

The thermodynamics of a black hole in equilibrium implies the breakdown of Einstein equations on a macroscopic near-horizon shell

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ABSTRACT: We study a black hole of mass M , enclosed within a spherical box, in equilibrium with its Hawking radiation. We show that the spacetime geometry inside the box is described by the Oppenheimer-Volkoff equations for radiation, except for a *thin shell* around the horizon. We use the maximum entropy principle to show that the invariant width of the shell is of order \sqrt{M} , its entropy is of order M and its temperature of order $1/\sqrt{M}$ (in Planck units). Thus, the width of the shell is *much larger* than the Planck length. Our approach is to insist on thermodynamic consistency when classical general relativity coexists with the Hawking temperature in the description of a gravitating system. No assumptions about an underlying theory are made and no restrictions are placed on the origins of the new physics near the horizon. We only employ classical general relativity and the principles of thermodynamics. Our result is strengthened by an analysis of the trace anomaly associated to the geometry inside the box, i.e., the regime where quantum field effects become significant correspond to the shells of maximum entropy around the horizon.

KEYWORDS: Black Holes, Classical Theories of Gravity

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1 Introduction

The attribution of thermodynamic properties to black holes is incompatible with classical general relativity (GR) [1, 2]. The derivation of Hawking radiation requires a quantum treatment of matter degrees of freedom. For this reason, the origin of black hole thermodynamics is commonly sought at the quantum gravity level.

In this article, we focus on the thermodynamic level of description for black holes. Thermodynamics is a theory for macroscopic coarse-grained variables and it can be consistently formulated without any reference to the underlying physics. In particular, thermodynamics applies even if the underlying physics is fully quantum. For this reason, we believe that it is possible to formulate a thermodynamic description of black holes that *incorporates the quantum effects of matter* within a classical theory of gravity. In ref. [3], we showed that the thermodynamics of gravitating systems in equilibrium is holographic at the classical level, in the sense that all thermodynamic properties are fully specified by variables defined on the system's boundary. In ref. [4], we constructed a consistent thermodynamic description of solutions to Einstein's equations that correspond to radiation in a box.

In this work, we employ these solutions in order to describe a black hole of mass M inside a box, in thermal equilibrium with its Hawking radiation. We find that the breakdown of classical GR takes place in a thin shell around the horizon. Since the principles

of thermodynamics are insensitive to the microscopic underlying dynamics, we identify the shell’s physical characteristics by employing the maximum entropy principle. We find that the shell is characterized by high temperature (of order $1/\sqrt{M}$) and its invariant thickness is of order \sqrt{M} . Hence, the width of the shell around the horizon is *much larger* than the Planck scale. This is unlike most existing models that postulate a shell or membrane of Planck-length around the horizon — for example [5, 6]. The invariant width of the shell derived here is also larger than the invariant distance of $M^{1/3}$ from the horizon characterized by strong gravitational interactions due to the “atmosphere” of high angular-momentum particles, derived in ref. [7].

The electromagnetic (EM) field is an excellent example by which to demonstrate our perspective. The quantum EM field has a consistent statistical mechanical description, while the classical EM field has none. Nonetheless, the thermodynamics of the classical EM field is well defined: the equation of state follows from the classical action, and the entropy functional is inferred from the equation of state. The only imprint of quantum theory is the Stefan-Boltzmann constant that appears as a phenomenological parameter in the entropy functional. In analogy, when seeking an integrated description of black hole thermodynamics and GR at the macroscopic level, we expect that quantum effects are incorporated into phenomenological parameters of the thermodynamic potentials.

The structure of this article is the following. In section 2, we describe the background for studying a black hole inside a box. In section 3, we present the properties of the solutions to the Oppenheimer-Volkoff equation inside the box. In section 4, we derive the condition for the breakdown of the Oppenheimer-Volkoff equation showing that it is restricted into a thin shell around the horizon. In section 5, we implement the maximum-entropy principle, using minimal modeling assumptions, in order to identify the properties of the shell.

2 The equilibrium black hole

A black hole in an asymptotically flat spacetime is not an equilibrium system because it radiates. However, a black hole enclosed within a perfectly reflecting spherical box is an equilibrium system because it involves two competing processes: emission of Hawking quanta, and their re-absorption after reflection from the boundary. One expects that the equilibrium state corresponds to the black hole coexisting with its Hawking radiation. This system has been studied before [8, 9], albeit with simplifying assumptions.

Since the Hawking emission of massive particles is exponentially suppressed [2], radiation is well described by the thermodynamic equations for ultra-relativistic particles:

$$\rho = bT^4, \quad P = \frac{1}{3}\rho, \quad s = \frac{4}{3}b^{1/4}\rho^{3/4}, \quad (2.1)$$

where ρ is the energy density, P is the pressure, T is the temperature and s is the entropy density; b is the Stefan-Boltzmann constant that takes the value $\frac{\pi^2}{15}$ for pure EM radiation. (We use Planck units, $\hbar = c = G = 1$.) Particle numbers are not preserved in the processes of black hole formation and evaporation; thus, they do not define thermodynamic variables and the associated chemical potentials vanish.

Assuming spherical symmetry, the metric outside the box is a Schwarzschild solution with Arnowitt-Deser-Misner (ADM) mass M . An observer outside the box has access to several macroscopic variables that are constant in absence of external intervention. Such variables are the mass M , the area $4\pi R^2$ of the box, the boundary temperature T_R and the boundary pressure P_R . The internal energy of a spherically symmetric system coincides with the ADM mass M [3]. A change δR of the boundary radius corresponds to work $-P_R(4\pi R^2)\delta R/\sqrt{1-2M/R}$ as measured by a local static observer, or $-P_R(4\pi R^2)$ to an observer at infinity. The first law of thermodynamics then becomes

$$\delta M = T_\infty \delta S - P_R(4\pi R^2)\delta R \quad (2.2)$$

where $T_\infty = T_R/\sqrt{1-2M/R}$ is the temperature at infinity. The first law above implies that the thermodynamic state space of the system consists of the variables M and R .

This physical system is characterized by two phases, the radiation phase and the black-hole phase. For fixed R , and for sufficiently small values of M , the box contains only radiation; for higher values of M the box contains a black hole coexisting with its Hawking radiation. A heuristic description of the two phases is the following. For $2M/R \ll 1$, gravity is negligible in the radiation phase, the system is homogeneous with constant density $\rho = m/(\frac{4}{3}\pi R^3)$ and the entropy is

$$S_{\text{rad}} = \frac{4}{3}\pi R^3 s = \frac{4}{3} \left(\frac{4}{3}\pi b \right)^{1/4} M^{3/4} R^{3/4}. \quad (2.3)$$

For $2M/R$ approaching unity, almost all the mass is contained in the black hole of radius $2M$, hence, the Bekenstein-Hawking formula for the black hole entropy applies, $S_{\text{BH}} = 4\pi M^2$. The black hole phase is entropically favored if $S_{\text{BH}} > S_{\text{rad}}$, i.e., for

$$M^5 R^{-3} > \frac{4b}{3^5 \pi^3}. \quad (2.4)$$

The radiation phase was studied in ref. [4]. In what follows, we construct the thermodynamics of the black-hole phase through the following steps: (i) we derive the geometry inside the box using classical GR; (ii) since radiation cannot coexist in equilibrium with a horizon in GR, we identify the spacetime region where Einstein's equations break down; (iii) we find an effective macroscopic description for the physics of this region by using the maximum-entropy principle.

