Security of an Identity-Based Cryptosystem and the Related Reductions^{*}

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Abstract. Recently an efficient solution to the discrete logarithm problem on elliptic curves over $\mathbf{F}_p$ with p points (p : prime), so-called *anomalous curves*, was independently discovered by Semaev [14], Smart [17], and Satoh and Araki [12]. Since the solution is very efficient, i.e., $O(|p|^3)$, the Semaev-Smart-Satoh-Araki (SSSA) algorithm implies the possibility of realizing a trapdoor for the discrete logarithm problem, and we have tried to utilize the SSSA algorithm for constructing a cryptographic scheme. One of our trials was to realize an identity-based cryptosystem (key-distribution) which has been proven to be as secure as a primitive problem, called the Diffie-Hellman problem on an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$ ($n = pq$, p and q are primes) where E_p and E_q are anomalous curves (anomalous E_n -Diffie-Hellman problem). Unfortunately we have found that the anomalous E_n -Diffie-Hellman problem is not secure (namely, our scheme is not secure). First, this paper introduces our trial of realizing an identity-based cryptosystem based on the SSSA algorithm, and then shows why the anomalous E_n -Diffie-Hellman problem is not secure. In addition, we generalize the observation of our breaking algorithm and present reductions of factoring n to computing the order² of an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$. (These reductions roughly imply the equivalence of intractability between factoring and computing elliptic curve's order.) The algorithm of breaking our identity-based cryptosystem is considered to be a special case of these reductions, and the essential reason why our system was broken can be clarified through these reductions: E_n in our system is a very specific curve such that the order of E_n (i.e., n) is trivially known.

1 Introduction

1.1 Background

One of the most interesting open problems in cryptography is the realization of a trapdoor on the discrete logarithm, in which to solve the discrete logarithm is hard only if published parameters are used, while it is easy by using a secret key (trapdoor key).

One of the possible candidates to solve the problem is: a pair of primes, (p, q) , is a secret key, and $n = pq$ is the corresponding public-key. (1) the discrete

^{*} This is an updated version. The original version accepted to the conference had a security weakness.

² We define the order of an elliptic curve, E_n , over $\mathbf{Z}/n\mathbf{Z}$ by $\text{lcm}(\#E_p, \#E_q)$, where $n = pq$ (p, q : primes), E_p and E_q are elliptic curves over $\mathbf{F}_p$ and $\mathbf{F}_q$, and E_n is the Chinese remainder composite of E_p and E_q

logarithm $\text{mod } n$ is intractable against an adversary who knows no factors of n and (2) a person who knows a trapdoor, (p, q) , can compute the discrete logarithm $\text{mod } n$ efficiently.

However, this trapdoor trick does not seem to work well. This is because: in order to make the trapdoor for computing the discrete logarithm $\text{mod } n$ efficiently (i.e., to make the discrete logarithm $\text{mod } p$ be easy), $p - 1$ should consist of all (fairly) small primes. Such n is (fairly) easily factored by the $p - 1$ method. Of course, we can make some delicate computational complexity gap between computing the discrete logarithm $\text{mod } p$ and factoring n [7, 8], but it seems hard to make a computationally feasible gap such that one is a polynomial time and the other an (assumed) super-polynomial time.

1.2 New hope: anomalous elliptic curves

The discrete logarithm problem can be defined on various finite groups as well as the multiplicative group over a finite field $\mathbf{F}_q$. A typical example except the multiplicative group is the discrete logarithm problem on elliptic curves over $\mathbf{F}_q$, and many cryptographic schemes are constructed on the elliptic curve discrete logarithm problems.

Recently an efficient solution of the discrete logarithm problem on elliptic curves over $\mathbf{F}_p$ with p points (p : prime), so-called *anomalous curves*, was independently discovered by Semaev [14], Smart [17], and Satoh and Araki [12]. The solution is very efficient, i.e., $O(|p|^3)$, and is almost as efficient as typical practical public-key encryption and decryption algorithms such as those of the RSA scheme. The Semaev-Smart-Satoh-Araki (SSSA) algorithm, therefore, implies the possibility of realizing a trapdoor for the discrete logarithm problem on elliptic curves over $\mathbf{Z}/n\mathbf{Z}$, (say E_n) where E_n is the Chinese remainder composition of anomalous elliptic curves, E_p and E_q . Here, the discrete logarithm over E_n seems hard without knowing (p, q) , while if the trapdoor information (p, q) is known, it can be easily calculated by using the SSSA algorithm for E_p and E_q .

