RECEIVED: October 11, 2013

REVISED: November 19, 2013

ACCEPTED: December 11, 2013

PUBLISHED: January 16, 2014

Non-Abelian vortices with an Aharonov-Bohm effect

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ABSTRACT: The interplay of gauge dynamics and flavor symmetries often leads to remarkably subtle phenomena in the presence of soliton configurations. Non-Abelian vortices — vortex solutions with continuous internal orientational moduli — provide an example. Here we study the effect of weakly gauging a $U(1)_R$ subgroup of the flavor symmetry on such BPS vortex solutions. Our prototypical setting consists of an $SU(2) \times U(1)$ gauge theory with $N_f = 2$ sets of fundamental scalars that break the gauge symmetry to an “electromagnetic” $U(1)$. The weak $U(1)_R$ gauging converts the well-known $\mathbf{CP}^1$ orientation modulus $|B|$ of the non-Abelian vortex into a parameter characterizing the strength of the magnetic field that is responsible for the Aharonov-Bohm effect. As the phase of B remains a genuine zero mode while the electromagnetic gauge symmetry is Higgsed in the interior of the vortex, these solutions are superconducting strings.

KEYWORDS: Nonperturbative Effects, Spontaneous Symmetry Breaking, Duality in Gauge Field Theories, Sigma Models

ARXIV EPRINT: [1310.1224](https://arxiv.org/abs/1310.1224)

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1 Introduction

Topological solitons play a fundamental role in various physical systems, especially in those characterized by gauge interactions. The interplay of the classical or quantum gauge dynamics with the global symmetries in such systems leads to remarkably rich and subtle phenomena. Of particular interest among these is a class of vortex solutions carrying continuous orientational zero modes: non-Abelian vortices [1]–[32].

These typically arise when the (complete) gauge symmetry breaking supports vortex solutions, and at the same time the vacuum is invariant under a color-flavor diagonal group so that the system is in the so-called color-flavor locked phase [33]. As the individual vortex breaks the exact global symmetry, it develops orientational zero modes which can fluctuate on the vortex world sheet. At low energies these fluctuations can be described by an

appropriate $2D$ sigma model that has its own nontrivial, large-distance, quantum dynamics. A longstanding goal concerning non-Abelian vortices is to find solutions describing a non-Abelian vortex-monopole complex [2, 4, 29], [24]–[26].

Here we examine yet another, little studied, question regarding non-Abelian vortex systems: what happens if a part or the whole of the exact global (color-flavor diagonal) symmetry is weakly gauged? In ref. [34], we have initiated the study of such effects by weakly gauging the entire exact flavor symmetry of these vortex solutions. In a sense this was a simpler question: the answer is that now any color-flavor rotation of a given solution is a genuine global gauge transformation, rendering all charge one vortex solutions gauge equivalent whatever their orientations. Stated differently, a mini-Higgs mechanism is at work in the vortex worldsheet, transforming the orientational zero modes into light, propagating modes. As a result of the non-Abelian nature of the gauge groups involved, these light modes are unstable and decay into massless $4D$ gauge bosons.

Here we turn our attention to the case where only a part of the global symmetry group is gauged. As a simple concrete model we take an $SU(2)_L \times U(1)$ gauge theory with two complex scalars in the fundamental representation, in which a $U(1)_R \subset SU(2)_R$ subgroup of the flavor symmetry is weakly gauged.¹ The $SU(2)_L \times U(1)$ and $U(1)_R$ gauge symmetries are broken by the scalar VEVs, but a $U(1)^{\text{em}}$ subgroup remains unbroken. These $4D$ massless gauge modes interact with the $2D$ zero modes on the vortex worldsheet. The result is rather unexpected and elegant. We find that the two-dimensional vortex moduli (partially) survive the $U(1)_R$ gauging, but the complex parameter B inherited from the original $\mathbf{CP}^1$ vortex moduli acquires a new physical meaning. The modulus $|B|$ can be still interpreted as a truly $2D$ vortex modulus which now characterizes the $U(1)^{\text{em}}$ magnetic flux carried by the vortex: a particle with unit $U(1)_R$ charge (and hence unit $U(1)^{\text{em}}$ charge) will experience a well-defined Aharonov-Bohm (AB) effect [35] going around the vortex far from the core. The phase $\text{Arg}(B)$, on the other hand, is eaten by the $4D$ gauge bosons. An AB effect on vortices have been studied before [36]. Compared to that case, a peculiar feature of our case is that the AB phase depends on the modulus $|B|$.

The question of $U(1)$ gauging was originally raised in the context of dense quark matter. QCD in the high baryon density limit is expected to be in the color-flavor locked phase with diquark condensation [33], where the $SU(3)$ color symmetry and the $SU(3)$ flavor symmetry of the three light quarks are spontaneously broken to the diagonal subgroup. In this context, non-Abelian vortices appear [37]–[39] which differ from local non-Abelian vortices for which the $U(1)$ group is a global symmetry. Nevertheless, they possess normalizable orientational zero modes [40, 41]. Taking into account the $U(1)$ electromagnetic coupling corresponds to gauging a $U(1)$ subgroup of the flavor $SU(3)$ symmetry, some consequences of which were studied in refs. [42]–[44]. See ref. [45] for a recent review.

1.1 Orientational $\mathbf{CP}^1$ modes of the standard non-Abelian vortex

Before introducing the weak gauging of a part of the flavor symmetry, let us first briefly review a few salient features of the non-Abelian vortex. The simplest example is an $SU(2) \times$

¹Here R means “acting from the right”, i.e., on the flavor indices and is unrelated to the R symmetry of supersymmetric theories.

U(1) gauge theory with two scalar fields transforming in the fundamental representation, $Q = (q^1 \ q^2)$, written in a color-flavor mixed 2×2 matrix form. The action is

$$S = \int d^4x \left\{ \frac{1}{4g^2} (F_{\mu\nu}^0)^2 + \frac{1}{4g^2} (F_{\mu\nu}^a)^2 + |\nabla_\mu q^A|^2 + \frac{g^2}{8} (\bar{q}^A \tau^a q^A)^2 + \frac{g^2}{8} (|q^A|^2 - 2\xi)^2 \right\}, \quad (1.1)$$

where

$$\nabla_\mu q^A = \partial_\mu q^A + \frac{i}{2} A_\mu^0 q^A + i \frac{\tau^a}{2} A_\mu^a q^A, \quad A = 1, 2. \quad (1.2)$$

As is often done in the recent literature, the matter content and the potential terms are chosen such that the model above can be extended to have an $\mathcal{N} = 2$ supersymmetry. As a consequence, the scalar quartic couplings are set to the critical value so that the classical equations for the soliton configurations become first order differential equations, in the “self-adjoint”, or BPS, form.

In the presence of a nonzero parameter ξ , the system is in a Higgs vacuum:

$$Q_{\text{vev}} = q_i^A = \begin{pmatrix} \sqrt{\xi} & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}. \quad (1.3)$$

The gauge and flavor symmetries are completely broken, but there remains an unbroken color-flavor diagonal $\text{SU}(2)_{C+F}$ global symmetry (i.e., it is in color-flavor locked phase). As

$$\pi_1 \left(\frac{\text{SU}(2) \times \text{U}(1)}{\mathbf{Z}_2} \right) = \mathbb{Z}, \quad (1.4)$$

the system possesses stable, nonsingular vortices. An individual vortex solution breaks the $\text{SU}(2)_{C+F}$ global symmetry to a U(1) subgroup and so it develops an orientational modulus $B \in \mathbf{CP}^1 = \text{SU}(2)/\text{U}(1)$. Indeed, the vortex solution with a generic orientation and in the regular gauge takes the form

$$Q = U \begin{pmatrix} e^{i\varphi} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = \frac{e^{i\varphi} \phi_1(r) + \phi_2(r)}{2} \mathbf{1}_2 + \frac{e^{i\varphi} \phi_1(r) - \phi_2(r)}{2} U T U^{-1}, \quad (1.5)$$

$$A_i = -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} [(f(r) - 1) \mathbf{1}_2 + (f_{\text{NA}}(r) - 1) U T U^{-1}], \quad i = 1, 2 \quad (1.5)$$

$$T = \text{diag} (1, -1) = \tau^3, \quad (1.6)$$

where the boundary conditions are

$$\phi_{1,2}(\infty) = \sqrt{\xi}, \quad \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad (1.7)$$

$$f(\infty) = f_{\text{NA}}(\infty) = 0, \quad f(0) = f_{\text{NA}}(0) = 1. \quad (1.8)$$

The “reducing matrix” U has the form

$$U = \begin{pmatrix} \mathbf{1} & -B^\dagger \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & Y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ B & \mathbf{1} \end{pmatrix} = \begin{pmatrix} X^{-\frac{1}{2}} & -B^\dagger Y^{-\frac{1}{2}} \\ B X^{-\frac{1}{2}} & Y^{-\frac{1}{2}} \end{pmatrix}, \quad (1.9)$$

with the matrices X and Y defined by

$$X \equiv \mathbf{1} + B^\dagger B, \quad Y \equiv \mathbf{1} + BB^\dagger. \quad (1.10)$$

The vortex tension,

$$T = 2\pi\xi$$

does not depend on the $\mathbf{CP}^1$ coordinate B .

More generally, perturbations of these solutions can be described by promoting the modulus B to a collective coordinate which depends upon the worldsheet coordinates of the vortex. The fluctuations of B are then described by a worldsheet $\mathbf{CP}^1$ sigma model:

$$S_{\text{eff}} = \frac{4\pi}{g_L^2} \int dt dz \frac{1}{(1 + |B|^2)^2} \partial_\alpha B^* \partial^\alpha B. \quad (1.11)$$

Our main interest below is to determine the fate of these $\mathbf{CP}^1$ collective coordinates in the presence of an external, weak $U(1)_R$ gauge field.