3 Classical geometry inside the box

The region inside the box where the black hole coexists with its Hawking radiation corresponds to a static solution to Einstein's equations with radiation,

$$ds^2 = - \left(1 - \frac{2M}{R} \right) \sqrt{\frac{\rho(R)}{\rho(r)}} dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2, \quad (3.1)$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$ and (t, r, θ, ϕ) are the standard coordinates. The mass function $m(r)$ satisfies $\frac{dm}{dr} = 4\pi r^2 \rho$, and the energy density $\rho(r)$ satisfies the Oppenheimer-Volkoff (OV) equation

$$\frac{d\rho}{dr} = -\frac{4\rho}{r^2} \frac{(m + \frac{4}{3}\pi r^3 \rho)}{1 - \frac{2m}{r}}. \quad (3.2)$$

We change the variables to

$$\xi := \ln \frac{r}{R}, \quad (3.3)$$

$$u := \frac{2m(r)}{r}, \quad (3.4)$$

$$v := 4\pi r^2 \rho, \quad (3.5)$$

to obtain

$$\frac{du}{d\xi} = 2v - u \quad \frac{dv}{d\xi} = \frac{2v(1 - 2u - \frac{2}{3}v)}{1 - u}. \quad (3.6)$$

Eq. (3.6) is to be integrated from the boundary ($\xi = 0$, or $r = R$) inwards, because the thermodynamic variables M and R are defined at the boundary. We denote the values of u and v at the boundary as u_R and v_R , respectively. Thus, $u_R = 2M/R$ and $v_R = 4\pi b R^2 T_R^4$.

There are two classes of solutions to eq. (3.6) that are distinguished by their behavior as $r \rightarrow 0$ [4, 10–12]. The first class contains solutions with a conical singularity at the center. They satisfy $\rho(0) = 0$ and $m(0) = -M_0$, for some constant $M_0 > 0$. The solutions in the second class are regular (everywhere locally Minkowskian). They satisfy $m(0) = 0$ and $\rho(0) = \rho_c$, for some constant $\rho_c > 0$.

The integration of eq. (3.6) from the boundary inwards does not encounter a horizon ($u = 1$), except for the trivial case of $v_R = 0$ that corresponds to a Schwarzschild horizon and no radiation inside the box [4]. However, there is a sub-class of singular solutions with $u \simeq 1$ near a surface $r = r_*$. These solutions arise for $v_R \ll u_R$, i.e., for low density at the boundary. We call these geometries Approximate-Horizon (AH) solutions.

Next, we study the properties of the AH solutions. Plots of u and v as a function of r are given in figure 1. A typical AH solution is characterized by three regions

- (i) In region *I*, u increases and v decreases with decreasing r . P is the local minimum of v .
- (ii) In region *II*, u keeps increasing with decreasing r until it reaches a maximum *very close to unity* at O_* ($r = r_* \simeq 2M$); v also increases with decreasing r in region *II* and equals $\frac{1}{2}$ at O_* . By eq. (3.5), the density

$$\rho_* \simeq \frac{1}{32\pi M^2}, \quad (3.7)$$

and the local temperature

$$T_* = \frac{1}{(32\pi b)^{1/4} \sqrt{M}} \quad (3.8)$$

at O_* depend only on M , within an excellent approximation.

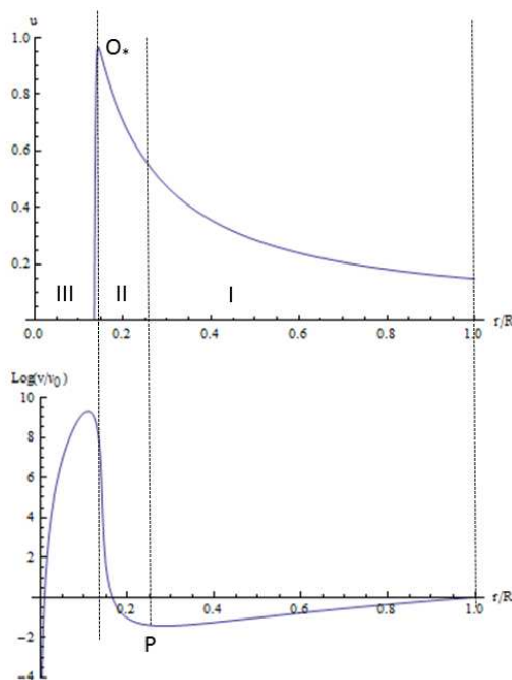


Figure 1. u and v as functions of r for an AH solution with $u_R = 0.15$ and $v_R = 0.01$. Note that we have to use a logarithmic scale for v due to its rapid increase near O_* .

- (iii) Region *III* corresponds to decreasing u ; v increases dramatically shortly after O_* , but then drops to zero at $r = 0$.

An approximate evaluation of the AH solutions is described in appendix A. Every AH solution is characterized by the parameter

$$\epsilon_* = 1 - u(r_*) \ll 1 \tag{3.9}$$

that defines the maximal blue-shift at O_* . In appendix A, we express ϵ_* as a function of the boundary variables,

$$\epsilon_* = \frac{16}{9} u_R (1 - u_R) \sqrt{2v_R}. \tag{3.10}$$

Solving eq. (3.10) for v_R and using eq. (3.5), we relate the boundary temperature T_R to ϵ_* ,

$$T_R = \frac{3\sqrt{\epsilon_*}}{4\sqrt{2M}(8\pi b)^{1/4}\sqrt{1-u_R}}. \tag{3.11}$$

In the appendix A, we also prove a relation between radial coordinate at O_* and ϵ_* ,

$$r_* = 2M \left(1 + \frac{3\epsilon_*}{8} \right), \tag{3.12}$$

and a relation between the value of the mass function at O_* , $m_* := m(r_*)$ and ϵ_* ,

$$m_* = M \left(1 - \frac{5\epsilon_*}{8} \right). \tag{3.13}$$

In the vicinity of O_* , the metric eq. (3.1) becomes

$$ds^2 = -N_*^2 dt^2 + \frac{dx^2}{\epsilon_* + \frac{x^2}{3M^2\epsilon_*}} + r_*^2 d\Omega^2 \quad (3.14)$$

where $x = r - r_*$, and $N_* = \frac{3}{4}\sqrt{\epsilon_*}$ is the lapse function. Interestingly, the proper acceleration at O_* equals $(4M)^{-1}$, i.e., it coincides with the surface gravity of a Schwarzschild black hole of mass M .