1.3 Identity-based cryptosystems

We have tried to utilize this trapdoor trick based on the Semaev-Smart-Satoh-Araki (SSSA) algorithm for realizing a cryptographic scheme. One of our trials was to realize an identity-based cryptosystem (key-distribution).

In 1984, Shamir proposed a new concept, identity-based cryptosystem, to solve the authentication problem of standard public-key cryptosystems [15]. Here the authentication problem is: a public-key file managed by a trusted authority or authority's certificates of public-keys should be employed in order to check the validity of public-keys (i.e., to check the correct linkage between a public-key and a user).

In the identity-based cryptosystem, there exists an authority, who generates users' secret keys by using users's identities and the authority's secret key, and distributes them to users through secure channels. When user A with A 's secret key, s_A , wants to send a ciphertext to B , user A encrypts the message, m , into a ciphertext, c , using B 's identity, ID_B , and s_A , in place of B 's public-key. Since B 's identity can be assumed to be publicly known (i.e., to be known by A), such an identity-based cryptosystem does not require A to interact with any party to get information regarding B (e.g., B 's public-key) when A sends

a ciphertext to B (i.e., A needs no interaction with any party, after A gets s_A from the authority), while B 's public-key should be transmitted to A through a public-key file or directly from B , in the standard public-key cryptosystems. Note that the security of identity-based cryptosystems depends much more on the security of the authority than the standard public-key cryptosystems, since the authority knows all users' secret keys.

Although Shamir gave a concrete realization of identity-based digital signature schemes, he left it open as to how to realize identity-based cryptosystems (or identity-based key-distributions). Several schemes have been proposed regarding this problem, but no satisfactory solution to Shamir's open problem has been proposed so far.

1.4 Trial

We obtained a *solution* to this open problem, by using the trapdoor technique based on anomalous elliptic curves. The identity-based cryptosystem (key-distribution) has the following properties:

- This scheme has no interaction and no threshold of collusion (i.e., truly follows Shamir's concept).
- The required computational power for the authority is feasible, i.e., $O(|n|^3)$ per one user key generation.
- The security assumption is clear and feasible: the security of the system is reduced to the intractability of a reasonable primitive problem, the anomalous E_n -Diffie-Hellman problem, under the random oracle model .

A rough idea of our system is as follows: n and (parameters of) E_n are the public system parameters and (p, q) is the secret of the authority, who can compute the discrete logarithm, s_A , using the SSSA algorithm, where ID_A is the identity of user A , G and $F(ID_A)$ are points on E_n (i.e., F is a function to embed ID_A to a point on E_n) and $s_A G = F(ID_A)$. The authority sends s_A to user A secretly, and A can compute a shared key, K_{AB} , with user B by using s_A and $F(ID_B)$ such that $K_{AB} = s_A F(ID_B) = s_A (s_B G) = s_B (s_A G) = s_B F(ID_A)$. Therefore, B can also compute the same key, K_{AB} , from s_B and $F(ID_A)$. Here note that the authority with secret-key (p, q) can compute s_A efficiently (with running time of $O(|p|^3)$) using the SSSA algorithm, while it is considered to be hard to compute it without knowing (p, q) .

1.5 Failure

Unfortunately we have found that the anomalous E_n -Diffie-Hellman problem is not secure (namely, our scheme is not secure). We have also found an another example of insecure cryptosystems based on anomalous curves. We will observe why the anomalous E_n -Diffie-Hellman problem is not secure and why it is hard to utilize anomalous curves for realizing a secure cryptographic scheme.

1.6 Reductions

Through the observation on our algorithm of breaking the anomalous E_n -Diffie-Hellman problem and on the reason why it is hard to utilize anomalous curves

for constructing secure cryptographic schemes, we have realized a relationship between the breaking algorithm and some reductions.

For this purpose, we will show reductions of factoring n to computing the order of an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$. (See the footnote in Abstract for the meaning of “order”.) These reductions roughly imply the equivalence of intractability between factoring and computing elliptic curve’s order. Here, the algorithm of breaking our ID-based cryptosystem is considered to be a special case of these reductions, since the order of E_n in our identity-based cryptosystem (i.e., in the anomalous E_n -Diffie-Hellman problem) is trivial, i.e., n . The reductions imply that factoring n is trivially easy if and only if the order of E_n is trivially obtained.

