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On marginal deformations and non-integrability

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Abstract: We study the interplay between a particular marginal deformation of $\mathcal{N}=4$ super Yang-Mills theory, the β deformation, and integrability in the holographic setting. Using modern methods of analytic non-integrability of Hamiltonian systems, we find that, when the β parameter takes imaginary values, classical string trajectories on the dual background become non-integrable. We expect the same to be true for generic complex β parameter. By exhibiting the Poincaré sections and phase space trajectories for the generic complex β case, we provide numerical evidence of strong sensitivity to initial conditions. Our findings agree with expectations from weak coupling that the complex β deformation is non-integrable and provide a rigorous argument beyond the trial and error approach to non-integrability.

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Contents

1 Introduction

The AdS/CFT correspondence is a powerful tool that has provided a bridge connecting field theory and gravity [1–3]. In its most powerful setting it implies a mathematical identification between strings in $\mathrm{AdS}_5 \times S^5$ with Ramond-Ramond flux and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (sYM). It is particularly insightful in the strong coupling limit of field theories.

Integrability plays a fundamental role in the AdS/CFT correspondence, and has brought us closest to the potential solution of $\mathcal{N} = 4$ sYM in the planar limit [4]. In the planar limit of $\mathcal{N} = 4$ sYM and in the dual string theory on $\mathrm{AdS}_5 \times S^5$ integrability is beyond doubt [4]. The advances achieved using integrability naturally beg the question of whether these techniques can be applied in a wider context. For example, beyond the planar limit of $\mathcal{N} = 4$ sYM, or to the deformations of sYM. It is, therefore, particularly instructive to bring the topic of non-integrability to weigh on in particular modifications of $\mathcal{N} = 4$ sYM to place a bound on the power of integrability in the wider context.

To motivate the role of non-integrability from a more general perspective, we first recall that some of the key advances in our understanding of the AdS/CFT correspondence have been propelled by semiclassical quantization. Semiclassical quantization is the best way to approach backgrounds with RR charges where standard techniques in string theory fail. Classical trajectories of strings, and branes, have played a fundamental role in developments within the AdS/CFT correspondence. For example, semiclassical quantization played a central role in the study of BMN [5], GKP [6] and rotating strings [7] which can all be understood as classical trajectories of the string with the corresponding fluctuations. The phase

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space of most mechanical systems is not integrable and thus the role of chaotic classical trajectories needs to be revisited in the context of semiclassical quantization of strings, as originally advocated in [8]. Several applications and examples have recently been provided [9–14], as will be summarized in section 2. Certainly much remains to be elucidated, including working out explicitly some entries in the AdS/CFT dictionary under chaotic motion.

A particularly natural class of deformations of field theories are those that are marginal. Leigh and Strassler obtained powerful results pertaining to the marginal deformations of $\mathcal{N} = 4$ sYM [15]. Their work is purely in field theory, where the superpotential of the initial theory is deformed with appropriate exponentials related to global charges. One of the first attempts to construct the gravitational dual of these exactly marginal deformations was presented in [16]. The problem yielded to an explicit and elegant solution in the particular case of the so-called β deformation in [17]. The gravity dual background can, in general, be constructed by applying a transformation of the form $S_{\tilde{\sigma}} T s_{\tilde{\gamma}} T S_{\tilde{\sigma}}^{-1}$ in the original $AdS_5 \times S^5$ theory. The inner $T s_{\tilde{\gamma}} T$ transformation refers to a series of T-duality, a shift with parameter $\tilde{\gamma}$ and a T-duality on an appropriate combination of two of U(1) angles of the metric, and produces the real β dual background. The additional $S_{\tilde{\sigma}}$ deformation denotes an $SL(2, R)$ transformation with a parameter $\tilde{\sigma}$ to generate the Lunin-Maldacena (LM) background with complex β. In other words, starting from the $AdS_5 \times S^5$ space, a generic appropriate $SL(3, R)$ transformation can be applied to the eight dimensional theory to obtain the complex background.

Having a holographic dual to the β deformation of $\mathcal{N} = 4$ sYM opens the gate to answering many questions explicitly. In fact, the real β-deformed theory has been studied extensively (see [18] for a review in the context of AdS/CFT integrability) and, in several cases, has been found to resemble closely, although non-trivially, the results of its undeformed parent theory. For example, the expectation values of particular BPS-like Wilson loops [19] remain undeformed. Moreover, a Lax pair for the real β-deformation was explicitly constructed in [20] by relating the deformed system to the undeformed one, therefore establishing integrability. On the field theory side, the appropriate twist of the integrable structure was discussed in [21].

However, for *complex* β the existence of integrability was doubtful from the very beginning. In [22], it was shown that the 2-scalar field sector enjoys 1-loop integrability, but in [23] 2-loop integrability was argued to be problematic. In the 3-scalar sector, already at one-loop the dilatation operator was shown not to map to an integrable hamiltonian [24]. On the gravity side, the intuition [23] is that the above-mentioned S-dualities necessary to obtain the complex-β background introduce string interactions in intermediate stages, which (as in the study of non-planar effects, see [25] for a review) are believed to break integrability. Thus the current consensus is that the complex deformation is not integrable. However a rigorous proof on the strong-coupling side remains lacking.