4 Breakdown of the OV equation

The regions *I* and *II* of an AH solution describe the geometry of the black hole phase at some distance from the horizon. Since the OV equation cannot account for the presence of a horizon, it must break down somewhere in region *II*, and close to O_* . It must be substituted by a different equation that is compatible with the formation of an horizon. However, any such modification must be very drastic: the OV equation is compatible with a horizon only for matter configurations with negative pressure [13].

It is conceivable that the equation of state for radiation needs is modified near O_* in order to incorporate quantum effects of non-gravitational origin, such as QED vacuum polarization. However, such modifications are unlikely to lead to the negative pressures that are necessary for the formation of a horizon. For a solar mass black hole, $\rho_* \sim 10^{16}\rho_{H_2O}$, where ρ_{H_2O} is the density of water. Hence, ρ_* is of the same order of magnitude with the density at the center of a neutron star. The corresponding local temperature T_* is of the order of $10^{12}K$, which is a typical temperature for quark-gluon plasma. No existing model of strong/nuclear interactions suggests the possibility of negative pressure in these regimes. For super-massive black holes, $\rho_* \sim 10^2\rho_{H_2O}$; negative pressures are even more implausible in this regime. For this reason, we expect that quantum effects at high densities may cause quantitative changes in the thermodynamics of self-gravitating radiation, but they are not strong enough to generate a black hole phase. In further support of this assertion, we note that any contribution from quantum effects would have strong and complex dependence on the mass M , involving masses and thresholds from high energy physics. The resulting thermodynamics would not manifest the simplicity and universality of the Bekenstein-Hawking entropy.

Since high densities or temperature cannot lead to the formation of a horizon, the main cause for the breakdown of the OV equation in region *II* is the extreme blue-shift $\epsilon_*^{-1/2}$. At extreme blue-shifts, the description of matter in terms of hydrodynamic variables (e.g, energy density) fails because the hydrodynamic description is not fundamentally continuous but presupposes a degree of coarse-graining.

In Minkowski spacetime, the energy density ρ is defined as $\rho = U/L^3$, where U is the energy in a cube of size L . L defines the degree of spatial coarse-graining and it cannot be arbitrarily small.¹ The energy density can be treated as a continuous field only when

¹For thermal radiation at temperature T , the requirement that the energy fluctuations are much smaller than the mean energy in a volume L^3 implies that $LT \gg 1$. At higher temperatures, the Compton wavelength of the electron defines an absolute lower limit to L .

measured at scales much larger than L . The hydrodynamic description fails when the fluid dynamics generate length-scales of order L . Then, either the consideration of fluctuations or a microscopic treatment is necessary.

In curved spacetimes, the coarse-graining scale L is defined with respect to the local rest frame, so it represents a proper length. By eq. (3.14), the coarse-graining scale L corresponds to a radial distance $\Delta r \sim L\sqrt{\epsilon_*}$ near O_* . Hence, if $|r_* - 2m_*| \sim L\sqrt{\epsilon_*}$, or, equivalently, if

$$M\sqrt{\epsilon_*} \sim L \tag{4.1}$$

the hydrodynamic fluctuations obscure any distinction of O_* from a genuine horizon. We note that eq. (4.1) does not require L to be a constant. A temperature dependence of L is equivalent to a dependence on the mass M , because the local temperature at O_* depends only on M . Then, eq. (4.1) still provides an estimate of ϵ_* as a function of M .

An alternative justification of eq. (4.1) is the following. In a hydrodynamic system, local densities and temperature are meaningfully defined only if they vary at scales significantly larger than the coarse-graining scale L ; the variation within a shell of volume L^3 must be a small fraction of the averaged value. Tolman's law implies that the product of the local temperature T and the lapse function N is constant. Using eq. (A.24) for the lapse at O_* ,

$$\left| \frac{\nabla_r T}{T} \right| = \frac{2}{3M\epsilon_*}. \tag{4.2}$$

The coordinate distance Δr corresponding to proper length L near O_* is $\Delta r = L\sqrt{\epsilon_*}$.

When the variation of temperature in a cell of proper length L is of the same order of magnitude as the temperature, the hydrodynamic description breaks down. The relevant condition is $|\nabla_r T/T|\Delta r \sim 1$, which implies eq. (4.1).

In section 6, we will show that in the regime that corresponds to the thermodynamically stable black hole, contributions to the stress-energy tensor from QFT in curved spacetime (the trace anomaly) become important. The present analysis is compatible with this result, because in this regime quantum fluctuations of the stress-energy tensor are very strong, and thus, no classical hydrodynamic variables can be defined — see, section 6.3.

5 Maximum-entropy principle

The fundamental thermodynamic variables of the system are the ADM mass M and the box radius R . However, the solutions to Einstein equations depend on *three* independent parameters, which can be chosen as the mass M , the box radius R , and the blue-shift parameter ϵ_* . By eq. (3.11), the dependence on ϵ_* is equivalent to a dependence on the boundary temperature T_R . The equilibrium configuration is determined by the *maximum-entropy principle*: the value assumed by any unconstrained parameter in a thermodynamic system is the one that maximizes the entropy subject to the system's constraints [14].

The thermodynamic constraints for an *isolated* box is the constancy of M and R ; the blue-shift parameter ϵ_* is unconstrained. Hence, the equilibrium configuration for fixed M

and R corresponds to the value of ϵ_* that maximizes the entropy functional. We expect that the entropy functional has one local maximum for each phase.

The radiation phase maximum has the larger value of ϵ_* . For $\sqrt{\epsilon_*} \gg L/M$, the OV equation holds everywhere and we recover the thermodynamics of self-gravitating radiation [4]. Smaller values of ϵ_* correspond to the black hole phase. For $\sqrt{\epsilon_*} \sim L/M$, the OV equation breaks down near the surface O_* . This breakdown is accompanied by a formation of a horizon H near O_* , at $r = r_H < r_*$. The violation of the OH equations is restricted to a thin shell around O_* with a radial width $\delta r := r_* - r_H$ of order $\epsilon_* M$. All properties of the shell depend on ϵ_* , and they are fully specified once ϵ_* is fixed by the maximum-entropy principle.

We model the spacetime geometry of the black-hole phase as follows. In the region between the bounding box and the surface O_* , the metric is described by an AH solution. A horizon is formed at $r = r_H < r_*$ and a thin shell where the OV equation does not apply extends from r_H to r_* . We write

$$r_H = 2M(1 - \lambda\epsilon_*), \tag{5.1}$$

where $\lambda > \frac{5}{8}$ is an unspecified constant of order unity. The simplification involved in this model is that we assume the breakdown of the OV equation to occur sharply at O_* , rather than considering a gradual degradation. This approximation should not affect the order-of-magnitude estimate of the shell's properties. Note that we need not assume that the shell extends up to the horizon $r = r_H$. This would be problematic because points of the horizon are causally disconnected from the interior. For the subsequent analysis, it suffices that the shell extends up to a distance from the horizon that is much smaller than $r_* - r_H$.

