$D = 3$ unification of curious supergravities

M.J. Duff,^{a,b} S. Ferrara^{c,d,e} and A. Marrani^{f,g,c}

^aTheoretical Physics, Blackett Laboratory, Imperial College London,
London SW7 2AZ, U.K.

^bMathematical Institute University of Oxford, Andrew Wiles Building,
Woodstock Road, Radcliffe Observatory Quarter, Oxford, OX2 6GG, U.K.

^cTheoretical Physics Department, CERN,
CH-1211 Geneva, Switzerland

^dINFN — Laboratori Nazionali di Frascati,
Via Enrico Fermi 40, I-00044 Frascati, Italy

^eDepartment of Physics and Astronomy and
Mani L. Bhaumik Institute for Theoretical Physics, UCLA,
Los Angeles CA 90095-1547, U.S.A.

^fMuseo Storico della Fisica e Centro Studi e Ricerche “Enrico Fermi”,
Via Panisperna 89A, I-00184, Roma, Italy

^gDipartimento di Fisica e Astronomia “Galileo Galilei”,
Università di Padova and INFN, Sez. di Padova,
Via Marzolo 8, I-35131 Padova, Italy

E-mail: m.duff@imperial.ac.uk, sergio.ferrara@cern.ch,
Alessio.Marrani@pd.infn.it

ABSTRACT: We consider the dimensional reduction to $D = 3$ of four maximal-rank supergravities which preserve minimal supersymmetry in $D = 11, 7, 5$ and 4 . Such “curious” theories were investigated some time ago, and the four-dimensional one corresponds to an $\mathcal{N} = 1$ supergravity with 7 chiral multiplets spanning the seven-disk manifold. Recently, this latter theory provided cosmological models for α -attractors, which are based on the disk geometry with possible restrictions on the parameter α . A unified picture emerges in $D = 3$, where the Ehlers group of General Relativity merges with the S -, T - and U -dualities of the $D = 4$ parent theories.

KEYWORDS: Supergravity Models, Extended Supersymmetry, M-Theory, Differential and Algebraic Geometry

ARXIV EPRINT: [1610.08800](https://arxiv.org/abs/1610.08800)

Contents

1	Introduction	1
2	$E_{8(8)}$ and the eight-disk manifold	3
3	The M-theory path	3
4	The Ehlers path	5
5	Conclusion	7

1 Introduction

Among compactifications of $D = 11$ supergravity on a 7-manifold to $D = 4$, an interesting $\mathcal{N} = 1$ theory emerges, whose spectrum consists of seven chiral (Wess-Zumino) multiplets living in the seven-disk manifold

$$\left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]^{\otimes 7}. \quad (1.1)$$

This theory, proposed in [1] has some peculiar properties. It is the smallest member of a family of four “*left curious*” supergravities, defined in $D = (11, 7, 5, 4)$ dimensions, having a scalar manifold of (maximal) rank $(0, 4, 6, 7)$, respectively, and endowed with a minimal number ν of supersymmetries in the corresponding dimensions, $\nu = (32, 16, 8, 4)$, respectively. Such theories couple naturally to supermembranes and admit these membranes as solutions. In [7] the seven-disk manifold (1.1) was considered as providing possible restrictions on the parameter α of the cosmological α -attractors models for inflation, depending on the embeddings of the single one-disk into (1.1).

When compactified on a 7-manifold X^7 with independent Betti numbers $(b_0, b_1, b_2, b_3) = (b_7, b_6, b_5, b_4)$, the number of fields of spin $s = (2, 3/2, 1, 1/2, 0)$ in the resulting $D = 4$ supergravity is given by $n_s = (b_0, b_0 + b_1, b_1 + b_2, b_2 + b_3, 2b_3)$, and we may loosely associate Betti numbers with any supergravity with n_s fields of spin s , whether or not manifolds with these Betti numbers actually exist. We may then define a generalized mirror transformation [1]

$$(b_0, b_1, b_2, b_3) \rightarrow (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2), \quad (1.2)$$

under which

$$\rho(X^7) := \sum_{k=0}^7 (-1)^{k+1} (k+1) b_k = 7b_0 - 5b_1 + 3b_2 - b_3, \quad (1.3)$$

changes sign:

$$\rho \rightarrow -\rho \quad (1.4)$$

(In the special case $b_1 = 0$, ρ reversal reduces to the reflection symmetry of G_2 manifolds defined by Joyce [2, 3]). Generalised self-mirror theories are here defined to be those for which ρ vanishes. Under further toroidal compactification to $D = 4$, the four curious supergravities have $\mathcal{N} = 8, 4, 2, 1$ supersymmetries and Betti numbers $(b_0, b_1, b_2, b_3) = (1, \mathcal{N} - 1, n, 3n - 5\mathcal{N} + 12)$ and thus are all self-mirror. (The $\mathcal{N} = 2$ theory is just the self-mirror *stu* model [4, 5].) Similarly, we may define a generalized mirror transformation for 6-manifolds X^6 [1] with independent Betti numbers $(c_0, c_1, c_2, c_3) = (c_6, c_5, c_4, c_3)$

$$(c_0, c_1, c_2, c_3) \rightarrow (c_0, c_1, c_2 - \chi/2, c_3 + \chi) \quad (1.5)$$

under which

$$c(X^6) := \sum_{k=0}^6 (-1)^k c_k = 2c_0 - 2c_1 + 2c_2 - c_3 \quad (1.6)$$

changes sign:

$$\chi \rightarrow -\chi \quad (1.7)$$

(In the special case $c_1 = 0$, χ reversal reduces to ordinary mirror symmetry of Calabi-Yau [6]). Generalised self-mirror theories are here defined to be those for which χ vanishes. In the special case $X^7 = X^6 \times S^1$, $\rho = \chi$ and the two symmetries coincide.

Given the unusual properties and possible cosmological applications of these curious supergravities, in the present note we give a $D = 3$ three-way unified picture in terms of

- 1) compactifications of M -theory in terms of toroidal moduli;
- 2) dimensional reduction of the four curious supergravities $D = (11, 7, 5, 4)$ to $D = 3$;
- 3) dimensional reduction of 4 curious supergravities in $D = 4$ to $D = 3$. In particular, the resulting $\mathcal{N} = 2$, $D = 3$ supergravity has the scalar manifold given by the eight-disk manifold

$$\left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]^{\otimes 8}, \quad (1.8)$$

which can be regarded as the unification of S -, T - and U - dualities of the $\mathcal{N} = 1$, $D = 4$ corresponding theory mentioned above, augmented by the disk manifold $\frac{\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}}{\mathrm{U}(1)}$ pertaining to the $D = 4$ Ehlers group $\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}$.

The paper is organized as follows.

In section 2 we recall the embedding of $[\mathrm{SL}(2, \mathbb{R})]^{\otimes 8}$ into $E_{8(8)}$. In section 3 we give an interpretation of the four curious supergravities in terms of sequential reductions of M -theory on an eight-manifold with only toroidal moduli of T^8 , $T^4 \times T^4$, and $T^2 \times T^2 \times T^2 \times T^2$ (“*M-theoretical path*”). Then, in section 4 we consider the so-called “*Ehlers path*”, by compactifying these theories from $D = 4$ to $D = 3$. Finally, section 5 contains some concluding remarks.

2 $E_{8(8)}$ and the eight-disk manifold

Almost all exceptional Lie algebras $\mathcal{E}$ enjoy a rank-preserving (generally non-maximal nor symmetric) embedding of the type

$$\mathcal{E} \supset [\mathfrak{sl}(2)]^{\oplus r}, \quad r := \text{rank}(\mathcal{E}). \quad (2.1)$$

This holds for $\mathcal{E} = \mathfrak{e}_8, \mathfrak{e}_7, \mathfrak{f}_4, \mathfrak{g}_2$, with $r = 8, 7, 4, 2$, respectively. The unique exception¹ is provided by the rank-6 exceptional algebra $\mathfrak{e}_6$, which embeds only $[\mathfrak{sl}(2)]^{\oplus 4}$, and not $[\mathfrak{sl}(2)]^{\oplus 6}$.

In the following treatment, we will focus on the maximally non-compact (i.e., split) real form $\mathfrak{e}_{8(8)}$ of $\mathfrak{e}_8$, considering it at the Lie group level ($E_{8(8)} \supset [\text{SL}(2, \mathbb{R})]^{\otimes 8}$), in the context of $D = 3$ supergravity theories.