Using methods of analytic non-integrability, in this paper we show that a particular complex beta deformation, where the deformation parameter is taken to be imaginary, *is not integrable*. We find a coherent picture of the interplay between the marginal β deformation and integrability. Namely, we analytically prove non-integrability of string motion on the LM background corresponding to deforming $\mathcal{N} = 4$ sYM with an imaginary β pa-

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rameter. We also perform a numerical analysis of the dynamics of string motion for general complex beta, which shows chaotic-like motion while a similar analysis for the real-β deformation shows no signs of non-integrability. Our work provides the most conclusive yet answer to the question of integrability of the marginal deformations of $\mathcal{N} = 4$ sYM. As part of the proof of non-integrability in the complex deformed background, we find certain new string solutions. It is worth remarking that due to the fact that the complex deformed background has an overall conformal factor in the AdS part of the metric and in the complex deformed 5-sphere, as well as a B-field having components along several directions, its system of string equations has major differences compared to that of the real β deformed [23] and to that of the undeformed theory. A recent realization of this appeared in the computation of the generalized cusp anomalous dimension [26] where in the gravity side for real β parameter the system of string equations can be partially mapped to the undeformed $\mathcal{N} = 4$ sYM, while for the complex β this is not possible. Another computation pointing out such differences, is the Lax pair construction for the real β deformed background, which is crucially based on a correspondence between the deformed and undeformed string equations [20, 27], while this mapping can not be made in the complex β deformed background.

We should emphasise that our results do not exclude integrability in certain subsectors of string motion. In particular, for the complex β deformation, a certain subsector of $\mathcal{N} = 4$ sYM consisting of two holomorphic and one antiholomorphic scalar has been shown to be integrable at one-loop level [28] and, recently, fast spinning string motion in this subsector, for the purely imaginary deformation, was argued to be consistent with integrability [29]. Our string Ansatz, being more general than that of [29], is sufficiently generic to exhibit the non-integrability of the background. The search for and classification of integrable sub-sectors of non-integrable theories remains a very important element in mapping the transition to non-integrability and we hope that our results will provide new impetus for such work.

The paper is organized as follows. In section 2 we review the essential statements and results in the field of analytic non-integrability of Hamiltonian systems. Then we review the supergravity background dual to the β-deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills in section 3. We consider the string sigma model in section 4, and in section 5 we find an integrable geodesic solution. We analytically prove that the background is non-integrable in section 6, by finding and studying an extended string solution. In section 7, we study numerically the sensitivity of string motion in the complex β deformed background with respect to initial conditions. The behavior we find is in line with our analytic results. We conclude in section 8 with a summary of our results and pointing out some interesting new directions to explore.

2 Analytic non-integrability in Hamiltonian systems

In this section, we briefly review the main statements in the area of analytic[1] non-integrability [30–32]. Proving non-integrability of a system of differential equations

[1]By analytic we mean meromorphic. A meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all D except a set of isolated points, which are poles of the function.

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$\dot{\vec{x}} = \vec{f}(\vec{x})$ is based on the analysis of the variational equation around a particular solution $\bar{x} = \bar{x}(t)$. The variational equation around $\bar{x}(t)$ is a linear system obtained by linearizing the vector field around $\bar{x}(t)$. If the original nonlinear system admits some first integrals, the variational equation does so as well. It follows that showing that the variational equation does not admit any first integrals within a given class of functions implies that the original nonlinear system is non-integrable. In particular, we are working in the analytic setting where inverting the solution $\bar{x}(t)$ one obtains a (noncompact) Riemann surface Γ given by integrating $dt = dw/\dot{\bar{x}}(w)$ with the appropriate limits. By linearizing the system of differential equations around the straight line solution we obtain the *Normal Variational Equation* (NVE), which is the component of the linearized system describing the variational normal to the surface Γ.

The method described here applies to Hamiltonian systems. The relevance to AdS/CFT arises because, in the context of classical strings in the conformal gauge, the Virasoro constraint provides a Hamiltonian for the systems we consider. The classical string system can be, in certain cases, reduced consistently to a 2-d Hamiltonian system. Given a Hamiltonian system, Ziglin's theorems [33, 34] connect the existence of a first integral of motion with the monodromy matrices around the straight line solution. In [35–37] a major improvement on Ziglin's theory was proposed by introducing techniques of differential Galois theory. It was found that the identity component of the differential Galois group of the variational equations normal to an integrable plane of solutions is Abelian. The calculation of the Galois group is rather intricate, but it can be simplified considerably by applying an algorithm due to Kovacic [38], by using the fact that an Abelian identity element in the Galois group is equivalent to finding Liouvillian solutions to the NVE. Liouvillian solutions are those that can be written as combinations of integrals of exponentials, logarithms, rational functions and algebraic expressions. Kovacic's algorithm is an algorithmic implementation of Picard-Vessiot theory for second-order homogeneous linear differential equations with polynomial coefficients, and provides, in a constructive manner, an answer to the existence of integrability by quadratures.

The above approach to declaring systems non-integrable has been successfully applied to various situations, with some interesting examples including [39–41]. It has also been applied to cosmological models as well as to theories arising in the context of gauge/gravity duality. Signs of chaotic behavior have been found in Schwarzschild black holes [8, 42]. More recently the method has been applied to certain classical string configurations relevant in the context of the AdS/CFT correspondence [9–11], where for example integrability in the Sasaki-Einstein spaces has been ruled out [12, 13], or has provided evidence that rules out integrability of $N = 4$ sYM beyond the planar limit [14], although at special large N limits some integrability does appear [43, 44].