2 The model, BPS equations and vortex solutions

2.1 The model and BPS vortex equations

The model that we consider in this paper is the same $SU(2)_L \times U(1)_0$ gauge theory with $N_f = 2$ flavors of scalar fields as was described above, but we weakly gauge a $U(1)_R \subset SU(2)_R$ subgroup of the flavor symmetry. The action is then

$$\begin{aligned} S &= \int d^4x \left\{ \frac{1}{4} (F_{\mu\nu}^0)^2 + \frac{1}{4} (F_{\mu\nu}^a)^2 + |\nabla_\mu q^A|^2 + \frac{g_L^2}{8} (\bar{q}^A \tau^a q^A)^2 + \frac{g_0^2}{8} (|q^A|^2 - 2\xi)^2 \right\} + \\ &\quad + \frac{1}{4} (F_{\mu\nu}^{R3})^2 + \frac{g_R^2}{8} (q^A \tau^{R3} \bar{q}^A)^2 = \\ &= \int d^4x \text{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + |\nabla_\mu Q|^2 + \frac{g^2}{4} (\bar{Q}Q - \xi)^2 + \frac{1}{2} (F_{\mu\nu}^R)^2 \right\} + \frac{g_R^2}{8} (\text{Tr} (\bar{Q}Q \tau^3))^2, \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} F_{\mu\nu} &= F_{\mu\nu}^a \frac{\tau^a}{2}, \quad F_{\mu\nu}^R = F_{\mu\nu}^{R3} \frac{\tau^3}{2}, \quad A_\mu = A_\mu^a \frac{\tau^a}{2}, \quad A_\mu^R = A_\mu^R \frac{\tau^3}{2}; \\ \nabla_\mu Q &= \partial_\mu Q + ig_0 A_\mu^{(0)} Q + ig_L A_\mu Q + ig_R Q A_\mu^R. \end{aligned} \quad (2.2)$$

The vacuum is the same as in eq. (1.3),

$$\langle Q \rangle = \langle q_i^A \rangle = \begin{pmatrix} \sqrt{\xi} & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}. \quad (2.3)$$

The $U(1)_0$, $SU(2)_L$ and $U(1)_R$ gauge groups are all broken, but a combination of $U_{\tau^3}(1) \subset SU(2)_L$ and $U(1)_R$ remains unbroken. A gauge boson

$$A_\mu^{\text{em}} = \frac{g_L}{\sqrt{g_L^2 + g_R^2}} A_\mu^R - \frac{g_R}{\sqrt{g_L^2 + g_R^2}} A_\mu^{L3}, \quad (2.4)$$

remains massless in the bulk: we call it “electromagnetic” in analogy with the situation in the Weinberg-Salam theory.² All other gauge bosons, the orthogonal combination

$$B_\mu = \frac{g_L}{\sqrt{g_L^2 + g_R^2}} A_\mu^{L3} + \frac{g_R}{\sqrt{g_L^2 + g_R^2}} A_\mu^R, \quad (2.5)$$

$A_\mu^{L\pm}$, and $A_\mu^{(0)}$ via the Higgs mechanism acquire a mass of order of $M \sim g_L \sqrt{\xi} \sim g_0 \sqrt{\xi}$. The inverse of (2.4) and (2.5) is

$$A_\mu^R = \frac{g_L}{\sqrt{g_L^2 + g_R^2}} A_\mu^{\text{em}} + \frac{g_R}{\sqrt{g_L^2 + g_R^2}} B_\mu, \quad A_\mu^{L3} = \frac{g_L}{\sqrt{g_L^2 + g_R^2}} B_\mu - \frac{g_R}{\sqrt{g_L^2 + g_R^2}} A_\mu^{\text{em}}. \quad (2.6)$$

Ordinarily, this would be the end of the story: $B_\mu, A_\mu^{L\pm}, A_\mu^{(0)}$ are massive fields and cannot propagate farther than $1/M$; the long-distance physics is dominated by the massless photon A_μ^{em} . However in the presence of a vortex, a one dimensional infinitely long “hole” in the condensate, the situation is a little subtler. Of course, one knows from the standard ANO vortex that a “massive” gauge boson can become massless along the vortex core, the associated magnetic field penetrating the vortex core and giving rise to important physical effects such as the vortex tension, a magnetic flux, vortex interactions, etc. In the case of a non-Abelian vortex these features are also present, because in the simplest context of a $U(N)$ theory the latter may be considered to be an ANO vortex embedded in a corner of the color-flavor mixed space. The interesting phenomena related to the internal orientational zero modes and their dynamics arise from the fluctuation of this embedding direction.

In the present case, where $U(1)_R$ weak external gauge interactions break the color-flavor rotational symmetry, yet one more, perhaps less familiar, effect arises. In a vortex solution of a generic orientation, some combination of scalar fields Q vanish along the vortex core, meaning that some of the “massive” gauge bosons become massless. Part of the effect goes, as in the ANO vortex, into producing a magnetic flux and the associated vortex energy (tension). On the other hand, far from the core another combination of the gauge fields, the electromagnetic field, becomes massless allowing a nontrivial Wilson loop at infinity. In our setting we will find that the value of this Wilson loop is *unrelated* to the vortex tension. As there are no electrically charged condensates at infinity, the electromagnetic Wilson loop can have any value and so it is observable via an Aharonov-Bohm (AB) effect: electrically charged particles circumnavigating the vortex acquire an AB phase.

Which components of the gauge fields become massless at the vortex core depends on the particular solution considered, and it might appear that it is quite a complicated matter to disentangle these two effects. Fortunately the BPS nature of our systems allows us to determine in detail the solutions, with the asymptotic behavior of all gauge fields explicitly

²As we need several coupling constants here is the summary. The unbroken electromagnetic field is coupled with the coupling

$$e \equiv \frac{g_R g_L}{\sqrt{g_L^2 + g_R^2}};$$

while the broken gauge field B_μ has coupling $g' = \sqrt{g_L^2 + g_R^2}$. Throughout, we take g_0 and g_L to be of the same order of magnitude, whereas $g_R \ll g_L$ hence $e \ll g_L, g_0$.

exhibited. Knowing them, and if we are interested only in the observable effects far from the vortex, we shall be able to describe unambiguously the new AB effect associated with the weak $U(1)_R$ gauge field and the massless A_μ^{em} .

In obtaining the minimum tension vortex solutions, we will take advantage of the basic fact is that it is possible to write a BPS completion, even in the presence of the $U(1)_R$ gauge field, for static vortex configurations:

$$S = \int d^2x \text{Tr} \left\{ \left(F_{12} + \frac{g_L}{2} (Q\bar{Q} - \xi) \right)^2 + \left(F_{12}^R + \frac{g_R}{2} \tau^{R3} \text{Tr} \left(\bar{Q}Q \frac{\tau^{R3}}{2} \right) \right)^2 + \right. \\ \left. + |\nabla_1 Q + i\nabla_2 Q|^2 + g_L \xi F_{12} - \frac{1}{2} \epsilon_{ij} \partial_i (i\nabla_j Q \bar{Q} - Q i\nabla_j \bar{Q}) \right\} \quad (2.7)$$

where the tension depends only on the winding of $U(1)_0$. The BPS equations read

$$F_{12} = -\frac{g_L}{2} (Q\bar{Q} - \xi) ; \quad (2.8)$$

$$F_{12}^R = -\frac{g_R}{2} \tau^{R3} \text{Tr} (\bar{Q}Q \tau^{R3}/2) ; \quad (2.9)$$

$$\bar{D}Q = \nabla_1 Q + i\nabla_2 Q = 0 . \quad (2.10)$$

2.2 BPS solution with $B = 0$

Several BPS solutions can be found by inspection. A vortex solution in which the scalar fields take the color-flavor diagonal form,

$$Q = \begin{pmatrix} e^{i\varphi} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} , \quad (2.11)$$

whereas $A_i^\pm \equiv 0$ (as in the “ $B = 0$ ” solution of the standard non-Abelian vortex, eq. (1.5)), can be found easily. The profile functions satisfy the boundary conditions, eq. (1.7). The solution takes the form

$$A_{Ri}(x) = \frac{g_R}{\sqrt{g_R^2 + g_L^2}} B_i(x) ; \quad A_{Li}^3(x) = \frac{g_L}{\sqrt{g_R^2 + g_L^2}} B_i(x) . \quad (2.12)$$

In other words, eq. (2.9) and the non-Abelian part of eq. (2.8) are identical, whereas only the combination

$$g_L A_L^3 + g_R A_R = g' B_i \quad (2.13)$$

enters the third BPS equation. The Abelian $U(1)_0$ field $A_\mu^{(0)}$ is as in eq. (1.5).

In order to have a finite energy configuration the kinetic term

$$\nabla_i Q = (\partial_i + ig_0 A_\mu^{(0)} + ig' B_i) Q , \quad g' = \sqrt{g_L^2 + g_R^2}$$

must approach zero asymptotically. This means that non-vanishing gauge fields (eq. (2.5)) must, *in the regular gauge*, approach

$$g_0 A_i^{(0)} \rightarrow -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad (2.14)$$

$$g' B_i = g_L A_i^{L3} + g_R A_i^R \rightarrow -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (2.15)$$

In fact the condition is even stronger: A_i^{L3} and A_i^R are proportional, $A_i^{L3}/g_L = A_i^R/g_R$ (eq. (2.12)). Combining this and eq. (2.15), one finds that at large r

$$A_i^R \rightarrow \frac{g_R}{g_L^2 + g_R^2} \left(-\frac{1}{2}\right) \epsilon_{ij} \frac{x^j}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_i^{L3} \rightarrow \frac{g_L}{g_L^2 + g_R^2} \left(-\frac{1}{2}\right) \epsilon_{ij} \frac{x^j}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.16)$$

When a probe particle carrying a unit $U(1)_R$ charge $+1$ follows a large circle around the vortex, it will acquire an AB phase equal to

$$\oint dx^i A_i^R = \int d^2x F_{12}^R = -2\pi \frac{g_R}{g_L^2 + g_R^2}. \quad (2.17)$$

2.3 $|B| = 1$ solution

The solution with $|B| = 1$ can also be found easily. Setting

$$B = e^{i\delta}, \quad (2.18)$$

in eqs. (1.5)–(1.9), one finds the scalar field configuration

$$\begin{aligned} Q &= U \begin{pmatrix} e^{i\varphi} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi} \phi_1 + \phi_2 & e^{-i\delta} (e^{i\varphi} \phi_1 - \phi_2) \\ e^{i\delta} (e^{i\varphi} \phi_1 - \phi_2) & e^{i\varphi} \phi_1 + \phi_2 \end{pmatrix} \\ &= \frac{e^{i\varphi} \phi_1 + \phi_2}{2} \mathbb{1} + \frac{e^{i\varphi} \phi_1 - \phi_2}{2} (\tau^1 \cos \delta + \tau^2 \sin \delta). \end{aligned} \quad (2.19)$$

As $\bar{Q}Q$ is orthogonal to the direction τ^3 of the right $U(1)_R$,

$$\text{Tr } \bar{Q}Q \tau_R^3 = 0,$$

one sees from eq. (2.7) that a BPS solution can be constructed by setting $A_{Ri} \equiv 0$ and by choosing $A_i^L = A_i$ and $A_i^{(0)}$ as in (1.5):

$$g_L A_i^L(x) = -\frac{1}{2} (\tau^1 \cos \delta + \tau^2 \sin \delta) \epsilon_{ij} \frac{x_j}{r^2} [1 - f_3(r)], \quad (2.20)$$

$$A_i^{(0)}(x) = -\epsilon_{ij} \frac{x_j}{r^2} [1 - f(r)]. \quad (2.21)$$

As $A_i^R(x) \equiv 0, \forall x$, there is no AB effect associated with the $U(1)_R$ gauge interactions.

