The a-function in six dimensions

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ABSTRACT: The a-function is a proposed quantity defined in even dimensions which has a monotonic behaviour along RG flows, related to the β-functions via a gradient flow equation. We study the a-function for a general scalar theory in six dimensions, using the β-functions up to three-loop order for both the $\overline{\text{MS}}$ and MOM schemes (the latter presented here for the first time at three loops).

Contents

1 Introduction

There has been considerable recent interest over the possibility (first raised by Cardy [1]) of a four-dimensional generalisation of Zamolodchikov's c-theorem [2] in two dimensions; namely a function $a(g)$ of the couplings g^I which has monotonic behaviour under renormalisation-group (RG) flow or which is defined at fixed points such that $a_{\mathrm{UV}} - a_{\mathrm{IR}} > 0$. These two possibilities are referred to as the strong or weak a-theorem, respectively. A proof of the weak a-theorem has been proposed by Komargodski and Schwimmer [3, 4] with further analysis in ref. [5].

The strong a-theorem has been proved valid for small values of the couplings [6, 7], using Wess-Zumino consistency conditions for the response of the theory defined on curved spacetime, and with x-dependent couplings $g^I(x)$, to a Weyl rescaling of the metric [8]. A function A is defined which satisfies the crucial equation

$$\partial_I A = T_{IJ}\beta^J \,, \tag{1.1}$$

for a function T_{IJ} which is defined in terms of RG quantities and may in principle be computed perturbatively within the theory extended to curved spacetime and x-dependent g^I. Eq. (1.1) implies

$$\mu\frac{d}{d\mu}A = \beta^I\frac{\partial}{\partial g^I}A = G_{IJ}\beta^I\beta^J \tag{1.2}$$

where $G_{IJ} = T_{(IJ)}$; thus verifying the strong a-theorem so long as G_{IJ} is positive-definite, a property which holds at least for weak couplings in four dimensions. It is clear that if A satisfies an equation of the form eq. (1.1) then so does

$$A' = A + g_{IJ}\beta^I\beta^J \tag{1.3}$$

for any g_{IJ} (for a different T_{IJ}, of course).

Further extensions of this general framework have been explored in refs. [9–11]. We should mention explicitly here that for theories with a global symmetry, β^I in these equations should be replaced by a B^I which is defined, for instance, in ref. [7]. However, it was shown in refs. [12, 13] that the two quantities only begin to differ at three loops for theories in four dimensions.

It was shown in ref. [14] that equations of a similar form to the above may be derived (in a similar manner) for a renormalisable theory in six dimensions;[1] though the definition of A and T_{IJ} as renormalisation-group quantities is of course different. For instance,

$$A = 6a + b_1 - \frac{1}{15}b_3 + W_I\beta^I \tag{1.4}$$

where W_I, like T_{IJ}, has a definition in terms of RG quantities, a is the β-function corresponding to the six-dimensional Euler density

$$E_6 = \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\epsilon_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6}R^{\mu_1\mu_2\nu_1\nu_2}R^{\mu_3\mu_4\nu_3\nu_4}R^{\mu_5\mu_6\nu_5\nu_6} \tag{1.5}$$

and b_1 and b_3 are the β-functions corresponding respectively to

$$L_1 = -\frac{1}{30}K_1 + \frac{1}{4}K_2 - K_6, \quad L_3 = -\frac{37}{6000}K_1 + \frac{7}{150}K_2 - \frac{1}{75}K_3 + \frac{1}{10}K_5 + \frac{1}{15}K_6, \tag{1.6}$$

where (following the notation of ref. [14])

$$K_1 = R^3, \quad K_2 = RR^{\kappa\lambda}R_{\kappa\lambda}, \quad K_3 = RR^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}, \quad K_4 = R^{\kappa\lambda}R_{\lambda\mu}R^{\mu}{}_{\kappa},$$
$$K_5 = R^{\kappa\lambda}R_{\kappa\mu\nu\lambda}R^{\mu\nu}, \quad K_6 = R^{\kappa\lambda}R_{\kappa\mu\nu\rho}R_{\lambda}{}^{\mu\nu\rho}. \tag{1.7}$$

We use the notation A to avoid confusion with the $\tilde{a}$ of ref. [14] which differs by a factor of 6. In six dimensions, G_{IJ} has recently been computed to be negative definite at leading order for a multiflavor ϕ^3 theory [19, 20]. The six-dimensional case has also been considered in more general terms in ref. [21]. Our purpose here is to extend the results of ref. [19, 20] to higher orders, again for the multiflavor ϕ^3 theory. However, we shall do this by using the β-functions together with eq. (1.1) to construct the quantities A and T_{IJ} order by order (rather than by a direct perturbative computation). We shall compute the function A up to 5 loop order in the standard $\overline{\text{MS}}$ renormalisation scheme, requiring a knowledge of the three-loop $\overline{\text{MS}}$ β-function. We shall find that a solution for A and T_{IJ} is only possible if the β-function coefficients satisfy a set of consistency conditions, and we shall be able to show

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[1]See ref. [15] for an analogous extension to three dimensions; and ref. [16] for a recent explicit construction of an a-function in three dimensions. See also ref. [17, 18] for attempts to derive a weak a-theorem in six and general d dimensions using the methods of ref. [3, 4].

that these conditions are invariant under the coupling redefinitions which must relate any pair of renormalisation schemes. We illustrate the redefinition process using the example of the MOM (momentum subtraction) scheme. We accordingly present for the first time the three-loop β-functions for MOM (the three-loop $\overline{\text{MS}}$ β-functions may be read off from the results presented in ref. [22], although they were not written down explicitly there[2]). We shall also provide full details of the three-loop calculation for the $\overline{\text{MS}}$ β-functions, and then give a precise definition of the MOM scheme, explaining how the calculation may be adapted for this case. In the general case we shall be considering, the theory has a global symmetry; and just as in four dimensions, we shall find that at three loops the consistency conditions can only be satisfied if we replace β^I by the quantity B^I defined in ref. [7].

The layout of the paper is as follows: in section 2 we present the one and two loop results for the β-functions and also the lowest-order results for A and T_{IJ}. In section 3 we give an explanation of our computational methods and how they may be applied to the computation of the β-functions in both the $\overline{\text{MS}}$ and MOM schemes, and then go on to list the results for the three-loop $\overline{\text{MS}}$ β-function. In section 4 we present our results for $A^{(5)}$ together with consistency conditions on the β-function coefficients, which must be satisfied in any scheme in order for eq. (1.1) to hold. In section 5 we discuss the implementation of renormalisation scheme changes in general terms and then go on to focus on the case of the MOM scheme. We present our concluding remarks in section 6, and finally the explicit three-loop MOM β-function together with some calculational details are given in the appendices.

2 One and two-loop results

We consider the theory

$$L = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{3!}g^{ijk}\phi^i\phi^j\phi^k, \tag{2.1}$$

involving a multiplet of fields ϕ^i coupled via the tensor g^{ijk}. The one and two loop β-functions are given by

$$\begin{aligned}
\beta^{(1)} &= -g_{(1a)} + \frac{1}{12}g_{(1A)}, \\
\beta^{(2)} &= c_{(2B)}g_{(2B)} + c_{(2C)}g_{(2C)} + c_{(2b)}g_{(2b)} + c_{(2c)}g_{(2c)} + c_{(2d)}g_{(2d)},
\end{aligned} \tag{2.2}$$

where the tensor structures are defined by

$$\begin{aligned}
g_{(1a)}^{ijk} &= g^{ilm}g^{jmn}g^{knl}, & g_{(2b)}^{ijk} &= g^{jpq}g^{kpr}g_{(1a)}^{iqr}, \\
g_{(2c)}^{ijk} &= g^{ipr}g^{jpq}g_{1A}^{qs}g^{ksr}, & g_{(2d)}^{ijk} &= g^{imn}g^{jpq}g^{krs}g^{nqs}g^{mpr},
\end{aligned} \tag{2.3}$$

with also

$$g_{(1A)}^{ij} = g^{ikl}g^{jkl}, \quad g_{(2B)}^{ij} = g^{ipq}g_{(1a)}^{jpq}, \quad g_{(2C)}^{ij} = g^{imn}g^{jmq}g_{(1A)}^{nq}. \tag{2.4}$$

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[2]Earlier three-loop results were presented in ref. [23, 24], but these do not determine the results for the general theory unambiguously, as we shall explain later.
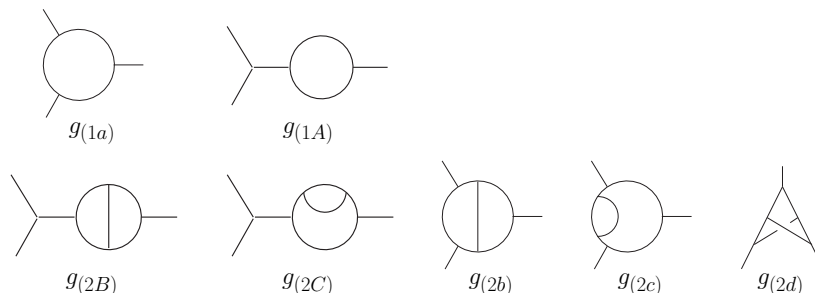
Table 1. Tensor structures appearing in the one- and two-loop β-functions.

We also define 3-index quantities corresponding to the 2-index quantities of eq. (2.4) by

$$g_{(1A)}^{ijk} = g_{(1A)}^{il} g^{ljk}, \quad \text{etc.} \tag{2.5}$$

We have therefore given the same label to both a two-index and a three-index tensor, but we hope that it will always be clear from the context which is meant. For structures which are not three-fold symmetric, we list one symmetrisation but it is to be understood that it is accompanied by its two symmetrised partners. We shall also, wherever possible, suppress indices as we have done in eq. (2.2). We display the tensor structures appearing in eq. (2.2) in table 1 (in which the index i is always at the right, except for $g_{(2d)}$, which is completely symmetric in i, j and k).

The coefficients in eq. (2.2) are given in $\overline{\text{MS}}$ by

$$c_{(2B)} = \frac{1}{18}, \quad c_{(2C)} = -\frac{11}{432}, \quad c_{(2b)} = -\frac{1}{4}, \quad c_{(2c)} = \frac{7}{72}, \quad c_{(2d)} = -\frac{1}{2}. \tag{2.6}$$

Here and elsewhere we suppress a factor of $(64\pi^3)^{-1}$ for each loop order.

It was shown in refs. [19, 20] that eq. (1.1) was valid at leading order with

$$A^{(3)} = -\frac{1}{4}\lambda g^{ijk}(g_{(1a)}^{ijk} - \frac{1}{4}g_{(1A)}^{ijk}) = -\frac{1}{4}\lambda(g^{klm}g^{knp}g^{lpq}g^{mqn} - \frac{1}{4}g_{(1A)}^{mn}g_{(1A)}^{mn}),$$
$$T_{IJ}^{(2)} = G_{IJ}^{(2)} = \lambda\delta_{IJ}, \tag{2.7}$$

and

$$\lambda = -\frac{1}{3240}. \tag{2.8}$$

As mentioned earlier, our definition of A differs from that of $\tilde{a}$ in ref. [14] by a factor of 6, introduced for convenience. As the notation implies, $G_{IJ}^{(2)}$ would require a two-loop perturbative calculation using the methods of ref. [6, 7], which was performed explicitly in refs. [19, 20]; $A^{(3)}$ would correspondingly require a three-loop calculation, but its value was inferred by imposing eq. (1.1). This is the technique we shall apply to obtain $A^{(4)}$, $A^{(5)}$ later in this paper. We should note that A has no explicit two-loop contributions; however there is a one-loop (free-field) contribution [19, 20].[3] At this point we have all the information required for our computation of $A^{(4)}$, but we shall postpone this to section 4 where we shall explain the general method we shall use to compute both $A^{(4)}$ and $A^{(5)}$.

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[3]See refs. [25–27] for earlier perturbative calculations on six-dimensional curved spacetime.

3 Three-loop results

In this section we shall give our explicit results for the three-loop β-functions for the theory eq. (2.1), computed using $\overline{\text{MS}}$. We start by describing the computational methods used,[4] and we shall also take the opportunity to describe the MOM scheme (which will feature in later sections) and how to adapt our methods to obtain the MOM β-functions. The reader uninterested in the technical details of the Feynman diagram calculations may skip to the paragraph containing eq. (3.6).