More specifically, starting from² $E_{8(8)}$ we will analyze two paths yielding the same $\mathcal{N} = 2$, $D = 3$ supergravity theory,³ coupled to 8 matter multiplets, whose scalars coordinatize the completely factorized rank⁴-8 Hodge-Kähler symmetric, *eight-disk manifold* (1.8).

3 The M -theory path

The first path starts from M -theory (or, more appropriately, $\mathcal{N} = 1$, $D = 11$ supergravity), and performs iterated compactifications on tori T^8 , $T^4 \times T^4$, and on $T^2 \times T^2 \times T^2 \times T^2$; this corresponds to the following chain of maximal and symmetric embeddings:

$$E_{8(8)} \supset \text{SO}(8, 8) \quad (3.1)$$

$$\supset \text{SO}(4, 4) \times \text{SO}(4, 4) \quad (3.2)$$

$$\supset [\text{SO}(2, 2)]^{\otimes 4} \cong [\text{SL}(2, \mathbb{R})]^{\otimes 8}. \quad (3.3)$$

Each step of this chain has an interpretation in terms of truncations of the massless spectrum of M -theory dimensionally reduced to $D = 3$, such as to preserve $\mathcal{N} = 16, 8, 4, 2$ local supersymmetries. As we discuss below, the last three are obtained keeping only the geometric moduli of the tori T^8 , $T^4 \times T^4$ and $T^2 \times T^2 \times T^2 \times T^2$, respectively. It is worth here recalling that the classical moduli space of a d -dimensional torus is $(I, J = 1, \dots, d)$

$$M_d := \mathbb{R}^+ \times \frac{\text{SL}(d, \mathbb{R})}{\text{SO}(d)}, \quad \text{spanned by } g_{IJ} = g_{(IJ)}, \quad (3.4)$$

¹It should be here pointed that $\mathfrak{e}_6$ stands on its own among exceptional Lie algebras for *at least* another reason: it is the unique exceptional Lie algebra which does not embed maximally its principal (Kostant's) $\mathfrak{sl}(2)_P$ [8] algebra. Indeed, while all Lie algebras maximally embed $\mathfrak{sl}(2)_P$ ($\mathfrak{e}_8$ and $\mathfrak{e}_7$ actually maximally embed three and two $\mathfrak{sl}(2)$'s, respectively), $\mathfrak{e}_6$ embeds its $\mathfrak{sl}(2)_P$ through the chain of maximal embeddings $\mathfrak{e}_6 \supset \mathfrak{f}_4 \supset \mathfrak{sl}(2)_P$ (in other words, $\mathfrak{e}_6$ "inherits" the $\mathfrak{sl}(2)_P$ of $\mathfrak{f}_4$).

² $E_{8(8)}$ belongs to the so-called *exceptional $E_{n(n)}$ -sequence* [9–11] of symmetries of maximal supergravities in $11 - n$ dimensions.

³For a thorough analysis of the geometric structure of scalar manifolds of $D = 3$ supergravity theories, see [12].

⁴The *rank* of a manifold is defined as the maximal dimension (in $\mathbb{R}$) of a flat (i.e., with vanishing Riemann tensor), totally geodesic submanifold (see e.g. §6, page 209 of [13]).

whereas the quantum one (in a stringy sense) reads

$$\mathcal{M}_d := \frac{\mathrm{SO}(d, d)}{\mathrm{SO}(d) \times \mathrm{SO}(d)}, \text{ spanned by } g_{IJ} = g_{(IJ)} \text{ and } B_{IJ} = B_{[IJ]}. \quad (3.5)$$

The first, starting step of the *M-theoretical path* (3.1)–(3.3) corresponds to:⁵

$$M\text{-theory} \xrightarrow{T^8 \text{ (geom+non-geom)}} \mathcal{N} = 16, D = 3 : \frac{E_{8(8)}}{\mathrm{SO}(16)}, \quad (B, F) = (128, 128), \quad (3.6)$$

namely a compactification retaining *both* geometric (g_{IJ} , $A_{\mu IJ}$;) and non-geometric ($g_{\mu I}$, A_{IJK}) moduli of T^8 , down to maximal supergravity in $D = 3$ [15] ($I, J, K = 1, \dots, 8$, and $\mu = 0, 1, 2$); note that the 128 bosonic massless degrees of freedom can be organized in $\mathrm{SO}(8)$ irreps. as follows:

$$\begin{matrix} g_{IJ}, & A_{\mu IJ}, & g_{\mu I}, & A_{IJK}, \\ \mathbf{35+1} & \mathbf{28} & \mathbf{8} & \mathbf{56} \end{matrix}, \quad (3.7)$$

where the 1-form $A_{\mu IJ} = A_{\mu[IJ]}$ (playing the role of the “*M*-theoretical *B*-field”) gets then dualized to scalar fields A_{IJ} in $D = 3$.