We will show two types of reductions: one (Reduction 1) is the reduction of the *elliptic curve factoring problem* (i.e., given (E_n, n) , factoring n) to computing the order of an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$; the other (Reduction 2) is the reduction of the *factoring problem* (given n , factoring n) to computing the order of an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$.

In Reduction 1, we assume that E_n is selected under the condition that $\#E_p$ and $\#E_q$ have large prime factors. (Note that n can be trivially factored from E_n if either $\#E_p$ or $\#E_q$ has only small prime factors.) The cryptographic implication of this reduction is: When E_n is selected by a key generator of a cryptographic scheme (i.e., the key generator checks whether $\#E_p$ and $\#E_q$ have large prime factors), computing the order of E_n , given (E_n, n) , is as hard as the elliptic curve factoring problem. On the other hand, in Reduction 2, we assume that E_n is randomly selected from a non-negligible fraction of elliptic curves over $\mathbf{Z}/n\mathbf{Z}$. The cryptographic implication of this reduction is: When E_n is randomly selected from a non-negligible fraction, computing the order of E_n , given (E_n, n) , is as hard as the factoring problem.

If E_n is selected under special cryptographic conditions (or selected from a negligible fraction of elliptic curves over $\mathbf{Z}/n\mathbf{Z}$), then only Reduction 1 works, but gives no reduction from the factoring problem. Since E_n for our ID-based cryptosystem (E_p and E_q are anomalous) is specific (i.e., is selected from a negligible fraction), the breaking algorithm is a special case of Reduction 1, but not of Reduction 2.

Remark (Related Works)

Miller and Long showed the reduction of factoring n to computing the Euler function $\phi(n)$ [9, 6]. Kunihiro and Koyama have recently presented the reduction (say the “KK reduction”) of the factoring problem to computing the order of an elliptic curve, E_n , over $\mathbf{Z}/n\mathbf{Z}$ [5]. Hence our Reduction 2 corresponds to the KK reduction. In addition Reduction 1 corresponds to the reduction as a variant of the KK reduction described in Remarks (3), in [5]. Both the KK reduction and our Reduction 2 work for non-negligible fractions of $\#E_n$. The difference between these reductions are the failure cases of reduction. The KK reduction does not work when the order of $\#E_n$ has only large prime factors, while our Reduction 2 works for any non-negligible fractions. On the other hand, our Reduction 2 does not consider the case where n has more than two prime factors, while the KK reduction carefully considers such a case. Therefore, for some cases the KK reduction works while our Reduction 2 does not work, and for the other cases our Reduction 2 works while the KK reduction does not work. This difference depends on the underlying techniques in the reductions: x -coordinate addition formula and twist curve technique play essential roles in our Reductions 1 and 2.

2 Anomalous Elliptic Curves and the Discrete Logarithm Algorithm

This section introduces the underlying key techniques and notations employed in our scheme: anomalous elliptic curves, the SSSA algorithm, elliptic curves over $\mathbf{Z}/n\mathbf{Z}$, and the addition formula for x -coordinates.

2.1 Anomalous Elliptic Curves

Let K be a finite field with characteristics $\neq 2, 3$.

An elliptic curve over K (in affine coordinates), denoted by $E(K)$, is the set of all solutions $(x, y) \in K \times K$ to the equation

$$E: y^2 = x^3 + ax + b \quad (a, b \in K, 4a^3 + 27b^2 \neq 0),$$

together with a special point $\mathcal{O}_p$, called the point at infinity.

The set forms a finite Abelian group, and the group law formula [16] (usually we call it the addition, and use the notation $+$) is defined over the points on K .

Let p be a prime ($p > 5$), and $\mathbf{F}_p$ be a finite field with p elements. Now, let $\#E(\mathbf{F}_p)$ be the number of points on $E(\mathbf{F}_p)$, and $\#E(\mathbf{F}_p) = p + 1 - t$, where t is called trace and $-2\sqrt{p} \leq t \leq 2\sqrt{p}$. An elliptic curve, $E(\mathbf{F}_p)$, is called *anomalous* if $\#E(\mathbf{F}_p) = p$ (i.e., trace t is equal to 1.)