We base our calculations on the original work of refs. [23, 24], although our couplings are more general than those articles. However, one can always make contact with the results of refs. [23, 24] by setting $g^{ijk} = gd^{ijk}$ where g was the coupling constant and d^{ijk} was a group valued object which is completely symmetric in its indices like g^{ijk}. For example, if ϕ^i took values in the adjoint representation of $\text{SU}(N_c)$ then d^{ijk} would be the corresponding totally symmetric rank 3 colour tensor of that group. From the point of view, however, of constructing the RG functions it is the evaluation of the underlying Feynman integrals which is required. For this aspect it is appropriate to focus for the moment on the basic ϕ^3 Lagrangian

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{6} \phi^3 \ . \tag{3.1}$$

Previously this was considered in refs. [23, 24, 29] to three loops and the wave function and coupling constant renormalisation constants were deduced from the divergences in the 2- and 3-point functions. For the latter the pole structure was determined by exploiting specific properties of the six dimensional spacetime. Briefly it was possible to nullify the external momentum of one of the legs of the 3-point functions to reduce the evaluation of the graph to a 2-point function. Such integrals are more straightforward to determine through knowledge of the 2-point evaluation. The point is that ordinarily the nullification of an external momentum leads to infrared problems. For instance, a propagator in a Feynman integral of the form $1/(k^2)^2$ leads immediately to infrared singularities in the associated four-dimensional integral where k is the loop momentum. In six dimensions, however, this is not the case. The corresponding propagator where such an infrared issue would appear in that dimension is $1/(k^2)^3$. Therefore, the nullification used in ref. [24], which is a simple application of the infrared rearrangement technique, is perfectly valid in six dimensions for eq. (2.1) and eq. (3.1). Moreover, this method is sufficient to determine the RG functions in the $\overline{\text{MS}}$ scheme.

However, as our focus here will not be restricted to $\overline{\text{MS}}$ but will include the momentum subtraction (MOM) scheme we will have to determine the Feynman integrals contributing to the vertex function for a non-nullified external momentum and to the finite part in ϵ (where $d = 6 - 2\epsilon$). Ahead of the description of the computational tools we use it is appropriate to recall the definition of the MOM scheme as this informs the integral evaluation. The particular MOM scheme we use is that developed in ref. [30] for Quantum Chromodynamics. However, we note that it was used prior to that at three loops for eq. (2.1) in

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[4]Similar methods have recently been applied to the computation of the four-loop $\overline{\text{MS}}$ β-function for this theory [28].

ref. [31] when the specific coupling tensor corresponded to an $SU(3) \times SU(3)$ symmetry group. First, we recall that in the minimal subtraction scheme the renormalisation constants are defined at a subtraction point in such a way that only the poles with respect to the regularizing parameter are included in the renormalisation constant. The $\overline{\text{MS}}$ scheme is a variant on this where a specific finite part is also absorbed into the renormalisation constants. This additional number, which is $\ln(4\pi e^{-\gamma})$ where γ is the Euler-Mascheroni constant, in effect corresponds to a trivial rescaling of the coupling constant. The renormalisation constants of the MOM scheme by contrast are defined at a particular subtraction point in such a way that after renormalisation at that point there are no $O(g)$ contributions to the Green's functions [30]. In other words the Green's functions are set to their tree values at the subtraction point [30]. The specific subtraction point used in ref. [30] for the 3-point vertex functions is that where the squared momenta of all the external legs are equal. Moreover, they are set equal to $(-\mu^2)$ where μ is the mass scale introduced to ensure the coupling constant or tensor is dimensionless in general d dimensions. To achieve this for eq. (2.1) requires evaluating all the 2-point function and vertex graphs to the finite parts in ϵ; the latter being a more involved computation than the former. While this is the canonical definition of the MOM scheme we note that one can have variations on it. For instance, it is acceptable to have a scheme where the renormalisation constants associated with 2-point functions are defined in an $\overline{\text{MS}}$ way but the coupling-constant renormalisations are determined using the MOM definition or vice versa. While it is possible to study such hybrid schemes in order to consider different applications of our general formalism, we will concentrate here purely on the $\overline{\text{MS}}$ and MOM schemes. However, we will provide enough details in the evaluation of the symmetric-point 3-point integrals to allow an interested reader to explore these hybrid schemes independently. In mentioning that a MOM scheme had been considered in ref. [31] we need to clarify this in light of our discussion. The renormalisation of ref. [31] was at three loops in a MOM scheme but for a very specific coupling tensor. The consequence of the choice of the $SU(3) \times SU(3)$ colour group is that there were no one or two loop vertex graphs to be evaluated at the symmetric point. At three loops due to the special symmetry properties there was only one graph to evaluate at the symmetric point, and since it was primitively divergent the evaluation was straightforward [31]. Therefore, in considering the general cubic theory in six dimensions eq. (2.1) we are filling in the gap in the computation of the lower loop integrals for the momentum subtraction scheme analysis.

One of our aims is the study of the a-theorem up to third order in various RG schemes; it may appear a formidable task to actually *calculate* the MOM β-function for eq. (2.1) at this order given the previous discussion. However, it is possible to determine it purely from the *two* loop renormalisation of the vertex functions in this scheme. To illustrate the process, we consider for the moment the simpler theory eq. (3.1), postponing the general case of eq. (2.1) to section 5. After choosing a renormalisation scheme for a given theory, any RG quantity will be a function of the couplings in that scheme. However, the expressions in different schemes must be related and this is achieved by a conversion function. If we denote the coupling constant in one scheme by g and that in another by $\bar{g}$

then the conversion function $C_g(g)$ is defined by

$$C_g(g) = \frac{\partial \bar{g}(g)}{\partial g} \qquad (3.2)$$

where the bare coupling constant g_{o} is related to the two renormalized coupling constants by

$$g_{\mathrm{o}} = \mu^\epsilon Z_g \, g = \mu^\epsilon \bar{Z}_g \, \bar{g} \; . \qquad (3.3)$$

$\bar{g}(g)$ has the form

$$\bar{g}(g) = g + \sum_{n=1}^{\infty} h_n g^{2n+1} \qquad (3.4)$$

where the coefficients h_n are related to the finite parts of the Z_g and $\bar{Z}_g$. Once the expansion has been established from the explicit renormalisation then the respective β-functions are related by

$$\bar{\beta}(\bar{g}) = [\beta(g)C_g(g)]_{g \to \bar{g}} \qquad (3.5)$$

where the mapping means that the coupling constant g is mapped back to $\bar{g}$ via the inverse of eq. (3.4). Now returning to our problem of computing the MOM β-function, if $\bar{g}$ is the MOM coupling constant and g is that in the $\overline{\text{MS}}$ scheme, it is clear from eq. (3.5) that only h_1 and h_2 are required to find $\bar{\beta}(\bar{g})$ at three loops . These may be derived from the finite part of $\bar{Z}_g$ up to two loops which in turn derives from the finite parts of the one and two-loop vertex functions at the symmetric subtraction point. The coupling redefinition required to generate the three-loop MOM β-function will be presented in section 5, and the three-loop MOM β-function itself is given in appendix A.

We now turn to the algorithm we used to evaluate the two and three loop Feynman integrals to the requisite orders in ϵ to determine the RG functions in $\overline{\text{MS}}$ and MOM. The method we use is to apply the Laporta algorithm [32] to the two and three loop graphs contributing to the 2-point and vertex functions. This method systematically integrates by parts all the graphs in such a way that they are algebraically reduced to a basic set of what is termed master integrals. Then the ϵ-expansions of the latter are substituted to complete the computation. The masters have to be determined by direct methods or one which does not use integration by parts and this is the more demanding aspect of the calculation. The version of the Laporta algorithm which we used was REDUZE [33, 34]. To handle the surrounding tedious algebra we used the symbolic manipulation language FORM [35, 36]. The whole evaluation proceeded automatically by generating all the Feynman diagrams electronically with the QGRAF package [37]. While we have summarised what is now a standard procedure to carry out multiloop Feynman graph evaluation, the novel feature here is finding a method to access the six dimensional master integrals. The most straightforward way to proceed is to realise that if the problem was in four dimensions then the corresponding masters are already known. For instance, the three loop 2-point functions were developed for the MINCER package [38] which was encoded in FORM in ref. [39]. Equally the two loop 3-point function masters were determined over a period of years in refs. [40–44]. While the algorithms which were developed to determine such four-dimensional masters could in principle be extended to six dimensions, in practice this

would be tedious. Instead we have used the Tarasov method developed in refs. [45, 46]. This allows one to relate a scalar (master) integral in d-dimensions to integrals with the same propagator topology in $(d+2)$ dimensions. The caveat is that the integrals in the higher-dimensional case have increased powers of their propagators. However, the Laporta algorithm [32] can be applied to them in order to reduce them to the corresponding master in that dimension and integrals which involve masters where the number of propagators in the topology have been reduced. In summary a four dimensional master can be related to its unknown six-dimensional cousin plus already determined lower-level six-dimensional masters. Hence one can algebraically solve the system for the masters to the order in ϵ which they are required for the 2-point and vertex functions. One feature which invariably arises in the use of integration by parts is the appearance of spurious poles. Consequently in the determination of the masters in six dimensions as well as the reduction of the Feynman graphs contributing to the RG functions, one sometimes has to evaluate the masters *beyond* the finite part in ϵ. In the appendix we record the expressions for the nontrivial one and two loop 3-point masters; though we stress that the expressions given there are for the pure integral. There has been no subtraction of subgraphs as was the case in the results presented for graphs in refs. [22, 24]. This is primarily because in the automatic FORM programmes to determine the RG functions we use the algorithm of ref. [47] to implement the renormalisation automatically in our two schemes. Briefly this is achieved by performing the calculation in terms of the bare coupling constant or tensor. Then what would be termed counterterms are introduced automatically by rescaling the bare parameter to the renormalized one. The so-called constant of proportionality would ordinarily correspond to the coupling constant renormalisation constant which therefore appends the necessary counterterms. Finally, the definition of the renormalisation constant is implemented at this last stage. At whatever loop order one is working to, the remaining undetermined counterterm is defined according to whether the scheme is $\overline{\text{MS}}$ or MOM.

We can now present our three-loop results for $\overline{\text{MS}}$. At this order we require several new tensors. Those contributing to the one-particle irreducible (1PI) terms in the β-function are

$$
\begin{aligned}
g_{(3e)}^{ijk} &= g_{(2b)}^{ilm} g_{22}^{jlkm}, & g_{(3f)}^{ijk} &= g_{(2b)}^{lmi} g_{22}^{jlkm}, & g_{(3g)}^{ijk} &= g^{ipq} g_{(1a)}^{jpr} g_{(1a)}^{kqr}, \\
g_{(3h)}^{ijk} &= g_{(1a)}^{ilm} g_{(1A)}^{nq} g^{jln} g^{kmq}, & g_{(3i)}^{ijk} &= g_{(1A)}^{pq} g_{(1a)}^{ipr} g_{22}^{jqkr}, & g_{(3j)}^{ijk} &= g^{ilm} g^{jln} g^{kmq} g_{(2B)}^{nq}, \\
g_{(3k)}^{ijk} &= g_{(2c)}^{mil} g_{22}^{jlkm}, & g_{(3l)}^{ijk} &= g_{(2c)}^{ilm} g_{22}^{jlkm}, & g_{(3m)}^{ijk} &= g^{iln} g^{jmq} g^{klm} g_{(2C)}^{nq}, \\
g_{(3n)}^{ijk} &= g^{iln} g^{jmq} g^{klm} g_{(2D)}^{nq}, & g_{(3o)}^{ijk} &= g_{(1A)}^{pq} g_{(1A)}^{rs} g_{22}^{jpkr} g^{iqs}, & g_{(3p)}^{ijk} &= g^{ist} g_{(1A)}^{qt} g_{22}^{jrps} g_{22}^{kpqr}, \\
g_{(3q)}^{ijk} &= g_{(1a)}^{irs} g_{22}^{jqps} g_{22}^{kpqr}, & g_{(3r)}^{ijk} &= g^{ipq} g^{jrs} g_{22}^{knps} g_{(1a)}^{rnq}, & g_{(3s)}^{ijk} &= g_{(2d)}^{ilm} g_{22}^{jlkm}, \\
g_{(3t)}^{ijk} &= g^{ipq} g_{22}^{jrps} g_{22}^{knrl} g_{22}^{nqls}, & g_{(3u)}^{ijk} &= g^{kpq} g_{22}^{jrsn} g_{22}^{ispl} g_{22}^{rlqn},
\end{aligned}
\tag{3.6}
$$

where

$$
g_{22}^{ijkl} = g^{ijm} g^{klm}, \quad g_{(2D)}^{ij} = g_{(1A)}^{im} g_{(1A)}^{mj},
\tag{3.7}
$$

and $g_{(1a)}$, $g_{(1A)}$ etc are defined in eqs. (2.3), (2.4). The tensors contributing to the anoma-

lous dimension are

$$g_{(3D)}^{ij} = g^{ipq}g_{(2b)}^{qjp}, \qquad g_{(3E)}^{ij} = g^{ipq}g_{(2b)}^{jpq}, \qquad g_{(3F)}^{ij} = g^{ipq}g_{(2c)}^{jpq},$$

$$g_{(3G)}^{ij} = g^{imn}g_{(1A)}^{mp}g_{(1a)}^{pnj}, \qquad g_{(3G')}^{ij} = g_{(1a)}^{inp}g_{(1A)}^{mp}g^{jmn}, \qquad g_{(3H)}^{ij} = g_{22}^{ipjq}g_{(2B)}^{pq},$$

$$g_{(3I)}^{ij} = g^{ipq}g_{(2d)}^{jpq}, \qquad g_{(3J)}^{ij} = g^{imn}g_{(1A)}^{mp}g_{(1A)}^{nq}g^{jpq}, \qquad g_{(3K)}^{ij} = g_{22}^{ipjq}g_{(2C)}^{pq},$$

$$g_{(3L)}^{ij} = g_{22}^{injq}g_{(2D)}^{nq}. \tag{3.8}$$

The notation for the two-index tensors in eq. (3.8) matches the diagrams in figure 7 of ref. [22], so that $g_{(3D)}^{ij}$ corresponds to the tensor structure of figure 7(d), and so on. The notation for the three-index tensors $g_{(3e)}^{ijk} \ldots g_{(3u)}^{ijk}$ in eq. (3.6) similarly matches the diagrams in figures 8 and 9 of ref. [22]; and furthermore, the indices i, j, k are arranged to run anti-clockwise around the diagram, with i at the top. Finally, $g_{(3D)}^{ijk} \ldots g_{(3L)}^{ijk}$, as defined in eq. (2.5) in terms of $g_{(3D)}^{ij} \ldots g_{(3L)}^{ij}$, correspond to three-point diagrams with an insertion of the corresponding wave-function renormalisation diagram of figure 7 of ref. [22].