The next step corresponds to the first, maximal and symmetric embedding (3.1), which amounts to retaining only the geometric moduli of T^8 (i.e., to setting $g_{\mu I} = 0 = A_{IJK}$ in the bosonic sector), thus giving rise upon compactification to half-maximal supergravity coupled to $n = 8$ matter multiplets in $D = 3$:

$$M\text{-theory} \xrightarrow{T^8 \text{ (geom)}} \mathcal{N} = 8, D = 3, n = 8 : \frac{\mathrm{SO}(8, 8)}{\mathrm{SO}(8) \times \mathrm{SO}(8)}. \quad (B, F) = (64, 64) \quad (3.8)$$

The subsequent maximal and symmetric embedding (3.2) corresponds to a compactification on $T^4 \times T^4$ retaining only the corresponding geometric moduli ($i, j = 1, \dots, 4$, and $i', j' = 5, \dots, 8$):

$$g_{ij}, A_{\mu ij}, g_{i'j'}, A_{\mu i'j'}, \quad (3.9)$$

thus giving rise to the following $\mathcal{N} = 4$, $D = 3$ supergravity model:

$$M\text{-theory} \xrightarrow{T^4 \times T^4 \text{ (geom)}} \mathcal{N} = 4, D = 3, n = 8 : \frac{\mathrm{SO}(4, 4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)} \times \frac{\mathrm{SO}(4, 4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)}. \quad (B, F) = (32, 32) \quad (3.10)$$

The last step is given by the maximal and symmetric embedding (3.3), corresponding to a compactification on $T^2 \times T^2 \times T^2 \times T^2$ retaining only the related geometric moduli

$$g_{11}, g_{12}, g_{22}, A_{\mu 12}, \quad g_{33}, g_{34}, g_{44}, A_{\mu 34}, \quad g_{55}, g_{56}, g_{66}, A_{\mu 56}, \quad g_{77}, g_{78}, g_{88}, A_{\mu 78}, \quad (3.11)$$

thus giving rise to the $\mathcal{N} = 2$, $D = 3$ supergravity model whose scalar manifold is given by the eight-disk manifold (1.8):

$$M\text{-theory} \xrightarrow{T^2 \times T^2 \times T^2 \times T^2 \text{ (geom)}} \mathcal{N} = 2, D = 3 : \left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]^{\otimes 8}. \quad (B, F) = (16, 16) \quad (3.12)$$

Some comments are in order.

⁵”*B*” and “*F*” denote the number of bosonic and fermionic massless degrees of freedom throughout.

1. All symmetric scalar manifolds in (3.6), (3.8), (3.10) and (3.12) have rank 8, as a consequence of the fact that all embeddings of the chain (3.1)–(3.3) are rank-preserving.
2. The theories (3.6), (3.8), (3.10) and (3.12) are nothing but the $D = 3$ reduction of the four curious supergravities, studied in [1] and mentioned in section 1. These latter are defined in $D = q + 3 = 11, 7, 5, 4$ Lorentzian space-time dimensions (with $q := \dim_{\mathbb{R}} \mathbb{A} = 8, 4, 2, 1$, where $\mathbb{A} = \mathbb{O}$ (octonions), $\mathbb{H}$ (quaternions), $\mathbb{C}$ (complex numbers), $\mathbb{R}$ (reals) denote the four Hurwitz division algebras), with scalar manifolds of rank 0, 4, 6, 7 respectively. As observed in [1], such $\mathcal{N} = 8, 4, 2, 1$, $D = 4$ curious supergravities respectively correspond to $\mathcal{N} - 1 = 7, 3, 1, 0$ lines of the Fano plane, and hence they admit a division algebraic interpretation consistent with the so-called “black-hole/qubit” correspondence (cfr. e.g. [16] for an introduction and a list of refs.). By further compactifying them respectively on T^8 , T^4 , T^2 , $T^1 = S^1$ down to $D = 3$, the rank of the corresponding scalar manifold (after dualization) increase by 8, 4, 2, 1, so that all the resulting $D = 3$ theories have rank-8 scalar manifolds, as given by (3.6), (3.8), (3.10) and (3.12). They have $\mathcal{N} = 2^4, 2^3, 2^2, 2$ local supersymmetry in $D = 3$, with 2^8 , 2^7 , 2^6 and 2^5 total number of massless states, respectively. In this perspective, the dimensional reduction to $D = 3$ provides a *unified view of the curious supergravities*.