2.4 BPS solution with $B = \infty$

Using eq. (1.5) and eq. (1.9) one finds that the $B = \infty$ vortex has a squark condensate of the form,

$$Q = \begin{pmatrix} \phi_2(r) & 0 \\ 0 & e^{i\varphi} \phi_1(r) \end{pmatrix} \xrightarrow{r \rightarrow \infty} \sqrt{\xi} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}. \quad (2.22)$$

In the model without $U(1)_R$ weak gauge interactions this corresponds to the origin of the second patch of the $\mathbf{CP}^1$ vortex moduli space, the south pole. The vortex equations are symmetric under an exchange of the two flavors combined with a reflection of the moduli

space $\mathbf{CP}^1$, which maps $B \rightarrow 1/B$. Therefore this solution is essentially identical to the $B = 0$ vortex with the two flavors are interchanged. As a result the BPS solution can be constructed, in the presence of the $U(1)_R$ weak gauge fields, as in subsection 2.2, taking into account certain sign changes (i.e., in eqs. (2.15) and (2.16)). A probe particle carrying a unit $U(1)_R$ charge +1 going around the vortex will this time obtain an AB phase of

$$\oint dx^i A_i^R = \int d^2x F_{12}^R = 2\pi \frac{g_R}{g_L^2 + g_R^2}, \quad (2.23)$$

i.e., of the same magnitude as the $B = 0$ vortex but with the opposite sign.

2.5 A general BPS solution

To show the existence of other BPS solutions interpolating between the $B = 0$, $B = 1$ and $B = \infty$ solutions found above requires a more careful analysis, to be presented in the next section. Nevertheless, it is easy to obtain the $U(1)_R$ flux of such a general solution if one *assumes* that it exists and is given by the deformation in $g_R \neq 0$ of the unperturbed BPS vortices eqs. (1.5)–(1.10) characterized by the $\mathbf{CP}^1$ coordinate B . The point is that to find the BPS solution to first order in g_R it is sufficient (see eq. (2.4)) to know A_μ^{L3} to the zeroth order and A_μ^R to first order. The former is given by eqs. (1.5)–(1.10), whereas the latter is given by the BPS equation (2.9) in which the right hand side is replaced by the scalar fields in the zeroth-order solution eq. (1.5). One finds indeed

$$F_{12}^R \simeq -g_R \text{Tr}(\bar{Q}Q \tau^{R3}/2)|_{g_R=0} = -\frac{g_R}{2}(\phi_1^2 - \phi_2^2) \frac{1 - |B|^2}{1 + |B|^2}, \quad (2.24)$$

but by virtue of a zeroth order BPS equation (see for instance, eq. (3.8) of ref. [2], or eq. (16) of ref. [10])

$$\frac{1}{r} \partial_r f_{\text{NA}} = \frac{g_L^2}{2}(\phi_1^2 - \phi_2^2), \quad (2.25)$$

the F_{12}^R flux is given by

$$\int d^2x F_{12}^R = \frac{g_R}{g_L^2} \frac{1 - |B|^2}{1 + |B|^2} 2\pi \int dr r \frac{1}{r} \partial_r f_{\text{NA}} = \frac{2\pi g_R}{g_L^2} \frac{|B|^2 - 1}{|B|^2 + 1}, \quad 0 \leq |B| \leq \infty \quad (2.26)$$

(where eq. (1.8) was used). This correctly reproduces the results for the $B = 0, 1$, and ∞ solutions found above. It is clear from this derivation that the AB effect in eq. (2.26) is gauge invariant, both with respect to the $SU(2)_L \times U(1)_0$ and to the weak $U(1)_R$ gauge groups.

3 Moduli matrix and the master equations

Showing that there are indeed generic BPS vortex solutions interpolating between those with $B = 0$, $B = 1$ and $B = \infty$ is somewhat more difficult as the color-flavor rotations are no longer exact symmetry operations. Below we will instead appeal to the powerful moduli-matrix method developed in refs. [7, 11, 28]. Before beginning the analysis, let us note that the index for the dimension of the BPS vortex moduli space

$$\mathcal{I} = N N_f \nu, \quad (3.1)$$

(i.e., the number of the zero modes) [1, 21] does not get modified by the addition of the weak gauge $U(1)_R$ interactions under which the two squark fields have charge $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. ν is the $U(1)_0$ winding index depending on the gauge group considered: $U(1)_0 \times G'$, $G' = SU(N), SO(2N), USp(2N)$, etc. [20]. For $G' = SU(N)$, $\nu = \frac{k}{N}$ where k is the winding number of the vortex. As a result we expect, in our case, $N = N_f = 2$, $k = 1$, the two zero modes (the complex $\mathbf{CP}^1$ coordinate B) to persist somehow upon weak $U(1)_R$ gauging.

3.1 The origin of the vortex moduli

Following the familiar procedure [28] we now set

$$Q(z, \bar{z}) = S_L^{-1}(z, \bar{z}) H_0(z) S_R^{-1}(z, \bar{z}), \quad A_{\bar{z}} = i S_L^{-1} \partial_{\bar{z}} S_L, \quad A_{\bar{z}}^R = i S_R^{-1} \partial_{\bar{z}} S_R, \quad (3.2)$$

$$S_L = e^{\psi_0} \begin{pmatrix} e^{\psi_3/2} & 0 \\ e^{\psi_3/2} w & e^{-\psi_3/2} \end{pmatrix}, \quad S_R = \begin{pmatrix} e^{\psi_R/2} & 0 \\ 0 & e^{-\psi_R/2} \end{pmatrix}, \quad H_0 = \begin{pmatrix} z & 0 \\ B & 1 \end{pmatrix}, \quad (3.3)$$

where B is a complex number and $z \equiv x + iy$.³ We also recall that the moduli matrix $H_0(z)$ and S matrices in (3.2) are defined up to a “gauge choice” of the form

$$H_0(z) \rightarrow V_L H_0(z) V_R(z), \quad S_L \rightarrow V_L S_L, \quad S_R \rightarrow S_R V_R, \quad (3.4)$$

where the holomorphic matrices $V_L(z)$ and $V_R(z)$ belong to the complexifications of the $SU(2)_L \times U(1)$ and $U(1)_R$ gauge groups respectively. In the analysis of the vortex moduli spaces as complex manifolds, these V -equivalence relations play a fundamental role.

Remarks. We continue to use here the same letter B for the complex parameter characterizing the solutions, as in the model of subsection 1.1 without the $U(1)_R$ gauging, *even though* the physical meaning will be different. The normalizable $\mathbf{CP}^1$ zero modes B were massless Nambu-Goldstone bosons traveling along the vortex length in the absence of $U(1)_R$ gauge fields; on the other hand, with the $U(1)_R$ weak gauging, only $\text{Arg}(B)$ remains the true Nambu-Goldstone mode, but it is coupled with and eaten by the exact, asymptotically massless electromagnetic field which propagates in the $4D$ bulk, while $|B|$ will determine the (electro-)magnetic AB flux carried by the vortex. The change in the nature of the zero modes reflects the fact that now the non-Abelian vortex is coupled to an exact $4D$ gauge mode whose direction is nontrivially oriented with respect to the underlying ($g_R = 0$ theory) non-Abelian vortex orientation.

Eqs. (3.2) solve the BPS equation (2.10), while the other two BPS equations (2.8), (2.9) turn into the master equations

$$4 \partial_{\bar{z}} (\Omega \partial_z \Omega^{-1}) + g_L^2 \left(H_0 \Omega_R^{-1} H_0^\dagger \Omega^{-1} - \xi \right) = 0, \quad \Omega \equiv S S^\dagger, \quad (3.5)$$

$$4 \partial_{\bar{z}} (\Omega_R^{-1} \partial_z \Omega_R) + g_R^2 \left(\Omega_R^{-1} H_0^\dagger \Omega^{-1} H_0 - \frac{1}{2} \text{Tr} (\Omega_R^{-1} H_0^\dagger \Omega^{-1} H_0) \right) = 0, \quad \Omega_R \equiv S_R^\dagger S_R.$$

³ S_L can be and has been chosen to have a lower triangular form by an appropriate $SU(2)_L$ gauge transformation. Also, the functions ψ_0, ψ_3, ψ_R have been set to be real by an appropriate diagonal $U(1)_0 \times SU(2)_L \times U(1)_R$ gauge rotation.

According to eq. (3.2) the scalar field takes the form,

$$\begin{aligned} Q &= S_L^{-1} H_0(z) S_R^{-1} = \sqrt{\xi} e^{-\psi_0} \begin{pmatrix} z e^{-\psi_3/2} & 0 \\ -(zw + B) e^{\psi_3/2} & e^{\psi_3/2} \end{pmatrix} e^{-\psi_R \tau^3/2} \\ &= \sqrt{\xi} e^{-\psi_0} \begin{pmatrix} z e^{-\psi_+/2} & 0 \\ -(z\tilde{w} + \tilde{B}) e^{\psi_+/2} & e^{\psi_+/2} \end{pmatrix}, \end{aligned} \quad (3.6)$$

where⁴

$$\psi_+ \equiv \psi_3 + \psi_R, \quad \tilde{w} \equiv w e^{-\psi_R}, \quad \tilde{B} \equiv B e^{-\psi_R}. \quad (3.7)$$

After the $SU(2)_L \times U(1) \times U(1)_R$ gauge freedom is used to fix the form the the S matrices, eq. (3.3), there remains still an arbitrariness of the V_R transforms, (3.4). The latter can be used to fix the value of $\psi_R(0)$, for instance, to 0. This unambiguously defines the meaning of the parameter B . The physical meaning of the parameter B can be understood as the ratio of the scalar fields of the first and second flavors at the vortex core ($z = 0$).