When G is a point on $E(\mathbf{F}_p)$, mG means m times (elliptic curve) addition of G (i.e., $mG = G + \dots + G$ (m times)). Let $G = (x, y) \in E(\mathbf{F}_p)$, then mG is denoted by $(m * x, m * y) \in E(\mathbf{F}_p)$. (i.e., $m * x$ means the x -coordinate of mG .)

An efficient way to find the parameters of an anomalous elliptic curve over a prime field has been shown by [10].

2.2 Discrete Logarithm Algorithm of Anomalous Elliptic Curves

Very recently, Semaev [14], Smart [17] and Satoh and Araki [12] independently discovered an efficient ($O(|p|^3)$) algorithm for computing the discrete logarithm for *anomalous* elliptic curve $E(\mathbf{F}_p)$.

Definition 1. Let $E(\mathbf{F}_p)$ be an anomalous elliptic curve, and G, P be points on $E(\mathbf{F}_p)$. Given, G, P , and $E(\mathbf{F}_p)$, the discrete logarithm problem over $E(\mathbf{F}_p)$ is to find integer m ($0 \leq m < p$) such that $P = mG$.

Although Semaev's algorithm has a different flavor from the algorithm by Smart, Satoh, and Araki, in this paper we introduce Smart-Satoh-Araki's approach and call it the SSSA algorithm as follows (we mainly follow the notations of [12]):

[SSSA Algorithm]

Input: G, P , and $E(\mathbf{F}_p)$

Output: m

1. Find a curve, E_{p^2} , over $\mathbf{Z}$ lifted from $E(\mathbf{F}_p)$, through which non-zero map $\lambda_{E_{p^2}}: E(\mathbf{F}_p) \mapsto \mathbf{F}_p$ is constructed. (Such a curve can be found efficiently ($O(|p|^2)$) by Theorem 3.7 in [12].)

2. Let $\lambda_{E_{p^2}}$ be non-zero. Compute $\lambda_{E_{p^2}}(G)$ and $\lambda_{E_{p^2}}(P)$. (The computational complexity of computing $\lambda_{E_{p^2}}(G)$ is $O(|p|^3)$.) Compute $s = \lambda_{E_{p^2}}(P)/\lambda_{E_{p^2}}(G) \bmod p$.

[Computation of non-zero map $\lambda_{E_{p^2}}$]

Input: $\alpha = (s, t) \in E(\mathbf{F}_p) \setminus \{\mathcal{O}_p\}$, $E(\mathbf{F}_p)$, E_{p^2}

Output: $\lambda_{E_{p^2}}(\alpha)$

1. Find $A = (x, y) \in E_{p^2}(\mathbf{Z}/p^2\mathbf{Z})$ satisfying $x \bmod p = s, y \bmod p = t$.
2. Compute $((p-1) * x, (p-1) * y) = (p-1)A_p \in E_{p^2}(\mathbf{Z}/p^2\mathbf{Z})$.
3. Compute $\lambda_{E_{p^2}}(\alpha)$ as follows:

$$\lambda_{E_{p^2}}(\alpha) = \frac{1}{p} \frac{(p-1) * x - x}{(p-1) * y - y} \bmod p,$$

2.3 Addition Formula with x -coordinates

Here we show an addition formula only with x -coordinates, based on the division polynomials [2].

For $E(\mathbf{F}_p)$ $((a_p, b_p)$ be the parameters of $E(\mathbf{F}_p)$), $P = (x_p, y_p)$, and $(i * x_p, i * y_p) = iP$, if $i * y_p \not\equiv 0 \pmod{p}$, then

$$2i * x_p = \frac{((i * x_p)^2 - a_p)^2 - 8b_p(i * x_p)}{4((i * x_p)^3 + a_p(i * x_p) + b_p)} \bmod p. \quad (1)$$

In addition, if $i * x_p \not\equiv (i+1) * x_p \pmod{p}$ and $x_p \not\equiv 0 \pmod{p}$, then

$$(2i+1) * x_p = \frac{(a_p - (i * x_p)((i+1) * x_p))^2 - 4b_p(i * x_p + (i+1) * x_p)}{x_p(i * x_p - (i+1) * x_p)^2} \pmod{p}. \quad (2)$$

2.4 Elliptic Curves over the Ring

An addition formula with x -coordinates on $E(\mathbf{Z}/n\mathbf{Z})$ can be also defined in the same way as the addition formula on $E(\mathbf{F}_p)$. Hence, given $x \in \mathbf{Z}/n\mathbf{Z}$ with $P = (x, y) \in E(\mathbf{Z}/n\mathbf{Z})$, for any integer $m \in \mathbf{Z}/n\mathbf{Z}$, $m * x$ can be calculated in the running time of $O(|n|^3)$ through the addition chain of the above-mentioned addition formula with x -coordinates.