4 The Ehlers path

The second path yielding the $\mathcal{N} = 2$, $D = 3$ supergravity theory with scalar manifold (1.8) starts with the so-called *Ehlers embedding* (cfr. e.g. [17], and refs. therein) for maximal supergravity in $D = 4 \rightarrow D = 3$, and then proceeds with a chain of maximal, symmetric and rank-preserving embeddings which has already been considered in [7, 14, 18]:

$$E_{8(8)} \supset E_{7(7)} \times \mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}} \quad (4.1)$$

$$\supset \mathrm{SO}(6, 6) \times \mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}} \times \mathrm{SL}(2, \mathbb{R}) \quad (4.2)$$

$$\supset \mathrm{SO}(4, 4) \times [\mathrm{SL}(2, \mathbb{R})]^{\otimes 2} \times \mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}} \times \mathrm{SL}(2, \mathbb{R}) \quad (4.3)$$

$$\supset [\mathrm{SL}(2, \mathbb{R})]^{\otimes 8} \quad (4.4)$$

Since this path, which we name *Ehlers path*, starts with a $D = 4 \rightarrow D = 3$ dimensional reduction, it is immediate to realize that the $D = 3$ scalar manifolds given in (3.6), (3.8), (3.10) and (3.12) are nothing but the dimensional reduction of the $D = 4$ cosets of $\mathcal{N} = 8, 4, 2, 1$ curious supergravities with rank-7 scalar manifolds (after dualization; cfr. table XVIII of [1]).

While for $\mathcal{N} = 8, 4, 2$ the dimensional reduction $D = 4 \rightarrow D = 3$ is well-known from the study of Maxwell-Einstein systems coupled to non-linear sigma models ([19], thereby including the *c*-map [20, 21] relating projective special Kähler manifolds to quaternionic

manifolds), for $\mathcal{N} = 1$ the dimensional reduction reads

$$(B, F)=(16, 16) : \left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]_{\mathcal{N}=1, D=4, n_c=7, n_v=0}^{\otimes 7} \longrightarrow \left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]_{\mathcal{N}=2, D=3, n=8}^{\otimes 8}, \quad (4.5)$$

and it stands on a different footing. Indeed, the $\mathcal{N} = 1$, $D = 4$ supergravity theory is coupled only to 7 chiral multiplets, *with no vectors at all*. Therefore, under (spacelike) dimensional reduction $D = 4 \rightarrow D = 3$, the chiral multiplets' scalar manifold (1.1) gets enlarged only by a further factor manifold $\frac{\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}}{\mathrm{U}(1)}$, spanned by the axio-dilaton given by the S^1 -radius of compactification and by the dualization of the corresponding Kaluza-Klein vector. In other words, the added $\frac{\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}}{\mathrm{U}(1)}$ manifold pertains to the two degrees of freedom of the $D = 4$ massless graviton (since in $D = 3$ the graviton does not propagate any degree of freedom): as mentioned in section 1, the *seven-disk manifold* (1.1) [1, 7] gets enlarged to the *eight-disk manifold* (1.8) by including the $D = 4$ Ehlers group $\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}$.

