[INSPIRE](#)].
- [13] M. Beneke, A. Signer and V.A. Smirnov, *Two loop correction to the leptonic decay of quarkonium*, *Phys. Rev. Lett.* **80** (1998) 2535 [[hep-ph/9712302](#)] [[INSPIRE](#)].
- [14] A. Czarnecki and K. Melnikov, *Two loop QCD corrections to the heavy quark pair production cross-section in e^+e^- annihilation near the threshold*, *Phys. Rev. Lett.* **80** (1998) 2531 [[hep-ph/9712222](#)] [[INSPIRE](#)].
- [15] M. Beneke et al., *Leptonic decay of the $\Upsilon(1S)$ meson at third order in QCD*, *Phys. Rev. Lett.* **112** (2014) 151801 [[arXiv:1401.3005](#)] [[INSPIRE](#)].
- [16] A. Czarnecki and K. Melnikov, *Charmonium decays: $J/\psi \rightarrow e^+e^-$ and $\eta_c \rightarrow \gamma\gamma$* , *Phys. Lett. B* **519** (2001) 212 [[hep-ph/0109054](#)] [[INSPIRE](#)].
- [17] A.I. Onishchenko and O.L. Veretin, *Two loop QCD corrections to B_c meson leptonic constant*, *Eur. Phys. J. C* **50** (2007) 801 [[hep-ph/0302132](#)] [[INSPIRE](#)].

- [18] L.-B. Chen and C.-F. Qiao, *Two-loop QCD corrections to B_c meson leptonic decays*, *Phys. Lett. B* **748** (2015) 443 [[arXiv:1503.05122](#)] [[INSPIRE](#)].
- [19] F. Feng, Y. Jia and W.-L. Sang, *Can nonrelativistic QCD explain the $\gamma\gamma^* \rightarrow \eta_c$ transition form factor data?*, *Phys. Rev. Lett.* **115** (2015) 222001 [[arXiv:1505.02665](#)] [[INSPIRE](#)].
- [20] F. Feng, Y. Jia and W.-L. Sang, *Next-to-next-to-leading-order QCD corrections to the hadronic width of pseudoscalar quarkonium*, *Phys. Rev. Lett.* **119** (2017) 252001 [[arXiv:1707.05758](#)] [[INSPIRE](#)].
- [21] T. Hahn, *Generating Feynman diagrams and amplitudes with FeynArts 3*, *Comput. Phys. Commun.* **140** (2001) 418 [[hep-ph/0012260](#)] [[INSPIRE](#)].
- [22] A.V. Kotikov, *Differential equations method: new technique for massive Feynman diagrams calculation*, *Phys. Lett. B* **254** (1991) 158 [[INSPIRE](#)].
- [23] A.V. Kotikov, *Differential equation method: the calculation of N point Feynman diagrams*, *Phys. Lett. B* **267** (1991) 123 [Erratum *ibid.* **B 295** (1992) 409] [[INSPIRE](#)].
- [24] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601 [[arXiv:1304.1806](#)] [[INSPIRE](#)].
- [25] L.-B. Chen, Y. Liang and C.-F. Qiao, *Two-loop integrals for CP-even heavy quarkonium production and decays*, *JHEP* **06** (2017) 025 [[arXiv:1703.03929](#)] [[INSPIRE](#)].
- [26] L. Adams, E. Chaubey and S. Weinzierl, *Simplifying differential equations for multiscale Feynman integrals beyond multiple polylogarithms*, *Phys. Rev. Lett.* **118** (2017) 141602 [[arXiv:1702.04279](#)] [[INSPIRE](#)].
- [27] E. Remiddi and L. Tancredi, *Differential equations and dispersion relations for Feynman amplitudes. The two-loop massive sunrise and the kite integral*, *Nucl. Phys. B* **907** (2016) 400 [[arXiv:1602.01481](#)] [[INSPIRE](#)].
- [28] A. von Manteuffel and L. Tancredi, *A non-planar two-loop three-point function beyond multiple polylogarithms*, *JHEP* **06** (2017) 127 [[arXiv:1701.05905](#)] [[INSPIRE](#)].