Note that the electromagnetic gauge transformation acts on Q as

$$Q \rightarrow \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} Q \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}. \quad (3.8)$$

Keeping the functions ψ_0, ψ_3, ψ_R real via a combination of the V transformations, B can be seen to transform under the electromagnetic gauge transformations simply as

$$B \rightarrow e^{i\beta} B. \quad (3.9)$$

For a comparison with the physical approach of the previous section it is convenient to recall the relation between the gauge fields and various ψ functions

$$\bar{A}_R \equiv A_x^R - i A_y^R = -\frac{i}{g_R} \bar{\partial} \psi_R, \quad \bar{A}_0 \equiv A_x^{(0)} - i A_y^{(0)} = -\frac{i}{g_0} \bar{\partial} \psi_0; \quad (3.10)$$

the $SU(2)_L$ gauge fields are related to ψ 's and w through

$$g_L A_{\bar{z}} = i S_L^{-1} \partial_{\bar{z}} S_L. \quad (3.11)$$

Explicitly eq. (3.5) becomes

$$\begin{aligned} \frac{4}{g_0^2} \partial \bar{\partial} \psi_0 &= 2\xi - \xi e^{-2\psi_0} \left(|z|^2 e^{-\psi_3 - \psi_R} + (e^{\psi_R} + |zw + B|^2 e^{-\psi_R}) e^{\psi_3} \right); \\ \frac{4}{g_L^2} \partial \bar{\partial} \psi_3 - \frac{4}{g_L^2} |\partial \bar{w}|^2 e^{2\psi_3} &= \xi e^{-2\psi_0} \left(-|z|^2 e^{-\psi_3 - \psi_R} + (e^{\psi_R} + |zw + B|^2 e^{-\psi_R}) e^{\psi_3} \right); \\ \frac{1}{g_L^2} \partial (2 e^{2\psi_3} \bar{\partial} w) &= \xi \bar{z} (zw + B) e^{-2\psi_0 + \psi_3 - \psi_R}; \\ \frac{4}{g_R^2} \partial \bar{\partial} \psi_R &= \xi e^{-2\psi_0} \left(-|z|^2 e^{-\psi_3 - \psi_R} + (e^{\psi_R} - |zw + B|^2 e^{-\psi_R}) e^{\psi_3} \right). \end{aligned} \quad (3.12)$$

⁴For the sake of clarity we recall that the functions ψ_+ , ψ_3 , and ψ_R correspond to the gauge fields B_μ , A_μ^{L3} and A_μ^R respectively.

Let us first consider the $B = 0$ solution. By setting

$$B = 0, \quad w = 0, \quad \psi_R/g_R^2 = \psi_3/g_L^2, \quad (3.13)$$

the master equations reduce to the first two, which are exactly the equations for the familiar non-Abelian vortex with the $B = 0$ orientation. This solution was discussed in subsection 2.2.

The solutions with $B \neq 0$ are more interesting. Note that by redefining

$$w \rightarrow B w, \quad \psi_R \rightarrow \psi_R + \log |B|, \quad \psi_3 \rightarrow \psi_3 - \log |B|, \quad (3.14)$$

the parameter B can be eliminated from all four equations (3.12) and gets replaced by 1 everywhere. It might thus look as if the set of equations (3.12) involved a redundant set of functions and that there is actually only one solution for $B \neq 0$, equivalent to the $B = 1$ solution discussed in subsection 2.3.

Actually, this is not so. The coupled differential equations (3.12) define the solution only after a specific set of boundary conditions are imposed. The appropriate boundary conditions at large r are

$$\psi_0 \sim \frac{1}{2} \log |z|, \quad \psi_3 + \psi_R \sim \log |z|, \quad w \sim -\frac{B}{z}, \quad (3.15)$$

so that the squark fields approach the form,

$$Q \sim \sqrt{\xi} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.16)$$

But this does not uniquely fix the boundary conditions for ψ_3 and ψ_R separately: there remains an arbitrariness in the sharing of $\log |z|$ between them.

A general boundary condition is thus

$$\psi_R \sim \delta \log |z| + \dots, \quad \psi_3 \sim (1 - \delta) \log |z| + \dots. \quad (3.17)$$

Furthermore if we define⁵

$$\psi^{\text{em}} \equiv \frac{1}{g_L^2 + g_R^2} (g_L^2 \psi_R - g_R^2 \psi_3), \quad (3.18)$$

one has

$$\psi_+ = \psi_3 + \psi_R \sim \log |z|, \quad \psi^{\text{em}} \sim \left(\delta - \frac{g_R^2}{g_L^2 + g_R^2} \right) \log |z|. \quad (3.19)$$

In the next section we solve eqs. (3.12) explicitly for small g_R and determine δ : the asymptotic behaviors of ψ_R , ψ_+ , ψ_3 and ψ_0 .⁶

⁵Recalling that ψ functions contain the coupling constants, it can be seen that this combination corresponds exactly to A^{em} of Eq. (2.4).

⁶Notice here that the invariance of the master equation (3.12) under rescaling of the modulus of B can be regarded as a consequence of the fact that our vortex solutions are BPS saturated, thus preserving half of the supersymmetries when our model is embedded in an $\mathcal{N} = 2$ theory. In supersymmetric gauge theories, gauge symmetry is naturally extended to a complex symmetry. The phase symmetry in eq. (3.9) of the parameter B thus implies a symmetry under rescaling of the modulus of B . As discussed, however, boundary conditions are sensitive to the modulus of B , and this is responsible for the existence of a nontrivial set of (gauge inequivalent) vortex solutions.

3.2 Perturbative expansion in $\lambda = g_R/g_L$

We will now solve eq. (3.12) perturbatively in $\lambda \equiv g_R/g_L$. At zeroth order one must solve the equations for generic B at $g_R = 0$. Setting $g_R = 0$ one has

$$Q = S_L^{-1} H_0(z) = \sqrt{\xi} e^{-\psi_0} \begin{pmatrix} z e^{-\psi_3/2} & 0 \\ -(zw + B) e^{\psi_3/2} & e^{\psi_3/2} \end{pmatrix}, \quad (3.20)$$

and the functions appearing here satisfy

$$\begin{aligned} \frac{4}{g_0^2} \partial \bar{\partial} \psi_0 &= 2\xi - \xi e^{-2\psi_0} \left(|z|^2 e^{-\psi_3} + (1 + |zw + B|^2) e^{\psi_3} \right); \\ \frac{4}{g_L^2} \partial \bar{\partial} \psi_3 - \frac{4}{g_L^2} |\partial \bar{w}|^2 e^{2\psi_3} &= \xi e^{-2\psi_0} \left(-|z|^2 e^{-\psi_3} + (1 + |zw + B|^2) e^{\psi_3} \right); \\ \frac{1}{g_L^2} \partial (2 e^{2\psi_3} \bar{\partial} w) &= \xi \bar{z} (zw + B) e^{-2\psi_0 + \psi_3}. \end{aligned} \quad (3.21)$$

We first review the $B = 0$ solution, $\psi_0^{(0)} \equiv \psi_0|_{B=0}$, $\psi_3^{(0)} \equiv \psi_3|_{B=0}$, $w = 0$. In this case

$$\begin{aligned} \frac{4}{g_0^2} \partial \bar{\partial} \psi_0^{(0)} &= 2\xi - \xi e^{-2\psi_0^{(0)}} \left(|z|^2 e^{-\psi_3^{(0)}} + e^{\psi_3^{(0)}} \right); \\ \frac{4}{g_L^2} \partial \bar{\partial} \psi_3^{(0)} &= \xi e^{-2\psi_0^{(0)}} \left(-|z|^2 e^{-\psi_3^{(0)}} + e^{\psi_3^{(0)}} \right). \end{aligned} \quad (3.22)$$

To simplify the formulas somewhat, below we will set

$$g_0^2 = \frac{g_L^2}{2}; \quad 2g_L^2 \xi = 1, \quad (3.23)$$

the latter being simply the choice of the mass unit. We then go to the singular gauge via the change of the variables

$$\psi_0^{(0)} = \hat{\psi}_0^{(0)} + \frac{1}{2} \log |z|, \quad \psi_3^{(0)} = \hat{\psi}_3^{(0)} + \log |z|, \quad (3.24)$$

so that the equations become

$$\begin{aligned} 8 \partial \bar{\partial} \hat{\psi}_0^{(0)} &= \frac{1}{2} \left[2 - e^{-2\hat{\psi}_0^{(0)}} \left(e^{-\hat{\psi}_3^{(0)}} + e^{\hat{\psi}_3^{(0)}} \right) \right] + 2\pi \delta^2(x); \\ 4 \partial \bar{\partial} \hat{\psi}_3^{(0)} &= \frac{1}{2} \left[e^{-2\hat{\psi}_0^{(0)}} \left(-e^{-\hat{\psi}_3^{(0)}} + e^{\hat{\psi}_3^{(0)}} \right) \right] + 2\pi \delta^2(x). \end{aligned} \quad (3.25)$$

The difference between the two equations yields

$$2 \hat{\psi}_0^{(0)}(z) - \hat{\psi}_3^{(0)}(z) \equiv 0, \quad (3.26)$$

whereas the sum yields the Taubes' equation [46]

$$4 \partial \bar{\partial} \varphi(z) = \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \varphi(z) \right) = \frac{1}{2} (1 - e^{-2\varphi(z)}) + 2\pi \delta^2(x), \quad (3.27)$$

for

$$(2 \hat{\psi}_0^{(0)}(z) + \hat{\psi}_3^{(0)}(z))/2 = 2 \hat{\psi}_0^{(0)}(z) = \hat{\psi}_3^{(0)}(z) \equiv \varphi(z), \quad (3.28)$$

with the boundary conditions

$$\lim_{r \rightarrow 0} r\varphi' = -1, \quad \lim_{r \rightarrow \infty} \varphi = 0. \quad (3.29)$$

The solution $\varphi(r)$ behaves as

$$\varphi(r) = \begin{cases} -\log r + a & \text{for } r \ll 1 \\ q K_0(r) & \text{for } r \gg 1 \end{cases}, \quad (3.30)$$

with constants [47]

$$a = 0.50536\dots, \quad q = 1.707864\dots \quad (3.31)$$

The modified Bessel function of the second kind K_0 is exponentially damped at large r . This gives $Q|_{B=0}$.