Some observations are:

1. All symmetric scalar manifolds in (4.6), (4.7) and (4.8) have rank 7, as a consequence of the fact that all embeddings of the chain (4.1)–(4.4) are rank-preserving.
2. The chain of embeddings (4.1)–(4.4) has been used in [18] (also cfr. [14]) to study the tripartite entanglement of seven qubits inside E_7 . Moreover, it was recently exploited in [7] in order to obtain the $\mathcal{N} = 1$, $D = 4$ theory with 7 WZ multiplet given in the fourth line of (4.5).
3. The maximal and symmetric embedding (4.2) corresponds to the truncation of maximal $D = 4$ supergravity to half-maximal supergravity coupled to 6 matter (vector) multiplets:

$$\frac{E_{7(7)}}{\mathrm{SU}(8)}_{\mathcal{N}=8, D=4, (B, F)=(128, 128)} \longrightarrow \frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)}_{\mathcal{N}=4, D=4, n=6, (B, F)=(64, 64)} \times \frac{\mathrm{SO}(6, 6)}{\mathrm{SO}(6) \times \mathrm{SO}(6)}. \quad (4.6)$$

4. The subsequent step (4.3) corresponds to the truncation of half-maximal $D = 4$ supergravity coupled to 6 vector multiplets to the $\mathcal{N} = 2$, $D = 4$ *stu* model coupled to 4 hypermultiplets, whose quaternionic scalars coordinatize the symmetric scalar manifold $\frac{\mathrm{SO}(4, 4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)}$; since this latter is the *c*-map [20] of the corresponding vector-multiplets' projective special Kähler manifold $\left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]^{\otimes 3}$, this model is *self-mirror* (also cfr. e.g. [22]):

$$\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)}_{\mathcal{N}=4, D=4, n=6, (B, F)=(64, 64)} \times \frac{\mathrm{SO}(6, 6)}{\mathrm{SO}(6) \times \mathrm{SO}(6)} \longrightarrow \left[\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right]^{\otimes 3}_{\mathcal{N}=2, D=4, n_v=3, n_H=4, \text{ self-mirror stu model, } (B, F)=(32, 32)} \times \frac{\mathrm{SO}(4, 4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)}. \quad (4.7)$$

5. The last step (4.3) corresponds to the truncation of the self-mirror $D = 4$ *stu* model to an $\mathcal{N} = 1$, $D = 4$ theory with 7 WZ multiplets, whose scalars span the *seven-disk*

manifold (1.1) [1, 7]:

$$\left[\frac{\mathrm{SL}(2, R)}{\mathrm{U}(1)} \right]^{\otimes 3} \times \frac{\mathrm{SO}(4, 4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)} \longrightarrow \left[\frac{\mathrm{SL}(2, R)}{\mathrm{U}(1)} \right]^{\otimes 7} .$$

$\mathcal{N}=2, D=4, n_v=3, n_H=4, \text{ self-mirror } stu \text{ model, } (B, F)=(32, 32) \quad \mathcal{N}=1, D=4, n_c=7, n_v=0, (B, F)=(16, 16)$
(4.8)

This step is non-trivial for what concerns the retaining of an $\mathcal{N} = 1$ local supersymmetry in the gravity theory with non-linear sigma model given by (1.8). Besides the necessary truncation of the $\mathcal{N} = 1$ gravitino multiplet coming from the supersymmetric $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ reduction of the $\mathcal{N} = 2$ gravity multiplet, one has to truncate all $\mathcal{N} = 1$ vector multiplets coming from the supersymmetry reduction of the three $\mathcal{N} = 2$ vector multiplets; furthermore, a truncation of half of the $\mathcal{N} = 1$ chiral multiplets stemming from the supersymmetry reduction of the four $\mathcal{N} = 2$ hypermultiplets must be performed. This last step is particularly challenging for the consistency with local $\mathcal{N} = 1$ supersymmetry, which is however granted by the results⁶ in [23] (also cfr. [24]); see, in particular, the discussion around eq. (6.145) therein.

5 Conclusion

Summarizing, there exist (*at least*) three different ways to obtain the four $\mathcal{N} = 16, 8, 4, 2$ curious supergravities (3.6), (3.8), (3.10) and (3.12) with symmetric scalar manifolds of (maximal) rank 8 in $D = 3$:

1. Toroidal compactification of M -theory from $D = 11$ to $D = 3$, respectively retaining geometric and non-geometric moduli of T^8 , and then geometric moduli of T^8 , of $T^4 \times T^4$, and of $T^2 \times T^2 \times T^2 \times T^2$. This is given by the *M-theoretical path* (3.1)–(3.3) discussed in section 3.
2. Toroidal compactification of the four curious supergravities [1] (defined in 11, 7, 5, 4 dimensions) respectively on $T^8, T^4, T^2, T^1 = S^1$ down to $D = 3$; this is discussed at point 3 of section 3.
3. S^1 -dimensional reduction $D = 4 \rightarrow D = 3$ of the $\mathcal{N} = 8, 4, 2, 1$, $D = 4$ curious supergravities with rank-7 scalar manifolds (after dualization; cfr. table XVIII of [1]). This is given by the *Ehlers path* (4.1)–(4.4) discussed in section 4.

