18 HOOKS AND SKEW-HOOKS

Hooks play an important part in the representation theory of $\mathfrak{S}_n$, but it is not clear in terms of modules why they have a rôle at all! For example, it would be nice to have a direct proof of the Hook formula for dimensions (section 20), without doing all the work required for the standard basis of the Specht module.

The (i,j)-hook may be regarded as the intersection of an infinite Γ shape (having the (i,j)-node at its corner) with the diagram.

18.1 EXAMPLE

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X X X X          The (2,2)-hook is   X X X X
X X X X                              X ⊠ ⊠ ⊠
X X X                                X ⊠ X
```

and the hook graph is

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6 5 4 2
5 4 3 1
3 2 1
```

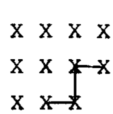
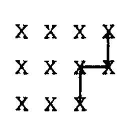
18.2 DEFINITIONS

(i) The <u>(i,j)-hook</u> of [μ] consists of the (i,j)-node along with the $\mu_i - j$ nodes to the right of it (called the <u>arm</u> of the hook) and the $\mu'_j - i$ nodes below it (called the <u>leg</u> of the hook).

(ii) The <u>length</u> of the (i,j)-hook is $h_{ij} = \mu_i + \mu'_j + 1 - i - j$

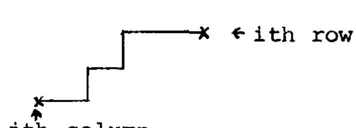
(iii) If we replace the (i,j)-node of [μ] by the number h_{ij} for each node, we obtain the <u>hook graph</u>.

(iv) A <u>skew-hook</u> is a connected part of the rim of [μ] which can be removed to leave a proper diagram.

18.3 EXAMPLE  and  show the only two skew 4-hooks in $[4^2,3]$. The diagram also has one skew 6-hook, two skew 5-hooks, two skew 3-hooks, two skew 2-hooks, and two skew 1-hooks. Comparing this with the hook graph, we have illustrated:

18.4 LEMMA <u>There is a natural 1-1 correspondence between the hooks of [μ] and the skew-hooks of [μ].</u>

<u>Proof</u>: The skew hook



corresponds to the (i,j)-hook.