As in section 2, we write the three-loop β-function as

$$\beta^{(3)} = c_{(3e)}g_{(3e)} + \ldots + c_{(3u)}g_{(3u)} + c_{(3D)}g_{(3D)} + \ldots + c_{(3L)}g_{(3L)}. \tag{3.9}$$

The coefficients in eq. (3.9) corresponding to anomalous dimension contributions are given by

$$c_{(3D)} = \frac{7}{864}, \qquad c_{(3E)} = \frac{71}{1728}, \qquad c_{(3F)} = -\frac{103}{10368}, \qquad c_{(3G)} = c_{(3G')} = -\frac{1}{108},$$

$$c_{(3H)} = -\frac{121}{5184}, \qquad c_{(3I)} = \frac{7}{96} - \frac{1}{24}\zeta(3), \qquad c_{(3J)} = \frac{23}{62208}, \qquad c_{(3K)} = \frac{103}{7776},$$

$$c_{(3L)} = -\frac{13}{31104}, \tag{3.10}$$

where $\zeta(z)$ is the Riemann ζ-function, and the remaining coefficients are given by

$$c_{(3e)} = -\frac{3}{8}, \qquad c_{(3f)} = \frac{1}{4},$$

$$c_{(3g)} = \frac{5}{16}, \qquad c_{(3h)} = -\frac{47}{864}, \qquad c_{(3i)} = -\frac{47}{432}, \qquad c_{(3j)} = \frac{23}{288},$$

$$c_{(3k)} = \frac{5}{27}, \qquad c_{(3l)} = \frac{11}{216}, \qquad c_{(3m)} = -\frac{19}{324}, \qquad c_{(3n)} = \frac{11}{1728},$$

$$c_{(3o)} = \frac{11}{1728}, \qquad c_{(3p)} = \frac{11}{144}, \qquad c_{(3q)} = -\frac{1}{16}, \qquad c_{(3r)} = -\frac{23}{24} + \zeta(3),$$

$$c_{(3s)} = -\frac{29}{48} + \frac{1}{2}\zeta(3), \qquad c_{(3t)} = -1, \qquad c_{(3u)} = \frac{1}{3} - \zeta(3). \tag{3.11}$$

We have computed all the coefficients in eqs. (3.10), (3.11) explicitly and independently; and we have checked that we reproduce the wave function and β-function results of refs. [22] and [23, 24] (in the latter case, after we specialize to the corresponding restriction on the group theory structure used there). Although in ref. [22] the final β-function results are given for two particular theories, the general results can be constructed from the individual diagrammatic results. This is largely the case for ref. [23, 24] too; however, the results for the pairs $\{c_{(3e)}, c_{(3f)}\}$, $\{c_{(3k)}, c_{(3l)}\}$, and $\{c_{(3D)}, c_{(3E)}\}$, are presented together and cannot be separated.
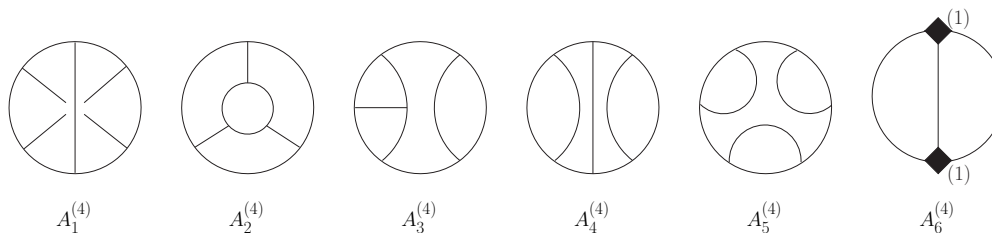
Table 2. Contributions to $A^{(4)}$.

4 The a-function beyond leading order

We now turn to the derivation of $A^{(4)}$, $A^{(5)}$. At two and three loops, we have avoided any diagrammatic computations using the methods of ref. [6, 7], as explored in the six-dimensional context in refs. [19, 20]; instead we have proceeded to infer A by imposing eq. (1.1). Beyond leading order (corresponding to eq. (2.7)) we need to take into account potential higher order corrections to T_{IJ}. We suppress the details here but the calculation proceeds along similar lines to those presented in full in the four-dimensional case in refs. [48, 49]; a similar method was used in the pioneering work of Wallace and Zia [50]. At next-to-leading order the general form of the a-function is given by

$$A^{(4)} = \lambda \left[-\frac{1}{12} A_1^{(4)} + a_2 A_2^{(4)} + a_3 A_3^{(4)} + a_4 A_4^{(4)} + a_5 A_5^{(4)} + \alpha_1 \beta^{(1)ijk} \beta^{(1)ijk} \right], \qquad (4.1)$$

where λ is defined in eq. (2.8). The individual contributions to $\tilde{A}^{(4)}$, depicted above in table 2, are given by

$$\begin{aligned}
A_1^{(4)} &= g^{ijk} g_{(2d)}^{ijk}, & A_2^{(4)} &= g^{ijk} g_{(2b)}^{kij}, & A_3^{(4)} &= g^{ijk} g_{(2c)}^{ijk}, \\
A_4^{(4)} &= g^{ijk} g_{(2C)}^{ijk}, & A_5^{(4)} &= g_{(1A)}^{ij} g_{(1A)}^{jk} g_{(1A)}^{ki},
\end{aligned} \qquad (4.2)$$

with the tensor structures again defined in eqs. (2.3), (2.4), (2.5), and (from eq. (2.2))

$$\beta^{(1)ijk} \beta^{(1)ijk} = A_2^{(4)} - \frac{1}{2} A_3^{(4)} + \frac{1}{24} A_4^{(4)} + \frac{1}{48} A_5^{(4)}. \qquad (4.3)$$

Correspondingly the tensor T_{IJ} in eq. (1.1) is automatically symmetric at this order (so that $G_{IJ} = T_{IJ}$) and may be written in the form

$$T_{ijk,lmn}^{(3)} = \sum_{\alpha=1}^{4} t_\alpha^{(3)} \left(T_\alpha^{(3)} \right)_{ijk,lmn}, \qquad (4.4)$$

where the individual structures which may arise are depicted in table 3. Here the diagrams represent $t_\alpha^{(3)} (T_\alpha^{(3)})_{ijk,lmn} \beta^{ijk} (dg)^{lmn}$ for $\alpha = 1 \ldots 4$. A cross denotes $(dg)^{ijk}$ and a diamond represents β^{ijk}.

A careful analysis leads to a system of linear equations whose solution imposes a single consistency condition on the β-function coefficients:

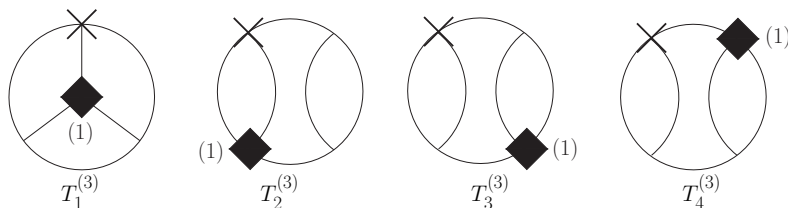$$6c_{(2C)} + c_{(2c)} + c_{(2B)} = 0. \qquad (4.5)$$

Table 3. Next-to-leading-order metric terms $T^{(3)}$.

This is satisfied by the $\overline{\text{MS}}$ coefficients as given by eq. (2.6). Similar integrability conditions (on three-loop β-function coefficients) were found in ref. [48]. In eq. (4.1), α_1 is arbitrary, reflecting the general freedom expressed in eq. (1.3); and in particular for $\overline{\text{MS}}$ we have

$$a_2 = -\frac{1}{8} + \alpha_1, \quad a_3 = \frac{7}{48} - \frac{1}{2}\alpha_1, \quad a_4 = -\frac{7}{288} + \frac{1}{24}\alpha_1, \quad a_5 = -\frac{5}{864} + \frac{1}{48}\alpha_1. \quad (4.6)$$

Our methods therefore specify A up to the freedom expressed in eq. (1.3). The metric coefficients in eq. (4.4) are given by

$$t_2^{(3)} = -\frac{7}{24}\lambda + \frac{1}{2}\alpha_1, \quad t_3^{(3)} + t_4^{(3)} = -\frac{1}{8}\lambda + \alpha_1, \quad t_1^{(3)} = -6\alpha_1, \quad (4.7)$$

where once more λ is defined in eq. (2.8). The metric coefficients therefore also reflect the freedom expressed in eq. (1.3); but there is an additional arbitrariness since only the combination $t_3^{(3)} + t_4^{(3)}$ is determined.

At the next order (corresponding to the three-loop β-function) we need to face the possibility that β^I in eq. (1.1) should be replaced by a generalisation B^I due to the invariance of the Lagrangian eq. (2.1) under $O(N)$ transformations of the real fields ϕ^i. It was shown in ref. [7] that in this situation, in the general case with couplings g^I we have

$$\beta^I \rightarrow B^I = \beta^I - (vg)^I \quad (4.8)$$

where v is an element of the Lie algebra of the symmetry group. In the case at hand, this corresponds to

$$\beta^{ijk} \rightarrow B^{ijk} = \beta^{ijk} - v^{l(i}g^{jk)l} \quad (4.9)$$

where v is an antisymmetric matrix. In principle v could be computed using similar methods to those described in ref. [7] and carried out explicitly in the four-dimensional case in ref. [48]; but it is clear *a priori* that the relevant tensor structures are the same as those appearing in the three-loop anomalous dimension. Since most of those are symmetric, the only possible 1PI contributions to v correspond to

$$v = c_{(3G)}^v (g_{(3G)} - g_{(3G')}) \quad (4.10)$$

with $g_{(3G)}$, $g_{(3G')}$ as defined in eq. (3.8).