By comparing the two paths (3.1)–(3.3) and (4.1)–(4.4), it is evident that they exhibit different and features.

The *M-theoretical path* (3.1)–(3.3) is deeply rooted in M -theory, and it makes “*octality*”, pertaining to the symmetry of the fully factorised rank-8 Hodge-Kähler symmetric coset (*eight-disk manifold* (1.8)) in $D = 3$, completely manifest: the $\mathrm{SL}(2, \mathbb{R})$ ’s of T -duality (from the T^2 -factors of the 8-dimensional internal manifold), the $\mathrm{SL}(2, \mathbb{R})$ ’s of S -duality

⁶In a different framework, more pertaining to the first path (3.1)–(3.3) to (1.8), $\mathcal{N} = 1$ local supersymmetry for the $D = 4$ theory with scalar manifold (1.8) was obtained in [1] by considering M -theory compactified on a suitable 7-dimensional manifold with G_2 -structure.

and U -duality, and the $D = 4$ Ehlers group $\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}$ (of gravitational origin) get *unified*, and they stand on the same footing.

On the other hand, the *Ehlers path* (4.1)–(4.4), makes only “*septality*”, pertaining to the full-fledged symmetry of the fully factorised rank-7 Hodge-Kähler symmetric coset in $D = 4$ (*seven-disk manifold* (1.1)), completely manifest: *only* the $\mathrm{SL}(2, \mathbb{R})$ ’s of S -, T - and U - dualities get unified.

However, notwithstanding the first step (4.1) which seems to single out the $D = 4$ Ehlers group $\mathrm{SL}(2, \mathbb{R})_{\mathrm{Ehlers}}$, a *complete equivalence* between the two paths is reached at their final steps. It would be worth pursuing an E_{11} interpretation [25] of these four maximal rank theories preserving minimal supersymmetry in $D = 11, 7, 5, 4$.

We also recall that in $D = 4$ the four curious supergravities with $\mathcal{N} = 8, 4, 2, 1$ are associated with $7, 3, 1, 0$ vertices of the Fano plane [1]. Similarly, in $D = 3$ the $\mathcal{N} = 16, 8, 4, 2$ theories are associated with the $7, 3, 1, 0$ quadrangles of the Fano plane and the dual Fano plane.⁷

Acknowledgments

We are grateful to Renata Kallosh, for useful discussions and related collaboration. MJD is grateful to the Leverhulme Trust for an Emeritus Fellowship and to Philip Candelas for hospitality at the Mathematical Institute, Oxford. This work was supported by the STFC under rolling grant ST/G000743/1. The work of SF is supported in part by CERN TH Dept. and INFN-CSN4-GSS. AM wishes to thank the CERN Theory Division, for the kind hospitality during the realization of this work. MJD and SF acknowledge the hospitality of the GGI institute in Firenze, where this work was completed during the workshop ‘Supergravity: What Next?’.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] M.J. Duff and S. Ferrara, *Four curious supergravities*, *Phys. Rev. D* **83** (2011) 046007 [[arXiv:1010.3173](https://arxiv.org/abs/1010.3173)] [[INSPIRE](#)].
- [2] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2 . I*, *J. Diff. Geom.* **43** (1996) 291.
- [3] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2 . II*, *J. Diff. Geom.* **43** (1996) 329.
- [4] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality*, *Nucl. Phys. B* **459** (1996) 125 [[hep-th/9508094](https://arxiv.org/abs/hep-th/9508094)] [[INSPIRE](#)].
- [5] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W.K. Wong, *STU black holes and string triality*, *Phys. Rev. D* **54** (1996) 6293 [[hep-th/9608059](https://arxiv.org/abs/hep-th/9608059)] [[INSPIRE](#)].