The solutions at $B \neq 0$ can be found by a color-flavor rotation of $Q|_{B=0}$,

$$Q = U Q|_{B=0} U^{-1}, \quad U = \frac{1}{\sqrt{|B|^2 + 1}} \begin{pmatrix} 1 & -B^* \\ B & 1 \end{pmatrix}, \quad (3.32)$$

followed by an appropriate $SU(2)_L$ gauge transformation to bring Q back to the form, (3.20). The answer is (see appendix B):

$$\begin{aligned} \psi_0 &= \psi_0^{(0)}; \\ \psi_3 &= \psi_3^{(0)} + \log \left\{ \frac{1 + |z|^2 |B|^2 e^{-2\psi_3^{(0)}}}{1 + |B|^2} \right\}; \\ w &= -\bar{z} B e^{-\psi_3^{(0)} - \psi_3} = \frac{-\bar{z} B (1 + |B|^2) e^{-\psi_3^{(0)}}}{e^{\psi_3^{(0)}} + |B|^2 |z|^2 e^{-\psi_3^{(0)}}}. \end{aligned} \quad (3.33)$$

Once one knows these solutions with $\lambda = g_R/g_L = 0$ for generic B , the solutions for $\lambda \ll 1$ can be found by perturbation theory. Using eq. (3.33), the fourth BPS equation (3.12) yields

$$\begin{aligned} \frac{4}{g_R^2} \partial \bar{\partial} \psi_R &\simeq \xi e^{-2\psi_0} \left(-|z|^2 e^{-\psi_3} + (1 - |zw + B|^2) e^{\psi_3} \right) \\ &= -\xi \frac{|B|^2 - 1}{|B|^2 + 1} (1 - e^{-2\varphi}) = -\frac{1}{2g_L^2} \frac{|B|^2 - 1}{|B|^2 + 1} (1 - e^{-2\varphi}). \end{aligned} \quad (3.34)$$

Next, using the Taubes' equation (3.25) for φ , one finds

$$\frac{1}{r} (r\psi_R')' = -\lambda \frac{|B|^2 - 1}{|B|^2 + 1} \times \frac{1}{r} (r\varphi')' + \mathcal{O}(\lambda^2); \quad (3.35)$$

this means that

$$\psi_R(r) = -\lambda \frac{|B|^2 - 1}{|B|^2 + 1} (\varphi(r) + \log r - a) + \mathcal{O}(\lambda^2) \quad (3.36)$$

where we have recalled eq. (3.29) and have appropriately taken into account the boundary conditions for ψ_R .⁷

The large r behavior

$$\psi_R \sim \delta \log r, \quad \delta = -\frac{g_R^2 |B|^2 - 1}{g_L^2 |B|^2 + 1} \quad (3.37)$$

follows thus from eq. (3.30). Recalling (eq. (3.15), eq. (3.16)) that $\psi_+ = \psi_3 + \psi_R \sim \log r$, one finds to this order that

$$\psi_3 \sim (1 - \delta) \log r = (1 + \frac{g_R^2 |B|^2 - 1}{g_L^2 |B|^2 + 1}) \log r. \quad (3.38)$$

For completeness, we get for ψ^{em} (eq. (3.19))

$$\psi^{\text{em}} \sim \left(\delta - \frac{g_R^2}{g_L^2 + g_R^2} \right) \log r \sim -\frac{g_R^2}{g_L^2} \frac{2|B|^2}{|B|^2 + 1} \log r, \quad (3.39)$$

whereas the $U_0(1)$ gauge fields winds half (eq. (3.15)):

$$\psi_0 \sim \frac{1}{2} \log r. \quad (3.40)$$

The asymptotic behavior of the gauge fields A_i^{L3} and A_i^R is then

$$A_i^{L3} = -\frac{1}{g_L} \epsilon_{ij} \partial_j \psi_3 \sim -\frac{1 - \delta}{g_L} \epsilon_{ij} \frac{x_j}{r^2}, \quad A_i^R = -\frac{1}{g_R} \epsilon_{ij} \partial_j \psi_R \sim -\frac{\delta}{g_R} \epsilon_{ij} \frac{x_j}{r^2}. \quad (3.41)$$

For completeness, the “broken” gauge field B_μ behaves asymptotically as

$$B_i = -\frac{1}{g'} \epsilon_{ij} \partial_j \psi_+ \sim \frac{1}{g'} \epsilon_{ij} \frac{x_j}{r^2}. \quad (3.42)$$

This leads, for a particle carrying only the unit charge with respect to $U(1)_R$, to an AB phase

$$g_R \oint dx_i A_i^R = g_R \int d^2x F_{12}^R = \frac{2\pi g_R^2}{g_L^2} \frac{|B|^2 - 1}{|B|^2 + 1}, \quad (3.43)$$

as it goes around the vortex, in accordance with eq. (2.26); one sees that the three explicit solutions we found earlier, with $|B| = 0, 1, \infty$, are interpolated by the modulus $|B|$.

4 Aharonov-Bohm effect

The result (3.43) is well defined and gauge invariant. A particle with a unit $U(1)_R$ charge but with no charges with respect to the $SU(2)_L \times U_0(1)$ will experience the AB effect (3.43). More generally, a particle in a definite representation of the $SU(2)_L \times U_0(1) \times U(1)_R$ gauge

⁷Note that $\varphi = \hat{\psi}_3^{(0)}(z)$ was defined in the singular gauge, see eqs. (3.30) and (3.31), whereas ψ_R represents the original right gauge field (eq. (3.10)) and therefore is regular at the vortex core. $\psi_R(0)$ can be set to 0 without losing generality, by an appropriate choice of the V -gauge (eq. (3.4)).

group will get a definite AB phase after encircling around the vortex. Some examples are shown in table 1. We note that, in the case of the $U(1)_R$ charge one particle (particle K in table 1), the AB phase is maximum (in the magnitude) for the vortices $B = 0$ and $B = \infty$, where the vortex orientation and the external gauging are aligned, whereas for the vortices $|B| = 1$ (the vortex solutions along the equator of $CP^1 \sim S^2$) the orientations are orthogonal and there is no AB effect.

As the $U(1)_R$ gauge symmetry is a spontaneously broken symmetry, it might be thought that any physical effect at large distances (and far from the vortex core) should be describable in terms of the coupling of the particle to the massless gauge field, A_μ^{em} . For instance, a particle carrying a unit charge with respect to $U(1)_R$ (but with no other charges) interacts through the covariant derivative

$$\left(\partial_\mu + \frac{i}{2}g_R A_\mu^R\right) K = \left(\partial_\mu + \frac{i}{2}e A_\mu^{\text{em}} + \frac{i}{2}\frac{g_R^2}{\sqrt{g_L^2 + g_R^2}}B_\mu\right) K. \quad (4.1)$$

It would seem natural to assume that the long-distance physics is dominated by the coupling to the massless “photon” field A_μ^{em} , the interactions with the massive B_μ field providing some small corrections (the “weak interactions”) calculable in perturbation theory.

In the presence of vortex, this is not quite so. As one can explicitly see from (3.42) also the massive gauge fields contribute to the AB effect to the same order, as is well known [36]. As a result, two particles with the same electromagnetic charge but with different couplings to the broken gauge field B_μ such as K and ψ_1 in table 1 (i.e., two particles belonging to different representations of $SU(2)_L \times U_0(1) \times U(1)_R$) experience different AB effect going around the vortex, however far from it.

One must also be somewhat careful in deriving the physical AB effect from the results of the calculations in the preceding sections, as one is working with a spontaneously broken $SU(2)_L$ gauge theory. The situation is somewhat analogous to that of the Weinberg-Salam theory. In the Weinberg-Salam theory the neutrino and electron are usually described by the upper and lower components of an $SU_L(2)$ doublet field; nevertheless both describe physical particles of definite mass and charge. Of course, the solution of this apparent puzzle is well known: the identification of the two leptons with the two components of a doublet is correct in a gauge in which the upper component of the Higgs doublet gets the nonvanishing VEV. More appropriately, the electron and neutrino must be associated with some appropriate $SU_L(2)$ gauge-invariant composite fields involving the left-handed lepton and the Higgs scalar fields [51].

In our case, $SU(2)_L \times U_0(1) \times U(1)_R$ gauge symmetry is broken in the bulk by the scalar VEVs,

$$\langle Q \rangle = \sqrt{\xi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.2)$$

(see eq. (2.3)) to $U^{\text{em}}(1)$. The identification of

$$A_\mu^{\text{em}} = \frac{g_L}{\sqrt{g_L^2 + g_R^2}}A_\mu^R - \frac{g_R}{\sqrt{g_L^2 + g_R^2}}A_\mu^{L3} \quad (4.3)$$

with the massless “photon” (and B_μ with the massive field, see eqs. (2.4)–(2.6)) is correct in the gauge in which the scalar VEVs take the form, (4.2).

The vortex solutions also depend on the gauge used to solve the field equations. The solution in subsections 3.1, 3.2 has been obtained by working in the gauge where the scalar fields have the asymptotic form

$$Q \sim \sqrt{\xi} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} \quad (4.4)$$

(eq. (3.16)). A_i^R and A_i^{L3} are found to behave as in (3.41), (3.37) in this gauge. Note that the identification of the combination (4.3) with the massless field in the bulk is appropriate in this gauge also.⁸

Even if choosing a gauge is unavoidable as in any other gauge theory calculations, clearly the concept of the AB phase which a particle in any definite representation of the $SU(2)_L \times U(1) \times U(1)_R$ group will acquire in encircling the vortex far from it, is a physical one. The only unusual aspect is that, as pointed out above, the AB effect does not depend only on the “electromagnetic” charge but also on the “weak” charge. The calculation can be done in any gauge, of course, but when the result is transformed back to the conventional gauge where the scalar VEV’s approach a color-flavor unit matrix form, in order to relate it to the physical effect, the answer is the same.

