Published for SISSA by Springer

Received: *November 10, 2015*
Accepted: *December 28, 2015*
Published: *January 18, 2016*

JHEP01(2016)089

Modular properties of characters of the W$_3$ algebra

Nicholas J. Iles and Gérard M.T. Watts

Department of Mathematics, King's College London,
Strand, London, WC2R 2LS U.K.

E-mail: nicholas.iles@kcl.ac.uk, gerard.watts@kcl.ac.uk

Abstract: In a previous work, exact formulae and differential equations were found for traces of powers of the zero mode in the W_3 algebra. In this paper we investigate their modular properties, in particular we find the exact result for the modular transformations of traces of W_0^n for $n = 1, 2, 3$, solving exactly the problem studied approximately by Gaberdiel, Hartman and Jin. We also find modular differential equations satisfied by traces with a single W_0 inserted, and relate them to differential equations studied by Mathur et al. We find that, remarkably, these all seem to be related to weight 0 modular forms with expansions with non-negative integer coefficients.

Keywords: Conformal and W Symmetry, AdS-CFT Correspondence

ArXiv ePrint: 1411.4039

doi:10.1007/JHEP01(2016)089

Contents

1 Introduction

In a previous paper [1], we found formulae for characters of the W_3 algebra with insertions of powers of the zero mode W_0 (see appendix B for the definition of the W_3 algebra). These characters with insertions have been of interest recently since they are involved in counting the states of black holes in (2+1)-dimensional AdS higher-spin gravity. There are two regimes which are related by modular transformation, and it is of particular interest to understand the properties of the characters under modular transformation. In this paper we study the properties of the characters we found in [1] under modular transformations.

JHEP01(2016)089

We studied two classes of W_3-algebra representations in [1], Verma modules and minimal model representations. We found that traces over the minimal model representations with a single W_0 insertion satisfied differential equations and in section 3 here we show that these are modular, that is, they are covariant under modular transformations and this implies that the solutions (the traces) have particular modular properties.

In general, there are many fewer non-zero solutions than there are minimal model representations, and so the non-zero traces transform in a smaller representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$ than the full set of minimal model characters, but we show that they are in fact compatible with a general result of Gaberdiel et al. [2] that the traces of zero-modes of primary fields transform as modular forms of particular weights with a standard matrix representation.

In section 4, we use this result of [2], combined with the form of a particular primary field, to find the exact formula for the transform of the trace with the insertion of $(W_0)^2$ for which an approximate result was found in [2]. We verify our result by performing the exact calculation from [2], and further use their method to find the transformation of a trace with $(W_0)^3$ inserted.

In section 5 we consider the result from [1] for the trace of W_0^2 over a Verma module. Firstly we express this in terms of Eisenstein series and then use this formulation to investigate its modular properties.

Finally, in section 6 we consider the transform of the trace of a $(W_0)^2$ insertion in the limit where the central charge c becomes large. Looking at the result for a Verma module from section 5, interestingly, we get a different result to that in [2]. We find that both results can be reproduced from the exact results of section 4 under two different assumptions on the state of lowest conformal weight in the theory. We exhibit two models in which the alternative assumption holds and they do indeed reproduce the behaviour of section 5 as opposed to that of [2].

2 Modular transformations, characters and differential equations

We are concerned in this paper with the modular properties of traces over W-algebra highest weight representations. It will be helpful first to review some general facts.

A conformal field theory has a chiral algebra spanned by the modes of holomorphic fields. This algebra has representations, L_i, and characters (sometimes called "specialised characters")

$$\chi_i(q) = \mathrm{Tr}_{L_i}(q^{L_0 - c/24}) \; . \tag{2.1}$$

The factor $q^{-c/24}$ is necessary for the characters to transform nicely under the modular transformation $q \to \hat{q}$,

$$q = \exp(2\pi i\tau) \; , \qquad \hat{q} = \exp(-2\pi i/\tau) \; . \tag{2.2}$$

If the conformal field theory is "minimal", then the set of representations is finite-dimensional and there is a "standard" modular S-matrix S_{ij}, running over all the representations

JHEP01(2016)089

of the chiral algebra, and the characters satisfy

$$\chi_i(\hat{q}) = \sum_j S_{ij}\,\chi_j(q)\;. \tag{2.3}$$

If a representation is not self-conjugate, then both the representation and its conjugate will have the same character, so the space of characters is of smaller dimension than the space of representations.

Perhaps the simplest example is the Wess-Zumino-Witten model based on a_2 at level 1, with symmetry algebra the affine algebra $(a_2^{(1)})_1$. The highest weight representations of $(a_2^{(1)})_1$ are determined by their Dynkin labels, and are [100], [010] and [001] (see section 14.3.1, [7]). The representations [001] and [010] are conjugate and so have the same (reduced) character. In this particular case the characters have the especially simple forms

$$\chi_{[100]} = q^{-1/12}\frac{\sum_{m,n} q^{m^2+n^2-mn}}{\prod_{m=1}(1-q^m)^2}\;, \tag{2.4}$$

$$\chi_{[001]} = \chi_{[010]} = q^{1/4}\frac{\sum_{m,n} q^{m^2+n^2+m-mn}}{\prod_{m=1}(1-q^m)^2}\;. \tag{2.5}$$

The standard modular S-matrix is [7]

$$S = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \kappa & \kappa^2 \\ 1 & \kappa^2 & \kappa \end{pmatrix}\;, \qquad \kappa = e^{2\pi i/3}\;, \tag{2.6}$$

but the characters themselves transform in the two-dimensional representation

$$S' = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\;. \tag{2.7}$$

The fact that the characters span a two-dimensional space which is invariant under the modular transformation is reflected in the fact that they satisfy a particular differential equation [3],

$$\left[\left(q\frac{\mathrm{d}}{\mathrm{d}q} - \frac{2}{12}E_2(q)\right) q\frac{\mathrm{d}}{\mathrm{d}q} - \frac{1}{48}E_4(q)\right]\chi(q) = 0\;, \tag{2.8}$$

where $E_{2n}(q)$ are Eisenstein series, whose definitions and properties are collected in appendix A. These series have well-defined modular transformation properties which ensure that the differential equation (2.8) is invariant under $q \to \hat{q}$, that is, if $\chi(q)$ satisfies eq. (2.8), then the same function $\chi(q)$ satisfies the differential equation with q replaced by $\hat{q}$, that is

$$\left[\left(\hat{q}\frac{\mathrm{d}}{\mathrm{d}\hat{q}} - \frac{2}{12}E_2(\hat{q})\right) \hat{q}\frac{\mathrm{d}}{\mathrm{d}\hat{q}} - \frac{1}{48}E_4(\hat{q})\right]\chi(q) = 0\;. \tag{2.9}$$

The particular combinations $q(\mathrm{d}/\mathrm{d}q) - (r/12)E_2(q)$ act as covariant derivatives, mapping modular forms of weight r to forms of weight $(r+2)$, and we shall denote them by $D^{(r)}$ — see appendix A for details. Differential equations of the form

$$\left[D^{(2)}D^{(0)} + \mu E_4(q)\right]\chi(q) = 0 \tag{2.10}$$

have been studied intensively [3–6] as the typical defining equations for characters of conformal field theories with only two independent characters, and the solutions can be found as hypergeometric functions and (as a consequence) the modular transformation matrix found explicitly.

3 Modular differential equations for $\mathrm{Tr}(W_0)$ in minimal models

In [1], we found that the traces $\mathrm{Tr}_L\left(W_0\, q^{L_0}\right)$, where L is one of the irreducible representations of the W_3 algebra at $c = 4/5$, satisfied a second-order differential equation in q. We can rewrite the equation in [1] as a different equation for $\mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right)$ and we find that it becomes

$$\left[D^{(5)}D^{(3)} - \frac{299}{3600}E_4\right] \mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right) = 0\,. \tag{3.1}$$

This equation automatically implies that the traces transform as weight 3 modular forms. As a consequence, the combinations $f_L(q) = \eta(q)^{-6}\,\mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right)$ transform as regular weight 0 modular forms (where $\eta(q)$ is the Dedekind eta function), and satisfy the differential equation

$$\left[D^{(2)}D - \frac{299}{3600}E_4\right] f_L(q) = 0\,. \tag{3.2}$$

It comes as a bit of a surprise that the particular equation (3.2) is in fact exactly one of the equations of the form (2.10) studied in [3], corresponding to the Wess-Zumino-Witten model for the exceptional algebra f_4 at level 1, with affine Kac-Moody algebra symmetry $(f_4^{(1)})_1$ and central charge 26/5: the traces in the 3-state Potts model with W_0 inserted are proportional to the characters of the $(f_4^{(1)})_1$ algebra. To be precise, the algebra $(f_4^{(1)})_1$ has two irreducible highest weight representations, the vacuum representation $L_{[10000]}$ and the fundamental representation $L_{[00001]}$ with highest weight space the 26-dimensional representation of F_4 and we find

$$\mathrm{Tr}_{[11;12]}(W_0 q^{L_0-c/24}) = w[11;12]\,\eta^6\ \mathrm{Tr}_{[10000]}(q^{L_0-c/24})\,, \tag{3.3}$$
$$\mathrm{Tr}_{[11;13]}(W_0 q^{L_0-c/24}) = w[11;12]\,\eta^6\ \mathrm{Tr}_{[00001]}(q^{L_0-c/24})\,. \tag{3.4}$$

where $[rs;r's']$ label W_3-algebra representations, see appendix B for details. This means that the modular properties of the W_0-traces in the 3-state Potts model are already known, as are exact expressions for these W_0-traces in terms of the Kac-Weyl character formula [7], as well as hypergeometric functions and contour integrals, using directly the results from [3].