The solution of eq. (1.1) leads to a complex system of linear equations whose solution imposes several consistency conditions on the β-function coefficients:

$$c_{(3q)} - c_{(3s)} - 12c_{(3I)} = -6c_{(2B)},$$

$$c_{(3r)} - 2c_{(3s)} + 12c_{(3p)} = 12c_{(2c)},$$

$$c_{(3e)} - c_{(3g)} - 24c_{(3h)} - 144c_{(3o)} - 72Z = -3(c_{(2B)} + 2c_{(2c)}),$$

$$c_{(3e)} - c_{(3g)} - 6c_{(3i)} + 6c_{(3k)} + 72Z = 3c_{(2B)} + 144c_{(2c)}^2,$$

$$c_{(3j)} + 12c_{(3n)} - c_{(3E)} + 12c_{(3H)} + 36c_{(3K)} + 72c_{(3L)}$$
$$-12(c_{(3\rho)} + c_{(3\sigma)}) - 72(c_{(3\tau)} + c_{(3\chi)}) + 6Z = 12(2c_{(2c)}^2 + 2c_{(2c)}c_{(2B)} - c_{(2B)}^2),$$

$$2c_{(3h)} + 6c_{(3m)} - 12c_{(3n)} + 18c_{(3o)} + c_{(3D)} + 12c_{(3F)} + 72c_{(3J)}$$
$$+36c_{(3K)} - 72c_{(3L)} = \frac{11}{144}[1 + 24(c_{(2B)} - c_{(2c)})],$$

$$c_{(3e)} - \frac{1}{2}c_{(3f)} + 6c_{(3k)} - 12c_{(3l)} = 0,$$

$$c_{(3j)} + 6c_{(3m)} + 6c_{(3H)} + 36c_{(3K)} = 12c_{(2B)}c_{(2c)},$$

$$c_{(3h)} - c_{(3i)} + c_{(3l)} - c_{(3D)} - 12c_{(3F)} + 12Z = 12c_{(2c)}(c_{(2B)} + 2c_{(2c)}), \quad (4.11)$$

and

$$12c_{(3G)}^v + 6(c_{(3G)} - c_{(3G')}) + 12(c_{(3\rho)} - c_{(3\sigma)}) + 72(c_{(3\tau)} - c_{(3\chi)}) = c_{(3j)} + 6c_{(3m)} + 12c_{(2c)}^2, \quad (4.12)$$

where

$$Z = c_{(3G)} + c_{(3G')} - c_{(3o)} + \frac{1}{6}c_{(3E)} - 2c_{(3F)} + 12c_{(3J)}. \quad (4.13)$$

We have included in our calculations potential contributions to the three-loop β-functions defined according to eq. (2.5) in terms of one-particle-reducible (1PR) anomalous dimension structures given by

$$g_{(3\rho)}^{ij} = g_{(2B)}^{il}g_{(1A)}^{lj}, \quad g_{(3\sigma)}^{ij} = g_{(1A)}^{il}g_{(2B)}^{lj}, \quad g_{(3\tau)}^{ij} = g_{(2C)}^{ik}g_{(1A)}^{kj}, \quad g_{(3\chi)}^{ij} = g_{(1A)}^{ik}g_{(2C)}^{kj}, \quad (4.14)$$

using again tensors defined earlier in eq. (2.4). Such contributions cannot of course arise in $\overline{\text{MS}}$ but are potentially present in other schemes, in particular the MOM scheme which we shall be considering later as an example. (There are other potential 1PR contributions in a general scheme, but we have included only the ones which are relevant for MOM.) In deriving eqs. (4.11), (4.12), we have imposed the two-loop consistency condition eq. (4.5) and also used the $\overline{\text{MS}}$ values of eq. (2.6) for all two-loop β function coefficients except $c_{(2B)}$ and $c_{(2c)}$, since as we shall see later (in eq. (5.5)), these are the only scheme-dependent values. The conditions in eq. (4.11) are readily checked to be satisfied by the $\overline{\text{MS}}$ three-loop β-function coefficients given in eq. (3.10), (3.11); however, eq. (4.12) may only be satisfied within $\overline{\text{MS}}$ by taking a non-zero value of $c_{(3G)}^v$, namely

$$\left(c_{(3G)}^v\right)_{\overline{\text{MS}}} = -\frac{137}{10368}. \quad (4.15)$$

We shall assume this value from now on, though of course it would be reassuring to compute it directly along the lines of ref. [13].

Given that the consistency conditions are satisfied, we may solve eq.(1.1) for the a-function. With the three-loop $\overline{\text{MS}}$ coefficients we find

$$A^{(5)} = \sum_{i=1}^{16} a_i^{(5)} A_i^{(5)},\tag{4.16}$$

where

$$
\begin{aligned}
A_1^{(5)} &= g^{ijk} g_{(3I)}^{ijk} = g^{ijk} g_{(3p)}^{ijk}, & A_9^{(5)} &= g^{ijk} g_{(3M)}^{ijk} = g^{ijk} g_{(3N)}^{ijk} = g^{ijk} g_{(3n)}^{ijk}, \\
A_2^{(5)} &= g^{ijk} g_{(3q)}^{ijk} = g^{ijk} g_{(3r)}^{ijk} = g^{ijk} g_{(3s)}^{ijk}, & A_{10}^{(5)} &= g^{ijk} g_{(3j)}^{ijk}, \\
A_3^{(5)} &= g^{ijk} g_{(3u)}^{ijk}, & A_{11}^{(5)} &= g^{ijk} g_{(3H)}^{ijk} = g^{ijk} g_{(3m)}^{ijk}, \\
A_4^{(5)} &= g_{(1A)}^{ij} g_{(1A)}^{jk} g_{(1A)}^{kl} g_{(1A)}^{li}, & A_{12}^{(5)} &= g^{ijk} g_{(3e)}^{ijk} = g^{ijk} g_{(3f)}^{ijk} = g^{ijk} g_{(3g)}^{ijk}, \\
A_5^{(5)} &= g^{ijk} g_{(3L)}^{ijk} = g^{ijk} g_{(3P)}^{ijk} = g^{ijk} g_{(3Q)}^{ijk}, & A_{13}^{(5)} &= g^{ijk} g_{(3D)}^{ijk} = g^{ijk} g_{(3h)}^{ijk} = g^{ijk} g_{(3k)}^{ijk} = g^{ijk} g_{(3l)}^{ijk}, \\
A_6^{(5)} &= g^{ijk} g_{(3K)}^{ijk}, & A_{14}^{(5)} &= g^{ijk} g_{(3E)}^{ijk} = g^{ijk} g_{(3i)}^{ijk}, \\
A_7^{(5)} &= g^{ijk} g_{(3J)}^{ijk}, & A_{15}^{(5)} &= g^{ijk} g_{(3o)}^{ijk} = g^{ijk} g_{(3G)}^{ijk}, \\
A_8^{(5)} &= g^{ijk} g_{(3t)}^{ijk}, & A_{16}^{(5)} &= g^{ijk} g_{(3F)}^{ijk},
\end{aligned}
\tag{4.17}
$$

with the tensor structures $g_{(3p)}^{ijk}$ etc. as defined in eq. (3.6). The invariants $A_1^{(5)} \dots A_{16}^{(5)}$ are depicted in table 4. The coefficients $a_i^{(5)}$ in eq. (4.16) are given by

$$
\begin{aligned}
a_1^{(5)} &= \left(\frac{9}{64} - \frac{1}{16}\zeta(3)\right)\lambda - \frac{1}{4}\alpha_1, & a_9^{(5)} &= \frac{47}{1152}\lambda + \frac{1}{36}\alpha_1 + \frac{1}{144}\tilde{\alpha}_1 - \frac{1}{6}\tilde{\alpha}_2 - \frac{1}{6}\tilde{\alpha}_3, \\
a_2^{(5)} &= \left(-\frac{29}{48} + \frac{1}{2}\zeta(3)\right)\lambda + \alpha_1, & a_{10}^{(5)} &= -\frac{23}{576}\lambda - \frac{1}{3}\alpha_1 + \tilde{\alpha}_3, \\
a_3^{(5)} &= \left(\frac{1}{8} - \frac{3}{8}\zeta(3)\right)\lambda, & a_{11}^{(5)} &= -\frac{7}{128}\lambda + \frac{5}{24}\alpha_1 - \frac{1}{3}\tilde{\alpha}_3, \\
a_4^{(5)} &= -\frac{145}{82944}\lambda + \frac{1}{144}\tilde{\alpha}_2 + \frac{1}{144}\tilde{\alpha}_3, & a_{12}^{(5)} &= \frac{107}{96}\lambda + \frac{3}{2}\alpha_1 + \tilde{\alpha}_1, \\
a_5^{(5)} &= -\frac{5}{41472}\lambda - \frac{11}{864}\alpha_1 + \frac{1}{24}\tilde{\alpha}_2 + \frac{1}{36}\tilde{\alpha}_3, & a_{13}^{(5)} &= -\frac{5}{32}\lambda - \frac{5}{6}\alpha_1 - \frac{1}{3}\tilde{\alpha}_1, \\
a_6^{(5)} &= \frac{29}{2304}\lambda - \frac{11}{432}\alpha_1 + \frac{1}{36}\tilde{\alpha}_3, & a_{14}^{(5)} &= -\frac{35}{128}\lambda - \frac{1}{8}\alpha_1 - \frac{1}{6}\tilde{\alpha}_1 + \tilde{\alpha}_2, \\
a_7^{(5)} &= \frac{1}{72}\tilde{\alpha}_2, & a_{15}^{(5)} &= \frac{101}{3456}\lambda + \frac{7}{72}\alpha_1 + \frac{1}{24}\tilde{\alpha}_1 - \frac{1}{3}\tilde{\alpha}_2, \\
a_8^{(5)} &= -\frac{1}{8}\lambda, & a_{16}^{(5)} &= -\frac{5}{2304}\lambda + \frac{7}{144}\alpha_1 + \frac{1}{72}\tilde{\alpha}_1,
\end{aligned}
\tag{4.18}
$$

where λ is again defined in eq. (2.8) and we explicitly display the freedom as expressed in eq. (1.3)

$$
\begin{aligned}
A \to {}& A + 2\alpha_1 \beta^{(1)ijk} \beta^{(2)ijk} + \tilde{\alpha}_1 \beta^{(1)ijk} \beta^{(1)klm} g_{22}^{iljm} + \tilde{\alpha}_2 \beta^{(1)ikl} \beta^{(1)jkl} g_{(1A)}^{ij} \\
& + \tilde{\alpha}_3 \beta^{(1)ijk} \beta^{(1)lmn} g^{jkl} g^{imn},
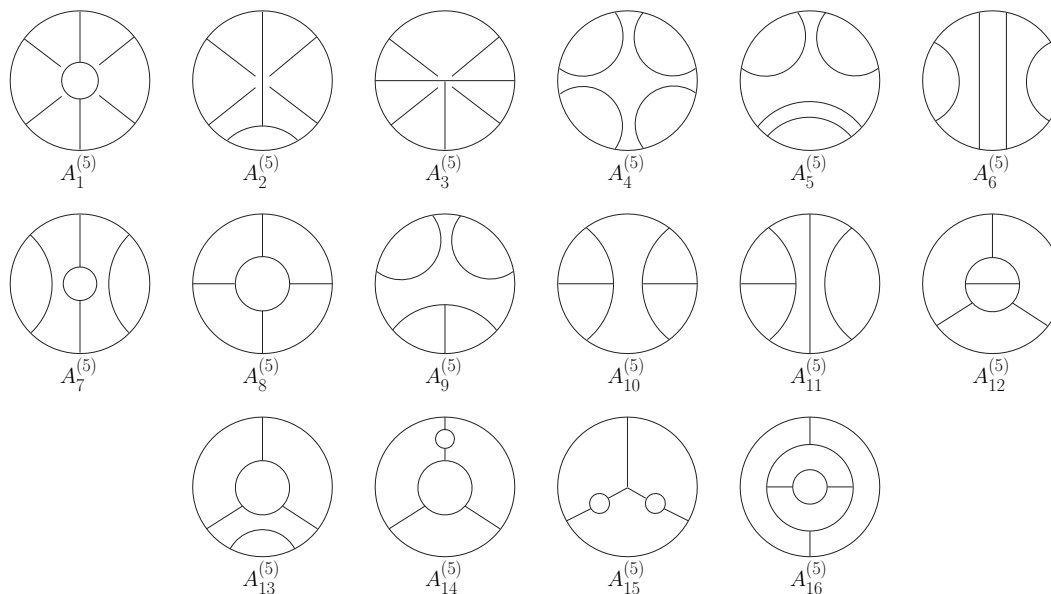\end{aligned}
\tag{4.19}
$$

Table 4. Contributions to $A^{(5)}$.

using the results of eqs. (2.2), (2.6). The terms with $\tilde{\alpha}_1$-$\tilde{\alpha}_3$ correspond to the terms with $T_1^{(3)}$-$T_3^{(3)}$ in table 3 (note that $T_3^{(3)}$, $T_4^{(3)}$ have the same effect at this order, as may be observed in eq. (4.7)).