As a further remark on the gauge invariant nature of our results, let us note that, as ’t Hooft’s observation for the electroweak theory [51], a particle described by a gauge variant field such as ψ_1 (in our conventional gauge) can be regarded as a gauge invariant object associated with an $SU(2)_L$ singlet composite field such as $\bar{Q}^{(1)}\psi$ or $\epsilon_{ij}Q_i^{(2)}\psi_j$. The physical AB phase is indeed the same in all these descriptions, see table 1.

5 Low-energy effective action

When the moduli parameters B are allowed to fluctuate along the vortex length and in time, i.e., in the vortex worldsheet, the associated collective coordinates become dynamical, described by a long-wavelength effective action. In the model of section 1 (without the $U(1)_R$ gauging) this is just a $2D$ $\mathbf{CP}^1$ sigma model, eq. (1.11), with Kähler potential [1]–[10],

$$K^{(\mathbf{CP})} = (4\pi/g_L^2) \log(|B|^2 + 1) , \quad (5.1)$$

corresponding to the Fubini-Study metric.

In the presence of $4D$ bulk zero modes — an exact unbroken $U(1)^{\text{em}}$ gauge symmetry — coupled to the vortex dynamics, the straightforward approach of appendix A cannot be applied. Although there are nontrivial mixings of A^{em} with other broken gauge fields inside of the vortex, the vortex configurations themselves can be approximated by setting $g_R = e = 0$ to lowest order. One then finds [34]

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} \int d^4x (F_{\mu\nu}^{\text{em}})^2 + \frac{4\pi}{g_L^2} \int dt dz \frac{1}{(|B|^2 + 1)^2} \nabla_\alpha \bar{B} \nabla_\alpha B + O(e^3) , \quad \alpha = 3, 0 \quad (5.2)$$

⁸We thank Chandrasekhar Chatterjee for discussions on this point.

Particle	$U(1)_{L3} \subset SU(2)_L$		$U(1)_0$	$U(1)_R$	$U^{\text{em}}(1)$	$U_B(1)$	AB phase
K	0	$\underline{1}$	0	1	$\frac{1}{2}$	$\frac{\gamma}{2}$	$-\frac{\delta}{2}$
L	0	$\underline{1}$	0	-1	$-\frac{1}{2}$	$-\frac{\gamma}{2}$	$\frac{\delta}{2}$
ψ_1	1	$\underline{2}$	0	0	$-\frac{1}{2}$	$\frac{1-\gamma}{2}$	$\frac{\delta-1}{2}$
ψ_2	-1	$\underline{2}$	0	0	$\frac{1}{2}$	$-\frac{1-\gamma}{2}$	$\frac{1-\delta}{2}$
χ_1	1	$\underline{2}$	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$
χ_2	-1	$\underline{2}$	0	1	1	$-\frac{1}{2}$	$\frac{1}{2} - \delta$
$Q_1^{(1)}$	1	$\underline{2}$	1	1	0	$\frac{1}{2}$	0
$Q_2^{(1)}$	-1	$\underline{2}$	1	1	1	γ	$-\delta$
$Q_1^{(2)}$	1	$\underline{2}$	1	-1	-1	$-\gamma$	$+\delta$
$Q_2^{(2)}$	-1	$\underline{2}$	1	-1	0	$-\frac{1}{2}$	0
$\bar{Q}^{(1)}\psi$	0	$\underline{1}$	-1	-1	$-\frac{1}{2}$	$-\frac{\gamma}{2}$	$\frac{\delta-1}{2}$
$\epsilon_{ij}Q_i^{(1)}\psi_j$	0	$\underline{1}$	1	1	$\frac{1}{2}$	$\frac{\gamma}{2}$	$\frac{1-\delta}{2}$
$\bar{Q}^{(2)}\psi$	0	$\underline{1}$	-1	1	$\frac{1}{2}$	$\frac{\gamma}{2}$	$-\frac{\delta+1}{2}$
$\epsilon_{ij}Q_i^{(2)}\psi_j$	0	$\underline{1}$	1	-1	$-\frac{1}{2}$	$-\frac{\gamma}{2}$	$\frac{\delta+1}{2}$
$\bar{Q}^{(1)}\chi$	0	$\underline{1}$	-1	0	0	0	$-\frac{1}{2}$
$\epsilon_{ij}Q_i^{(1)}\chi_j$	0	$\underline{1}$	1	2	1	γ	$\frac{1}{2} - \delta$
$\bar{Q}^{(2)}\chi$	0	$\underline{1}$	-1	2	1	γ	$-\frac{1}{2} - \delta$
$\epsilon_{ij}Q_i^{(2)}\chi_j$	0	$\underline{1}$	1	0	0	0	$\frac{1}{2}$

Table 1. $\delta = -\frac{g_R^2}{g_L^2} \frac{|B|^2-1}{|B|^2+1}$; $\gamma \equiv \frac{g_R^2}{g_L^2+g_R^2}$. In the table are some fields with their charges under the various gauge groups, and the associated AB phase (given in the unit of 2π and *modulo* 2π , so that $-\frac{1}{2} \sim \frac{1}{2}$). The extra factor $\frac{1}{2}$'s in the last three columns take into account of the $\tau^3/2$ in front of the $A_\mu^{L,3}$ and A_μ^R . The electromagnetic charge represents the coupling to A_μ^{em} in the unit of the coupling constant e ; the coupling to the broken gauge field B^μ is in the unit of $g' = \sqrt{g_L^2 + g_R^2}$.

where

$$\nabla_\alpha B = (\partial_\alpha + ieA_\alpha^{\text{em}})B. \quad (5.3)$$

This form is dictated by electromagnetic gauge invariance, under which the B and A^{em} fields transform as (see eq. (3.9)):

$$B \rightarrow e^{i\beta} B, \quad A_\alpha^{\text{em}} \rightarrow A_\alpha^{\text{em}} - \frac{1}{e} \partial_\alpha \beta. \quad (5.4)$$

The current carried along the vortex is given by

$$J_\alpha = -\frac{8\pi e}{g_L^2} \frac{\bar{B} \nabla_\alpha B - B \nabla_\alpha \bar{B}}{2i(1+|B|^2)^2} + \mathcal{O}(e^3) \quad (5.5)$$

and is proportional to

$$\frac{e^2}{g_L^2} \frac{|B|^2}{(|B|^2 + 1)^2} , \quad (5.6)$$

which is conserved along the vortex,

$$\partial^\alpha J_\alpha = 0 . \quad (5.7)$$

Dynamical aspects of the vortex zero modes (their fluctuations) are subtle, as the $2D$ vortex zero modes interact nontrivially with the $4D$ bulk zero modes (the “photon”). Eq. (5.2) appears to give rise to the familiar Higgs mechanism, giving the photon a small nonvanishing mass,

$$m_\gamma^2 \sim \frac{e^2}{g_L^4} \frac{|B|^2}{(|B|^2 + 1)^2} . \quad (5.8)$$

However the B condensate lives only inside the vortex, whereas the photon is massless in the $4D$ bulk outside the vortex. As the action is quadratic in A_α the effect of integrating out the photon field can be determined. For instance, a constant time variation $\partial_0 B$ would give rise to the charge density along the vortex:

$$q \sim \frac{8\pi e}{g_L^2} \frac{|B|^2}{(|B|^2 + 1)^2} \partial_0 \text{Arg} B . \quad (5.9)$$

The electromagnetic potential of an infinite string of such charge density is given by

$$A_0 \sim \frac{q}{2\pi} \log r , \quad (5.10)$$

so that

$$\nabla A_0 = \hat{r} \partial_r A_0 \sim \hat{r} \frac{q}{2\pi r} . \quad (5.11)$$

Substituting this into eq. (5.2) one finds a (divergent) energy

$$\frac{1}{2} \int d^3x (\partial_i A_0)^2 \sim \frac{1}{2} \int dz 2\pi \int \frac{dr}{r} \frac{q^2}{4\pi^2} = \log \Lambda \int dz \frac{16\pi e^2}{g_L^4} \frac{|B|^2}{(|B|^2 + 1)^4} \partial_0 \bar{B} \partial_0 B \quad (5.12)$$

where an infrared cutoff in the transverse plane at $\sqrt{x^2 + y^2} = \Lambda$ has been introduced.

This physical discussion explains the result which one would obtain if one were to formally apply the standard method of calculation to the effective action in the presence of a massless $4D$ field: one would find that the Kähler potential is given by (appendix A.3)

$$K = K_{\text{core}} + K_{\text{bulk}} , \quad (5.13)$$

$$K_{\text{core}} \simeq K^{(\text{CP})} = \frac{4\pi}{g_L^2} \log(|B|^2 + 1) ; \quad \frac{\partial^2 K_{\text{bulk}}}{\partial B \partial \bar{B}} = \frac{16\pi e^2}{g_L^4} \frac{|B|^2}{(|B|^2 + 1)^2} \log \Lambda , \quad (5.14)$$

namely a finite part which coincides (in the limit $g_R = 0$) with the $\mathbf{CP}^1$ Kähler potential and a divergent part. As the above discussion shows, the latter is caused by the coupling of B with the $4D$ massless modes, and it is such an interaction that effectively makes the $|B|$ mode non-normalizable.

As a potential physical application, one may consider a vortex loop of finite size, as in Witten’s cosmological string model [48] or a vorton [49, 50], and study the resulting finite physical effects. We shall leave the study of these issues to a separate work.

6 Conclusion

The AB effect found above is a result of the mismatch between the the fixed τ_R^3 weak gauging direction and the generic vortex orientation B in the color-flavor $SU(2)$ space. A particle with unit $U(1)_R$ charge, such as those in table 1, will get an AB phase as it travels along a large circle around the vortex.

Although we restricted ourselves in this paper to the simplest non-trivial prototype model based on $SU(2)_L \times U_0(1) \times U(1)_R$ gauge symmetry for the sake of clarity of presentation, it is indeed quite straightforward and rather interesting, to extend our analysis to more general gauge groups and patterns of partial weak gauging (in preparation). Also, even though our derivation and the persistence of the vortex moduli space upon $U(1)_R$ gauging both depend on the BPS nature of the model considered, the occurrence of the “electromagnetic” AB effect itself has a clear physical explanation, and is independent of the BPS approximation.