This remarkable coincidence is repeated for the other W-algebra minimal model with two independent traces: the W-minimal model $WM(3,8)$, which has central charge -23, has 7 representations, of which 3 are uncharged and there are two pairs of charged representations. Since the traces $\mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right)$ over self-conjugate representations are zero and over conjugate representations differ just by a sign, these traces are spanned by two independent functions. Just as in the 3-state Potts model, there is a null state at level 7 in the vacuum representation which leads to the traces satisfying the differential equation

$$\left[D^{(5)}D^{(3)} - \frac{5}{576}E_4\right] f(q) = 0\,, \tag{3.5}$$

or, equivalently,

$$\left[D^{(2)}D - \frac{5}{576}E_4\right]\left(\frac{f(q)}{\eta(q)^6}\right) = 0\ . \tag{3.6}$$

This is the same differential equation satisfied by the characters of the WZW model $(a_1^{(1)})_1$. This model has central charge $c = 1$ and two representations $L_{[0]}$ and $L_{[1]}$; again the traces in the W-algebra are proportional to the characters of the affine algebra, with the same proportionality constant $w[11; 13]$:

$$\begin{aligned}\tilde{\chi}_0(q) \equiv \mathrm{Tr}_{[11;13]}(W_0 q^{L_0-c/24}) &= w[11;13]\,\eta^6\ \mathrm{Tr}_{[0]}(q^{L_0-c/24})\ ,\\ \tilde{\chi}_1(q) \equiv \mathrm{Tr}_{[11;12]}(W_0 q^{L_0-c/24}) &= w[11;13]\,\eta^6\ \mathrm{Tr}_{[1]}(q^{L_0-c/24})\ ,\end{aligned} \tag{3.7}$$

where we defined $\tilde{\chi}_i(q)$ for convenience. The modular transformation of the traces is

$$\tilde{\chi}_i(\hat{q}) = \tau^3 \sum_j S'_{ij}\tilde{\chi}_j(q)\ , \qquad S' = \frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\ . \tag{3.8}$$

Finally, the simplest of all is the minimal model $WM(3,7)$ which has 5 representations and only two conjugate representations with non-zero charge. We find their traces $\mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right)$ satisfy

$$D^{(3)}f(q) = 0\ , \tag{3.9}$$

or, equivalently,

$$q\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{f(q)}{\eta^6}\right) = 0\ , \tag{3.10}$$

with explicit solutions

$$\mathrm{Tr}_{[11;12]}(W_0\, q^{L_0-c/24}) = w[11;12]\,\eta(q)^6\cdot 1\ , \qquad \mathrm{Tr}_{[11;21]}(W_0\, q^{L_0-c/24}) = -w[11;12]\,\eta(q)^6\cdot 1\ , \tag{3.11}$$

where 1 can be interpreted as the trace over the one-dimensional representation of the Virasoro algebra at $c = 0$. As there are only two charged representations of $WM\,(3,7)$, and these are conjugate, the S' 'matrix' is 1×1 and we have

$$\mathrm{Tr}_{[11;12]}(W_0\,\hat{q}^{L_0-c/24}) = w[11;12]\,\eta(\hat{q})^6 = w[11;12]\,i\tau^3\eta(q)^6 = i\tau^3\,\mathrm{Tr}_{[11;12]}(W_0\,q^{L_0-c/24}) \tag{3.12}$$

i.e. $S' = i$.[1]

We have checked further and so far in every case, the functions $\eta(q)^{-6}\,\mathrm{Tr}_L\left(W_0\,q^{L_0-\frac{c}{24}}\right)$ in the W-minimal models satisfy modular differential equations of the appropriate order and can be normalised (with the same normalisation factor for each representation L in a given model) so that they each have expansions with non-negative integer coefficients. This last point seems a very surprising fact as there is not anything obvious that these functions are counting, but the functions do not seem to be easily identifiable with the sets of characters of known conformal field theories, so that the identifications with the characters of $(f_4^{(1)})_1$ and $(a_1^{(1)})_1$ seem rather coincidental.

[1]Notice that this satisfies $S'^2 = C$, the charge conjugation matrix ($C = -1$ in this case), and $S'^\dagger S' = 1$, as should be the case for an S-matrix.

4 Exact results for modular transformations of $\mathrm{Tr}(W_0^n)$ for $n = 1, 2, 3$

The results from the previous section are surprising, but the fact that the traces $\mathrm{Tr}_L\left(W_0\, q^{L_0 - \frac{c}{24}}\right)$ transform as weight 3 modular forms is not, as is explained in [2]. It is shown there that the trace of the zero mode a_0 of a holomorphic primary field $a(z)$ of weight h transforms as a modular form of weight h with the "standard" S-matrix.

Recall from the previous discussion that if the irreducible representations of the symmetry algebra are L_i and the characters are $\chi_i(q) = \mathrm{Tr}_{L_i}(q^{L_0 - c/24})$, then

$$\chi_i(\hat{q}) = \sum_j S_{ij}\, \chi_j(q)\ . \tag{4.1}$$

It is shown in [2] that[2]

$$\mathrm{Tr}_{L_i}(a_0\, \hat{q}^{L_0 - c/24}) = \tau^h \sum_j S_{ij}\ \mathrm{Tr}_{L_j}(a_0\, q^{L_0 - c/24})\ . \tag{4.2}$$

4.1 Case $n = 1$

We can apply equation (4.2) to the field $W(z)$ of weight 3, we get

$$\mathrm{Tr}_{L_i}(W_0\, \hat{q}^{L_0 - c/24}) = \tau^3 \sum_j S_{ij}\ \mathrm{Tr}_{L_j}(W_0\, q^{L_0 - c/24})\ , \tag{4.3}$$

or

$$\left[\frac{\mathrm{Tr}_{L_i}(W_0\, \hat{q}^{L_0 - c/24})}{\eta(\hat{q})^6}\right] = -i \sum_j S_{ij} \left[\frac{\mathrm{Tr}_{L_j}(W_0\, q^{L_0 - c/24})}{\eta(q)^6}\right]\ . \tag{4.4}$$

Note that this is the "standard" S-matrix; in our case, the traces over self-conjugate fields will be zero and over conjugate fields will differ by just a sign, so the actual dimension of the representation will be smaller.

For example, in the case of $WM(3,8)$ mentioned above, there are 7 representations so that S_{ij} is a 7×7 matrix, but there are only two independent traces $\tilde{\chi}_i(q)$ as in (3.7), transforming with a 2×2 matrix S' given in equation (3.8). This is entirely consistent with the "standard" transformation properties as conjugation is an automorphism of the Hilbert space and so one can restrict the modular S-matrix to non-self-conjugate representations and still obtain a representation of the modular group.

We can also check the result (4.3) directly. In appendix D we show this holds for the modular transformation of traces over Verma module representations with $h > (c-2)/24$.

4.2 Case $n = 2$

In the case $n = 2$, there is no primary field $M(z)$ such that $\mathrm{Tr}_L\left(M_0\, q^{L_0 - \frac{c}{24}}\right) = \mathrm{Tr}_L\left(W_0^2\, q^{L_0 - \frac{c}{24}}\right)$, and so we cannot apply (4.2) directly to calculate the modular transform of $\mathrm{Tr}_L\left(W_0^2\, q^{L_0 - \frac{c}{24}}\right)$. However, we can apply equation (4.2) to the field $M(z)$ that

[2] As T is not primary, this expression is not valid for L_0. Instead, the insertion of L_0 in a trace is equivalent to the action of the differential operator $L_0(q) = q\frac{\mathrm{d}}{\mathrm{d}q} + \frac{c}{24}$ and we can transform $\hat{q} \to q$ directly, which gives $L_0(\hat{q}) = \tau^2 L_0(q) + \frac{c}{24}\left(1 - \tau^2\right)$.