At this loop order the tensor T_{IJ} is *not* inevitably symmetric. Analogously to the previous order, the metric may be expressed in the form

$$T_{ijk,lmn}^{(4)} = \sum_{\alpha=1}^{30} t_\alpha^{(4)} \left(T_\alpha^{(4)} \right)_{ijk,lmn}. \tag{4.20}$$

The structures which may arise are depicted in table 5, using a similar convention to the previous order. As before, a cross denotes $(dg)^{ijk}$ and a diamond represents β^{ijk}. We see immediately that $T_1^{(4)} \ldots T_{16}^{(4)}$ are individually symmetric; the remaining diagrams are grouped in pairs whose coefficients should be equal for symmetry. Solving eq. (1.1) certainly does not guarantee that $t_{17}^{(4)} = t_{18}^{(4)}$, $t_{19}^{(4)} = t_{20}^{(4)}$ etc. However it turns out that we can impose symmetry on $T_{IJ}^{(4)}$ provided the additional condition

$$c_{(3m)} - 2c_{(3n)} + \frac{1}{6}c_{(3E)} - c_{(3H)} - 12c_{(3L)} - Z = \frac{11}{36}(c_{(2B)} - 2c_{(2c)}) \tag{4.21}$$

is satisfied. The values in eq. (3.10), (3.11) do indeed satisfy this condition and we obtain a symmetric metric with the values

$$t_1^{(4)} = \frac{13}{8}\lambda - \frac{3}{2}\lambda\zeta(3) - 3\alpha_1 - t_3^{(4)},$$

$$t_2^{(4)} = \frac{1}{4}\lambda - 2\alpha_1,$$

$$t_4^{(4)} = -\frac{161}{48}\lambda + \frac{11}{2}\alpha_1 + 24\tilde{\alpha}_2,$$

$$t_5^{(4)} = -\frac{89}{24}\lambda + 11\alpha_1 + 48\tilde{\alpha}_2,$$

$$t_6^{(4)} = -\frac{31}{24}\lambda + 4\alpha_1 + 24\tilde{\alpha}_2,$$

$$t_7^{(4)} = -\frac{13}{24}\lambda + \frac{1}{3}\alpha_1 - 2\tilde{\alpha}_2,$$

$$t_8^{(4)} = \frac{49}{144}\lambda - \frac{7}{12}\alpha_1 - 4\tilde{\alpha}_2,$$

$$t_9^{(4)} = -\frac{11}{96}\lambda - 2\tilde{\alpha}_2,$$

$$t_{10}^{(4)} = \frac{1}{3}\tilde{\alpha}_2,$$

$$t_{11}^{(4)} = \frac{391}{1728}\lambda - \frac{11}{72}\alpha_1 + \frac{1}{3}\tilde{\alpha}_2,$$

$$t_{12}^{(4)} = \frac{11}{432}\lambda + \frac{4}{3}\tilde{\alpha}_2 - t_{13}^{(4)},$$

$$t_{14}^{(4)} = \frac{1}{192}\lambda + \frac{1}{3}\tilde{\alpha}_2,$$

$$t_{15}^{(4)} = -\frac{299}{1728}\lambda - t_{16}^{(4)} - 2t_{17}^{(4)} + \frac{11}{36}\alpha_1 + \frac{5}{3}\tilde{\alpha}_2,$$

$$t_{17}^{(4)} = t_{18}^{(4)},$$

$$t_{19}^{(4)} = t_{20}^{(4)} = -\frac{59}{72}\lambda + \frac{2}{3}t_4^{(3)} - 12(t_{16}^{(4)} + t_{17}^{(4)}) + \frac{11}{12}\alpha_1 + 6\tilde{\alpha}_2,$$

$$t_{21}^{(4)} = t_{22}^{(4)} = \frac{115}{288}\lambda - \frac{7}{6}\alpha_1 - 8\tilde{\alpha}_2,$$

$$t_{23}^{(4)} = t_{24}^{(4)} = \frac{73}{48}\lambda - \frac{2}{3}t_4^{(3)} + 12(t_{16}^{(4)} + t_{17}^{(4)}) - \frac{25}{12}\alpha_1 - 12\tilde{\alpha}_2,$$

$$t_{25}^{(4)} = t_{26}^{(4)} = \frac{101}{288}\lambda - \frac{7}{12}\alpha_1 - 6\tilde{\alpha}_2,$$

$$t_{27}^{(4)} = t_{28}^{(4)} = \frac{373}{1728}\lambda - \frac{11}{36}t_4^{(3)} + 2(t_{16}^{(4)} + t_{17}^{(4)}) - \frac{11}{72}\alpha_1 - \tilde{\alpha}_2,$$

$$t_{29}^{(4)} = t_{30}^{(4)} = -\frac{11}{48}\lambda + \frac{11}{36}t_4^{(3)} - 2(t_{16}^{(4)} + t_{17}^{(4)}) + \frac{11}{72}\alpha_1 + 2\tilde{\alpha}_2. \tag{4.22}$$

The values of $t_3^{(4)}$, $t_{13}^{(4)}$, $t_{16}^{(4)}$, $t_{17}^{(4)}$ remain arbitrary, in a similar fashion to the previous order where in eq. (4.7) only $t_3^{(3)} + t_4^{(3)}$ was determined. Before imposing symmetry, the $T_{IJ}^{(4)}$ coefficients would also display the freedom eq. (1.3) as expressed at this order in eq. (4.19); however, the general redefinition eq. (4.19) is not compatible with symmetry of $T_{IJ}^{(4)}$. Nevertheless, as may be seen in eq. (4.22), there is still a residual two-parameter freedom expressed in α_1, $\tilde{\alpha}_2$, corresponding to choosing

$$\tilde{\alpha}_1 = -12\tilde{\alpha}_2 - \frac{7}{2}\alpha_1, \quad \tilde{\alpha}_3 = 2\tilde{\alpha}_2 + \frac{11}{24}\alpha_1. \tag{4.23}$$

It is worth remarking that the freedom in α_1 is in general only preserved in the symmetric case providing eq. (4.5) is satisfied and is therefore somewhat non-trivial. We note that in the four-dimensional case, the requirement of symmetry of T_{IJ} was more restrictive and was not possible within $\overline{\text{MS}}$ [48].
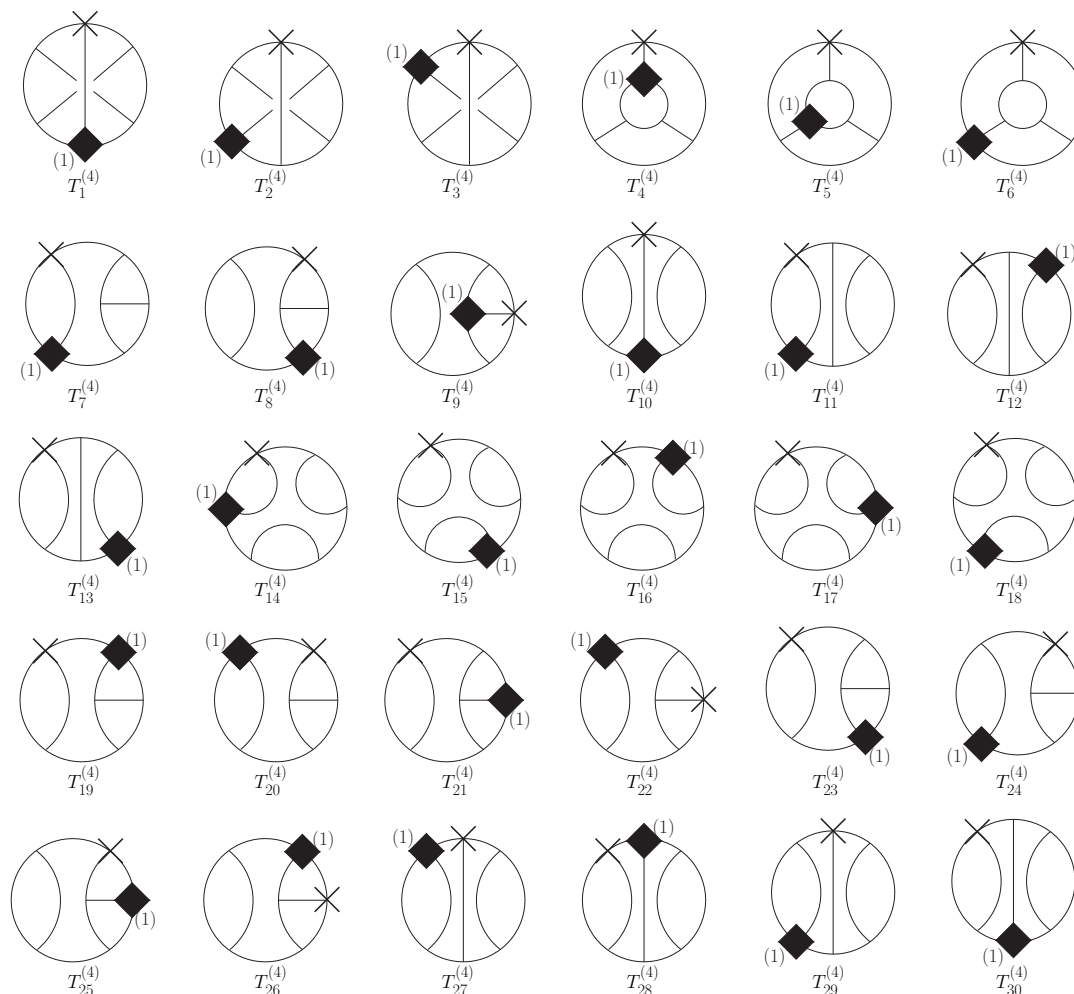
Table 5. Next-to-next-to-leading-order metric terms $T^{(4)}$.

5 Scheme changes

In this section we shall turn to a fuller discussion of scheme changes such as that from $\overline{\text{MS}}$ to MOM. As we mentioned in section 3, we have obtained the three-loop MOM β-function by implementing the appropriate scheme change, avoiding a separate three-loop Feynman diagram calculation for MOM. Here we wish to consider the effect of more general scheme changes, in order to demonstrate the scheme-invariance of the consistency conditions on the β-function coefficients, eqs. (4.11); we shall therefore give our results in full generality. We now rewrite the coupling redefinition of eq. (3.4), which implements the change of scheme, in the form

$$g^{ijk} \to \overline{g}^{ijk} \equiv \overline{g}^{ijk}(g), \tag{5.1}$$

returning to the general couplings of eq. (2.1) and a general scheme change. The effects of eq. (5.1) may be computed from the generalisation of eq. (3.5),

$$\overline{\beta}^{ijk}(\overline{g}) = \mu \frac{d}{d\mu} \overline{g}^{ijk} = \beta \cdot \frac{\partial}{\partial g} \overline{g}^{ijk}(g) \tag{5.2}$$
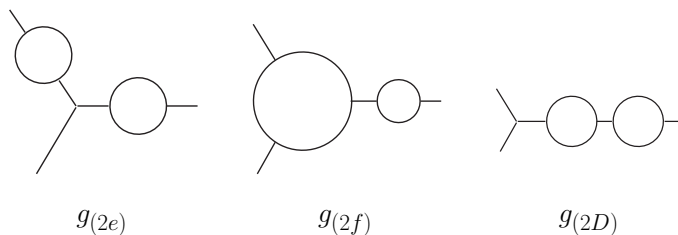
Table 6. Two-loop 1PR structures arising from coupling redefinitions.

(where $\beta \cdot \frac{\partial}{\partial g} \equiv \beta^{klm} \frac{\partial}{\partial g^{klm}}$) which to lowest order may be written

$$\delta \beta^{ijk} = \beta \cdot \frac{\partial}{\partial g} \delta g^{ijk} - \delta g \cdot \frac{\partial}{\partial g} \beta^{ijk}. \tag{5.3}$$

The effect of a one-loop change

$$\delta g = \delta_1 g_{(1a)} + \delta_2 g_{(1A)} \tag{5.4}$$

on the two-loop β-functions is easily computed as

$$\delta c_{(2B)} = -\frac{1}{6}\Delta, \quad \delta c_{(2c)} = \frac{1}{6}\Delta, \quad \Delta = \delta_1 + 12\delta_2 \tag{5.5}$$

where $c_{(2B)}$ and $c_{(2c)}$ are defined in eq. (2.2). It is readily checked that the consistency condition eq. (4.5) is invariant under eq. (5.5), as expected.

At three loops we consider redefinitions

$$\delta g = \epsilon_1 g_{(2b)} + \epsilon_2 g_{(2c)} + \epsilon_3 g_{(2d)} + \epsilon_4 g_{(2e)} + \epsilon_5 g_{(2f)} + \epsilon_6 g_{(2B)} + \epsilon_7 g_{(2C)} + \epsilon_8 g_{(2D)}, \tag{5.6}$$

where

$$g_{(2e)} = g^{ilm} g_{(1A)}^{lj} g_{(1A)}^{mk}, \quad g_{(2f)}^{ijk} = g_{(1a)}^{ijl} g_{(1A)}^{lk}, \quad g_{(2D)}^{ijk} = g^{ijl} g_{(1A)}^{lm} g_{(1A)}^{mk} \tag{5.7}$$

as depicted in table 6, and the remaining tensor structures are defined in eqs. (2.3); and we also need to consider the effect at this order of the lower-order redefinitions given by eqs. (5.4), (5.5).