Based on the above, to apply our method we need to consistently reduce the string equations to a 2-d Hamiltonian system, say with variables $\alpha(\tau)$ and $\theta(\tau)$. Then a solution needs to be found where an invariant plane is chosen, for instance of the form $\theta = c$ and $\dot{\alpha} = F(\alpha, \tau)$. $F(\alpha, \tau)$ is a generic function of time and c is a constant. Usually, to investigate the integrability of the solution we do not need the full solution of the equations of motion. A full set of variations may be taken, where consistently it should be

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possible to keep as non-zero only the variation along the normal direction to the invariant plane. We end up with a second order homogeneous differential equation for the normal variations with respect to time:

$$\ddot{\eta}(t) + F_1(t,\alpha)\dot{\eta}(t) + F_2(\alpha,t)\eta(t) = 0 \tag{2.1}$$

This is the NVE mentioned above. In case the coefficients are not rational, in order to proceed one needs to look for an appropriate redefinition $z = f(\alpha)$ leading to rational coefficients, where it should be noted that only the equation of motion for α is required, and not the whole solution. In the resulting second order homogeneous differential equations with rational coefficients we need to look for Liouvillian solutions by applying the Kovacic algorithm. If there exist no such solutions the truncated Hamiltonian system is not integrable and hence the full initial system is not integrable. If we do find such a solution, then the process is inconclusive, since we can not rule out non-integrability of the full system just by finding an integrable truncated Hamiltonian.

3 The gravity dual metric of the complex β-deformed theory

In the notation of [23], the gravity dual background of the complex β-deformed $\mathcal{N} = 4$ sYM theory [17] takes the following form:

$$\begin{aligned}
\mathrm{d}s^2 &= R^2\sqrt{H}\left[\mathrm{d}s^2_{\mathrm{AdS}_5} + \sum_{i=1}^{3}(\mathrm{d}\rho_i^2 + G\rho_i^2\mathrm{d}\phi_i^2) + (\tilde{\gamma}^2+\tilde{\sigma}^2)G\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\mathrm{d}\phi_i\right)^2\right], && (3.1)\\
B &= R^2(\tilde{\gamma}Gw_2 - 12\tilde{\sigma}w_1\mathrm{d}\psi)\,, \quad \psi = \frac{1}{3}(\phi_1+\phi_2+\phi_3), && (3.2)\\
w_2 &= \rho_1^2\rho_2^2\mathrm{d}\phi_1\mathrm{d}\phi_2 - \rho_1^2\rho_3^2\mathrm{d}\phi_1\mathrm{d}\phi_3 + \rho_2^2\rho_3^2\mathrm{d}\phi_2\mathrm{d}\phi_3, \quad \mathrm{d}w_1 = \cos\alpha\sin^3\alpha\sin\theta\cos\theta\mathrm{d}\alpha\mathrm{d}\theta,\\
e^{\Phi} &= e^{\Phi_0}\sqrt{G}H,
\end{aligned}$$

where the metric, the NS-NS B field and the dilaton have been given. The RR-fields are not presented, since they do not couple directly to the bosonic part of the classical string action. The functions we have used are the following

$$G = \frac{1}{1+(\tilde{\gamma}^2+\tilde{\sigma}^2)Q}\,, \quad \text{with} \quad Q = \rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_1^2\rho_3^2, \tag{3.3}$$

and

$$H = 1 + \tilde{\sigma}^2 Q\,, \tag{3.4}$$

where we have split the real and imaginary parts of β as

$$\beta = \tilde{\gamma} - i\tilde{\sigma}\,. \tag{3.5}$$

The Cartesian coordinates ρ_i satisfy $\sum\rho_i^2 = 1$ and we choose to parametrize them as

$$\rho_1 = \sin\alpha\cos\theta\,, \quad \rho_2 = \sin\alpha\sin\theta\,, \quad \rho_3 = \cos\alpha\,. \tag{3.6}$$

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Note that then the functions take a more convenient form

$$Q = \frac{1}{4}(\sin^2 2\alpha + \sin^4\alpha \sin^2 2\theta) \tag{3.7}$$

and the metric can be written as

$$\begin{aligned} ds^2 = &\sqrt{H}\big(-\cosh^2\rho dt^2 + d\rho^2\big) + \sqrt{H}\left(d\alpha^2 + \sin^2\alpha d\theta^2 + !G \sum_{i,(j<k)} \rho_i^2\big(1 + (\tilde{\gamma}^2 + \tilde{\sigma}^2)\rho_j^2\rho_k^2\big)d\phi_i^2\right) \\ &+ 2\sqrt{H}G\big(\tilde{\gamma}^2 + \tilde{\sigma}^2\big)\rho_1^2\rho_2^2\rho_3^2(d\phi_1 d\phi_2 + d\phi_1 d\phi_3 + d\phi_2 d\phi_3)\,, \end{aligned} \tag{3.8}$$

where from the AdS part we have kept only the elements interesting us, and ρ is the radial direction. The indices i, j, k take values from 1 to 3, and the sum $\sum_{i,(j<k)}$ is used for presentation purposes, and defines a summation in i, while j and k take the only remaining allowed values. By taking the smooth limit $\tilde{\sigma} \to 0$ we obtain the usual real β-deformed theory. If the further smooth limit $\tilde{\gamma} \to 0$ is taken then we are left with the undeformed $\mathcal{N} = 4$ sYM. For more details on this background we refer the reader to [23].

Without loss of generality we integrate w_1 as[2]

$$w_1 = \frac{1}{4}\sin^4\alpha \cos\theta \sin\theta \mathrm{d}\theta\ . \tag{3.9}$$

For the B-field we thus find

$$B = R^2\left(\tilde{\gamma}G\sum_{i<j}\rho_i\rho_j\ \mathrm{d}\phi_i \wedge \mathrm{d}\phi_j - \tilde{\sigma}\rho_1\rho_2\big(1-\rho_3^2\big)\,(\mathrm{d}\theta \wedge (\mathrm{d}\phi_1 + \mathrm{d}\phi_2 + \mathrm{d}\phi_3))\right)\ . \tag{3.10}$$

Notice that in certain cases for presentation and convenience purposes we might use the ρ_i notation, while in some other their parametrization in angles given in (3.6).