As noted in the Introduction, weakly gauging a $U(1)$ subgroup of the flavor symmetry occurs in the color-locked phase of dense quark matter. Therefore, non-Abelian vortices in such a phase should also possess AB fluxes. If such a CFL phase is realized in the cores of neutron stars, non-Abelian vortices will be created by a rapid rotation. Consequently, there may be significant AB effects on particles charged under the asymptotically unbroken gauge symmetry, which could give considerable effects on evolutions of neutron stars. In a non-BPS set-up, the tension of the vortex will generically depend on the value of the modulus $|B|$, and the magnitude of the AB effect will depend upon the which value of $|B|$ corresponds to the vortex solution with lowest tension. In ref. [44] it was shown, in the case of high density QCD, that the vortex solutions with the smallest tension are those corresponding to $|B| = 0, \infty$, which, as shown in this paper, have a non-trivial AB effect for the unbroken electromagnetic field.

In conclusion, as a result of the AB effect an external weak gauging G_W of part of the exact color-flavor symmetry converts some of the well-known internal orientational moduli of a non-Abelian vortex into a new observable: an AB phase associated with G_W . This can be regarded as a novel physical property of non-Abelian vortices. The value of the phase depends on the particular solution considered, even though in our BPS systems the vortex tension is independent of the solution.

The extension of our results to more general gauge groups and weak gauging subgroups, including supersymmetric systems and roles of the fermions, will be discussed in a forthcoming paper.

Acknowledgments

K.K. thanks the Department of Physics, Osaka City University, for a warm hospitality during his visit there where part of the work was done. K.O. and M.N. thank INFN, Pisa, and the Department of Physics, University of Pisa, for a warm hospitality during their visit there where part of the work was done. We thank Simone Giacomelli for useful discussions. J.E. is supported by the Chinese Academy of Sciences Fellowship for Young International

Scientists grant number 2010Y2JA01. M.N. is supported in part by Grant-in-Aid for Scientific Research (No. 25400268) and by the “Topological Quantum Phenomena” Grant-in-Aid for Scientific Research on Innovative Areas (No. 25103720) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. W.V. is supported by the Global Engagement for Global Impact programme funded by EPSRC, and with support from ARO grant number W911NF-12-1-0523. K.K. is supported by INFN special project, “Nonperturbative Dynamics in Gauge Theories and in String Theory (PI14)”.

A Moduli matrix formalism

A.1 BPS equations

The standard covariant derivative has the form,

$$\mathcal{D}_\mu q^A = \partial_\mu q^A + i(A_\mu)^A{}_B q^B, \quad (A_\mu)^A{}_B = \sum_I A_\mu^I g_I (T_I)^A{}_B. \quad (\text{A.1})$$

The BPS equations for chiral fields

$$\mathcal{D}_{\bar{z}} q^A = 0, \quad (\text{A.2})$$

where

$$z \equiv x + iy, \quad \bar{z} \equiv x - iy, \quad \partial = \partial_z \equiv \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} \equiv \frac{1}{2}(\partial_x + i\partial_y)$$

as usual, are solved by ($\bar{A} \equiv A_x + iA_y$)

$$i\bar{A} = S^{-1}\bar{\partial}S, \quad q^A = (S^{-1})^A{}_B H_0^B(z), \quad \partial_{\bar{z}} H_0^A(z) = 0. \quad (\text{A.3})$$

Introducing a derivative *operator* $\hat{\partial}$, the covariant derivative can be set in a more compact form,

$$\mathcal{D}_{\bar{z}} = \bar{\partial} + i\bar{A} = S^{-1}\hat{\partial}S. \quad (\text{A.4})$$

$$\begin{aligned} F_{12} &= -i[\mathcal{D}_1, \mathcal{D}_2] = -2[\mathcal{D}_z, \mathcal{D}_{\bar{z}}] = -2iF_{z\bar{z}} \\ &= -2[S^\dagger \hat{\partial} S^{\dagger-1}, S^{-1} \hat{\partial} S] = -2S^\dagger \left[\hat{\partial}, S^{\dagger-1} S^{-1} \hat{\partial} S S^\dagger \right] S^{\dagger-1} \\ &= -2S^\dagger \left[\hat{\partial}, \Omega^{-1} \hat{\partial} \Omega \right] S^{\dagger-1} = -2S^\dagger \partial (\Omega^{-1} \bar{\partial} \Omega) S^{\dagger-1}, \end{aligned} \quad (\text{A.5})$$

where we used $C[A, B]C^{-1} = [CAC^{-1}, CBC^{-1}]$. BPS equations for gauge fields

$$F_{12}^I - g_I \left(q_A^\dagger (T_I q)^A - \xi_I \right) = 0 \quad (\text{A.6})$$

can be rewritten to

$$\begin{aligned} 2\partial (\Omega^{-1} \bar{\partial} \Omega) &= -S^{\dagger-1} (F_{12}^I g_I T_I) S^\dagger \\ &= g_I^2 S^{\dagger-1} \xi_I T_I S^\dagger - g_I^2 (S^{\dagger-1} T_I S^\dagger) \left(H_0^\dagger S^{\dagger-1} T_I S^{-1} H_0 \right) \\ &= g_I^2 \xi_I T_I - g_I^2 T_I \left(H_0^\dagger T_I \Omega^{-1} H_0 \right) \end{aligned} \quad (\text{A.7})$$

where we assume that, under complexified gauge transformation with $C \in G^{\mathbb{C}}$, ζ is invariant as

$$C\zeta C^{-1} = \zeta, \quad \zeta \equiv \sum_I g_I^2 \xi_I T_I, \quad (\text{A.8})$$

and a tensor $\sum_I T_I \otimes T_I$ is also invariant

$$\sum_I C T_I C^{-1} \otimes C T_I C^{-1} = \sum_I T_I \otimes T_I \quad (\text{A.9})$$

Therefore we find that the master equation

$$2\partial(\Omega^{-1}\bar{\partial}\Omega) = \zeta - g_I^2 T_I \left(H_0^\dagger T_I \Omega^{-1} H_0 \right). \quad (\text{A.10})$$

is equivalent to

$$2\bar{\partial}(\partial\Omega\Omega^{-1}) = \zeta - g_I^2 T_I \left(H_0^\dagger \Omega^{-1} T_I H_0 \right), \quad (\text{A.11})$$

where we used the following identity

$$\Omega\partial(\Omega^{-1}\bar{\partial}\Omega)\Omega^{-1} = \bar{\partial}(\partial\Omega\Omega^{-1}). \quad (\text{A.12})$$

A.2 Zero modes and Gauss's law

Following ref. [13], let us now lift the moduli parameters in the moduli matrix H_0^A to chiral fields

$$H_0^A(z, \phi^X) \rightarrow H_0(z, \phi^X(x^\alpha)), \quad x^\alpha = x^3, x^0. \quad (\text{A.13})$$

By assumption H_0 contains no anti-chiral field so that

$$\delta_\alpha^\dagger H_0^A = 0 \quad (\text{A.14})$$

where $\delta_\alpha, \delta_\alpha^\dagger$ are defined by

$$\delta_\alpha = \partial_\alpha \phi^X \frac{\delta}{\delta \phi^X}, \quad \delta_\alpha^\dagger = \partial_\alpha \bar{\phi}^X \frac{\delta}{\delta \bar{\phi}^X}. \quad (\text{A.15})$$

We can show that with the following gauge fields parallel to the vortex configuration ($\alpha = 0, 3$),

$$iA_\alpha = S^{-1}(\delta_\alpha^\dagger S) + S^\dagger(\delta_\alpha S^{\dagger-1}), \quad \mathcal{D}_\alpha = S^{-1}\hat{\delta}_\alpha^\dagger S + S^\dagger\hat{\delta}_\alpha S^{\dagger-1} \quad (\text{A.16})$$

satisfy the Gauss law. First we find

$$\mathcal{D}_\alpha q = S^\dagger \delta_\alpha (\Omega^{-1} H_0) \quad (\text{A.17})$$

and

$$\begin{aligned} iF_{\alpha\bar{z}} &= [\mathcal{D}_\alpha, \mathcal{D}_{\bar{z}}] = [S^{-1}\hat{\delta}_\alpha^\dagger S, S^{-1}\hat{\partial}S] + [S^\dagger\hat{\delta}_\alpha S^{\dagger-1}, S^{-1}\hat{\partial}S] \\ &= S^{-1}[\hat{\delta}_\alpha^\dagger, \hat{\partial}]S + S^\dagger[\hat{\delta}_\alpha, \Omega^{-1}\hat{\partial}\Omega]S^{\dagger-1} \\ &= S^\dagger \delta_\alpha (\Omega^{-1}\bar{\partial}\Omega) S^{\dagger-1}. \end{aligned} \quad (\text{A.18})$$

By using these, next, we find

$$\begin{aligned} \left(q^\dagger g_I T_I \mathcal{D}_\alpha q \right) \otimes g_I T_I &= \left(H_0^\dagger g_I^2 T_I \delta_\alpha (\Omega^{-1} H_0) \right) \otimes (S^\dagger T_I S^{\dagger-1}) \\ &= -2S^\dagger \delta_\alpha \left(\partial(\Omega^{-1} \bar{\partial} \Omega) \right) S^{\dagger-1} \end{aligned} \quad (\text{A.19})$$

and

$$\begin{aligned} 2i\mathcal{D}F_{\alpha\bar{z}}^I(g_I T_I) &= 2i\mathcal{D}F_{\alpha\bar{z}} \\ &= 2 \left[S^\dagger \hat{\partial} S^{\dagger-1}, S^\dagger \delta_\alpha (\Omega^{-1} \bar{\partial} \Omega) S^{\dagger-1} \right] \\ &= 2S^\dagger \left[\hat{\partial}, \delta_\alpha (\Omega^{-1} \bar{\partial} \Omega) \right] S^{\dagger-1} = 2S^\dagger \partial \delta_\alpha (\Omega^{-1} \bar{\partial} \Omega) S^{\dagger-1}. \end{aligned} \quad (\text{A.20})$$

This shows that (half of) Gauss's law

$$q^\dagger g_I T_I \mathcal{D}_\alpha q + 2i\mathcal{D}F_{\alpha\bar{z}}^I = 0 \quad (\text{A.21})$$

is indeed satisfied.