The total entropy within the box is a sum of three terms,

$$S_{\text{tot}} = S_H + S_{\text{sh}} + S_{\text{rad}}, \tag{5.2}$$

where

- (i) S_H is the Bekenstein-Hawking entropy of the horizon:

$$S_H = \pi r_H^2 \simeq 4\pi M^2 - 8\pi\lambda\epsilon_* M^2. \tag{5.3}$$

- (ii) The entropy S_{sh} of the thin shell is expected to depend only on the local temperature at O_* (hence, on M) and on the shell width δr . For $\delta r = 0$, there is no shell, so $S_{\text{sh}} = 0$. It follows that

$$S_{\text{sh}}(M, \delta r) = \frac{\partial S_{\text{sh}}}{\partial \delta r}(M, 0)\delta r + O[(\delta r)^2], \tag{5.4}$$

so we write

$$S_{\text{sh}} \simeq \epsilon_* MB, \tag{5.5}$$

where B is a function of M to be determined later.

(iii) The entropy of radiation S_{rad} is the volume integral of the entropy density s in the regions I and II,

$$S_{\text{rad}} = \frac{4}{3}(4\pi b)^{1/4} \int_{r_*}^R \frac{r^{1/2} v^{3/4}}{\sqrt{1-u}} dr. \quad (5.6)$$

In the appendix B, we show that

$$S_{\text{rad}} = \frac{1}{12}(8\pi b)^{1/4}(2M)^{3/2}\sqrt{\epsilon_*}[1 + O(\epsilon_*)]. \quad (5.7)$$

Hence, in the regime of small ϵ_* , the total entropy is

$$S_{\text{tot}} = 4\pi M^2 + \frac{1}{12}(8\pi b)^{1/4}(2M)^{3/2}\sqrt{\epsilon_*} - (8\pi\lambda M^2 - B)\epsilon_* + O(\epsilon_*^{3/2}), \quad (5.8)$$

i.e., it is approximated by a polynomial of second order with respect to $\sqrt{\epsilon_*}$.

In an isolated box, the values of M and R are constrained, while ϵ_* may fluctuate. Hence, the equilibrium configuration is defined as the maximum of the total entropy S_{tot} with respect to ϵ_* . The maximum occurs for

$$\sqrt{\epsilon_*} = (8\pi b)^{1/4} \frac{\sqrt{2M}}{12(8\pi\lambda M - B)}. \quad (5.9)$$

By eq. (3.11), the corresponding boundary temperature is

$$T_R = \frac{1}{16(8\pi\lambda M - B)\sqrt{1-u_R}}. \quad (5.10)$$

The boundary temperature should coincide with the Hawking temperature $T_\infty = \frac{1}{8\pi M}$, blue-shifted by a factor $\sqrt{1-u_R}$. It is a non-trivial check of our model that the R dependence of eq. (5.10) is compatible with such an identification for $B = (8\lambda - \frac{1}{2})\pi M$. Then, the entropy functional, eq. (5.8) is expressed solely in terms of known parameters,

$$S_{\text{tot}}(M, R, \epsilon_*) = 4\pi M^2 + \frac{(2\pi b)^{1/4} M^{3/2}}{3} \sqrt{\epsilon_*} - \frac{\pi M^2}{2} \epsilon_*, \quad (5.11)$$

and the equilibrium configuration corresponds to

$$\sqrt{\epsilon_*} = \frac{(2\pi b)^{1/4}}{3\pi\sqrt{M}}. \quad (5.12)$$

Eq. (5.12) implies that $N_* T_* = T_\infty$, i.e., Tolman's law is satisfied for the Hawking temperature at infinity. This agrees with the results of refs. [3, 15], where Tolman's law is derived solely from the maximum-entropy principle and it is independent of the dynamics of GR.

The equilibrium configuration eq. (5.12) must also satisfy the condition $L \gtrsim \sqrt{\epsilon_*} M$ for the existence of a black hole phase. By eq. (5.12), $L \gtrsim \sqrt{M}$, i.e., the coarse-graining scale L defines an upper bound to the mass of a black hole that can be nucleated in a box. This

bound is not particularly restrictive: it is satisfied even by super-massive black holes for L in the atomic scale.

The width δr of the shell in the equilibrium configuration is

$$\delta r = \left(\frac{3}{8} + \lambda\right) \frac{2\sqrt{2\pi b}}{9\pi^2}, \quad (5.13)$$

i.e., it is of the order of the Planck length. However, the proper width l of the shell is by no means Planckian. Eq. (3.14) implies that $l \sim \delta r / \sqrt{\epsilon_*} \sim \sqrt{M}$.

The entropy of the shell in the equilibrium configuration is

$$S_{\text{sh}} = \left(8\lambda - \frac{1}{2}\right) \frac{\sqrt{2\pi b}}{9\pi} M. \quad (5.14)$$

We estimate the internal energy E of the shell by treating the shell as a single thermodynamic object of temperature $T_{\text{sh}} = (\partial S_{\text{sh}} / \partial E)^{-1}$. In thermal equilibrium, T_{sh} should coincide with the local temperature T_* of radiation, eq. (3.8). Hence, we obtain

$$E = \frac{(8\lambda - \frac{1}{2})(2\pi b)^{1/4}}{9\pi} \sqrt{M}, \quad (5.15)$$

i.e., the internal energy of the shell is proportional to \sqrt{M} modulo a constant of order unity.

6 Physical origins of the shell

The results of the previous section follow solely from thermodynamic arguments. Here, we examine the physical origins of the breakdown of Einstein's equations near O_* . First, we examine the classical geometry of the shell by interpolating between the approximate and the true horizon H . Then, we examine the possibility that the breakdown of the geometry is due to quantum vacuum fluctuations. We show that the conformal anomaly becomes comparable to the classical stress energy tensor near O_* , for $\epsilon_* \sim M^{-1}$, a result that is non-trivially compatible with the condition (5.12) that follows from the maximum entropy principle. We also argue that the thermodynamic approach presented here can, in principle, resolve existing problems in the consistent formulation of the quantum back-reaction to the black hole geometry.

6.1 The classical geometry of the shell

The thermodynamic analysis of section 5 estimates the proper length of the shell to be of order \sqrt{M} , and hence, much larger than the Planck length. This implies that, in spite of the breakdown of Einstein's equations near O_* , a description of the shell in terms of a classical geometry is still possible.

For this reason, we consider a spherically symmetric metric, with a mass function that interpolates between the horizon H at $r = r_{\text{H}}$ and the approximate horizon O_* at $r = r_*$. We assume a power-law interpolation,

$$m(r) = \frac{1}{2}r_{\text{H}} + k(r - r_{\text{H}})^{a+1}, \text{ for } r_{\text{H}} < r < r_*, \quad (6.1)$$

where k and a are positive constants.