4 The string sigma-model

Let us consider the general ansatz describing a classical string in the deformed dual theory

$$t = t(\tau), \qquad \rho = \rho(\tau)\,, \qquad \alpha = \alpha(\tau,\sigma), \qquad \theta = \theta(\tau,\sigma)\,, \qquad \phi_i = \phi_i(\tau,\sigma)\ . \tag{4.1}$$

The string is not allowed to have any extension in the space where the field theory lives, while in the internal space the most generic motion is allowed. We are ultimately interested in understanding some consistent particular solutions of this ansatz but it is instructive to start with this level of generality, since we need to verify that the particular string configurations satisfy the full system of equations of motion.

Let us start with the classical string σ-model action on a general background including an NS-NS B-field:

$$S = -\frac{R^2}{2}\int \mathrm{d}\tau \frac{\mathrm{d}\sigma}{2\pi}\left[\gamma^{\alpha\beta}G_{MN}\partial_\alpha X^M\partial_\beta X^N - \epsilon^{\alpha\beta}B_{MN}\partial_\alpha X^M\partial_\beta X^N\right] \tag{4.2}$$

[2]There is clearly an ambiguity at this level since for instance $w_1 = (1/2)\cos\alpha\sin^3\alpha\sin^2\theta d\alpha$ is also a solution, but it just amounts to a gauge choice that we can make in a convenient way in order to simplify our equations. See also [29] for a relevant discussion.

where $\epsilon^{01} = 1$ and in conformal gauge $\gamma^{01} = \text{diag}(-1, 1)$. We have set $\alpha' = 1$ and extracted the R^2 factor from the metric and B-field to emphasise that this is a heavy, classical string.

Substituting our general ansatz above, we arrive at:

$$\begin{aligned} S = & -\frac{R^2}{4\pi}\int d\sigma d\tau \sqrt{H}\left(\cosh^2\rho\, \dot{t}^2 + (\rho'^2 - \dot{\rho}^2)\right) + \sqrt{H}\left((\alpha'^2 - \dot{\alpha}^2) + \sin^2\alpha\left(\theta'^2 - \dot{\theta}^2\right)\right) \\ & + \sum_i G_{ii}\left(\phi_i'^2 - \dot{\phi}_i^2\right) + 2\sum_{i,j,(i<j)} G_{ij}\left(\phi_i'\phi_j' - \dot{\phi}_i\dot{\phi}_j\right) - 2\sum_{i,j,(i<j)} B_{ij}\left(\dot{\phi}_i\phi_j' - \phi_i'\dot{\phi}_j\right) \\ & -2\sum_i B_{\theta i}\left(\dot{\theta}\phi_i' - \theta'\dot{\phi}_i\right). \end{aligned} \tag{4.3}$$

The general equations of motion for the non-cyclic coordinates α and θ are given by

$$\begin{aligned} & \partial_{\alpha,\theta}\sqrt{H}\left(\cosh^2\rho\dot{t}^2 + \rho'^2 - \dot{\rho}^2\right) + \partial_{\alpha,\theta}\sqrt{H}\left(\alpha'^2 - \dot{\alpha}^2\right) + \partial_{\alpha,\theta}\left(\sqrt{H}\sin^2\alpha\right)\left(\theta'^2 - \dot{\theta}^2\right) \\ & +\partial_{\alpha,\theta}G_{ii}\left(\phi_i'^2 - \dot{\phi}^2\right) + 2\sum_{i,j,(i<j)}\partial_{\alpha,\theta}G_{ij}\left(\phi_i'\phi_j' - \dot{\phi}_i\dot{\phi}_j\right) - 2\sum_{i,j,(i<j)}\partial_{\alpha,\theta}B_{ij}\left(\dot{\phi}_i\phi_j' - \phi_i'\dot{\phi}_j\right) \\ & -2\sum_i \partial_{\alpha,\theta}B_{\theta i}\left(\dot{\theta}\phi_i' - \theta'\dot{\phi}_i\right) + \\ & + \begin{cases} +2\partial_0\left(\sqrt{H}\dot{\alpha}\right) - 2\partial_1\left(\sqrt{H}\alpha'\right) = 0 \quad \text{for } \alpha\,, \\ +2\partial_0\left(\sqrt{H}\sin^2\alpha\dot{\theta} + B_{\theta i}\phi_i'\right) - 2\partial_1\left(\sqrt{H}\sin^2\alpha\theta' + 2B_{\theta i}\dot{\phi}_i\right) = 0 \quad \text{for } \theta\,. \end{cases} \end{aligned} \tag{4.4}$$

The equations for the cyclic coordinates ϕ_i take the simpler form

$$\partial_0\left(-\sum_j G_{ij}\dot{\phi}_j - \sum_j B_{ij}\phi_j' + B_{\theta i}\theta'\right) + \partial_1\left(\sum_j G_{ij}\phi_j' + \sum_j B_{ij}\dot{\phi}_j - B_{\theta i}\dot{\theta}\right) = 0\,. \tag{4.5}$$

Finally the Virasoro constraints read

$$\begin{aligned} & \sqrt{H}\left(-\cosh^2\rho\dot{t}^2 + (\rho'^2 + \dot{\rho}^2)\right) + \sqrt{H}\left((\alpha'^2 + \dot{\alpha}^2) + \sin^2\alpha\left(\theta'^2 + \dot{\theta}^2\right)\right) + \sum_i G_{ii}\left(\phi_i'^2 + \dot{\phi}_i^2\right) \\ & +2\sum_{i,j,(i<j)} G_{ij}\left(\phi_i'\phi_j' + \dot{\phi}_i\dot{\phi}_j\right) = 0\,, \end{aligned} \tag{4.6}$$

$$\sqrt{H}\rho'\dot{\rho} + \sqrt{H}\left(\alpha'\dot{\alpha} + \sin^2\alpha\theta'\dot{\theta}\right) + \sum_i G_{ii}\phi_i'\dot{\phi}_i + \sum_{i,j,(i<j)} G_{ij}\left(\phi_i'\dot{\phi}_j + \phi_j'\dot{\phi}_i\right) = 0\,. \tag{4.7}$$

In the following sections we use the generic equations found here to obtain particular solutions.[3] Of course, we explicitly prove that the reduced string configurations we study satisfy the full system of the equations, as should be the case.