In general, in addition to modifying the coefficients already present in the $\overline{\text{MS}}$ β-function as defined in eq. (3.9) (which correspond to 1PI diagrams, or 1PI wave-function renormalisation diagrams attached as in eq. (2.5)) these redefinitions will generate tensor structures corresponding to 1PR diagrams given by

$$g_{(3\alpha)}^{ijk} = g_{(1A)}^{il} g_{(2b)}^{klj}, \qquad g_{(3\beta)}^{ijk} = g_{(1A)}^{jl} g_{(2c)}^{lik}, \qquad g_{(3\gamma)}^{ijk} = g_{(1A)}^{il} g_{(2c)}^{jlk}, \quad g_{(3\delta)}^{ijk} = g_{(2B)}^{il} g_{(1a)}^{ljk},$$

$$g_{(3\epsilon)}^{ijk} = g_{(2C)}^{il} g_{(1a)}^{ljk}, \qquad g_{(3\zeta)}^{ijk} = g_{(1A)}^{il} g_{(1A)}^{jm} g_{(1a)}^{lmk}, \quad g_{(3\eta)}^{ijk} = g_{(2D)}^{il} g_{(1a)}^{ljk}, \quad g_{(3\kappa)}^{ijk} = g_{(1A)}^{il} g_{(2B)}^{jm} g^{lmk},$$

$$g_{(3\lambda)}^{ijk} = g_{(1A)}^{im} g_{(2C)}^{jl} g^{lmk}, \quad g_{(3\mu)}^{ijk} = g_{(1A)}^{il} g_{(2D)}^{jm} g^{lmk}, \quad g_{(3\nu)}^{ijk} = g_{(1A)}^{il} g_{(2D)}^{lm} g^{mjk}, \tag{5.8}$$

the tensor structures on the right-hand sides again defined in eqs. (2.3). These 1PR structures are depicted in table 7; we denote the coefficients of these tensor structures in the β-function by $c_{3\alpha} \dots c_{3\chi}$, respectively.
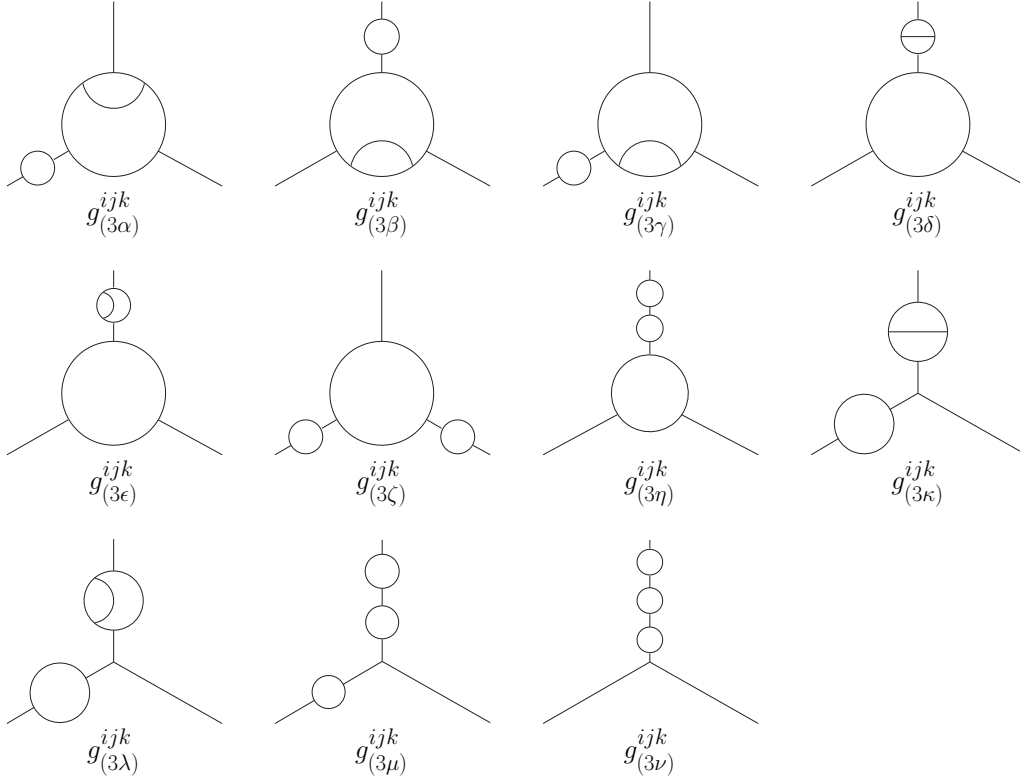
– 17 –

$g^{ijk}_{(3\alpha)}$ $g^{ijk}_{(3\beta)}$ $g^{ijk}_{(3\gamma)}$ $g^{ijk}_{(3\delta)}$

$g^{ijk}_{(3\epsilon)}$ $g^{ijk}_{(3\zeta)}$ $g^{ijk}_{(3\eta)}$ $g^{ijk}_{(3\kappa)}$

$g^{ijk}_{(3\lambda)}$ $g^{ijk}_{(3\mu)}$ $g^{ijk}_{(3\nu)}$

Table 7. Three-loop 1PR structures arising from coupling redefinitions.

We obtain using eq. (5.2) (expanding now to 2nd order in δ_1, δ_2 where necessary)

$$\delta c_{(3g)} = -2\epsilon_1 - 2c_{(2b)}\delta_1 + \delta_1^2,$$

$$\delta c_{(3h)} = \frac{1}{6}\epsilon_1 - \epsilon_2 - c_{(2c)}\delta_1 - 2c_{(2b)}\delta_2 - \frac{1}{6}\delta_1^2,$$

$$\delta c_{(3i)} = \frac{1}{3}\epsilon_1 - 2\epsilon_2 + 2\epsilon_5 - 2c_{(2c)}\delta_1 - 4c_{(2b)}\delta_2 - \frac{1}{3}\delta_1^2 - 2\delta_1\delta_2,$$

$$\delta c_{(3j)} = -2\epsilon_2 + 2\epsilon_6 - 2c_{(2c)}\delta_1 + 2c_{(2B)}\delta_1 - \frac{1}{3}\delta_1^2 - 4\delta_1\delta_2,$$

$$\delta c_{(3k)} = \frac{1}{3}\epsilon_1 + 2\epsilon_2 + 2c_{(2c)}\delta_1 - 4c_{(2b)}\delta_2,$$

$$\delta c_{(3l)} = \frac{1}{6}\epsilon_1 + \epsilon_2 + c_{(2c)}\delta_1 - 2c_{(2b)}\delta_2,$$

$$\delta c_{(3m)} = \frac{1}{3}\epsilon_2 + 2\epsilon_7 - 4c_{(2c)}\delta_2 + 2c_{(2C)}\delta_1 - 2\left(\frac{1}{3}\delta_1\delta_2 + 4\delta_2^2\right),$$

$$\delta c_{(3n)} = \frac{1}{3}\epsilon_2 + 2\epsilon_8 - 4c_{(2c)}\delta_2 - \frac{2}{3}\delta_1\delta_2 - 7\delta_2^2,$$

$$\delta c_{(3o)} = \frac{1}{3}\epsilon_2 + \epsilon_4 - 4c_{(2c)}\delta_2 - \frac{2}{3}\delta_1\delta_2 - 5\delta_2^2,$$

$$\delta c_{(3p)} = \frac{1}{3}\epsilon_3 - 4c_{(2d)}\delta_2, \qquad \delta c_{(3q)} = -\epsilon_3 - c_{(2d)}\delta_1,$$

$$\delta c_{(3r)} = -2\epsilon_3 - 2c_{(2d)}\delta_1, \quad \delta c_{(3s)} = \epsilon_3 + c_{(2d)}\delta_1, \tag{5.9}$$

and

$$\delta c_{(3D)} = -\frac{1}{3}\epsilon_1 - 2\epsilon_6 + 4c_{(2b)}\delta_2 - 2c_{(2B)}\delta_1 + 4\delta_1\delta_2 + \frac{1}{3}\delta_1^2,$$

$$\delta c_{(3E)} = -\frac{1}{6}\epsilon_1 - 2\epsilon_6 + 2c_{(2b)}\delta_2 - 2c_{(2B)}\delta_1 + \frac{1}{4}\delta_1^2 + 4\delta_1\delta_2,$$

$$\delta c_{(3F)} = -\frac{1}{6}\epsilon_2 + \frac{1}{6}\epsilon_6 + 2c_{(2c)}\delta_2 - 2c_{(2B)}\delta_2 + 4\delta_2^2 + \frac{1}{3}\delta_1\delta_2,$$

$$\delta c_{(3G)} = \delta c_{(3G')} = -\frac{1}{6}\epsilon_2 + \frac{1}{3}\epsilon_6 - \epsilon_7 - \frac{1}{6}\epsilon_5$$
$$+ 2c_{(2c)}\delta_2 - 4c_{(2B)}\delta_2 - c_{(2C)}\delta_1 + \frac{1}{2}\delta_1\delta_2 + 8\delta_2^2,$$

$$\delta c_{(3H)} = -\frac{1}{3}\epsilon_6 - 2\epsilon_7 + 4c_{(2B)}\delta_2 - 2c_{(2C)}\delta_1,$$

$$\delta c_{(3I)} = -\frac{1}{6}\epsilon_3 + 2c_{(2d)}\delta_2,$$

$$\delta c_{(3J)} = -\frac{1}{6}\epsilon_4 + \frac{1}{6}\epsilon_7 - 2c_{(2C)}\delta_2 - \frac{1}{6}\delta_2^2,$$

$$\delta c_{(3L)} = \frac{1}{3}\epsilon_7 - \frac{1}{3}\epsilon_8 - 4c_{(2C)}\delta_2 - \frac{1}{6}\delta_2^2, \tag{5.10}$$

and also for the 1PR structures

$$\delta c_{(3\alpha)} = -2\epsilon_5 + 2\delta_1\delta_2, \quad \delta c_{(3\beta)} = \frac{1}{6}\epsilon_5 - \frac{1}{6}\delta_1\delta_2, \quad \delta c_{(3\gamma)} = 2\epsilon_4 + \frac{1}{3}\epsilon_5 - \frac{1}{3}\delta_1\delta_2 - 2\delta_2^2,$$

$$\delta c_{(3\delta)} = -2\epsilon_5 + 2\delta_1\delta_2, \quad \delta c_{(3\epsilon)} = \frac{1}{3}\epsilon_5 - \frac{1}{3}\delta_1\delta_2, \quad \delta c_{(3\zeta)} = -\epsilon_4 + \delta_2^2,$$

$$\delta c_{(3\eta)} = \frac{1}{6}\epsilon_5 - \frac{1}{6}\delta_1\delta_2, \quad \delta c_{(3\kappa)} = -4\epsilon_4 + 4\delta_2^2, \quad \delta c_{(3\lambda)} = \frac{2}{3}\epsilon_4 - \frac{2}{3}\delta_2^2,$$

$$\delta c_{(3\mu)} = \frac{1}{3}\epsilon_4 - \frac{1}{3}\delta_2^2, \quad \delta c_{(3\nu)} = \frac{1}{6}\epsilon_8 - \frac{1}{4}\delta_2^2,$$

$$\delta c_{(3\rho)} = -\frac{1}{12}\epsilon_5 + \frac{1}{12}\epsilon_6 - 2\epsilon_8 - c_{(2B)}\delta_2 + 4\delta_2^2 + \frac{1}{12}\delta_1\delta_2,$$

$$\delta c_{(3\sigma)} = -\frac{1}{12}\epsilon_5 - \frac{1}{12}\epsilon_6 - 2\epsilon_8 + c_{(2B)}\delta_2 + 2\delta_2^2 + \frac{1}{12}\delta_1\delta_2,$$

$$\delta c_{(3\tau)} = -\frac{1}{6}\epsilon_4 + \frac{1}{12}\epsilon_7 + \frac{1}{3}\epsilon_8 - c_{(2C)}\delta_2 - \frac{1}{2}\delta_2^2$$

$$\delta c_{(3\chi)} = -\frac{1}{6}\epsilon_4 - \frac{1}{12}\epsilon_7 + \frac{1}{3}\epsilon_8 + c_{(2C)}\delta_2 - \frac{1}{6}\delta_2^2. \tag{5.11}$$

One may now verify that the consistency conditions eqs. (4.11) are invariant under the coupling redefinitions of eq. (5.6); and so is eq. (4.12), provided we assume that $c_{(3G)}^v$ is scheme-independent, as is natural for a quantity appearing for the first time at this loop order. Finally, it is interesting to note that the condition required for symmetry of $T_{IJ}^{(4)}$, eq. (4.21), is also invariant under these transformations so that a symmetric T_{IJ} can be obtained in any renormalisation scheme at this order.