JHEP01(2016)089

corresponds to the state (see appendix B for the definition of Λ_m and the commutation relations between these operators)

$$|M\rangle = [W_{-3}W_{-3} + aL_{-3}L_{-3} + bL_{-2}\Lambda_{-4} + 24\,d\,L_{-6} + 2eL_{-2}L_{-4}]\,|0\rangle\ . \tag{4.5}$$

This state is a Virasoro highest weight state for the choices[3]

$$a = -\frac{776 + 1978c + 225c^2}{2\,(-1+2c)\,(22+5c)\,(68+7c)}, \tag{4.6a}$$

$$b = -\frac{16\,(22+191c)}{3\,(-1+2c)\,(22+5c)\,(68+7c)}, \tag{4.6b}$$

$$d = -\frac{1472 - 832c - 114c^2 + 5c^3}{6\,(-1+2c)\,(22+5c)\,(68+7c)}, \tag{4.6c}$$

$$e = -\frac{2\left(-3672 + 1654c + 335c^2\right)}{5\,(-1+2c)\,(22+5c)\,(68+7c)}. \tag{4.6d}$$

In this case, $M(z)$ is a Virasoro primary field of weight 6 and consequently we can apply (4.2) to obtain

$$\mathrm{Tr}_i(M_0\,\hat{q}^{L_0-c/24}) = \tau^6 \sum_j S_{ij}\ \mathrm{Tr}_j(M_0\,q^{L_0-c/24})\ , \tag{4.7}$$

where Tr_i indicates the trace over the irreducible representation labelled by i. Using the results and methods of [1], we obtain

$$\begin{aligned}\mathrm{Tr}_L\Big(M_0\,q^{L_0-\frac{c}{24}}\Big) &= \mathrm{Tr}_L\Big(W_0^2\,q^{L_0-\frac{c}{24}}\Big) + f_3\ \mathrm{Tr}_L\Big(L_0^3\,q^{L_0-\frac{c}{24}}\Big) + f_2\ \mathrm{Tr}_L\Big(L_0^2\,q^{L_0-\frac{c}{24}}\Big)\\ &\quad + f_1\ \mathrm{Tr}_L\Big(L_0\,q^{L_0-\frac{c}{24}}\Big) + f_0\ \mathrm{Tr}_L\Big(q^{L_0-\frac{c}{24}}\Big)\end{aligned} \tag{4.8}$$

where, with $\beta = 16/(22+5c)$,

$$f_3 = b, \tag{4.9a}$$

$$f_2 = E_2\left(-\frac{\beta}{6} - \frac{b}{2}\right) + \left(\frac{37}{6}\beta + 4a + \frac{47}{10}b + 6e\right), \tag{4.9b}$$

$$\begin{aligned}f_1 &= E_4\left(\frac{c}{1440}b + \frac{b}{225\beta} + \frac{1}{360} - \frac{a}{60} + \frac{b}{72} + \frac{e}{60}\right) + E_2^2\left(\frac{\beta}{36} + \frac{b}{24}\right)\\ &\quad + E_2\left(\frac{c}{144}b - \frac{49}{45}\beta + \frac{2}{45\beta}b + \frac{2}{45} - \frac{2}{3}a - \frac{38}{45}b - e\right)\\ &\quad + \left(-\frac{11c}{1440}b + \frac{407}{180}\beta - \frac{11}{225\beta}b + \frac{487}{360} + \frac{401}{60}a + \frac{143}{90}b + 120d + \frac{1499}{60}e\right),\end{aligned} \tag{4.9c}$$

[3] The singularity at $22+5c=0$ corresponds to the Lee-Yang W_3 minimal model $WM(3,5)$. However, $-1+2c=0$ and $68+7c=0$ do not correspond to any W_3 minimal model.

and

$$f_0 = c\left\{E_6\left(-\frac{1}{90720}+\frac{a}{3024}-\frac{b}{4320}-\frac{e}{3024}\right)+E_2E_4\left(-\frac{\beta}{8640}\right)\right.$$
$$+E_4\left(\frac{37}{8640}\beta-\frac{1}{8640}+\frac{a}{288}+\frac{7}{2880}b+\frac{e}{288}\right)+E_2^2\left(-\frac{\beta}{864}-\frac{b}{576}\right)$$
$$+E_2\left(\frac{127}{2880}\beta-\frac{1}{1080}+\frac{a}{36}+\frac{47}{1440}b+\frac{e}{24}\right)$$
$$\left.+\left(-\frac{407}{8640}\beta+\frac{191}{181440}-\frac{191}{6048}a-\frac{143}{4320}b-\frac{271}{6048}e\right)\right\}. \tag{4.9d}$$

Simplifying this expression and using the modular transformation properties of L_0 and E_{2n}, we eventually obtain

$$\mathrm{Tr}_i\left(W_0^2\,\hat{q}^{L_0-\frac{c}{24}}\right)=\tau^6\sum_j S_{ij}\left\{\mathrm{Tr}_j\left(W_0^2\,q^{L_0-\frac{c}{24}}\right)+\frac{\beta}{i\pi\tau}\left[D^{(2)}D+\frac{c}{1440}E_4\right]\mathrm{Tr}_j\left(q^{L_0-\frac{c}{24}}\right)\right\}. \tag{4.10}$$

4.3 The GHJ calculation revisited: $n = 2$ again, $n = 3$

As stated above, $\mathrm{Tr}_i\left(W_0^2\hat{q}^{L_0-\frac{c}{24}}\right)$ was calculated in [2] to leading order in c and under the assumption that the dominant contribution comes from the vacuum. However, their techniques do allow the calculation to be performed without such assumptions, which we do here. We find precise agreement with (4.10).

In [2] the key tool for this calculation was developed: a recursion relation for traces containing zero modes, which for the weight-3 $W(z)$ field of interest here is

$$\begin{aligned}
&z_1^3\ldots z_n^3\,\mathrm{Tr}_i\left(W_0^l\,V(W,z_1)\ldots V(W,z_n)q^{L_0-\frac{c}{24}}\right)=\\
&\quad z_2^3\ldots z_n^3\,\mathrm{Tr}_i\left(W_0^{l+1}\,V(W,z_2)\ldots V(W,z_n)q^{L_0-\frac{c}{24}}\right)\\
&\quad+\sum_{k=0}^{l}\sum_{j=2}^{n}\sum_{m=0}^{\infty}\binom{l}{k}(2\pi i)^k\frac{(m-k)!}{m!}\partial_\tau^k P_{m+1-k}\left(\frac{z_j}{z_1},q\right)\\
&\quad\times z_2^3\ldots z_n^3\,\mathrm{Tr}_i\left(W_0^{l-k}\,V(W,z_2)\ldots V\left(\left\{(-1)^k(W[0])^kW\right\}[m]W,z_j\right)\ldots V(W,z_n)\,q^{L_0-\frac{c}{24}}\right)
\end{aligned} \tag{4.11}$$

where $V(a,z)=\sum_n a_n z^{-n-h_a}$ are vertex operators. Here we have used 'bracketed modes' $W[m]$, which are related to operator modes on the torus [10] rather than the complex plane, and (slightly non-standard) Weierstrass functions P_m:

$$W[m]=\frac{1}{(2\pi i)^{m+1}}\sum_{j\geq m-2}C_{3,j+2,m}\,W_j \qquad \text{and} \qquad P_m(x,q)=\frac{(2\pi i)^m}{(m-1)!}\sum_{n\neq 0}\frac{n^{m-1}x^n}{1-q^n}. \tag{4.12}$$

JHEP01(2016)089

The expansion coefficients C_{hjm} (not to be confused with the charge conjugation matrix) are defined through

$$(\log(1+z))^m (1+z)^{h-1} = \sum_{j\geq m} C_{hjm}\, z^j. \tag{4.13}$$

The modular transformation is then applied by

$$\mathrm{Tr}_i\left(W_0^n \hat{q}^{L_0-\frac{c}{24}}\right)=\sum_j S_{ij}\frac{\tau^{2n}}{(2\pi i)^n}\int_1^q \frac{\mathrm{d}z_1}{z_1}\ldots\int_1^q\frac{\mathrm{d}z_n}{z_n} z_1^3\ldots z_n^3 \mathrm{Tr}_j\left(V(W,z_1)\ldots V(W,z_n)q^{L_0-\frac{c}{24}}\right). \tag{4.14}$$

After some lengthy calculations, we find

$$\mathrm{Tr}_i\left(W_0^2\hat{q}^{L_0-\frac{c}{24}}\right) = \sum_j S_{ij}\left\{\tau^6\,\mathrm{Tr}_j\left(W_0^2 q^{L_0-\frac{c}{24}}\right)+\frac{\tau^5}{2\pi i}2\beta\left[D^{(2)}D+\frac{cE_4}{1440}\right]\mathrm{Tr}_j\left(q^{L_0-\frac{c}{24}}\right)\right\} \tag{4.15}$$

and

$$\begin{aligned}\mathrm{Tr}_i\left(W_0^3\hat{q}^{L_0-\frac{c}{24}}\right) = \sum_j S_{ij}\Bigg\{&\tau^9\,\mathrm{Tr}_j\left(W_0^3 q^{L_0-\frac{c}{24}}\right)+\frac{\tau^7}{(2\pi i)^2}18\beta\, D^{(3)}\mathrm{Tr}_j\left(W_0 q^{L_0-\frac{c}{24}}\right)\\ &+\frac{\tau^8}{2\pi i}6\beta\left[D^{(5)}D^{(3)}+\frac{E_2}{2}D^{(3)}+\frac{(c+30)E_4}{1440}\right]\mathrm{Tr}_j\left(W_0 q^{L_0-\frac{c}{24}}\right)\Bigg\}\end{aligned} \tag{4.16}$$

where we have used that, inside a trace, $L_0 = D^{(r)} + rE_2/12 + c/24$ for any r. Note that (4.15) is exactly the result we found using our method, given in (4.10).