A.3 The low-energy effective action

Next we calculate the effective action,

$$\mathcal{L}_{\text{eff}} = \int d^2x \left(\frac{1}{2} (F_{\alpha i}^I)^2 + \mathcal{D}_\alpha q^\dagger \mathcal{D}^\alpha q \right). \quad (\text{A.22})$$

Two terms on the right hand side are given by

$$\begin{aligned} \mathcal{D}_\alpha q^\dagger \mathcal{D}^\alpha q &= \delta_\alpha^\dagger \left(H_0^\dagger \Omega^{-1} \right) \Omega \delta_\alpha (\Omega^{-1} H_0) \\ &= \delta_\alpha^\dagger \left(H_0^\dagger \Omega^{-1} \right) \delta_\alpha H_0 - \sum_I \delta_\alpha^\dagger \left(H_0^\dagger \Omega^{-1} \right) g_I T_I H_0 (\delta_\alpha \Omega \Omega^{-1})^I \\ &= \delta_\alpha^\dagger \left(H_0^\dagger \Omega^{-1} \delta_\alpha H_0 \right) - \sum_I \delta_\alpha^\dagger \left(H_0^\dagger \Omega^{-1} T_I H_0 \right) g_I (\delta_\alpha \Omega \Omega^{-1})^I, \end{aligned} \quad (\text{A.23})$$

and

$$\begin{aligned} \frac{1}{2} (F_{\alpha i}^I)^2 &= 2|F_{\alpha\bar{z}}^I|^2 = \sum_I \frac{2}{g_I^2 c^2} \left| \text{Tr} \left[T_I S^\dagger \delta_\alpha (\Omega^{-1} \bar{\partial} \Omega) S^{\dagger-1} \right] \right|^2 \\ &= \sum_I \frac{2}{g_I^2 c^2} \text{Tr} \left[T_I S^\dagger \delta_\alpha (\Omega^{-1} \bar{\partial} \Omega) S^{\dagger-1} \right] \text{Tr} \left[T_I S^{-1} \delta_\alpha^\dagger (\partial \Omega \Omega^{-1}) S \right] \\ &= \sum_I \frac{2}{g_I^2 c^2} \text{Tr} \left[T_I \Omega \delta_\alpha (\Omega^{-1} \bar{\partial} \Omega) \Omega^{-1} \right] \text{Tr} \left[T_I \delta_\alpha^\dagger (\partial \Omega \Omega^{-1}) \right] \\ &= \sum_I \frac{2}{g_I^2 c^2} \text{Tr} \left[T_I \bar{\partial} (\delta_\alpha \Omega \Omega^{-1}) \right] \text{Tr} \left[T_I \delta_\alpha^\dagger (\partial \Omega \Omega^{-1}) \right] \\ &= \bar{\partial} \{ \Sigma_z \} + \Delta \mathcal{L}, \end{aligned} \quad (\text{A.24})$$

where

$$\begin{aligned}
 \Sigma_z &= \sum_I \frac{2}{g_I^2 c^2} \text{Tr} [T_I \delta_\alpha \Omega \Omega^{-1}] \text{Tr} [T_I \delta_\alpha^\dagger (\partial \Omega \Omega^{-1})] \\
 &= \sum_I \frac{2}{g_I^2 c^2} \text{Tr} [T_I \delta_\alpha \Omega \Omega^{-1}] \text{Tr} [T_I \Omega \partial (\Omega^{-1} \delta_\alpha^\dagger \Omega) \Omega^{-1}] \\
 &= \sum_I \frac{2}{g_I^2 c^2} \text{Tr} [T_I \Omega^{-1} \delta_\alpha \Omega] \partial \left(\text{Tr} [T_I \Omega^{-1} \delta_\alpha^\dagger \Omega] \right)
 \end{aligned} \tag{A.25}$$

and

$$\begin{aligned}
 \Delta \mathcal{L} &= - \sum_I \frac{2}{g_I^2 c^2} \text{Tr} [T_I \delta_\alpha \Omega \Omega^{-1}] \text{Tr} [T_I \delta_\alpha^\dagger \bar{\partial} (\partial \Omega \Omega^{-1})] \\
 &= \sum_I \frac{1}{c^2} \text{Tr} [T_I \delta_\alpha \Omega \Omega^{-1}] \text{Tr} [T_I T_J] \delta_\alpha^\dagger (H_0^\dagger \Omega^{-1} T_J H_0) \\
 &= \sum_I g_I (\delta_\alpha \Omega \Omega^{-1})^I \delta_\alpha^\dagger (H_0^\dagger \Omega^{-1} T_I H_0) .
 \end{aligned} \tag{A.26}$$

Here the following normalization of generator of G are used:

$$\text{Tr}[T_I T_J] = c \delta_{IJ} \tag{A.27}$$

and $X = X^I (g_I T_I)$, $Y = Y^I (g_I T_I)$,

$$X^I = \frac{1}{g_I c} \text{Tr}[T_I X] , \quad \sum_I \text{Tr}[T_I X] \text{Tr}[T_I Y] = c \text{Tr}[XY] . \tag{A.28}$$

We see that $\Delta \mathcal{L}$ cancels between the gauge and scalar kinetic terms, and the effective Lagrangian can be written in the form,

$$\mathcal{L}_{\text{eff}} = \frac{\partial^2 K_{\text{core}}}{\partial \phi^X \partial \bar{\phi}^Y} \partial_\alpha \phi^X \partial^\alpha \bar{\phi}^Y + \frac{\partial^2 K_{\text{bulk}}}{\partial \phi^X \partial \bar{\phi}^Y} \partial_\alpha \phi^X \partial^\alpha \bar{\phi}^Y , \tag{A.29}$$

where

$$\frac{\partial K_{\text{core}}}{\partial \phi^X} = \int d^2 x H_0^\dagger \Omega^{-1} \frac{\partial H_0}{\partial \phi^X} \tag{A.30}$$

and

$$\frac{\partial^2 K_{\text{bulk}}}{\partial \phi^X \partial \bar{\phi}^Y} \partial_\alpha \phi^X \partial^\alpha \bar{\phi}^Y = -\frac{i}{2} \int_{|z|=\Lambda} dz \Sigma_z , \tag{A.31}$$

see eqs. (A.23) and (A.24).

Applied to our concrete $\text{SU}(2) \times \text{U}(1)$ model, these give (read $\phi^X \rightarrow B$; $\bar{\phi}^Y \rightarrow \bar{B}$;))

$$K = K_{\text{core}} + K_{\text{bulk}} , \tag{A.32}$$

where

$$K_{\text{core}} \simeq K_{\text{core}}|_{g_R=0} = K^{(\text{CP})} = \frac{4\pi}{g_L^2} \log(|B|^2 + 1) , \tag{A.33}$$

while K_{bulk} is divergent:

$$\frac{\partial^2 K_{\text{bulk}}}{\partial B \partial \bar{B}} = \frac{4\pi}{e^2} \partial_{\bar{B}} \beta \partial_B \psi_{\text{em}} \Big|_{|z|=\Lambda} = \frac{16\pi e^2}{g_L^4} \frac{|B|^2}{(|B|^2 + 1)^2} \log \Lambda . \tag{A.34}$$

B Solutions without $U(1)_R$ weak gauging

Let us consider a solution for ψ_0, ψ_3, w where

$$S = e^{\psi_0} \begin{pmatrix} e^{\psi_3/2} & 0 \\ e^{\psi_3/2} w & e^{-\psi_3/2} \end{pmatrix}, \quad H_0 = \sqrt{\xi} \begin{pmatrix} z & 0 \\ -B & 1 \end{pmatrix}. \quad (\text{B.1})$$

We call the solution for $B = 0$ as $\psi_0 = \psi_0^{(0)}, \psi_3 = \psi_3^{(0)}, w = 0$:

$$S^{(0)} = e^{\psi_0^{(0)}} \begin{pmatrix} e^{\psi_3^{(0)}/2} & 0 \\ 0 & e^{-\psi_3^{(0)}/2} \end{pmatrix}, \quad H_0^{(0)} = \sqrt{\xi} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B.2})$$

$\psi_0^{(0)}$ and $\psi_3^{(0)}$ are determined explicitly in the main text (eqs. (3.26), (3.27)). The generic $B \neq 0$ solutions are generated by a color-flavor rotation

$$V_B H_0 = H_0^{(0)} U_B, \quad U_B = \frac{1}{\sqrt{1+|B|^2}} \begin{pmatrix} 1 & B^* \\ -B & 1 \end{pmatrix}, \quad (\text{B.3})$$

where the V-transformation V_B can be taken as

$$V_B = \frac{1}{\sqrt{1+|B|^2}} \begin{pmatrix} 1+|B|^2 & B^* z \\ 0 & 1 \end{pmatrix} \quad (\text{B.4})$$

so as to bring back S to the lower triangular form. Therefore there must be the following relation with an appropriate $U(2)_L$ gauge transformation U_L

$$V_B S = S^{(0)} U_L, \quad (\text{B.5})$$

or

$$S S^\dagger = V_B^{-1} S^{(0)} S^{(0)\dagger} (V_B^\dagger)^{-1}, \quad (\text{B.6})$$

that is

$$e^{\psi_0} \begin{pmatrix} e^{\psi_3} & e^{\psi_3} \bar{w} \\ e^{\psi_3} w & |w|^2 e^{\psi_3} + e^{-\psi_3} \end{pmatrix} = e^{\psi_0^{(0)}} \begin{pmatrix} \frac{e^{\psi_3^{(0)}} + |B|^2 |z|^2 e^{-\psi_3^{(0)}}}{1+|B|^2} & -z \bar{B} e^{-\psi_3^{(0)}} \\ -\bar{z} B e^{-\psi_3^{(0)}} & (1+|B|^2) e^{-\psi_3^{(0)}} \end{pmatrix}. \quad (\text{B.7})$$

Comparing the both sides we find:

$$\begin{aligned} \psi_0 &= \psi_0^{(0)}, & \psi_3 &= \log \left(\frac{e^{\psi_3^{(0)}} + |B|^2 |z|^2 e^{-\psi_3^{(0)}}}{1+|B|^2} \right), \\ w &= -\bar{z} B e^{-\psi_3 - \psi_3^{(0)}} = \frac{-\bar{z} B (1+|B|^2) e^{-\psi_3^{(0)}}}{e^{\psi_3^{(0)}} + |B|^2 |z|^2 e^{-\psi_3^{(0)}}}. \end{aligned} \quad (\text{B.8})$$

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