5 An integrable point-like solution

As a warm-up, let us consider a point-like string by taking the following ansatz:

$$t = t(\tau)\,, \quad \rho = \rho(\tau)\,, \quad \alpha = \alpha(\tau)\,, \quad \theta = \theta(\tau)\,, \tag{5.1}$$

[3] Some special string solutions of the above equations have previously been found in [23, 45] and [29].

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where the other embedding coordinates are constants. This ansatz gives a consistent motion for the string, while the motion along all the other coordinates is localized. The equations of motion for t and ρ give

$$\dot{t} = \frac{\kappa}{\sqrt{H}\cosh^2\rho}\,, \tag{5.2}$$

$$2\partial_0\left(\dot{\rho}\sqrt{H}\right) + \frac{\kappa^2}{\sqrt{H}}\partial_\rho \cosh^2\rho = 0\,, \tag{5.3}$$

where κ is the integration constant. The equations can be solved by turning off the motion in the ρ coordinate and localizing it to the bulk $\rho = 0$, resulting in the following equation for t

$$\dot{t} = \frac{\kappa}{\sqrt{H}}\;. \tag{5.4}$$

Then in the non-trivial Virasoro constraint (4.6) only one term from the AdS part contributes and becomes

$$-\frac{\kappa^2}{\sqrt{H}} + \sqrt{H}\dot{\alpha}^2 + \sqrt{H}\sin^2\alpha\dot{\theta}^2 = 0\;. \tag{5.5}$$

The equations of motion for α and θ become

$$\partial_\alpha\sqrt{H}\frac{\kappa^2}{H} - \partial_\alpha\sqrt{H}\dot{\alpha}^2 - \partial_\alpha\left(\sqrt{H}\sin^2\alpha\right)\dot{\theta}^2 + 2\partial_0\left(\dot{\alpha}\sqrt{H}\right) = 0, \tag{5.6}$$

$$\partial_\theta\sqrt{H}\frac{\kappa^2}{H} - \partial_\theta\sqrt{H}\dot{\alpha}^2 - \sin^2\alpha\partial_\theta\sqrt{H}\dot{\theta}^2 + 2\partial_0\left(\sqrt{H}\sin^2\alpha\dot{\theta}\right) = 0\;. \tag{5.7}$$

This system has an effective Lagrangian[4]

$$2L_{\text{eff}} = \frac{\kappa^2}{\sqrt{H}} + \sqrt{H}\dot{\alpha}^2 + \sqrt{H}\sin^2\alpha\dot{\theta}^2\,, \tag{5.8}$$

reducing to a 2-d particle Hamiltonian system. A solution to the equations of motion may be given by a further localization of the point-like string to the equator of the S^2

$$\alpha = \frac{\pi}{2}\,,\quad \dot{\theta}^2 = \frac{\kappa^2}{H}\;. \tag{5.9}$$

We proceed to the study of the normal fluctuations to this solution which is governed by the NVE. Taking the variation along the normal plane

$$\alpha = \frac{\pi}{2} + \eta(t)\,, \tag{5.10}$$

and keeping up to linear order in $\eta(t)$, the NVE becomes

$$2H_0^2\ddot{\eta} + \frac{\tilde{\sigma}^2}{2}H_0\sin 4\theta\;\dot{\theta}\dot{\eta} + \eta\left(H_0\left(2H_0 - \frac{\tilde{\sigma}^2}{2}(1+\cos^2 2\theta)\right)\dot{\theta}^2 + \tilde{\sigma}^2(1+\cos^2 2\theta)\frac{\kappa^2}{2}\right) = 0\,, \tag{5.11}$$

[4]This is the Lagrangian that leads to the above equations of motion, and can of course also be derived by appropriately substituting the cyclic coordinate t in the original action (4.3) following the Routhian procedure. Note that we will be setting the AdS_5 radius $R = 1$ from now on.

where $H_0 = 1 + \tilde{\sigma}^2 \sin^2 2\theta/4$. This NVE has non-rational coefficients. To bring it into a desirable form we switch to a new variable $z = \cos\theta$ to arrive at

$$2\kappa^2 H_0\big((1-z^2)\eta''(z) - z\eta'(z) + \eta(z)\big) = 0\,. \tag{5.12}$$

To obtain (5.12) we have used the equation (5.9), so that the NVE is calculated for the solution of $\theta(\tau)$. We note that the final NVE has no $\tilde{\sigma}$ or $\tilde{\gamma}$ dependence and it is undeformed, and therefore it is integrable admitting the Liouvillian solution

$$\eta(z) = c_1 z + c_2\sqrt{z^2-1}\,. \tag{5.13}$$

It is quite remarkable that the geodesic of the deformed theory, despite clearly being dependent on the deformation parameters, has an NVE that remains undeformed and is the same as in the $\mathcal{N}=4$ sYM theory. In the next section we note that a special limit of the extended string solution we obtain, is a point-like string which again has no $\tilde{\sigma}$ dependent NVE and is integrable, in contrast with the non-integrable extended string.