We have listed some of the intermediate results needed for checking (4.15) and (4.16) that are not given explicitly in [2] in appendix E.

We have performed a variety of checks, all of which support these results. Firstly, we have checked that applying the modular transformations twice gives back the original trace (that is, the modular transformation squares to the identity). This is a non-trivial check as cancellations are needed between different terms.

Further, for minimal models with small numbers of representations it is possible to calculate the traces numerically (using the formulae given in [1] as sums over the Weyl group of $su(3)$). We have checked that the formulae agree numerically, using the W_3 S-matrices which can be found in [9]. Next, we assumed that the equations take the form given in (4.15) and (4.16) but with an unspecified S-matrix and fitted the values of the S-matrix using various different values of τ, recovering the correct S-matrix. We have also used the same method to fit the coefficient of the E_4 terms, as these are not constrained by the requirement that the modular transformation squares to the identity, and again recovered the correct results with excellent numerical agreement. We are therefore satisfied that these formulae are correct.

5 Modular properties of traces over Verma modules

In [1], we found the exact formula for $\mathrm{Tr}_V\left(W_0^2\, q^{L_0-\frac{c}{24}}\right)$ over a Verma module, and so we can examine its modular transformation properties directly. We reproduce here the result from [1]:

$$\mathrm{Tr}_V\left(W_0^2\, q^{L_0-\frac{c}{24}}\right) = \frac{q^{h-\frac{c}{24}}}{\phi(q)^2}\left[\begin{array}{c} w^2 \;\; + \;\; \frac{4}{15}\sum_{r=1}^{\infty}\frac{r^2(r^2-4)q^r}{(1-q^r)^2} \\ +\,4\beta\sum_{r=1}^{\infty}\frac{r^2q^r}{(1-q^r)^2}\left[2h+\gamma(r)-2\frac{rq^{2r}}{1-q^{2r}}+4\sum_{k=1}^{r}\frac{kq^k}{1-q^k}\right] \\ +\,8\beta\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}\sum_{s>r/2}^{r-1}\frac{q^s}{1-q^s}\left[\frac{r(2s-r)}{1-q^r}+\frac{s(3s-2r)}{1-q^s}\right] \end{array}\right] \tag{5.1}$$

where

$$\phi(q) = \prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}\eta(q)\;, \qquad \gamma(n) = \begin{cases} -\frac{1}{20}(n^2-4) & n \text{ even} \\ -\frac{1}{20}(n^2-9) & n \text{ odd} \end{cases}. \tag{5.2}$$

This can be written much more simply in terms of Eisenstein series and the η function, for example

$$\mathrm{Tr}_V\left(W_0^2 q^{L_0-\frac{c}{24}}\right) = \frac{q^{\tilde{h}}}{\eta^2}\left[w^2 - \frac{\beta}{3}\tilde{h}E_2' + \frac{\beta}{9}E_2'' + \frac{\beta}{2}\frac{c+30}{1440}E_4'\right] \tag{5.3}$$

$$= \left[w^2 - \frac{\beta}{3}E_2'D - \beta\frac{1}{108}E_2E_2' + \beta\frac{3c+10}{8640}E_4'\right]\frac{q^{\tilde{h}}}{\eta^2} \tag{5.4}$$

where the details of the Eisenstein series and their derivatives can be found in appendix A, and we have defined

$$\tilde{h} = h - \frac{c-2}{24}. \tag{5.5}$$

From this result, we see that we may write

$$\frac{\mathrm{Tr}_V\left(W_0^2 q^{L_0-\frac{c}{24}}\right)}{\mathrm{Tr}_V\left(q^{L_0-\frac{c}{24}}\right)} = w^2 - \frac{\beta}{3}\tilde{h}E_2' + \frac{\beta}{9}E_2'' + \frac{\beta}{2}\frac{c+30}{1440}E_4'. \tag{5.6}$$

Using the known modular properties of Eisenstein series (see again appendix A), we can therefore write down an exact expression for the modular transformation of $\mathrm{Tr}_V\left(W_0^2 q^{L_0-\frac{c}{24}}\right)$,

$$\begin{aligned}\frac{\mathrm{Tr}_V\left(W_0^2 \hat{q}^{L_0-\frac{c}{24}}\right)}{\mathrm{Tr}_V\left(\hat{q}^{L_0-\frac{c}{24}}\right)} = w^2 &- \frac{\beta}{3}\tilde{h}\left(\tau^4E_2' + 2\frac{\tau^3}{2\pi i}E_2 + 12\frac{\tau^2}{(2\pi i)^2}\right) \\ &+ \frac{\beta}{9}\left(\tau^6E_2'' + 6\frac{\tau^5}{2\pi i}E_2' + 6\frac{\tau^4}{(2\pi i)^2}E_2 + 288\frac{\tau^3}{(2\pi i)^3}\right) \\ &+ \frac{\beta}{2}\frac{c+30}{1440}\left(\tau^6E_4' + 4\frac{\tau^5}{2\pi i}E_4\right).\end{aligned} \tag{5.7}$$

In the high-temperature limit, $\tau \to +i\infty$, we have $E_{2n} \to 1$ and $E_{2n}', E_{2n}'' \to 0$, and so

$$\frac{\mathrm{Tr}_V\left(W_0^2 \hat{q}^{L_0-\frac{c}{24}}\right)}{\mathrm{Tr}_V\left(\hat{q}^{L_0-\frac{c}{24}}\right)} = 2\beta\frac{c+30}{1440}\frac{\tau^5}{2\pi i} + O(\tau^4). \tag{5.8}$$

JHEP01(2016)089

6 The large c limit

A quantity of interest in black-hole holography is the ratio

$$\frac{\mathrm{Tr}\left(W_0^2\,\hat{q}^{L_0-\frac{c}{24}}\right)}{\mathrm{Tr}\left(\hat{q}^{L_0-\frac{c}{24}}\right)} \tag{6.1}$$

where the trace is taken over the whole Hilbert space. Of particular interest is the high temperature ($\hat{q}\to 1, \tau\to +i\infty$) behaviour of this ratio in the $c\to\infty$ limit.

The $c\to\infty$ limit was calculated by Gaberdiel et al. in [2] where they found the result

$$\frac{\sum_i \mathrm{Tr}_i\left(W_0^2\,\hat{q}^{L_0-\frac{c}{24}}\right)}{\sum_i \mathrm{Tr}_i\left(\hat{q}^{L_0-\frac{c}{24}}\right)} \sim \tau^5\frac{c}{180\pi i} \qquad \text{as} \qquad \hat{q}\to 1\,, \tag{6.2}$$

for the $W_\infty[\lambda]$ algebra.[4] This algebra reduces to the W_3 algebra when λ takes the value 3 and the particular result (6.2) is independent of λ.

We have found two more results.

Firstly we have a result for the trace taken over a Verma module, equation (5.8), for which the large c limit is

$$\frac{\mathrm{Tr}_V\left(W_0^2\,\hat{q}^{L_0-\frac{c}{24}}\right)}{\mathrm{Tr}_V\left(\hat{q}^{L_0-\frac{c}{24}}\right)} = \tau^5\frac{1}{450\pi i}+O(\tau^4)\,. \tag{6.3}$$

Secondly, we have the exact result (4.10)

$$\mathrm{Tr}_i\left(W_0^2\,\hat{q}^{L_0-\frac{c}{24}}\right) = \tau^6\sum_j S_{ij}\left\{\mathrm{Tr}_j\left(W_0^2\,q^{L_0-\frac{c}{24}}\right)+\frac{\beta}{i\pi\tau}\left[D^{(2)}D+\frac{c}{1440}E_4\right]\mathrm{Tr}_j\left(q^{L_0-\frac{c}{24}}\right)\right\}. \tag{6.4}$$

The question is, how can we reconcile (6.4) with both (6.2) and (6.3)?

The answer is that we can derive both Gaberdiel et al.'s result and our result for the Verma module from (6.4) under two different assumptions on the ground state in the theory: if we assume that the state of lowest energy has $h_{\min}=0$, then we recover (6.2), whereas we recover (6.3) if $h_{\min}=c/24+O(1)$.

If $h_{\min}=0$ then the leading term in (6.4) comes from the action of $D^{(2)}D$ on the vacuum character and we find agreement with the calculation of [2].

If, however, $h_{\min}=c/24+O(1)$ then the the action of $D^{(2)}D$ on the character with this weight is no longer the leading term, but instead it is the term in $c\,E_4/1440$, and we find agreement with (6.3).

It might be surprising that the trace over irreducible representations can be related to the trace over a Verma module, but there are two cases where this is indeed the case, and in both these $h_{\min}\sim c/24$. These are real coupling Toda theory, in which we can take $c\to+\infty$, and non-unitary minimal models, in which $c<2$ but we can take $c\to-\infty$.

[4]Note that our algebra generators are related to those of [2] by $W_{\mathrm{GHJ}}=\sqrt{10}\cdot W_{IW}$.

6.1 The $c \to \infty$ limit: real coupling Toda theory

One way to consider the $c \to \infty$ limit in the context of a well-known model is through real coupling Toda theory. In real coupling a_2 Toda theory, the central charge satisfies $c > 98$. Some properties are given in appendix C, and in particular the spectrum has a minimum value of h, $h_{\rm min} = (c-2)/24$ and for each weight occurring in the spectrum, the Verma module is in fact irreducible.