The coefficients $c_{(3\alpha)}$–$c_{(3\nu)}$ corresponding to 1PR diagrams, and $c_{(3\rho)}$–$c_{(3\chi)}$ corresponding to 1PR contributions to the anomalous dimension, are of course zero in $\overline{\text{MS}}$. It would be natural to restrict ourselves to transformations which take us to other schemes with

the same property; which indeed one might expect to be shared by any well-defined diagrammatic renormalisation scheme. Now it is clear from eq. (5.11) that any scheme change such that

$$\epsilon_4 = \delta_2^2, \quad \epsilon_5 = \delta_1 \delta_2, \quad \epsilon_8 = \frac{3}{2}\delta_2^2, \tag{5.12}$$

will preserve the vanishing of $c_{(3\alpha)} - c_{(3\nu)}$, but will then inevitably give

$$\delta c_{(3\rho)} = -\delta c_{(3\sigma)} = \frac{1}{12}\epsilon_6 + \delta_2^2 - c_{(2B)}\delta_2,$$

$$\delta c_{(3\tau)} = -\delta c_{(3\chi)} = \frac{1}{12}\epsilon_7 - \frac{1}{6}\delta_2^2 - c_{(2C)}\delta_2. \tag{5.13}$$

The values of $c_{(3\rho)}$, $c_{(3\sigma)}$ are opposite in sign, but not manifestly zero; likewise $c_{(3\tau)}$, $c_{(3\chi)}$. Of course one could decide to choose ϵ_6, ϵ_7 so as to ensure the vanishing of $c_{(3\rho)} - c_{(3\chi)}$ in eq. (5.13), but this seems rather restrictive and would not necessarily correspond to a natural diagrammatic renormalisation prescription. We shall therefore approach the issue from a different angle and consider in some detail the example of MOM, which certainly is defined by a diagrammatic prescription as described in detail in section 3.

The MOM and $\overline{\text{MS}}$ results first part company at two loops. The two-loop MOM β-function coefficients may readily be computed directly using the methods described in section 3. They are given by taking in eq. (2.2)

$$c_{(2B)}^{\text{MOM}} = \frac{1}{36} - \frac{2}{81}\pi^2 + \frac{1}{27}\psi'\left(\frac{1}{3}\right), \quad c_{(2c)}^{\text{MOM}} = \frac{1}{8} + \frac{2}{81}\pi^2 - \frac{1}{27}\psi'\left(\frac{1}{3}\right), \tag{5.14}$$

where $\psi(z)$ is the Euler ψ-function defined by

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z), \tag{5.15}$$

the other two-loop β-function coefficients remaining unchanged. Alternatively, we may compute the two-loop MOM β-function coefficients by effecting the appropriate scheme change as described above. Comparing eqs. (2.6), (5.14), we simply require to take in eq. (5.5)

$$\Delta = \frac{1}{6} + \frac{4}{27}\pi^2 - \frac{2}{9}\psi'\left(\frac{1}{3}\right) \tag{5.16}$$

to effect the change to the MOM values. Of course this does not specify δ_1, δ_2 in eq. (5.5) uniquely. However, calculating the coupling redefinition required for the change from $\overline{\text{MS}}$ to MOM as described in section 3, we find that the change to MOM corresponds to taking

$$\delta_1 = \frac{3}{2} + \frac{4}{27}\pi^2 - \frac{2}{9}\psi'\left(\frac{1}{3}\right), \quad \delta_2 = -\frac{1}{9}, \tag{5.17}$$

and it is easy to check that eqs. (5.5), (5.17) are compatible with the difference between eqs. (2.6) and (5.14).

The transformations required to change scheme from $\overline{\text{MS}}$ to MOM at the three-loop level may similarly be calculated using the methods described in section 3. They are given

by taking in eq. (5.6)

$$\epsilon_1 = \frac{51}{32} + \frac{11}{54}\pi^2 - \frac{11}{36}\psi'\left(\frac{1}{3}\right),$$

$$\epsilon_2 = -\frac{703}{1728} - \frac{41}{972}\pi^2 + \frac{41}{648}\psi'\left(\frac{1}{3}\right),$$

$$\epsilon_3 = \frac{59}{48} - \frac{1}{2}\zeta(3) - \frac{7}{27}\pi^2 + \frac{1}{144}\ln(3)^2\sqrt{3}\pi$$

$$- \frac{1}{12}\ln(3)\sqrt{3}\pi - \frac{29}{3888}\sqrt{3}\pi^3 + 3s_2\left(\frac{\pi}{6}\right) - 6s_2\left(\frac{\pi}{2}\right)$$

$$- 5s_3\left(\frac{\pi}{6}\right) + 4s_3\left(\frac{\pi}{2}\right) + \frac{7}{18}\psi'\left(\frac{1}{3}\right),$$

$$\epsilon_6 = -\frac{215}{864}, \qquad \epsilon_7 = \frac{791}{10368}, \tag{5.18}$$

together with ϵ_4, ϵ_5 and ϵ_8 as given in eqs. (5.12), with $\delta_{1,2}$ as defined in eq. (5.17). The explicit MOM results for the three-loop β-function coefficients obtained by combining eqs. (3.10), (3.11), (5.9), (5.10) are somewhat lengthy and are postponed to appendix A.

Unfortunately the values of ϵ_6, ϵ_7 shown in eq. (5.18), do not correspond to the vanishing of $\delta c_{(3\rho)}$, $\delta c_{(3\sigma)}$, $\delta c_{(3\tau)}$, $\delta c_{(3\chi)}$ in eq. (5.13). The scheme transformation therefore predicts non-vanishing MOM β-function contributions from these 1PR anomalous dimension structures, which seems somewhat counter-intuitive. Indeed, after a careful direct calculation using the standard definition of MOM given in Sect 3, and taking account of the fact that the relation between β-function coefficients and renormalisation constants is less trivial in MOM than in $\overline{\text{MS}}$, we obtain $c_{(3\chi)}^{\text{MOM}} = c_{(3\tau)}^{\text{MOM}} = 0$. It seems likely that the same applies to $c_{(3\rho)}^{\text{MOM}}$, $c_{(3\sigma)}^{\text{MOM}}$. We therefore have an apparent inconsistency between the MOM values of $c_{(3\rho)} \ldots c_{(3\chi)}$ obtained by the coupling redefinition process from $\overline{\text{MS}}$, and those obtained by direct calculation. We have checked the anomalous dimension coefficients $c_{(3J)}^{\text{MOM}}$, $c_{(3K)}^{\text{MOM}}$, $c_{(3L)}^{\text{MOM}}$ obtained by coupling redefinition as given in eq. (A.1) by a direct three-loop diagrammatic computation; it therefore appears likely that the discrepancy only affects the particular contriubtions $c_{(3\rho)} \ldots c_{(3\chi)}$ corresponding to 1PR anomalous dimension contributions. We also see from eq. (5.10) that this check of $c_{(3J)}^{\text{MOM}}$, $c_{(3L)}^{\text{MOM}}$ confirms the value of ϵ_7 and therefore fixes $\delta c_{(3\tau)} = -\delta c_{(3\chi)} \neq 0$ in eq. (5.13). This removes the possibility that there might be a different choice of $\epsilon_1 \ldots \epsilon_7$ in eq. (5.18) which would correctly reproduce all the directly-computed MOM coefficients including vanishing values for $c_{(3\rho)} \ldots c_{(3\chi)}$. It seems that one potential resolution of this problem lies in the use of a hybrid MOM scheme as alluded to briefly in section 3, in which the wave-function renormalisation constant is adjusted to give MOM values of $c_{(3\rho)} \ldots c_{(3\tau)}$ in agreement with the coupling redefinitions (without altering the values of any other β-function coefficients). In six dimensions the issue may appear to be simply a technicality, but it seems probable that similar features arise in the four-dimensional case, which is of more practical interest. We propose to return to the subject in a subsequent article where we shall give full details of the MOM calculations reported here and show how a hybrid scheme can resolve the apparent inconsistencies. We shall also show how our results extend to the four dimensional case. Furthermore, it seems conceivable that a similar adjustment of the wave function

renormalisation constant may be required in other schemes in order to match the results obtained by coupling redefinition to those obtained by direct computation, at least for 1PR anomalous dimension contributions.

We may now finally compute the MOM values of the coefficients in $A^{(4)}$, $A^{(5)}$. In order for this to be possible we know that the MOM coefficients should satisfy the appropriate consistency conditions derived in section 4. We have already remarked that the two-loop MOM coefficients satisfy eq. (4.5). Consequently, starting with $A^{(4)}$, the MOM values of $a_2 \ldots a_5$ in eq. (4.1) may be derived by again solving eq. (1.1), but with the MOM values as in eq. (5.14); or more easily using the fact that A transforms as a scalar under coupling redefinitions,

$$\overline{A}(\overline{g}) = A(g). \tag{5.19}$$

Using eqs. (5.4), (5.5), and (2.7), we find for a general one-loop redefinition a leading-order change in A given by

$$\delta A^{(4)} = \lambda \left[\Delta \left(A_2 - \frac{1}{4} A_3 \right) - 12 \delta_2 \beta^{(1)ijk} \beta^{(1)ijk} \right], \tag{5.20}$$

and consequently for MOM using the values in eq. (5.17) we may take

$$a_2^{\text{MOM}} = a_2 + \Delta = \frac{1}{24} + \frac{4}{27}\pi^2 - \frac{2}{9}\psi'\left(\frac{1}{3}\right), \quad a_3^{\text{MOM}} = a_3 - \frac{1}{4}\Delta = \frac{5}{48} - \frac{1}{27}\pi^2 + \frac{1}{18}\psi'\left(\frac{1}{3}\right), \tag{5.21}$$

with a_4 and a_5 unchanged. Clearly other choices are possible with corresponding adjustments of the value of α_1 in eq. (4.1).

At the next order, we have mentioned already that the three-loop MOM coefficients in eqs. (A.1), (A.2) satisfy all the consistency conditions in eqs. (4.11) and (4.12), provided we take the non-zero MOM values of $c_{(3\rho)} \ldots c_{(3\chi)}$ implied by eq. (5.13) (and listed in appendix A), and assume that $c_{(3G)}^v$ is scheme-independent, as is natural for a quantity appearing for the first time at this loop order. We have suggested above that these non-zero values correspond to a hybrid MOM scheme.[5] The MOM values of $a_{(1)}^{(5)} \ldots a_{(16)}^{(5)}$ in eq. (4.16) may then most easily be derived using eq. (5.19) and the explicit transformations given by taking eqs. (5.17), (5.18) in eqs. (5.4), (5.6) respectively. Again, one may also solve the equations using the MOM values of the β-function coefficients as given in eqs. (5.14), (A.1), (A.2); this will yield the same results, up to the freedom expressed in eq. (4.19).

6 Conclusions

We have shown that, as in four dimensions, the gradient-flow equation eq. (1.1) imposes constraints on the β-function coefficients, and we have shown that these constraints are satisfied by the explicit results as computed for the $\overline{\text{MS}}$ and MOM schemes up to three-loop order. We have demonstrated that the tensor T_{IJ} which appears in eq. (1.1) may be chosen

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[5]It is however interesting to observe that if we take $c_{(3\rho)} \ldots c_{(3\chi)}$ to be zero, as obtained by direct calculation in the standard MOM scheme, the MOM coefficients satisfy eq. (4.12) if we take $c_{(3G)}^v = 0$.

as symmetric up to this order. We have also shown that for a general scalar theory with an $O(N)$ global invariance, the β-functions on the right-hand side of eq. (1.1) must be replaced at three-loop order in the $\overline{\text{MS}}$ scheme by the generalised "B"-functions, as has also been observed in four dimensions. It would be useful to extend the analysis of ref. [21] along the lines of ref. [48] in order to understand the issues of theories with a global invariance further. This would have the benefit of enabling an explicit calculation of the "v" term in B (eq. (4.15)) and would also allow an understanding of its scheme dependence.

Finally our analysis of scheme dependence has raised issues concerning the relation of $\overline{\text{MS}}$ and MOM; specifically, the MOM values obtained for certain β-function coefficients corresponding to 1PR contributions to the anomalous dimension are different depending on whether they are obtained by direct calculation within the standard MOM scheme, or by coupling redefinition from $\overline{\text{MS}}$. We shall discuss this issue further in a subsequent article where we shall show that the apparent discrepancy can be avoided by using a hybrid MOM scheme; we shall also explore similar issues in four-dimensional theories.