We require that $m(r)$, eq. (6.1) is joined with an AH solution at O_* , such that the metric and its first derivatives are continuous. This implies that $m(r_*) = m_*$, eq. (A.17) and that $m'(r_*) = \frac{1}{2}$. The horizon is defined by the condition $2m(r_H) = r_H$. We further require that $m'(r_H) = 0$. This means that the effective ‘density’ on the horizon vanishes, because otherwise any matter on the horizon would be causally disconnected from other matter. The last condition implies that $a > 0$.

Then, we obtain

$$k = \frac{1}{2(a+1)(2M(1+a^{-1})\epsilon_*)^a} \tag{6.2}$$

$$r_H = 2M \left(1 - \frac{5}{8}\epsilon_* - \frac{1}{a}\epsilon_* \right). \tag{6.3}$$

Hence, the width of the shell is $\delta r = r_* - r_H = 2M\epsilon_*a^{-1}$.

The proper length l of the shell is

$$\begin{aligned} l &= \int_{r_H}^{r_*} \frac{dr}{\sqrt{1 - \frac{2m(r)}{r}}} = \int_0^{\delta r} dx \frac{\sqrt{r_H + x}}{\sqrt{x}\sqrt{1 - kx^a}} \\ &\simeq \sqrt{r_H} \int_0^{\delta r} \frac{dx}{\sqrt{x}\sqrt{1 - kx^a}} = C(a)\sqrt{2M\delta r}, \end{aligned} \tag{6.4}$$

where

$$C(a) = \int_0^1 \frac{dy}{\sqrt{y}\sqrt{1 - \frac{1}{2(a+1)}\left(\frac{y}{a+1}\right)^a}} \tag{6.5}$$

is a constant of order unity: for example, $C(\frac{1}{2}) = 2.16, C(1) \simeq 2.04, C(2) \simeq 2$.

For ϵ_* given by eq. (5.12), the proper length l is indeed of the order of \sqrt{M} .

Comparing eq. (6.1) with the OV equation, we can estimate an effective ‘equation of state’ that parameterizes the properties of the shell. The OV equation in the shell is well approximated by

$$\frac{dP}{dz} \simeq -\frac{(\rho + P)}{2z}(1 + 32\pi M^2 P), \tag{6.6}$$

where $z = r - r_H$. Numerical solution of eq. (6.6) leads to an effective equation of state, i.e., a relation between ρ and P , as shown in figure 2. We note that for $a \geq 1$, the effective equation of state is reasonably well approximated by a linear relation of the form $P = -w\rho$, where $w > 0$.

Near the horizon ($z = 0$), a linear equation of state with negative pressure is a good approximation for all a

$$P = -\frac{1}{2a+1}\rho, \tag{6.7}$$

Substituting eq. (6.7) into the continuity equation

$$\frac{dN}{N} = -\frac{dP}{\rho + P}, \tag{6.8}$$

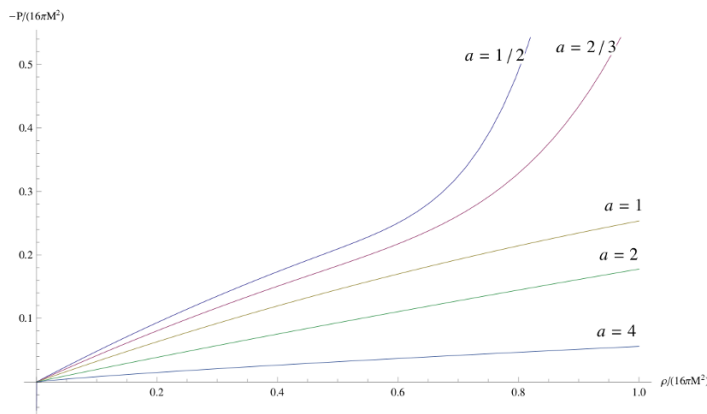


Figure 2. The effective equation of state inside the shell, the absolute value of the pressure $-P$ as a function of the density ρ , for different values of the interpolation exponent a .

we derive the lapse function near the horizon $N \sim \rho^{1/a} \sim \sqrt{r - r_H}$. N can be expressed as

$$N = \kappa x, \quad (6.9)$$

where $x = \sqrt{\frac{r - r_H}{8M}}$ and κ is the surface gravity of the horizon. Then, the geometry near the horizon

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + (2M)^2 d\Omega^2, \quad (6.10)$$

is of the Rindler type with acceleration κ .

We have no analytic expression for κ , but we expect it to be of order $1/M$. Indeed, if eq. (6.7) were a good approximation to the effective equation of state throughout the shell, we would obtain $\kappa = \frac{3\sqrt{a}}{16M}$.

6.2 Relation to the trace anomaly

In section 4, we gave a general argument that the usual hydrodynamic notion of the stress-energy tensor fails near the O_* , and, thus, the classical Einstein equations are not reliable near O_* . We did not discuss the physical mechanisms underlying this breakdown. The leading candidate is, of course, quantum phenomena that are expected to be amplified near a high-blue-shift surface.

Since radiation is scale invariant at the classical level, the stress energy tensor satisfies $T_\mu^\mu = 0$. Thus, the size of the quantum effects in curved spacetime is quantified by the trace anomaly [16–18],

$$\Theta = \langle \hat{T}_\mu^\mu \rangle_R, \quad (6.11)$$

i.e., the renormalized expectation value of the trace of a composite operator $\hat{T}_{\mu\nu}$ that represents the quantum stress-energy tensor.

For a conformally invariant quantum field,

$$\Theta = \alpha \mathcal{H} + \beta \mathcal{G} + \gamma \square R, \quad (6.12)$$

where

$$\mathcal{H} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2. \quad (6.13)$$

$$\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \quad (6.14)$$

are expressed in terms of the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R ; $\square = \nabla_\mu \nabla^\mu$.

The coefficients α, β and γ depend upon the spin of the field. Different methods give the same values to all coefficients except γ . In what follows, we will choose the value of γ given by dimensional regularization, $\gamma = \frac{2}{3}\alpha$. For the EM field, the coefficients α and β are

$$\alpha = \frac{1}{160\pi^2} \quad \beta = -\frac{31}{2880\pi^2}. \quad (6.15)$$

For a macroscopic black hole, we expect the quantum effects to be significant near the high blue-shift surface O_* . We write the metric as $ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2$, and we express the functions f and h as Taylor series in $(r - r_*)$, using eqs. (A.19)–(A.23).

The dominant contribution to the curvature tensors near O_* are

$$R_{trtr} = -\frac{1}{2}fhR, \quad R_{tt} = -\frac{1}{2}fR, \quad R_{rr} = \frac{1}{2}hR, \quad (6.16)$$

$$R = \frac{f''}{fh} - \frac{(f')^2}{2f^2h} - \frac{f'h'}{2fh^2}. \quad (6.17)$$

All other components of the Riemann tensor are smaller by a factor of ϵ_* .