Now that $h \sim c$, the leading c behaviour of (5.3) comes from

$$\mathrm{Tr}_V\left(W_0^2\, q^{L_0 - \frac{c}{24}}\right) \sim \frac{q^{\tilde{h}}}{\eta(q)^2}\left[w^2 + \frac{E_4'}{900}\right], \tag{6.5}$$

which gives the result advertised in (6.3),

$$\frac{\mathrm{Tr}_V\left(W_0^2\, \hat{q}^{L_0 - \frac{c}{24}}\right)}{\mathrm{Tr}_V\left(\hat{q}^{L_0 - \frac{c}{24}}\right)} \sim w^2 + \frac{1}{900}\left(\tau^6 E_4' + 4\frac{\tau^5}{2\pi i}E_4\right) \sim \tau^5 \frac{1}{450\pi i}, \tag{6.6}$$

where the final result comes from taking the $\tau \to +i\infty$ limit.

If we put the minimal value of h into (4.10) we see that the dominant contribution now comes from the term

$$\tau^5 \frac{\beta}{i\pi}\frac{cE_4}{1440} \sim \tau^5 \frac{1}{450\,\pi i}, \tag{6.7}$$

entirely in agreement with (6.3).

6.2 The $c \to -\infty$ limit: non-unitary minimal models

Whilst it is not possible to reach $c \to \infty$ for unitary minimal models, it is certainly possible to reach $c \to -\infty$ for non-unitary minimal models. To take a concrete series $W\!M(3, 3p+1)$ has central charge

$$c(3, 3p+1) = 50 - 8(3p+1) - \frac{72}{3p+1}. \tag{6.8}$$

In these models there is also a minimum value of h which is less than 0, which is

$$h_{\rm min} = h_{[11;pp]} = -\frac{(3p-1)(p-1)}{3p+1} = \frac{(c-2)}{24} + O(1/c^2), \tag{6.9}$$

and so again the leading term from (4.10) comes from the $(\beta/\pi i)(cE_4)/1440$ term and is in agreement with the analysis from the Verma module, (6.3).

This is perhaps not obvious, since for any particular non-unitary minimal model, a representation will have an infinite number of terms in its composition series, and this could in principle change the $\tau \to +i\infty$ behaviour. However, as $c \to -\infty$, all but a finite number of these become of ever higher weight and decouple from the spectrum; indeed for the representation of lowest weight, the character is $q^{\tilde{h}}(1 - 2q^p + \ldots)/\eta^2$ which tends to the character of the Verma module as $p \to \infty$, or equivalently as $c \to -\infty$, and so it is not surprising that the exact formula for the modular transform in these non-unitary minimal models with this choice of $h_{\rm min}$ reproduces the result for the Verma module.

The conclusion is that taking the $c \to \infty$ limit is slightly trickier than one might imagine.

JHEP01(2016)089

7 Conclusions

We have studied the modular transformations of traces of various powers of W_0. We have found that these have nice explicit forms. We have studied these using both the methods of Gaberdiel et al. in [2] as well as using a new method based on the explicit forms of these traces found in [1].

Firstly, we have shown that $\mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right)$ are vector-valued modular forms of weight 3. In the case of minimal models, we have also shown how the trace $\mathrm{Tr}_L\left(W_0\, q^{L_0-\frac{c}{24}}\right)$ obeys nice modular differential equations.

Next we found the exact modular transformation law of $\mathrm{Tr}_L\left(W_0^2\, q^{L_0-\frac{c}{24}}\right)$ and $\mathrm{Tr}_L\left(W_0^3\, q^{L_0-\frac{c}{24}}\right)$ using the methods of [2] and checked our results extensively. The calculations are somewhat cumbersome and we have not, as yet, extended them to higher powers. We also found a new method (based on Virasoro primary fields which exist for all values of c) and applied this to $\mathrm{Tr}_L\left(W_0^2\, q^{L_0-\frac{c}{24}}\right)$ and found agreement with the result using the method of [2]. This new method can be generalised to any trace $\mathrm{Tr}_L\left(W_0^n\, q^{L_0-\frac{c}{24}}\right)$. It is also somewhat cumbersome but (we think) conceptually simpler.

The results we have found are rather nice and it looks as though it might be possible to find the general term or exponentiate the results. They are more complicated than the analogous results for the trace of J_0^n in representations of an affine algebra, for example the trace of J_0^2 given in [2],

$$\mathrm{Tr}_i\left(J_0^2\, \hat{q}^{L_0-\frac{c}{24}}\right) = \tau^2 \sum_j S_{ij} \left\{ \mathrm{Tr}_j\left(J_0^2\, q^{L_0-\frac{c}{24}}\right) + \frac{k}{i\pi\tau}\, \mathrm{Tr}_j\left(q^{L_0-\frac{c}{24}}\right) \right\} , \tag{7.1}$$

but still simpler than might have been the case.

We have also shown that $c \to \infty$ limit is tricky. In [2], the authors consider the modular transform of the partition function and assume that the leading behaviour comes from the vacuum ($h = 0$) character. The $c \to \infty$ limit considered in [2] is relevant to the application to state counting in black holes, and their calculation agrees with the gravity calculation. Although [2] is concerned with the W_∞ algebra, the W_0^2 trace does not include contributions from modes with spin greater than 3, so it should agree with the same calculation in the W_3 algebra. We have reproduced the result of [2] from an exact calculation, under the assumption that the state of lowest weight has $h_{\min} = 0$.

We have tried to compare the result found in [2] with the calculation in the W_3 algebra for a trace over a Verma module, but by explicit examination we only get agreement between our exact calculation and our Verma module result if we break the assumption that $h_{\min} = 0$ and instead have $h_{\min} \sim c/24$. We have found two models which exhibit this alternative behaviour, namely Toda theory (for $c \to \infty$) and non-unitary minimal models (for the $c \to -\infty$ limit); if $h_{\min} \sim c/24$ then the exact formula indeed reproduces the result from consideration of a single Verma module.

On the other hand, in any unitary minimal model we always have $h_{\min} = 0$, and so the results of [2] are expected to hold in any large c limit of a unitary minimal model. For a

JHEP01(2016)089

W_N algebra, $c < N-1$ for minimal models, and so in order to investigate unitary minimal models of W_N algebras at large central charge one must take the $N \to \infty$ limit. Any discussion of a model with $c > 2$ in terms of the W_3 algebra would need the spectrum to include an infinite number of representations of the W_3 algebra, which might then modify the $\tau \to +i\infty$ limit in such a way that there is agreement with [2].

We will be investigating the relationship between these models and the W_3-algebra results presented here in more detail. In this context we feel that it is worth noting the partition functions considered in [11]. These are non-diagonal and include an infinite set of W-algebra representations, and so do not apply in the same arena as our calculations.[5]

In the future, it should be possible to study the modular transformation properties using the numerical evaluation of the traces in the minimal models. We only used this as a check of the results we found, but it should be possible to determine (numerically) the transformations of traces of higher powers, and check whether there is a pattern which could lead to a general formula.

Acknowledgments

We would like to thank Nadav Drukker, Matthias Gaberdiel, Juan Jottar, Sameer Murthy, Andreas Recknagel, and Volker Schomerus for their helpful comments. NJI was supported by an STFC doctoral training studentship.

A Eisenstein series, the η function etc.

The first few Eisenstein series, E_2, E_4 and E_6, are

$$E_2 = 1 - 24 \sum_{m\geq 1} \frac{mq^m}{1-q^m}\,, \qquad E_4 = 1 + 240 \sum_{m\geq 1} \frac{m^3 q^m}{1-q^m}\,, \qquad E_6 = 1 - 504 \sum_{m\geq 1} \frac{m^5 q^m}{1-q^m}\,. \tag{A.1}$$

For $n > 1$, E_{2n} is a modular form of weight $2n$ while E_2 is a holomorphic connection [8]. This means that under a modular transformation $\hat{q} = \exp(-2\pi i/\tau) \mapsto q = \exp(2\pi i\tau)$,

$$E_{2n}(\hat{q}) = \tau^{2n} E_{2n}(q)\,, \qquad n > 1\,, \qquad E_2(\hat{q}) = \tau^2 E_2(q) + \frac{6\tau}{\pi i}\,, \tag{A.2}$$

and so the combination

$$D^{(r)} = q\frac{\mathrm{d}}{\mathrm{d}q} - \frac{r}{12} E_2 \tag{A.3}$$

is a modular covariant derivative which maps forms of weight r to forms of weight $(r+2)$. We also define $D \equiv D^{(0)} = q(\mathrm{d}/\mathrm{d}q)$. The derivatives of E_2, E_4 and E_6 are

$$q\frac{\mathrm{d}}{\mathrm{d}q} E_2 = \frac{E_2^2 - E_4}{12}, \qquad q\frac{\mathrm{d}}{\mathrm{d}q} E_4 = \frac{E_2 E_4 - E_6}{3}, \qquad q\frac{\mathrm{d}}{\mathrm{d}q} E_6 = \frac{E_2 E_6 - E_4^2}{2}, \tag{A.4}$$

and so we can find their modular transformations,

$$\widehat{E_2'} = \tau^4 E_2' + \frac{2\tau^3}{2\pi i} + \frac{12\tau^2}{(2\pi i)^2}, \qquad \widehat{E_4'} = \tau^6 E_4' + \frac{4\tau^5}{2\pi i} E_4, \qquad \widehat{E_6'} = \tau^8 E_6' + \frac{6\tau^7}{2\pi i} E_6, \tag{A.5}$$

where the prime $'$ denotes $q\,\mathrm{d}/\mathrm{d}q$.