6 Non-integrable string solution extended along a U(1) direction

In this section we study extended string solutions in the internal space. We show how integrability is lost when the point like string becomes extended. Moreover we show how the integrability is restored when the deformation complex parameter, $\tilde{\sigma}$, goes to zero. A generic initial ansatz considered for study for the string motion is

$$t = t(\tau)\,, \quad \alpha = \alpha(\tau)\,, \quad \theta = \theta(\tau)\,, \quad \phi_1 = \phi_1(\sigma), \quad \phi_2 = \phi_2(\sigma), \quad \phi_3 = \phi_3(\sigma)\,, \tag{6.1}$$

which corresponds to a string moving along the internal space directions and extended along the U(1) angles. The solution in the AdS part is still given by the equation (5.4).

First, we solve the equations for the cyclic coordinates ϕ_i by constraining the string extension along them further

$$\phi_1 = \phi_3 = 0\,, \qquad \text{and} \qquad \phi_2 = m\sigma\,, \tag{6.2}$$

where m is a constant, and by setting the real part of deformation parameter to zero, $\tilde{\gamma} = 0$. We are thus looking for solutions on the imaginary-β background.

The remaining equations take the form

$$\begin{aligned}&\partial_{\alpha,\theta}\sqrt{H}\tfrac{\kappa^2}{H} - \partial_{\alpha,\theta}\sqrt{H}\dot{\alpha}^2 - \partial_{\alpha,\theta}\Big(\sqrt{H}\sin\alpha^2\Big)\dot{\theta}^2 + \partial_{\alpha,\theta}(G_{22})\phi_2'^2 - 2\partial_{\alpha,\theta}B_{\theta\phi_2}\dot{\theta}m + \\ &\qquad + \begin{cases} +2\partial_0\Big(\sqrt{H}\dot{\alpha}\Big) = 0 \quad \text{eom for } \alpha\,, \\ +2\partial_0\Big(\sqrt{H}\sin\alpha^2\dot{\theta} + B_{\theta i}m\Big) = 0 \quad \text{eom for } \theta\,,\end{cases}\end{aligned} \tag{6.3}$$

and the Virasoro constraint reads

$$-\frac{\kappa^2}{\sqrt{H}} + \sqrt{H}\dot{\alpha}^2 + \sqrt{H}\sin^2\alpha\dot{\theta}^2 + G_{22}m^2 = 0\,, \tag{6.4}$$

with $G_{22} = \sqrt{H} G \rho_2^2 [1 + (\tilde{\gamma}^2 + \tilde{\sigma}^2) \rho_1^2 \rho_3^2]$. The effective Lagrangian of the system then reads

$$2L_{\text{eff}} = \frac{\kappa^2}{\sqrt{H}} + \sqrt{H} \dot{\alpha}^2 + \sqrt{H} \sin^2 \alpha \dot{\theta}^2 - G_{22} m^2 + 2 B_{\theta \phi_2} \dot{\theta} m \,, \tag{6.5}$$

and the relevant Hamiltonian

$$2\mathcal{H} = -\frac{\kappa^2}{\sqrt{H}} + \sqrt{H} \dot{\alpha}^2 + \sqrt{H} \sin^2 \alpha \dot{\theta}^2 + G_{22} m^2 \,, \tag{6.6}$$

is constrained to zero. The solution $\theta = 0$ defines an invariant plane, where the equation of motion for the angle θ is satisfied trivially and the equation of motion for α after some manipulation can be written as a total integral

$$\partial_0 \left(-\frac{\kappa^2}{\sqrt{H}} + \sqrt{H} \dot{\alpha}^2 \right) = 0, \tag{6.7}$$

becoming identical to the Virasoro constraint

$$\dot{\alpha}^2 = \frac{\kappa^2}{H} \,. \tag{6.8}$$

Varying the equation of motion of θ (6.3), by setting $\theta = 0 + \eta(t)$ and keeping up to first order in $\eta(t)$, we obtain the following NVE

$$\ddot{\eta} + \left(2 \cot \alpha + \frac{\tilde{\sigma}^2 \sin 4\alpha}{4 H_0} \right) \frac{\kappa}{\sqrt{H_0}} \dot{\eta} + \left(m - \frac{2 \tilde{\sigma} \kappa \sin 2\alpha}{H_0} \right) m \eta = 0 \,, \tag{6.9}$$

where $H_0 = 1 + \tilde{\sigma}^2 \sin^2 2\alpha / 4$. To obtain an NVE with rational coefficients we perform a change of variables $z = \tan \alpha$ which gives the following equation

$$\eta(z)'' + \frac{2}{z} \eta(z)' + \frac{m \left[m \left(\left(\tilde{\sigma}^2 + 2 \right) z^2 + z^4 + 1 \right) - 4 \kappa \tilde{\sigma} z \left(z^2 + 1 \right) \right]}{\kappa^2 \left(z^2 + 1 \right)^4} \eta(z) = 0 \,. \tag{6.10}$$

This NVE with rational coefficients does not admit any Liouvillian solutions for generic values of the parameters. More precisely the solutions are in terms of the Heun double confluent functions and the system is not integrable.

By taking the limit $m \to 0$, we localize the string and we end up with a geodesic which differs from the one in the previous section. The resulting NVE does not depend on $\tilde{\sigma}$, so it is undeformed, and as a result the differential equation has Liouvillian solutions. By taking the smooth limit $\tilde{\sigma} \to 0$, the extended string solution (and the corresponding NVE) becomes an equation in the undeformed $AdS_5 \times S^5$ space and the integrability of the Hamiltonian system is restored as expected, since it gives Liouvillian solutions.

7 Explicit numerical analysis

In this section we analyze the string equations of motion numerically. The standard analysis is best organized in the Hamiltonian formulation. The aim is to explore its behavior as a dynamical system, in particular its Poincaré sections and the behavior of phase-space trajectories as the deformation parameters are varied.