Substituting into eqs. (6.13)–(6.14), we obtain

$$\mathcal{H} = \frac{1}{3}R^2 \quad (6.18)$$

$$\mathcal{G} = 0 \quad (6.19)$$

$$\square R = \frac{1}{h}R'' + \left(\frac{f'}{2fh} - \frac{h'}{2h^2} \right) R'. \quad (6.20)$$

We calculate the Ricci scalar and its derivatives at O_* ,

$$R_* = 0 \quad R'_* = \frac{16}{27}(2M)^{-3}\epsilon_*^{-2} \quad R''_* = -\frac{64}{27}(2M)^{-4}\epsilon_*^{-3}. \quad (6.21)$$

It follows that in the vicinity of O_* , the trace anomaly Θ is of the order of $(M^4\epsilon_*^2)^{-1}$. For concreteness, we compute the value of Θ at O_* ,

$$\Theta_* = -\frac{160}{81} \frac{\gamma}{(2M)^4\epsilon_*^2} = -\frac{320}{243} \frac{\alpha}{(2M)^4\epsilon_*^2}. \quad (6.22)$$

By eq. (3.7), we find the ratio

$$\frac{\Theta_*}{\rho_*} = -\frac{640\pi\alpha}{243M^2\epsilon_*^2}. \quad (6.23)$$

When Θ_*/ρ_* becomes of order unity, Einstein's equations fail near O_* due to the quantum effects associated to the trace anomaly. By eq. (6.23), the violation of Einstein's equations

occurs for $\epsilon_* \sim M^{-1}$. Remarkably, this estimation is in agreement with the value of ϵ_* determined from the maximum entropy principle, eq. (5.12). Hence, the configuration that maximizes entropy is also characterized by the onset of violations to Einstein's equations due to the trace anomaly. We conclude that *two very different types of argument suggest the same order of magnitude for ϵ_** , and, implicitly, the same order of magnitude for the width of the shell.

Substituting the value eq. (5.12) into eqs. (6.23) and (6.15), we find for pure EM radiation

$$\frac{\Theta_*}{\rho_*} = -\frac{5}{2}, \tag{6.24}$$

which suggests that the approximation of a sharply defined shell is only good for order of magnitude estimations. A realistic treatment ought to take into account the gradual deterioration of Einstein's equations as ϵ becomes smaller.

6.3 Problems in formulating back-reaction

In the previous section, we showed that the trace anomaly is of the correct magnitude to account for the breakdown of the classical Einstein equations near the horizon. The question then arises how to formulate the constitutive equations for the system that incorporate the contribution of the trace anomaly. This is the well-known back-reaction problem for QFT in curved spacetime.

A common proposal for the treatment of back-reaction involves the use of the semiclassical Einstein equations

$$G_{\mu\nu} = 8\pi \left(T_{\mu\nu} + \langle \hat{T}_{\mu\nu} \rangle_R \right). \tag{6.25}$$

In eq. (6.25), one includes the expectation value of the renormalized quantum stress-energy tensor as source of the gravitational field in addition to a classical distribution of matter. This is clearly an approximation and not a fundamental theory [19], since eq. (6.25) equates a classical observable with a quantum expectation value.

We believe that this approach does not work for the problem at hand, for the following reasons.

1. The approximation involved in eq. (6.25) requires that the higher moments of the stress-energy tensor are negligible in comparison to the mean value $\langle \hat{T}_{\mu\nu} \rangle_R$. This is not true, in general. The ratio of energy-density variance to the mean value of the energy density may become of order unity and larger [20]; in particular, this is the case for the stress-energy fluctuations in Schwarzschild spacetime [21, 22]. This behavior is not particular to curved spacetimes, but rather, it is a general feature of the quantum definition of the stress-energy tensor. As such, it persists even in the non-relativistic regime [23]. One proposed resolution to this problem is to include the quantum fluctuations as a stochastic force in the semiclassical Einstein equations — see, [24] and references therein.

In fact, the existence of strong quantum fluctuations in the stress-tensor near O_* is compatible with our analysis in section 4 of the breakdown of classical hydrodynamics

on O_* . Classical hydrodynamics presupposes a coarse-grained level of description at which quantum fluctuations are negligible; thus it is incompatible with a regime where quantum fluctuations dominate.

In conventional thermodynamics, the fluctuations of hydrodynamic variables are assumed to be negligible — as long as the observables are averaged within a sufficiently large volume. Typically, the relative size of fluctuations decreases with $N^{-1/2}$, where N is the number of particles in the averaging volume. However, this option is not available when considering *vacuum* fluctuations encoded in the trace anomaly. In our opinion, a thermodynamically consistent treatment of such fluctuations requires the definition of a coarse-grained version of the quantum stress-energy tensor operator $\hat{T}_{\mu\nu}^{coarse}$, possibly significantly different from the standard definitions of $\hat{T}_{\mu\nu}$ in the context of QFT in curved spacetime. The key conditions in the definition of $\hat{T}_{\mu\nu}^{coarse}$ are (i) $\hat{T}_{\mu\nu}^{coarse}$ should be a quasi-classical variable, i.e., a coarse-grained variable that satisfies classical evolution equations [25–27], and (ii) the associated fluctuations should be relatively small so that thermodynamic variables can be properly defined. For examples of quasi-classical hydrodynamic variables defined in quantum systems, see, ref. [28–30] and for a discussion of back-reaction in relation to quasi-classical variables, see ref. [31].

2. The trace anomaly, eq. (6.12) involves terms up to fourth order of the metric, while Einstein equations involve up to second order derivatives. The space of solutions of eq. (6.25) contains therefore additional variables that correspond to the values of the third and fourth derivative of the metric. However, such variables do not have an obvious physical significance; in particular, they have no interpretation in terms of known thermodynamic variables. We have no criterion for assigning values to them at the boundary, and thus the solutions to the back-reaction equations are severely under-determined.
3. Einstein’s equations for a static spacetime correspond to the maximum of the entropy for fixed boundary conditions [3]. An *ad hoc* modification of Einstein’s equations (especially one that involves higher derivatives of the metric) is not guaranteed to satisfy this property. This is problematic, because it implies that the geometry obtained from the solution of eq. (6.25) may not be stable under microscopic fluctuations.

Nonetheless, it is instructive to compute the renormalized expectation value of the stress-energy tensor $\langle \hat{T}_{\mu\nu} \rangle_R$. Closed expressions for $\langle \hat{T}_{\mu\nu} \rangle_R$ for static spacetimes have been computed in the bibliography [32–34], as well as expressions particular to static spherically symmetric spacetimes [35]. In what follows, we employ the expression for a thermal stress-energy tensor by Page [32]. This is obtained from a Gaussian path-integral approximation to the field propagator [36].

The quantum expectation value of the stress energy tensor consists of two terms. One term contains a logarithm of the lapse function N , the other one does not. The presence of the logarithmic term implies that $\langle \hat{T}_{\mu\nu} \rangle_R$ is not invariant under a constant conformal transformation $g_{\mu\nu} \rightarrow cg_{\mu\nu}$ for some constant c . For massless fields, this implies

an ambiguity in the definition of $\langle \hat{T}_{\mu\nu} \rangle_R$, equivalent to the introduction of an undetermined renormalization mass μ . It turns out that $\langle \hat{T}_{\mu\nu} \rangle_R$ contains a term proportional to $\log(N\mu)$.