⁷See the decompositions of the 56 under $E_7 \supset [\mathrm{SL}(2)]^{\otimes 7}$ and the 248 under $E_8 \supset [\mathrm{SL}(2)]^{\otimes 8}$ in [14].

- [6] Y.S. Tung, *Essays on mirror manifolds*, Int. Press, Hong Kong (1992).
- [7] S. Ferrara and R. Kallosh, *Seven-disk manifold, α -attractors and B-modes*, [arXiv:1610.04163](#) [[INSPIRE](#)].
- [8] B. Kostant, *The three-dimensional sub-group and the Betti numbers of a complex simple Lie group*, *Amer. J. Math.* **81** (1959) 973.
- [9] B. Julia, *Group disintegrations*, in *Superspace and supergravity*, S.W. Hawking and M. Rocek eds., Cambridge Univ. Press, Cambridge U.K. (1981) [[INSPIRE](#)].
- [10] E. Cremmer, *Supergravities in 5 dimensions*, in *Superspace and supergravity*, S.W. Hawking and M. Rocek eds., Cambridge Univ. Press, Cambridge U.K. (1981) [[INSPIRE](#)].
- [11] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, invited talk at *AMS-SIAM summer seminar on applications of group theory in physics and mathematical physics*, Chicago U.S.A. July 6–16 1982 [[INSPIRE](#)].
- [12] B. de Wit, A.K. Tollsten and H. Nicolai, *Locally supersymmetric $D = 3$ nonlinear σ -models*, *Nucl. Phys. B* **392** (1993) 3 [[hep-th/9208074](#)] [[INSPIRE](#)].
- [13] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York U.S.A. (1978).
- [14] L. Borsten, D. Dahanayake, M.J. Duff, H. Ebrahim and W. Rubens, *Black holes, qubits and octonions*, *Phys. Rept.* **471** (2009) 113 [[arXiv:0809.4685](#)] [[INSPIRE](#)].
- [15] N. Marcus and J.H. Schwarz, *Three-dimensional supergravity theories*, *Nucl. Phys. B* **228** (1983) 145 [[INSPIRE](#)].
- [16] L. Borsten, M.J. Duff and P. Lévy, *The black-hole/qubit correspondence: an up-to-date review*, *Class. Quant. Grav.* **29** (2012) 224008 [[arXiv:1206.3166](#)] [[INSPIRE](#)].
- [17] S. Ferrara, A. Marrani, M. Trigiante, A. Marrani and M. Trigiante, *Super-Ehlers in any dimension*, *JHEP* **11** (2012) 068 [[arXiv:1206.1255](#)] [[INSPIRE](#)].
- [18] M.J. Duff and S. Ferrara, *E_7 and the tripartite entanglement of seven qubits*, *Phys. Rev. D* **76** (2007) 025018 [[quant-ph/0609227](#)] [[INSPIRE](#)].
- [19] P. Breitenlohner, D. Maison and G.W. Gibbons, *Four-dimensional black holes from Kaluza-Klein theories*, *Commun. Math. Phys.* **120** (1988) 295 [[INSPIRE](#)].
- [20] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys. A* **4** (1989) 2475 [[INSPIRE](#)].
- [21] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys. B* **332** (1990) 317 [[INSPIRE](#)].
- [22] S. Ferrara and A. Marrani, *$N = 8$ non-BPS attractors, fixed scalars and magic supergravities*, *Nucl. Phys. B* **788** (2008) 63 [[arXiv:0705.3866](#)] [[INSPIRE](#)].
- [23] L. Andrianopoli, R. D’Auria and S. Ferrara, *Supersymmetry reduction of N extended supergravities in four-dimensions*, *JHEP* **03** (2002) 025 [[hep-th/0110277](#)] [[INSPIRE](#)].
- [24] S. Ferrara, R. Kallosh and A. Marrani, *Degeneration of groups of type E_7 and minimal coupling in supergravity*, *JHEP* **06** (2012) 074 [[arXiv:1202.1290](#)] [[INSPIRE](#)].
- [25] P. West, *A brief review of E theory*, *Int. J. Mod. Phys. A* **31** (2016) 1630043 [[arXiv:1609.06863](#)] [[INSPIRE](#)].