Using the Chinese remainder theorem, every element c of $\mathbf{Z}/n\mathbf{Z}$ can be represented uniquely as a pair $[c_p, c_q]$, where $c_p = c \bmod p$ and $c_q = c \bmod q$. Thus every point $P = (x, y)$ on $E(\mathbf{Z}/n\mathbf{Z})$ can be represented uniquely as a pair $[P_p, P_q] = [(x_p, y_p), (x_q, y_q)]$, where $P_p \in E(\mathbf{F}_p)$ and $P_q \in E(\mathbf{F}_q)$, $x = [x_p, x_q]$, and $y = [y_p, y_q]$. Here we denote $E(\mathbf{Z}/n\mathbf{Z}) = [E(\mathbf{F}_p), E(\mathbf{F}_q)]$ (or $E_n = [E_p, E_q]$), where parameters, a and b , of $E(\mathbf{Z}/n\mathbf{Z})$ satisfy the following: $a = [a_p, a_q]$, $b = [b_p, b_q]$, (a_p, b_p) be the parameters of $E(\mathbf{F}_p)$, and (a_q, b_q) be the parameters of $E(\mathbf{F}_q)$.

3 The Proposed Non-interactive Identity-Based Key-Distribution Scheme

3.1 The Proposed Scheme

The proposed system consists of the trusted authority, T , and users, A , B etc.. Authority T generates T 's secret-key, s_T , and publishes system public parameters. T generates user A 's secret key, s_A , from A 's identity, ID_A , by using s_T , and sends s_A to A through a secure channel. User A can non-interactively generate a shared secret key with user B , K_{AB} , by using A 's secret key s_A and B 's identity, ID_B . Then, when user A sends a ciphertext to B , user A encrypts message m to ciphertext $c = Enc(m, K_{AB})$, where Enc is an arbitrary secret-key encryption function with secret-key K_{AB} .

Set-up procedure Authority T generates primes p and q , and (parameters, (a_p, b_p) and (a_q, b_q) , of) anomalous elliptic curves, E_p and E_q , where $\#E_p = p$ and $\#E_q = q$. Without loss of generality, we assume $p > q$.

T , then, generates and publishes $n = pq$, and (parameters, (a, b) , of) $E_n = [E_p, E_q]$ (i.e., $a \equiv a_p \pmod{p}$, $a \equiv a_q \pmod{q}$, $b \equiv b_p \pmod{p}$, and $b \equiv b_q \pmod{q}$).

T also determines points, G_p and G_q , on E_p and E_q , respectively, such that $G_p \neq \mathcal{O}_p$, $G_q \neq \mathcal{O}_q$. T , then, finds non-zero maps, $\lambda_{E_p^2} : E_p(\mathbb{F}_p) \mapsto \mathbb{F}_p$ and $\lambda_{E_q^2} : E_q(\mathbb{F}_q) \mapsto \mathbb{F}_q$. T publishes x -coordinate, X_{G_n} , of $G_n = [G_p, G_q]$.

Therefore, the public and secret parameters are as follows:

- [Public system parameters] n, E_n, X_{G_n}
- [Secret key of authority T] p, q

Secret key generation for each user Let h be a random function, $h : \{0, 1\}^* \mapsto \{0, 1\}^{k-1}$. (See [1] for the definition of a random function. Practically, we can replace it by a practical one-way function such as SHA and MD5.) Authority T finds the smallest integer r_A ($r_A \geq 0$) such that $h(ID_A || r_A)$ is the x -coordinate of a point on E_n . Let F be a point on E_n whose x -coordinate is $h(ID_A || r_A)$ (there exist four possible solutions of F and one of them is randomly assigned as F), and $F_p = F \bmod p$ and $F_q = F \bmod q$.

T then calculates $s_A = [s_{A,p}, s_{A,q}]$ such that

$$s_{A,p}G_p = F_p, \quad s_{A,q}G_q = F_q,$$

using the SSSA algorithm.