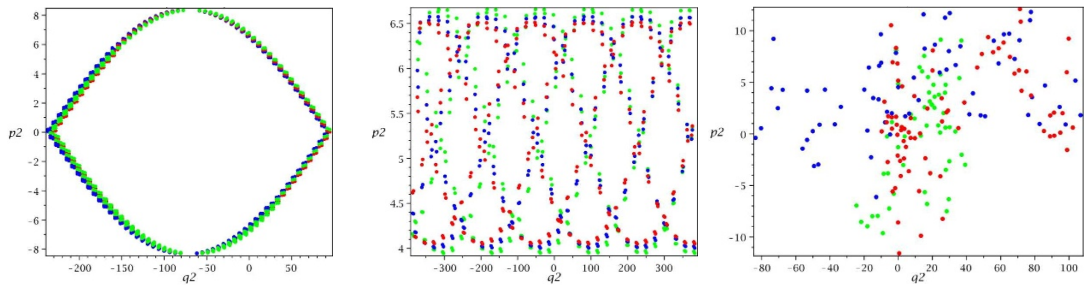

Figure 1. Three Poincaré sections for $\tilde{\gamma} = 1$ and $\tilde{\sigma} = 0.001$ (left panel) $\tilde{\sigma} = 2.0$ (center panel) and $\tilde{\sigma} = 10$ (right panel). The plots are for $m = 2$ and $\kappa = 10$.

Starting from the rather general sigma model action (4.3) and using the string configuration described by (6.1), (6.2), we rewrite the Hamiltonian (6.6) in terms of the conjugate momenta

$$\mathcal{H} = -\frac{\kappa^2}{2\sqrt{H}} + \frac{p_\alpha^2}{2\sqrt{H}} + \frac{p_\theta^2}{2\sqrt{H}\sin^2\alpha} - \frac{B_{\theta\phi_2} m}{\sqrt{H}\sin^2\alpha} p_\theta + \frac{1}{2} G_{22} m^2 + \frac{B_{\theta\phi_2}^2 m^2}{2\sqrt{H}\sin^2\alpha} \,. \tag{7.1}$$

The Hamiltonian is fixed to zero by the Virasoro constraint. To study it numerically, we fix the winding parameter m of the string and the constant κ. To study the Poincaré sections we further fix $\tilde{\gamma}$ to a non-zero value and vary $\tilde{\sigma}$. Increasing the $\tilde{\sigma}$ parameter leads to a destruction of the Kolmogorov-Arnold-Moser (KAM) tori which is a typical indicator of chaotic behavior. More specifically, in figure 1 we consider three initial conditions which are very close to each other in phase space and track their evolution for a fixed value of the Hamiltonian, drawing a point in the $(\theta, p_\theta) = (q_2, p_2)$ plane each time that the trajectory crosses this plane.[5] As it can be seen in the plot, for very small values of the complex deformation parameter $\tilde{\sigma} = 0.001$ the three originally nearby trajectories remain close for the whole evolution. In the case $\tilde{\sigma} = 2.0$ we start noticing quasi-periodic structure and finally for $\tilde{\sigma} = 10$ the initially nearby points get scattered all over the plane, signalling chaotic behavior, that is, strong sensitivity towards the choice of initial conditions.

It is also instructive to construct 3-dimensional plots of the phase space trajectories of our system, and compare the $\tilde{\gamma}$ deformation to the $\tilde{\sigma}$ deformation as the deformation parameters increase. Such trajectories are plotted in figure 2. While the integrability of the $\tilde{\gamma}$ deformation manifests itself in orderly and smooth trajectories, with little sensitivity to initial conditions, in the $\tilde{\sigma}$ deformation the trajectories quickly become irregular as $\tilde{\sigma}$ increases, and show a strong dependence on the initial conditions. This is a signal of chaotic behavior for the complex beta deformation, which directly implies non-integrability. The plots displayed here are for special values ($m = 2, \kappa = 10$), but we have checked that very similar behaviour arises for other choices. We have also checked that the chaotic behaviour persists for general complex β.

The results of this section provide strong evidence for the non-integrability of the complex β background and are completely in line with our analytical results in section 6.

[5]All the plots in this section have been created with the *Maple* `poincare` package.