[5] We thank Matthias Gaberdiel for bringing this work to our attention.

The Dedekind η function is defined as

$$\eta(q) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) \ , \tag{A.6}$$

and is a modular form of weight (1/2), satisfying

$$\eta(\hat{q}) = \sqrt{-i\tau}\, \eta(q) \ , \qquad D^{(1/2)}\eta = \left[q\frac{\mathrm{d}}{\mathrm{d}q} - \frac{1}{24}E_2 \right] \eta = 0 \tag{A.7}$$

i.e. $\eta' = \eta\, E_2/24$.

B The W_3 algebra, its representations and minimal models

The W_3 algebra is generated by modes W_m, L_m with commutation relations

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m \left(m^2 - 1\right) \delta_{m+n,0} \tag{B.1}$$

$$[L_m, W_n] = (2m-n)\, W_{m+n} \tag{B.2}$$

$$[W_m, W_n] = (m-n) \left[\frac{1}{15} (m{+}n{+}3)\, (m{+}n{+}2) - \frac{1}{6} (m{+}2)\, (n{+}2) \right] L_{m+n} \tag{B.3}$$

$$+\, \beta\, (m{-}n)\, \Lambda_{m+n} + \frac{c}{360} m \left(m^2{-}1\right) \left(m^2{-}4\right) \delta_{m+n,0} \tag{B.4}$$

where

$$\Lambda_n = \sum_{p\leq -2} L_p L_{n-p} + \sum_{p\geq -1} L_{n-p} L_p - \frac{3}{10}(n{+}2)(n{+}3) L_n \ , \qquad \beta = \frac{16}{22+5c} \ . \tag{B.5}$$

The minimal models of the W_3 algebra are well-understood. They occur for values of c such that

$$c = 50 - 24t - \frac{24}{t} \ , \tag{B.6}$$

where $t = p/p'$ is rational with p and p' co-prime integers, both greater than two. The W_3 algebra at this central charge has $\mathcal{N}$ representations, where

$$\mathcal{N}(p,p') = \frac{(p-2)(p-1)(p'-2)(p'-1)}{12} \ . \tag{B.7}$$

The possible modular invariant partition functions for minimal models have been classified by Beltaos and Gannon in [9]. We are not interested in the particular partition functions, but just the representations that arise, and so we shall simply call these $W\!M(p,p')$, for convenience.

The highest weight representations are parametrised by (h,w) which are the eigenvalues of L_0 and W_0 on the highest weight state $|h,w\rangle$.

The highest weight representations in the minimal model $W\!M(p,p')$ are parametrised by two weights of affine $a_2^{(1)}$, μ and μ', at levels $p{-}3$ and $p'{-}3$ respectively, modulo the action of Z_3. The representations are conventionally labelled $[rs;r's']$ where $\mu = (p - r -$

$s-1)\Lambda_0 + (r-1)\Lambda_1 + (s-1)\Lambda_2$ is a weight of $(a_2^{(1)})_{p-3}$, and $\mu' = (p'-r'-s'-1)\Lambda_0 + (r'-1)\Lambda_1 + (s'-1)\Lambda_2$ is a weight of $(a_2^{(1)})_{p'-3}$ (these Λ are the fundamental weights of $a_2^{(1)}$, they are not the same Λ as in (B.5) — see [7] for details of the weights of affine algebras). These W_3 algebra representations have conformal weight and W_0 eigenvalue

$$h_{[rs;r's']} = \frac{(r-r't)^2 + (r-r't)(s-s't) + (s-s't)^2 - 3(1-t)^2}{3t} , \tag{B.8}$$

$$w_{[rs;r's']} = \sqrt{\frac{2}{3}} \frac{\big(r-s-(r'-s')t\big)\big(2r+s-(2r'+s')t\big)\big(r+2s-(r'+2s')t\big)}{9t\sqrt{(5-3t)(5t-3)}} . \tag{B.9}$$

The vacuum representation of the W-algebra is hence labelled $[11;11]$. The Z_3 symmetry is given by $[rs;r's'] \equiv [(p-r-s)r;(p'-r'-s')r']$. The advantage of this labelling is that the representation $[rs;r's']$ has independent singular vectors at levels rs and $r's'$. *Conjugation* of a representation takes $(h,w) \to (h,-w)$; only representations with $w=0$ (also referred to as *uncharged* representations) are self-conjugate.

C Real coupling a_2 Toda theory

Quantum conformal Toda field theories can be associated to any finite-dimensional Lie algebra g and are theories constructed from $r = \mathrm{rank}(g)$ bosonic scalar fields which depend on a coupling constant (denoted variously β [12] and b [15]) and which have W-algebra symmetries [12]. When the coupling constant is imaginary, the central charge takes values $c \le r$, but for real coupling the central charge can become large and positive. The W_3 algebra is a symmetry of a_2 Toda theory, and the central charge is

$$c = 50 + 24u + \frac{24}{u} , \tag{C.1}$$

where $u = b^2$ and so $c \ge 98$.

The states in the a_2 Toda field theory are parametrised by a "momentum", denoted variously ω [13], m [14] and α [15]. If we use the notation of [15], then we can restrict the momenta to the values

$$\alpha = Q + i(a_1\lambda_1 + a_2\lambda_2) , \tag{C.2}$$

where (a_1, a_2) are two real numbers. With our normalisations, these are

$$h = \frac{a_1^2 + a_1a_2 + a_2^2}{3} + \frac{(c-2)}{24} , \qquad w = \sqrt{\frac{2}{3}} \frac{(a_1-a_2)(2a_1+a_2)(a_1+2a_2)}{9\sqrt{34+15u+15/u}} . \tag{C.3}$$

In this theory, the spectrum of fields is continuous but the conformal dimensions are bounded below by $h_{\min} = (c-2)/24 \ge 4$. These representations are all irreducible (they are the delta-normalised states of [14]) with character χ_h, and the partition function is given by the partition function for a pair of uncompactified free bosons,

$$Z = \frac{1}{\sqrt{3}} \iint \mathrm{d}a_1\, \mathrm{d}a_2\, |\chi_{h(a_1,a_2)}|^2 = \frac{1}{\mathrm{Im}(\tau)} \frac{1}{|\eta|^4} , \qquad \chi_h = \frac{q^{h-\frac{c-2}{24}}}{\eta(q)^2} . \tag{C.4}$$

This is clearly modular invariant.

JHEP01(2016)089

D Modular properties for traces over Verma modules

We can now discuss the modular properties of Toda theory and of a possible reduction.

We start with the modular S-matrix for two uncompactified free bosons. Consider the momentum state $|\mathbf{p}\rangle$ with $\mathbf{p} = (p_1, p_2)$ of conformal weight $h(\mathbf{p}) = \frac{1}{2}\mathbf{p}^2$. The character is

$$\chi_{\mathbf{p}} = \frac{q^{\mathbf{p}^2/2}}{\eta(q)^2} \,. \tag{D.1}$$

Using the Fourier transform

$$e^{-\pi i \mathbf{p}^2/\tau} = (-i\tau) \iint e^{-2\pi i \mathbf{p}\cdot\mathbf{p}'} e^{\pi i \tau \mathbf{p}'^2} \, \mathrm{d}^2 p' \,, \tag{D.2}$$

we have

$$\chi_{\mathbf{p}}(\hat{q}) = \iint e^{-2\pi i \mathbf{p}\cdot\mathbf{p}'} \chi_{\mathbf{p}'}(q) \, \mathrm{d}^2 p' \,. \tag{D.3}$$

We can now use this to find the modular transform of the characters of the Virasoro algebra that appear in Toda theory. If we take the momentum to be

$$\mathbf{p} = a_1 \lambda_{\mathbf{1}} + a_2 \lambda_{\mathbf{2}} = \left(\frac{a_1}{\sqrt{2}}, \frac{a_1}{\sqrt{6}} + \sqrt{\frac{2}{3}} a_2 \right) \,, \tag{D.4}$$

then

$$\frac{1}{2}\mathbf{p}^2 = h(a_1, a_2) - \frac{c-2}{24} \,, \qquad \chi_{h(a_1,a_2)}(q) = \frac{q^{\frac{1}{2}\mathbf{p}^2}}{\eta(q)^2} \,. \tag{D.5}$$

This means we can rewrite (D.3) as a modular transformation of the specialised W-algebra characters,

$$\chi_{h(a_1,a_2)}(\hat{q}) = \iint S_{(a_1,a_2),(b_1,b_2)} \, \chi_{h(b_1,b_2)}(q) \, \mathrm{d}b_1 \, \mathrm{d}b_2 \,, \tag{D.6}$$

where

$$S_{(a_1,a_2),(b_1,b_2)} = \exp\left(-\frac{2\pi i}{3} (2a_1 b_1 + 2a_2 b_2 + a_1 b_2 + a_2 b_1) \right) \,. \tag{D.7}$$