We calculate $\langle \hat{T}_{\mu\nu} \rangle_R$ at O_* as in [32] and obtain

$$\bar{\rho}_* = -\langle \hat{T}_t^t \rangle_R = \frac{1}{(2M)^4 \epsilon_*^2} \left[8\alpha \left[\frac{32}{243} - \frac{32}{81} \log(3\mu\sqrt{\epsilon_*}/4) \right] + \beta \frac{96}{81} + \gamma \frac{64}{81} \right] \quad (6.26)$$

$$\bar{P}_*^r = \langle \hat{T}_r^r \rangle_R = \frac{1}{(2M)^4 \epsilon_*^2} \left[-8\alpha \frac{16}{243} \log(3\mu\sqrt{\epsilon_*}/4) + \beta \frac{32}{81} + \gamma \frac{32}{243} \right] \quad (6.27)$$

$$\bar{P}_*^\theta = \langle \hat{T}_\theta^\theta \rangle_R = \frac{1}{(2M)^4 \epsilon_*^2} \left[8\alpha \left[\frac{16}{243} - \frac{40}{243} \log(3\mu\sqrt{\epsilon_*}/4) \right] + \beta \frac{32}{81} - \gamma \frac{160}{243} \right]. \quad (6.28)$$

The renormalization mass μ is expected to be smaller than the Planck scale, while in the physically relevant regime, $\epsilon_* \sim M^{-1}$. Thus, the logarithmic terms in eqs. (6.26)–(6.28) are of order $\log(\mu/\sqrt{M})$. For a solar mass black hole and taking $\mu = 1$, $\log(\mu/\sqrt{M}) \simeq -44$. Thus, for macroscopic black holes, the assumption that $\log(\mu/\sqrt{M}) < -10$ is very conservative. Given values of α and β as in eq. (6.15), this assumption implies that the logarithmic term dominates and renders all expectation values $\bar{\rho}_*$, \bar{P}_*^r and \bar{P}_*^θ positive.

If we interpret the expectation values of the stress-energy tensor as thermodynamic densities and pressures, these correspond to an anisotropic fluid with different pressures in the radial and in the tangential direction. However, these densities and pressures are positive. Therefore, even if they are included into the TOV equation (generalized for anisotropic fluid), they cannot lead to the formation of a horizon. Hence, the semi-classical Einstein eqs. (6.25) for back-reaction *cannot* describe an equilibrium black hole. A different method is needed that will provide a resolution to the problems of the semi-classical Einstein equations that we listed earlier.

We believe that the best method for the treatment of quantum back-reaction for equilibrium gravitating systems is to incorporate the quantum effects, including the trace anomaly, into the thermodynamic description of the system. This means that we should redefine the entropy functional in order to include contributions from the quantum effects associated to the trace anomaly. Then, we can construct an equation of state that takes these corrections into account and employ the classical Einstein's equations for this new equation of state. It is essential that this description is thermodynamically consistent; the constitutive equations of the system including back-reaction should correspond to maximum entropy solutions given boundary conditions similar to the ones employed in this paper.

7 Conclusions

We showed that the horizon of an equilibrium black hole is surrounded by a thin shell where the Einstein equations break down. The existence of the shell follows from the requirement that classical GR coexists with the quantum effect of Hawking radiation in a consistent thermodynamic description. The shell has proper width $l \sim \sqrt{M}$, temperature $T_{\text{sh}} \sim 1/\sqrt{M}$, internal energy $E \sim \sqrt{M}$ and entropy $S_{\text{sh}} \sim M$. The proper width of the shell is much larger than the Planck length. Hence, the breakdown of the equations of GR

is fundamentally not due to quantum gravity effects, but due to the quantum properties of matter (radiation). The shell's properties are independent of the box radius R . This strongly suggests that these properties persist even when the box is removed and the system evolves slowly out of equilibrium, i.e., to Schwarzschild black holes.

We emphasize the robustness of our conclusions. We made no assumptions about the quantum characteristics of the underlying theory (unitarity, CTP symmetry, holography). We placed no restrictions on the origin of the new physics near the horizon. In deriving the properties of the shell, we used *only* thermodynamic principles and classical GR. Nonetheless, the results are consistent with QFT in curved spacetime, in the sense that the regime in which quantum effects become significant is consistent with the shell properties derived by the maximum entropy principle.

A Analytic evaluation of the AH solutions

We present an approximate analytic expression for the AH solutions that is valid in the regions α and β of figure 1.

An AH solution is characterized by $v_R \ll u_R$. In the region I , u increases with decreasing r and v decreases with decreasing r . Hence, the condition $v \ll u$ applies to all points in region I . By continuity, the condition $v \ll u$ also applies in a part of region II .

In what follows, we denote derivative with respect to ξ by a prime.

For $v \ll u$, eq. (3.6) becomes

$$u' = -u \qquad v' = \frac{2v(1-2u)}{1-u} \tag{A.1}$$

Hence,

$$\frac{dv}{du} = -\frac{2v(1-2u)}{u(1-u)}, \tag{A.2}$$

The solution of eq. (A.2) with the boundary condition $v(u_R) = v_R$,

$$v = \frac{v_R u_R^2 (1-u_R)^2}{u^2 (1-u)^2}. \tag{A.3}$$

Eq. (A.1) implies that $u(\xi) = u_R e^{-\xi}$. Substituting into eq. (A.3), we derive

$$v(\xi) = \frac{v_R (1-u_R)^2 e^{2\xi}}{(1-u_R e^{-\xi})^2} \tag{A.4}$$

Next, we study the AH solution in the regime where $1-u(\xi) \ll 1$. For sufficiently small v_R , this condition applies to the whole of region II and to a segment of region I .

We set $u = 1 - \epsilon$. For $\epsilon \ll 1$, eq. (3.6) is approximated by

$$\epsilon' = 1 - 2v \tag{A.5}$$

$$v' = -\frac{2v(1+\frac{2}{3}v)}{\epsilon}. \tag{A.6}$$

Eqs. (A.6)–(A.6) imply that

$$\frac{d\epsilon}{dv} = -\frac{\epsilon(1-2v)}{2v(1+\frac{2}{3}v)} \quad (\text{A.7})$$

Eq. (A.7) has solutions of the form

$$\frac{v}{(v+\frac{3}{2})^4} = \frac{a}{\epsilon^2}, \quad (\text{A.8})$$

for some constant a .