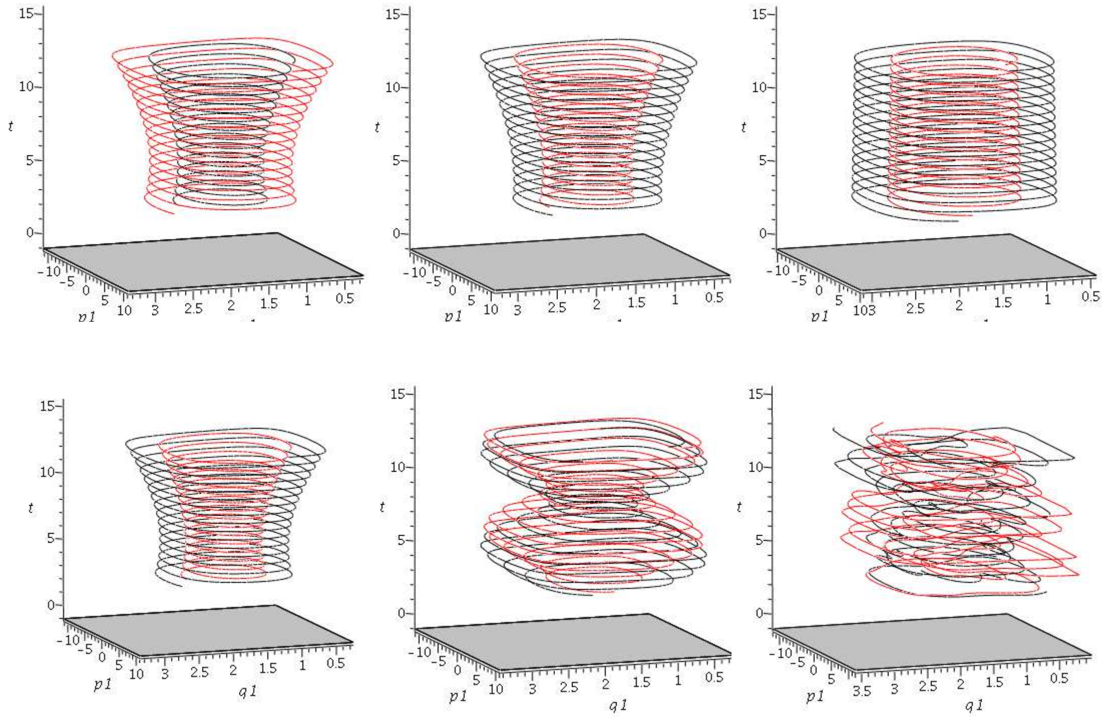

Figure 2. The top three plots show the phase space trajectories for $\tilde{\sigma} = 0$ and $\tilde{\gamma} = 0.01$ (left), $\tilde{\gamma} = 1$ (center) and $\tilde{\gamma} = 100$ (right). In the bottom three plots, the phase space trajectories for $\tilde{\gamma} = 0$ and $\tilde{\sigma} = 0.001$ (left), $\tilde{\sigma} = 1$ (center) and $\tilde{\sigma} = 10$ (right) are displayed. Each plot is for two nearby initial conditions (black and red trajectory). All plots are for $m = 2$ and $\kappa = 10$, with the energy constrained to 0. In the top three plots the trajectories remain regular and diverge very slowly from each other, as expected by integrability of the $\tilde{\gamma}$ deformation. In the bottom three plots, as $\tilde{\sigma}$ increases the trajectories become more irregular and quickly diverge from each other. Note that $(q_1, p_1) := (\alpha, p_\alpha)$.

Furthermore, since we are not constrained by the need to find an analytic solution, we can easily check that non-integrable behaviour is present for general complex β.

8 Conclusions

In this paper we have explicitly proved that string motion on the imaginary beta deformed LM background is not integrable. We showed that there exists at least one consistent truncation of the sigma model equations of motion of the imaginary beta deformed theory, that is not an integrable system of ordinary second-order differential equations and this is enough to prove non-integrability of the given sigma model. We arrived at this conclusion using analytical modern methods of Hamiltonian dynamical systems, which we described in detail. To further support our analytic study we explored numerically the motion of the system and found it to be chaotic. The chaotic motion arises due to some appropriate noise applied to usually well behaved solutions, resulting to a motion that is highly sensitive to to initial conditions. Non-integrability does not, necessarily, imply chaos. However, the appearance of chaos is evidence of the breakdown of integrability. In our study the chaotic motion of the strings is evident in the Poincaré sections and also in the phase space trajectories which show strong sensitivity to the initial conditions. Note that our analytical computations are restricted to the imaginary beta deformation, where it was possible to find an exact solution for string motion. However, our numerical results apply to the general complex case as well. We thus fully expect that a generalization of our analytical approach to the generic case of complex β will be able to show non-integrability there as well.

Our work motivates a number of further questions. The interplay between marginal deformations and integrability is one area that should be extensively reconsidered in the context of the AdS/CFT correspondence. We have taken a first step in this paper by addressing the β deformation. There is another marginal deformation, the h-deformation [15], which would be interesting to discuss in the same framework. In particular, although the expectation is that the general (β, h) deformation will not be integrable, at the one-loop level there do exist special integrable points in the (β, h)-plane [46, 47] (related to the real-β case through appropriate twists in the algebraic structure) and it would be interesting to check whether string motion on the dual background supports this. Unfortunately, the supergravity background for the h deformation is only known as an expansion to third order in the deformation parameter [16, 48], but the analytic non-integrability approach might still be applicable to this order and shed more light on the integrability of this background.

Another interesting question that permeates many areas concerns the quantum implications of classical chaos. The string trajectories that we are discussing should lead, at the quantum level, to excitations of very heavy stringy states. These states should then be identified with operators with large quantum numbers on the field theory side. A similar analysis was performed recently in the nonconformal case, where it was shown that the corresponding hadrons have a Wigner distribution similar to that of realistic hadrons [49, 50]. Given the generality of the results in quantum chaos, it is sensible to speculate about the existence of a quite universal sector where many field theories admitting gravity duals will have a similar spectrum of operators dual to highly excited strings.

In all cases that we considered, taking the point-like limit of our string σ-model led to integrable geodesic motion. We are not aware of general results on geodesic motion for the complex-deformed LM background, however the integrability or lack thereof of geodesics on several backgrounds arising in string theory was recently investigated in [14] and it would be worthwhile to apply those methods to the LM background as well.

Finally, marginal deformations can be applied to a wide range of field theories with gravity duals. It would be interesting to investigate the deformed gravity duals of these backgrounds on a wider swath of phase space, including non-integrable trajectories. The prototypical background is furnished by Sasaki-Einstein manifolds: $AdS_5 \times Y^{p,q}$ [51], where the method applied here can be used by generalizing the string solutions obtained in [52]. The un-deformed backgrounds are already known to be non-integrable [12, 13]. The hope is, rather, that as in the case of the pp-wave limit [5], even for these non-integrable backgrounds there is a universal sector that emerges corresponding to classically chaotic strings.

In conclusion, the methods of analytic non-integrability provide a useful and novel perspective on integrability in the AdS/CFT correspondence and we believe that further work along the lines we have discussed will be of great help in mapping the boundaries of integrability and understanding the mechanisms that lead to its breaking.

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