The integral in (D.6) is over all (b_1, b_2) and this overcounts the representations of the W algebra. The reason is that the weights (C.3) are invariant under the Weyl group of a_2, generated by

$$\omega_1 : (a_1, a_2) \mapsto (-a_1, a_1 + a_2) \,, \qquad \omega_2 : (a_1, a_2) \mapsto (a_1 + a_2, -a_2) \,. \tag{D.8}$$

Under the action of the Weyl group, the space of momenta splits into six Weyl chambers, with the fundamental Weyl chamber being $\{b_i \geq 0\}$, so that we can rewrite equation (D.6) as

$$\chi_{(a_1,a_2)}(\hat{q}) = \iint_{b_i \geq 0} \tilde{S}_{(a_1,a_2),(b_1,b_2)} \, \chi_{(b_1,b_2)}(q) \, \mathrm{d}b_1 \, \mathrm{d}b_2 \tag{D.9}$$

JHEP01(2016)089

where

$$\tilde{S}_{(a_1,a_2),(b_1,b_2)} = \sum_{\omega\in W} S_{(a_1,a_2),\omega\circ(b_1,b_2)}$$
$$= S_{(a_1,a_2),(b_1,b_2)} + S_{(a_1,a_2),(-b_1,b_1+b_2)} + S_{(a_1,a_2),(b_1+b_2,-b_2)}$$
$$+ S_{(a_1,a_2),(b_2,-b_1-b_2)} + S_{(a_1,a_2),(-b_1-b_2,b_1)} + S_{(a_1,a_2),(-b_2,-b_1)} \,. \tag{D.10}$$

We can now change variables from (a_2, a_2) to (h, w), and rewrite (D.9) as

$$\chi_h(\hat{q}) = \iint_{\mathcal{D}} \tilde{S}_{(a_1,a_2),(b_2,b_2)}\, \chi_{h'}(q) \left|\frac{\partial(b_1,b_2)}{\partial(h',w')}\right| \mathrm{d}h'\,\mathrm{d}w' \tag{D.11}$$

$$= \iint_{\mathcal{D}} S_{(h,w),(h',w')}\, \chi_{h'}(q)\, \mathrm{d}h'\,\mathrm{d}w' \;, \tag{D.12}$$

where the region $\mathcal{D}$ is given by (recall that $\tilde{h} = h - \frac{c-2}{24}$)

$$\mathcal{D} = \left\{ (h', w') \;\middle|\; \tilde{h}' \geq 0\;,\;\; w'^2 \leq \frac{4}{9}\beta\tilde{h}'^3 \right\} \;, \tag{D.13}$$

and the Jacobian is

$$\left|\frac{\partial(b_1,b_2)}{\partial(h',w')}\right| = \sqrt{\frac{3}{4\beta\tilde{h}'^3 - 9w'^2}}\;, \tag{D.14}$$

and consequently the S-matrix is

$$S_{(h,w),(h',w')} = \sqrt{\frac{3}{4\beta\tilde{h}'^3 - 9w'^2}}\; \tilde{S}_{(a_1,a_2),(b_1,b_2)} \;. \tag{D.15}$$

This is a generalisation of the modular S-matrices in [16], and just as in that paper, the modular transform (D.12) only includes representations with $h \geq (c-2)/24$.

Having found the formula (D.12), we can check that the modular transform of $\mathrm{Tr}_V(W_0 q^{L_0 - c/24})$ satisfies (4.3) with the same S-matrix.

This is straightforward if we start again from the the free-field S-matrix. We can introduce shifts $\mathbf{d}$ in (D.2),

$$\exp(-\pi i\mathbf{p}^2/\tau + \mathbf{d}\cdot\mathbf{p}) = (-i\tau) \iint \exp(\pi i\tau\mathbf{p}'^2 - 2\pi i\mathbf{p}\cdot\mathbf{p}') \exp\left(-\mathbf{d}^2\frac{\tau^2}{4\pi^2} + \mathbf{d}\cdot\mathbf{p}'\tau\right) \mathrm{d}^2p' \,. \tag{D.16}$$

We can then write the character $\mathrm{Tr}_V\left(W_0 q^{L_0 - \frac{c}{24}}\right)$ in terms of a differential operator on the shifted character,

$$w(\mathbf{p})e^{-\pi i\mathbf{p}^2/\tau} = \frac{1}{9}\sqrt{\frac{\beta}{2}} \left(3\frac{\partial^2}{\partial d_1^2} - \frac{\partial^2}{\partial d_2^2}\right) \frac{\partial}{\partial d_2} \left[e^{-\pi i\mathbf{p}^2/\tau + \mathbf{d}\cdot\mathbf{p}}\right]\Big|_{\mathbf{d}=0} \,. \tag{D.17}$$

Applying this to (D.16), after some calculation we get

$$w(\mathbf{p})e^{-\pi i\mathbf{p}^2/\tau} = \tau^3\,(-i\tau) \iint w(\mathbf{p}')\, e^{\pi i\tau\mathbf{p}'^2}\, e^{-2\pi i\mathbf{p}\cdot\mathbf{p}'}\, \mathrm{d}^2p' \;, \tag{D.18}$$

and so finally obtain

$$w\,\chi_h(\hat{q}) = \tau^3 \iint_{\mathcal{D}} S_{(h,w),(h',w')}\, w'\,\chi_{h'}(q)\, \mathrm{d}h'\,\mathrm{d}w' \;, \tag{D.19}$$

exactly in agreement with (4.3).

JHEP01(2016)089

E Some results needed for calculations in section 4.3

Here we state some useful intermediate results obtained in the calculation of $\mathrm{Tr}_i\left(W_0^n \hat{q}^{L_0-\frac{c}{24}}\right)$ for $n=2$ (4.15) and $n=3$ (4.16).

E.1 $n = 2$

After using the modular transformation (4.14) and applying the recursion relation (4.11) once to the result, we reach the expression (where here and below we use the shorthand notation $\int_{i\ldots k} = \int_1^q \frac{\mathrm{d}z_i}{z_i} \ldots \int_1^q \frac{\mathrm{d}z_k}{z_k}$)

$$\mathrm{Tr}_i(W_0^2 \hat{q}^{L_0-\frac{c}{24}}) = \sum_j S_{ij} \frac{\tau^4}{(2\pi i)^2} \int_{12} \left\{ \mathrm{Tr}_j(W_0^2 q^{L_0-\frac{c}{24}}) + \sum_{m\geq 0} P_{m+1}\left(\frac{z_2}{z_1}, q\right) \mathrm{Tr}_j((W[m]W)_0 q^{L_0-\frac{c}{24}}) \right\} \tag{E.1}$$

so we need

$$\int_{12} P_{m+1}\left(\frac{z_2}{z_1}, q\right) = \begin{cases} (2\pi i)^3 \frac{\tau}{2} (1-\tau) & \text{if} \quad m = 0 \\ (2\pi i)^3 \tau & \text{if} \quad m = 1 \\ 0 & \text{if} \quad m \geq 2 \end{cases} \tag{E.2}$$

and therefore the relevant traces are

$$(2\pi i)\, \mathrm{Tr}_j \left((W\,[0]\,W)_0\, q^{L_0-\frac{c}{24}} \right) = 0, \tag{E.3a}$$

$$\begin{aligned} (2\pi i)^2\, \mathrm{Tr}_j \left((W\,[1]\,W)_0\, q^{L_0-\frac{c}{24}} \right) &= 2\beta\, \mathrm{Tr}_j \left(L_0^2 q^{L_0-\frac{c}{24}} \right) - \beta \left(\frac{E_2}{3} + \frac{c}{6} \right) \mathrm{Tr}_j \left(L_0 q^{L_0-\frac{c}{24}} \right) \\ &\quad + \beta \left(\frac{cE_4}{720} + \frac{cE_2}{72} + \frac{c^2}{288} \right) \mathrm{Tr}_j \left(q^{L_0-\frac{c}{24}} \right) \\ &= 2\beta \left[D^{(2)} D + \frac{cE_4}{1440} \right] \mathrm{Tr}_j \left(q^{L_0-\frac{c}{24}} \right), \end{aligned} \tag{E.3b}$$

where in the last line we have used $L_0 = D^{(r)} + rE_2/12 + c/24$ inside a trace. If we now put these results into (E.1), we obtain (4.15).