- [29] R. Bonciani, V. Del Duca, H. Frellesvig, J.M. Henn, F. Moriello and V.A. Smirnov, *Two-loop planar master integrals for $Higgs \rightarrow 3$ partons with full heavy-quark mass dependence*, *JHEP* **12** (2016) 096 [[arXiv:1609.06685](#)] [[INSPIRE](#)].
- [30] L. Adams, C. Bogner, A. Schweitzer and S. Weinzierl, *The kite integral to all orders in terms of elliptic polylogarithms*, *J. Math. Phys.* **57** (2016) 122302 [[arXiv:1607.01571](#)] [[INSPIRE](#)].
- [31] L.-B. Chen, J. Jiang and C.-F. Qiao, *Two-loop integrals for CP-even heavy quarkonium production and decays: elliptic sectors*, [arXiv:1712.03516](#) [[INSPIRE](#)].
- [32] J. Vollinga and S. Weinzierl, *Numerical evaluation of multiple polylogarithms*, *Comput. Phys. Commun.* **167** (2005) 177 [[hep-ph/0410259](#)] [[INSPIRE](#)].
- [33] C.W. Bauer, A. Frink and R. Kreckel, *Introduction to the GiNaC framework for symbolic computation within the C++ programming language*, *J. Symb. Comput.* **33** (2000) 1 [[cs/0004015](#)] [[INSPIRE](#)].
- [34] H. Frellesvig, D. Tommasini and C. Wever, *On the reduction of generalized polylogarithms to Li_n and $Li_{2,2}$ and on the evaluation thereof*, *JHEP* **03** (2016) 189 [[arXiv:1601.02649](#)] [[INSPIRE](#)].
- [35] A.V. Smirnov, *FIESTA4: optimized Feynman integral calculations with GPU support*, *Comput. Phys. Commun.* **204** (2016) 189 [[arXiv:1511.03614](#)] [[INSPIRE](#)].

- [36] S. Borowka, G. Heinrich, S.P. Jones, M. Kerner, J. Schlenk and T. Zirke, *SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop*, *Comput. Phys. Commun.* **196** (2015) 470 [[arXiv:1502.06595](#)] [[INSPIRE](#)].
- [37] D.J. Broadhurst, N. Gray and K. Schilcher, *Gauge invariant on-shell Z_2 in QED, QCD and the effective field theory of a static quark*, *Z. Phys. C* **52** (1991) 111 [[INSPIRE](#)].
- [38] K. Melnikov and T. van Ritbergen, *The three loop on-shell renormalization of QCD and QED*, *Nucl. Phys. B* **591** (2000) 515 [[hep-ph/0005131](#)] [[INSPIRE](#)].
- [39] PARTICLE DATA GROUP collaboration, C. Patrignani et al., *Review of particle physics*, *Chin. Phys. C* **40** (2016) 100001 [[INSPIRE](#)].
- [40] K.G. Chetyrkin, J.H. Kuhn and M. Steinhauser, *RunDec: a Mathematica package for running and decoupling of the strong coupling and quark masses*, *Comput. Phys. Commun.* **133** (2000) 43 [[hep-ph/0004189](#)] [[INSPIRE](#)].
- [41] G.T. Bodwin, H.S. Chung, D. Kang, J. Lee and C. Yu, *Improved determination of color-singlet nonrelativistic QCD matrix elements for S-wave charmonium*, *Phys. Rev. D* **77** (2008) 094017 [[arXiv:0710.0994](#)] [[INSPIRE](#)].
- [42] H.S. Chung, J. Lee and C. Yu, *NRQCD matrix elements for S-wave bottomonia and $\Gamma[\eta_b(nS) \rightarrow \gamma\gamma]$ with relativistic corrections*, *Phys. Lett. B* **697** (2011) 48 [[arXiv:1011.1554](#)] [[INSPIRE](#)].
- [43] BESIII collaboration, M. Ablikim et al., *Evidence for $e^+e^- \rightarrow \gamma\eta_c(1S)$ at center-of-mass energies between 4.01 and 4.60 GeV*, *Phys. Rev. D* **96** (2017) 051101 [[arXiv:1705.06853](#)] [[INSPIRE](#)].