Acknowledgments

A   Three-loop MOM results

For the MOM scheme the three-loop β-function coefficients were computed by constructing the appropriate coupling redefinitions as described in section 3. Using the same notation as for the $\overline{\text{MS}}$ scheme, they are given by

$$
\begin{aligned}
c_{(3D)}^{\text{MOM}} &= \frac{1}{432} - \frac{1}{486}\pi^2 + \frac{16}{2187}\pi^4 + \frac{1}{324}\psi'(\tfrac{1}{3}) - \frac{16}{729}\psi'(\tfrac{1}{3})\pi^2 + \frac{4}{243}\psi'(\tfrac{1}{3})^2, \\
c_{(3E)}^{\text{MOM}} &= \frac{25}{432} - \frac{5}{972}\pi^2 + \frac{4}{729}\pi^4 + \frac{5}{648}\psi'(\tfrac{1}{3}) - \frac{4}{243}\psi'(\tfrac{1}{3})\pi^2 + \frac{1}{81}\psi'(\tfrac{1}{3})^2, \\
c_{(3F)}^{\text{MOM}} &= \frac{5}{5184} + \frac{1}{648}\pi^2 - \frac{1}{432}\psi'(\tfrac{1}{3}), \\
c_{(3G)}^{\text{MOM}} &= c_{(3G')}^{\text{MOM}} = -\frac{7}{432} + \frac{31}{5832}\pi^2 - \frac{31}{3888}\psi'(\tfrac{1}{3}), \\
c_{(3H)}^{\text{MOM}} &= -\frac{107}{2592} + \frac{11}{1458}\pi^2 - \frac{11}{972}\psi'(\tfrac{1}{3}), \\
c_{(3I)}^{\text{MOM}} &= -\frac{1}{48} + \frac{1}{24}\zeta(3) + \frac{7}{162}\pi^2 - \frac{1}{864}\ln(3)^2\sqrt{3}\pi + \frac{1}{72}\ln(3)\sqrt{3}\pi + \frac{29}{23328}\sqrt{3}\pi^3 \\
&\quad - \frac{1}{2}s_2(\tfrac{\pi}{6}) + s_2(\tfrac{\pi}{2}) + \frac{5}{6}s_3(\tfrac{\pi}{6}) - \frac{2}{3}s_3(\tfrac{\pi}{2}) - \frac{7}{108}\psi'(\tfrac{1}{3}), \\
c_{(3J)}^{\text{MOM}} &= \frac{103}{31104}, \quad c_{(3L)}^{\text{MOM}} = \frac{85}{15552},
\end{aligned}
\tag{A.1}
$$

for the anomalous dimension contributions, and

$$c_{(3g)}^{\mathrm{MOM}} = \frac{1}{8} + \frac{1}{9}\pi^2 + \frac{16}{729}\pi^4 - \frac{1}{6}\psi'(\tfrac{1}{3}) - \frac{16}{243}\psi'(\tfrac{1}{3})\pi^2 + \frac{4}{81}\psi'(\tfrac{1}{3})^2,$$

$$c_{(3h)}^{\mathrm{MOM}} = \frac{1}{24} - \frac{1}{81}\pi^2 - \frac{8}{2187}\pi^4 + \frac{1}{54}\psi'(\tfrac{1}{3}) + \frac{8}{729}\psi'(\tfrac{1}{3})\pi^2 - \frac{2}{243}\psi'(\tfrac{1}{3})^2,$$

$$c_{(3i)}^{\mathrm{MOM}} = \frac{1}{12} - \frac{2}{81}\pi^2 - \frac{16}{2187}\pi^4 + \frac{1}{27}\psi'(\tfrac{1}{3}) + \frac{16}{729}\psi'(\tfrac{1}{3})\pi^2 - \frac{4}{243}\psi'(\tfrac{1}{3})^2,$$

$$c_{(3j)}^{\mathrm{MOM}} = \frac{3}{16} - \frac{5}{486}\pi^2 - \frac{16}{2187}\pi^4 + \frac{5}{324}\psi'(\tfrac{1}{3}) + \frac{16}{729}\psi'(\tfrac{1}{3})\pi^2 - \frac{4}{243}\psi'(\tfrac{1}{3})^2,$$

$$c_{(3k)}^{\mathrm{MOM}} = \frac{1}{12} + \frac{1}{81}\pi^2 - \frac{1}{54}\psi'(\tfrac{1}{3}),$$

$$c_{(3l)}^{\mathrm{MOM}} = \frac{1}{162}\pi^2 - \frac{1}{108}\psi'(\tfrac{1}{3}),$$

$$c_{(3m)}^{\mathrm{MOM}} = -\frac{1}{16} - \frac{31}{2916}\pi^2 + \frac{31}{1944}\psi'(\tfrac{1}{3}),$$

$$c_{(3n)}^{\mathrm{MOM}} = -\frac{7}{288} - \frac{1}{324}\pi^2 + \frac{1}{216}\psi'(\tfrac{1}{3}),$$

$$c_{(3o)}^{\mathrm{MOM}} = -\frac{7}{288} - \frac{1}{324}\pi^2 + \frac{1}{216}\psi'(\tfrac{1}{3}),$$

$$c_{(3p)}^{\mathrm{MOM}} = \frac{19}{72} - \frac{1}{6}\zeta(3) - \frac{7}{81}\pi^2 + \frac{1}{432}\ln(3)^2\sqrt{3}\pi - \frac{1}{36}\ln(3)\sqrt{3}\pi - \frac{29}{11664}\sqrt{3}\pi^3 + s_2(\tfrac{\pi}{6})$$
$$- 2s_2(\tfrac{\pi}{2}) - \frac{5}{3}s_3(\tfrac{\pi}{6}) + \frac{4}{3}s_3(\tfrac{\pi}{2}) + \frac{7}{54}\psi'(\tfrac{1}{3}),$$

$$c_{(3q)}^{\mathrm{MOM}} = -\frac{13}{24} + \frac{1}{2}\zeta(3) + \frac{1}{3}\pi^2 - \frac{1}{144}\ln(3)^2\sqrt{3}\pi + \frac{1}{12}\ln(3)\sqrt{3}\pi + \frac{29}{3888}\sqrt{3}\pi^3 - 3s_2(\tfrac{\pi}{6})$$
$$+ 6s_2(\tfrac{\pi}{2}) + 5s_3(\tfrac{\pi}{6}) - 4s_3(\tfrac{\pi}{2}) - \frac{1}{2}\psi'(\tfrac{1}{3}),$$

$$c_{(3r)}^{\mathrm{MOM}} = -\frac{23}{12} + 2\zeta(3) + \frac{2}{3}\pi^2 - \frac{1}{72}\ln(3)^2\sqrt{3}\pi + \frac{1}{6}\ln(3)\sqrt{3}\pi + \frac{29}{1944}\sqrt{3}\pi^3 - 6s_2(\tfrac{\pi}{6})$$
$$+ 12s_2(\tfrac{\pi}{2}) + 10s_3(\tfrac{\pi}{6}) - 8s_3(\tfrac{\pi}{2}) - \psi'(\tfrac{1}{3}),$$

$$c_{(3s)}^{\mathrm{MOM}} = -\frac{1}{8} - \frac{1}{3}\pi^2 + \frac{1}{144}\ln(3)^2\sqrt{3}\pi - \frac{1}{12}\ln(3)\sqrt{3}\pi - \frac{29}{3888}\sqrt{3}\pi^3 + 3s_2(\tfrac{\pi}{6}) - 6s_2(\tfrac{\pi}{2})$$
$$- 5s_3(\tfrac{\pi}{6}) + 4s_3(\tfrac{\pi}{2}) + \frac{1}{2}\psi'(\tfrac{1}{3}), \tag{A.2}$$

for the 1PI contributions. In eqs. (A.1), (A.2), we define

$$s_n(z) = \frac{1}{\sqrt{3}}\mathcal{I}\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}}\right)\right], \tag{A.3}$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function. The values of the coefficients $c_{(3K)}$, $c_{(3e)}$, $c_{(3f)}$, $c_{(3t)}$, and $c_{(3u)}$ are identical in the two schemes. As mentioned in the main text, we have independently computed the values of $c_{(3J)}^{\mathrm{MOM}}$, $c_{(3K)}^{\mathrm{MOM}}$, $c_{(3L)}^{\mathrm{MOM}}$ *ab initio* and verified that we obtain the same values. Also, the coefficients corresponding to the 1PR anomalous dimension contributions obtained by coupling redefinition are

$$c_{(3\rho)} = -c_{(3\sigma)} = -\frac{23}{10368},$$
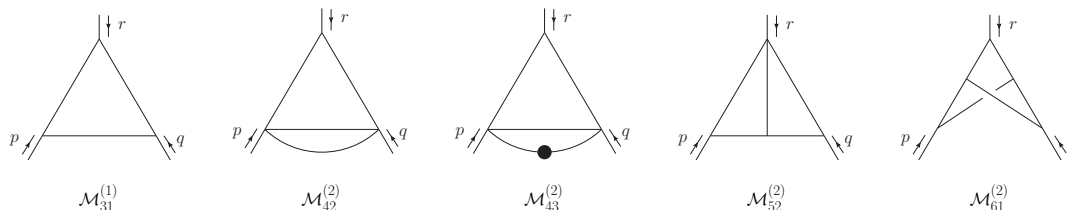$$c_{(3\tau)} = -c_{(3\chi)} = \frac{61}{41472}. \tag{A.4}$$

Table 8. One- and two-loop master integrals.

B   Master integrals

In this appendix we record the explicit values of the various one and two loop six-dimensional master integrals which were needed to perform the renormalisation in the MOM scheme. Their derivation is based on the values of the corresponding master integrals in four dimensions, which were given in ref. [44], and are reproduced above in table 8, where $\mathcal{M}^{(2)}_{43}$ involves the square of a propagator, denoted by a "dot". Using the same notation as ref. [44] to denote the various graphs, the one-loop triangle master integral is

$$
\begin{aligned}
\mathcal{M}^{(1)}_{31} &= \frac{1}{2\epsilon} + \frac{3}{2} + \frac{4}{27}\pi^2 - \frac{2}{9}\psi'(\tfrac{1}{3}) \\
&\quad + \left[\frac{7}{2} + \frac{23}{216}\pi^2 + 4s_3(\tfrac{\pi}{6}) - \frac{2}{9}\psi'(\tfrac{1}{3}) - \frac{35\pi^3}{324\sqrt{3}} - \frac{\ln(3)^2\pi}{12\sqrt{3}}\right]\epsilon + O(\epsilon^2) .
\end{aligned}
\tag{B.1}
$$

At two loops we have

$$
\begin{aligned}
\mathcal{M}^{(2)}_{42} &= \left[\frac{1}{144\epsilon^2} + \frac{65}{1728\epsilon} + \frac{1}{62208}[96\psi'(\tfrac{1}{3}) - 136\pi^2 + 8499] \right.\\
&\quad + [288\sqrt{3}\ln(3)^2\pi - 1728\sqrt{3}\ln(3)\pi + 32\sqrt{3}\pi^3 + 12000\psi'(\tfrac{1}{3}) \\
&\quad + 62208s_2(\tfrac{\pi}{6}) - 124416s_2(\tfrac{\pi}{2}) - 124416s_3(\tfrac{\pi}{6}) + 82944s_3(\tfrac{\pi}{2}) \\
&\quad \left. - 12680\pi^2 - 34560\zeta(3) + 318363]\frac{\epsilon}{746496} + O(\epsilon^2)\right]\mu^4 \\
\mathcal{M}^{(2)}_{43} &= \left[\frac{1}{24\epsilon^2} + \frac{5}{16\epsilon} + \frac{1}{2592}[-120\psi'(\tfrac{1}{3}) + 62\pi^2 + 3915] \right.\\
&\quad + [-72\sqrt{3}\ln(3)^2\pi - 216\sqrt{3}\ln(3)\pi - 136\sqrt{3}\pi^3 - 2664\psi'(\tfrac{1}{3}) \\
&\quad + 7776s_2(\tfrac{\pi}{6}) - 15552s_2(\tfrac{\pi}{2}) + 10368s_3(\tfrac{\pi}{2}) + 966\pi^2 - 4320\zeta(3) \\
&\quad \left. + 93555]\frac{\epsilon}{15552} + O(\epsilon^2)\right]\mu^2 \\
\mathcal{M}^{(2)}_{52} &= \left[\frac{1}{12\epsilon^2} + \frac{25}{48\epsilon} + \frac{205}{96} + \frac{7}{648}\pi^2 - \frac{1}{27}\psi'(\tfrac{1}{3}) + O(\epsilon)\right]\mu^2 \\
\mathcal{M}^{(2)}_{61} &= \frac{1}{4\epsilon} \\
&\quad + \left[\frac{59}{24} - \zeta(3) - \frac{28}{54}\pi^2 + 6s_2(\tfrac{\pi}{6}) - 12s_2(\tfrac{\pi}{2}) - 10s_3(\tfrac{\pi}{6}) + 8s_3(\tfrac{\pi}{2}) \right.\\
&\quad \left. + \frac{7}{9}\psi'(\tfrac{1}{3}) - \frac{29}{1944}\sqrt{3}\pi^3 - \frac{1}{6}\sqrt{3}\ln(3)\pi + \frac{1}{72}\sqrt{3}\ln(3)^2\pi\right] + O(\epsilon) .
\end{aligned}
\tag{B.2}
$$

The values for the remaining two-loop masters corresponding to $\mathcal{M}^{(1)}_{21}$, $\mathcal{M}^{(2)}_{31}$, $\mathcal{M}^{(2)}_{41}$ and $\mathcal{M}^{(2)}_{51}$ of ref. [44] are trivial to construct as they correspond to products of one-loop masters,

or the two-loop sunset graph in the case of $\mathcal{M}_{31}^{(2)}$. We note that the harmonic polylogarithms are based on the theory of cyclotomic polynomials [51]

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