E.2 $n = 3$

Now we use the modular transformation (4.14) followed by two applications of the recursion relation (4.11) to get

$$\begin{aligned} \mathrm{Tr}_i \left(W_0^3\, \hat{q}^{L_0-\frac{c}{24}} \right) = \sum_j S_{ij} \frac{\tau^6}{(2\pi i)^3} \int_{123} \Bigg\{ & \mathrm{Tr}_j \left(W_0^3\, q^{L_0-\frac{c}{24}} \right) \\ & + 3 \sum_{m\geq 0} P_{m+1} \left(\frac{z_3}{z_2}, q \right) \mathrm{Tr}_j \left(W_0\, (W[m]W)_0\, q^{L_0-\frac{c}{24}} \right) \\ & - \sum_{m\geq 1} \frac{2\pi i}{m}\, \partial_\tau P_m \left(\frac{z_3}{z_2}, q \right) \mathrm{Tr}_j \left([(W[0]W)[m]W]_0\, q^{L_0-\frac{c}{24}} \right) \end{aligned}$$

$$- \sum_{n\neq 0} \frac{2\pi i}{n} \left(\frac{z_3}{z_2}\right)^n \partial_\tau \left(\frac{1}{1-q^n}\right) \mathrm{Tr}_j \Big([(W[0]W)[0]W]_0 \, q^{L_0 - \frac{c}{24}} \Big)$$

$$+ \sum_{m\geq 0} \sum_{n\geq 0} P_{m+1}\left(\frac{z_2}{z_1}, q\right) P_{n+1}\left(\frac{z_3}{z_2}, q\right) \mathrm{Tr}_j \Big([(W[m]W)[n]W]_0 \, q^{L_0 - \frac{c}{24}} \Big)$$

$$+ \sum_{m\geq 0} \sum_{n\geq 0} P_{m+1}\left(\frac{z_3}{z_1}, q\right) P_{n+1}\left(\frac{z_3}{z_2}, q\right) \mathrm{Tr}_j \Big([W[n](W[m]W)]_0 \, q^{L_0 - \frac{c}{24}} \Big) \Bigg\}. \tag{E.4}$$

The integrals are, with the q argument suppressed in the P-functions and $i \neq j \neq k$,

$$\int_{ijk} P_{m+1}\left(\frac{z_j}{z_i}\right) = (2\pi i)^4 \, \tau^2 \qquad (m=1) \tag{E.5a}$$

$$\int_{ijk} \frac{2\pi i}{m} \, \partial_\tau P_m \left(\frac{z_j}{z_i}\right) = -\,(2\pi i)^5 \, \frac{\tau}{2} \qquad (m=1 \text{ or } m=2) \tag{E.5b}$$

$$\int_{ijk} P_{m+1}\left(\frac{z_j}{z_i}\right) P_{n+1}\left(\frac{z_k}{z_i}\right) = (2\pi i)^5 \, \tau \qquad (m=n=1) \tag{E.5c}$$

$$= (2\pi i)^5 \, \frac{\tau}{2} \, (1-\tau) \qquad (m=0, n=1 \text{ or } m=1, n=0) \tag{E.5d}$$

$$\int_{ijk} P_{m+1}\left(\frac{z_j}{z_i}\right) P_{n+1}\left(\frac{z_j}{z_i}\right) = (2\pi i)^5 \, \tau \qquad (m=n=1) \tag{E.5e}$$

$$= (2\pi i)^5 \, \frac{\tau}{2} \, (1-\tau) \qquad (m=0, n=1 \text{ or } m=1, n=0)\,. \tag{E.5f}$$

All integrals vanish for values of m and n higher than those shown here.

The relevant traces are

$$(2\pi i)^2 \, \mathrm{Tr}_j \Big(W_0 \, (W[1]\, W)_0 \, q^{L_0 - \frac{c}{24}} \Big)$$

$$= 2\beta \, \mathrm{Tr}_j \Big(L_0^2 \, W_0 \, q^{L_0 - \frac{c}{24}} \Big) - \beta \left(\frac{E_2}{3} + \frac{c}{6} \right) \mathrm{Tr}_j \Big(L_0 W_0 \, q^{L_0 - \frac{c}{24}} \Big)$$

$$+ \beta \left(\frac{cE_4}{720} + \frac{cE_2}{72} + \frac{E_4}{12} - \frac{E_2^2}{12} + \frac{c^2}{288} \right) \mathrm{Tr}_j \Big(W_0 q^{L_0 - \frac{c}{24}} \Big)$$

$$= 2\beta \left[D^{(5)} D^{(3)} + \frac{E_2}{2} D^{(3)} + \frac{E_4}{1440} \, (c+30) \right] \mathrm{Tr}_j \Big(W_0 \, q^{L_0 - \frac{c}{24}} \Big) \tag{E.6}$$

and

$$(2\pi i)^4 \mathrm{Tr}_j \Big([(W[1]W)[1]W]_0 q^{L_0 - \frac{c}{24}} \Big) = 12\beta \mathrm{Tr}_j \Big(L_0 W_0 q^{L_0 - \frac{c}{24}} \Big) - \beta \Big(3E_2 + \frac{c}{2} \Big) \mathrm{Tr}_j \Big(W_0 q^{L_0 - \frac{c}{24}} \Big)$$

$$= 12\beta D^{(3)} \mathrm{Tr}_j \Big(W_0 q^{L_0 - \frac{c}{24}} \Big) \tag{E.7a}$$

$$(2\pi i)^4 \mathrm{Tr}_j \Big([(W[0]W)[2]W]_0 q^{L_0 - \frac{c}{24}} \Big) = -12\beta D^{(3)} \mathrm{Tr}_j \Big(W_0 q^{L_0 - \frac{c}{24}} \Big) \tag{E.7b}$$

$$(2\pi i)^4 \mathrm{Tr}_j \Big([W[1](W[1]W)]_0 q^{L_0 - \frac{c}{24}} \Big) = 12\beta D^{(3)} \mathrm{Tr}_j \Big(W_0 q^{L_0 - \frac{c}{24}} \Big) \tag{E.7c}$$

$$(2\pi i)^4 \mathrm{Tr}_j \Big([W[2](W[0]W)]_0 q^{L_0 - \frac{c}{24}} \Big) = 12\beta D^{(3)} \mathrm{Tr}_j \Big(W_0 q^{L_0 - \frac{c}{24}} \Big) \tag{E.7d}$$

JHEP01(2016)089

where again we have used $L_0 = D^{(r)} + rE_2/12 + c/24$ when inside a trace. The remaining traces, over the zero modes of $(W[m]W)[n]W$ and $W[m](W[n]W)$ for $m+n \leq 1$, all vanish.

Finally, putting these results into (E.4), we recover (4.16).

References

[1] N.J. Iles and G.M.T. Watts, *Characters of the W_3 algebra*, *JHEP* **02** (2014) 009 [arXiv:1307.3771] [INSPIRE].

[2] M.R. Gaberdiel, T. Hartman and K. Jin, *Higher Spin Black Holes from CFT*, *JHEP* **04** (2012) 103 [arXiv:1203.0015] [INSPIRE].

[3] S. Mukhi, *Modular geometry and the classification of rational conformal field theories*, in the proceedings of the *3rd Regional Conference on Mathematical Physics: Superstrings & Relativity*, Islamabad, Pakistan, February 18–24 1989 [INSPIRE].

[4] S.D. Mathur, S. Mukhi and A. Sen, *On the Classification of Rational Conformal Field Theories*, *Phys. Lett.* **B 213** (1988) 303 [INSPIRE].

[5] S.D. Mathur, S. Mukhi and A. Sen, *Differential Equations for Correlators and Characters in Arbitrary Rational Conformal Field Theories*, *Nucl. Phys.* **B 312** (1989) 15 [INSPIRE].

[6] S.D. Mathur, S. Mukhi and A. Sen, *Reconstruction of Conformal Field Theories From Modular Geometry on the Torus*, *Nucl. Phys.* **B 318** (1989) 483 [INSPIRE].

[7] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer (1997).

[8] D. Zagier, *Introduction to Modular Forms*, in *From Number Theory to Physics*, M. Waldschmidt, P. Moussa, J.-M. Luck and C. Itzykson eds., Springer (1992).

[9] E. Beltaos and T. Gannon, *The W_N minimal model classification*, *Commun. Math. Phys.* **312** (2012) 337 [arXiv:1004.1205] [INSPIRE].

[10] M.R. Gaberdiel and C.A. Keller, *Modular differential equations and null vectors*, *JHEP* **09** (2008) 079 [arXiv:0804.0489] [INSPIRE].

[11] M.R. Gaberdiel and R. Gopakumar, *Higher Spins & Strings*, *JHEP* **11** (2014) 044 [arXiv:1406.6103] [INSPIRE].

[12] P. Mansfield and B.J. Spence, *Toda Theories, W-algebras, and minimal models*, *Nucl. Phys.* **B 362** (1991) 294 [INSPIRE].

[13] T.J. Hollowood and P. Mansfield, *Rational Conformal Field Theories At and Away From, Criticality as Toda Field Theories*, *Phys. Lett.* **B 226** (1989) 73 [INSPIRE].

[14] N. Drukker, D. Gaiotto and J. Gomis, *The Virtue of Defects in 4D Gauge Theories and 2D CFTs*, *JHEP* **06** (2011) 025 [arXiv:1003.1112] [INSPIRE].

[15] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $N = 2$ SU(N) quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [INSPIRE].

[16] J. Lacki, M. Ruiz-Altaba and P. Zaugg, *Modular transformations of $c \geq 1$ Virasoro characters*, *Phys. Lett.* **B 247** (1990) 493 [INSPIRE].

JHEP01(2016)089