The maximum value of u occurs for $\xi = \xi_*$, such that $u'(\xi_*) = 0$, or equivalently $v(\xi_*) = \frac{1}{2}$. The surface $\xi = \xi_*$ is the approximate horizon O_* . We denote by $\epsilon_* = \epsilon(\xi_*)$ the blue-shift parameter on the approximate horizon. Eq. (A.8) implies that $a = \epsilon_*^2/32$. Then, eq. (A.8) becomes

$$\frac{32v}{(v+\frac{3}{2})^4} = \left(\frac{\epsilon_*}{\epsilon}\right)^2. \quad (\text{A.9})$$

Using eqs. (A.9) and (A.6), we obtain a differential equation for $v(\xi)$

$$\left(v^{-1/2} + \frac{3}{2}v^{-3/2}\right)v' = -\frac{16\sqrt{2}}{3\epsilon_*}. \quad (\text{A.10})$$

Integrating from some reference point $\xi = \xi_r$ with $v(\xi_r) = v_r$, we find

$$2(\sqrt{v(\xi)} - \sqrt{v_r}) - 3\left(\frac{1}{\sqrt{v(\xi)}} - \frac{1}{\sqrt{v_r}}\right) = -\frac{16\sqrt{2}}{3\epsilon_*}(\xi - \xi_r) \quad (\text{A.11})$$

Eqs. (A.3) and (A.9) have different, but not disjoint, domains of validity. For sufficiently small v_R , both approximations are valid in a neighborhood of the point P (see, figure 1).

Comparing eqs. (A.3) and (A.9) near P , we find that

$$\epsilon_* = \frac{16}{9}u_R(1-u_R)\sqrt{2v_R}, \quad (\text{A.12})$$

which relates the parameter ϵ_* defined on the approximate horizon O_* to the boundary variables u_R and v_R .

Eq. (A.12) implies the following relation between the boundary temperature T_R and the parameter ϵ_* .

$$T_R = \frac{3\sqrt{\epsilon_*}}{4\sqrt{2M}(8\pi b)^{1/4}\sqrt{1-u_R}}. \quad (\text{A.13})$$

Using eq. (A.11) for a choice of the reference point $\xi = \xi_r$ lying in the domain of validity of eq. (A.4),

$$\xi = \log u_R + \frac{3\epsilon_*}{16\sqrt{2}} \left(\frac{3}{\sqrt{v(\xi)}} - 2\sqrt{v(\xi)} \right). \quad (\text{A.14})$$

Setting $\xi = \xi_*$ in eq. (A.14), we obtain

$$\xi_* = \log u_R + \frac{3\epsilon_*}{8}. \quad (\text{A.15})$$

Using the radial coordinate $r = Re^\xi$, we identify the radial coordinate r_* at the approximate horizon to leading order in ϵ_*

$$r_* = 2M \left(1 + \frac{3\epsilon_*}{8} \right). \quad (\text{A.16})$$

The corresponding value of the mass function $m_* = m(r_*)$ is

$$m_* = \frac{1}{2}r_*(1 - \epsilon_*) = M \left(1 - \frac{5\epsilon_*}{8} \right). \quad (\text{A.17})$$

We evaluate the metric in the vicinity of O_* . To this end, we write the metric eq. (3.1) as

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (\text{A.18})$$

in terms of functions $f(r)$ and $h(r)$. We express these functions as a Taylor series with respect to $(r - r_*)$ by computing their values and the values of their derivatives at O_* . To this end, we use eqs. (3.6)–(3.6), for u' and v' , which we differentiate successively in order to obtain equations for all derivatives of v and u . Expressing f and h in terms of u and v , we compute the derivatives of f and h at $r = r_*$. We list the derivatives that are necessary for the calculation of the trace anomaly.

$$f_* = \frac{9}{16}\epsilon_*, \quad h_* = \epsilon_*^{-1}, \quad (\text{A.19})$$

$$f'_* = \frac{3}{4}(2M)^{-1}, \quad h'_* = 0, \quad (\text{A.20})$$

$$f''_* = \frac{1}{2}(2M)^{-2}\epsilon_*^{-1}, \quad h''_* = -\frac{8}{3}\epsilon_*^{-1}(2M\epsilon_*)^{-2}, \quad (\text{A.21})$$

$$f'''_* = -\frac{2}{3}(2M)^{-3}\epsilon_*^{-2}, \quad h'''_* = \frac{80}{9}\epsilon_*^{-1}(2M\epsilon_*)^{-3}, \quad (\text{A.22})$$

$$f''''_* = \frac{2}{3}(2M)^{-4}\epsilon_*^{-3}, \quad h''''_* = \frac{32}{3}\epsilon_*^{-1}(2M\epsilon_*)^{-4}. \quad (\text{A.23})$$

The lapse function $N = \sqrt{f}$ near O_* is

$$N = \frac{3\sqrt{\epsilon_*}}{4} + \frac{1}{4M\sqrt{\epsilon_*}}(r - r_*) + \dots \quad (\text{A.24})$$

The acceleration $a_i = \nabla_i \log N$ is purely radial, with $a_r = \frac{1}{3M\sqrt{\epsilon_*}}$. Hence, the proper acceleration $a = \sqrt{a_\mu a^\mu}$ at O_* is

$$a = \frac{1}{4M}, \quad (\text{A.25})$$

i.e., it equals the surface gravity of a black hole of mass M .

B Evaluating the radiation entropy

We evaluate the entropy of radiation in the regions I and II of an AH solution, eq. (5.6). For solutions to the Oppenheimer-Volkoff equation, the integrand in eq. (5.6) is a total derivative, i.e.,

$$\frac{r^{1/2}v^{3/4}}{\sqrt{1-u}} = \frac{d}{dr} \left(\frac{v + \frac{3}{2}u}{6v^{1/4}\sqrt{1-u}} r^{3/2} \right). \quad (\text{B.1})$$

Hence, $S_{\text{rad}} = S_1 - S_*$ where

$$S_1 = \frac{2}{9}(4\pi b)^{1/4} \frac{v_R + \frac{3}{2}u_R}{v_R^{1/4}\sqrt{1-u_R}} R^{3/2} \quad (\text{B.2})$$

depends on field values at the boundary $r = R$, and

$$\begin{aligned} S_* &= \frac{4}{9}(8\pi b)^{1/4} \frac{1 - \frac{3}{4}\epsilon_*}{\sqrt{\epsilon_*}} r_*^{3/2} \\ &\simeq \frac{4}{9}(8\pi b)^{1/4} (2M)^{3/2} \left(\epsilon_*^{-1/2} - \frac{3}{16}\epsilon_*^{1/2} \right) \end{aligned} \quad (\text{B.3})$$

depends on the field values at $r = r_*$. Using eq. (A.12) to eliminate v_R from S_1 , we obtain

$$S_{\text{rad}} = \frac{(8\pi b)^{1/4}}{12} (2M)^{3/2} \sqrt{\epsilon_*} \left(1 + \frac{9}{16u_R^3(1-u_R)^2} \epsilon_* \right). \quad (\text{B.4})$$

Eq. (5.7) applies in the regime $K \ll 1$, where

$$K = \frac{R^3}{4\pi^2 M^4 (1-u_R)^2}, \quad (\text{B.5})$$

so that the second term inside the parenthesis in the r.h.s. of eq. (B.4) is negligible for the value of ϵ_* that maximizes the total entropy, eq. (5.12).

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