T sends (s_A, r_A) to A . A checks whether $h(ID_A || r_A) = s_A * X_{G_n}$ holds (using the addition formula of subsection 2.3) and r_A is the smallest non-zero integer such that $n * h(ID_A || r_A) = \infty$, and accepts (s_A, r_A) as A 's secret key, if it holds.

(Non-interactive) Key distribution between users Suppose that users A and B want to share a secret key, K_{AB} .

User A finds the smallest integer r_B ($r_B \geq 0$) such that $h(ID_B || r_B)$ is the x -coordinate of a point on E_n , by checking whether

$$((h(ID_B || r_B))^3 + a(h(ID_B || r_B)) + b)/n = 1, \text{ and } n * h(ID_B || r_B) = \infty.$$

A then calculates K_{AB} as follows:

$$K_{AB} = s_A * h(ID_B || r_B).$$

3.2 Security Condition

Definition 2. Let $E_p(\mathbf{F}_p)$ and $E_q(\mathbf{F}_q)$ be anomalous elliptic curves, $E_n(\mathbf{Z}/n\mathbf{Z}) = [E_p(\mathbf{F}_p), E_q(\mathbf{F}_q)]$, and G be a point on $E_n(\mathbf{Z}/n\mathbf{Z})$ such that $n = pq$ (p, q : primes), $G = [G_p \in E_p(\mathbf{F}_p), G_q \in E_q(\mathbf{F}_q)]$, and $G_p \neq \mathcal{O}_p$, $G_q \neq \mathcal{O}_q$. Given, n , $E(\mathbf{Z}/n\mathbf{Z})$, the x -coordinates of G , sG , and tG (s, t : integers), (say X_G , X_{sG} , and X_{tG}), the anomalous E_n -Diffie-Hellman problem is to find the x -coordinate of point $(st)G$, $X_{(st)G}$.

Definition 3. Let $\mathcal{G}(1^k) \rightarrow (n, E(\mathbf{Z}/n\mathbf{Z}), X_G, X_{sG}, X_{tG})$, where $k = |n|$, be the parameter generator such that the distribution of $(n, E(\mathbf{Z}/n\mathbf{Z}), X_G, X_{sG}, X_{tG})$ is the same as that of $(n, E_n, X_{G_n}, h(ID_A||r_A), h(ID_A||r_A))$ of the proposed scheme. (Note that $0 \leq X_{sG} < 2^{k-1}$, $0 \leq X_{tG} < 2^{k-1}$) The anomalous E_n -Diffie-Hellman problem is *intractable* if, for any probabilistic polynomial time machine, M_0 , for any constant c , for all sufficiently large k ,

$$\Pr[M_0(n, E(\mathbf{Z}/n\mathbf{Z}), X_G, X_{sG}, X_{tG}) \rightarrow X_{(st)G}] < 1/k^c.$$

The probability is taken over the coin flips of $\mathcal{G}$ and M_0 .

Definition 4. Let $\mathcal{G}'$ be a generator regarding the proposed scheme such that $\mathcal{G}'(1^k) \rightarrow (n, E(\mathbf{Z}/n\mathbf{Z}), X_{G_n})$, $(n, E(\mathbf{Z}/n\mathbf{Z}), X_{G_n})$ is public system parameters (with the same distribution of the proposed scheme), and $k = |n|$. Let ID_A and ID_B be given.

Let Adv be an adversary of the proposed scheme. Adv provides arbitrary $ID_1, \dots, ID_l$ (they are not ID_A nor ID_B) to the authority, and obtains $(s_1, r_1), \dots, (s_l, r_l)$, such that $h(ID_i||r_i) = s_i * X_{G_n}$. Adv then tries to find K_{AB} such that $K_{AB} = s_A * h(ID_B||r_B)$, and $h(ID_A||r_A) = s_A * X_{G_n}$.

The proposed scheme is *secure* if, for any probabilistic polynomial time adversary, Adv , for any constant c , for all sufficiently large k ,

$$\Pr[Adv(n, E(\mathbf{Z}/n\mathbf{Z}), X_{G_n}, ID_A, ID_B, (s_1, r_1), \dots, (s_l, r_l)) \rightarrow K_{AB}] < 1/k^c.$$

The probability is taken over the coin flips of $\mathcal{G}'$, random oracle h , and Adv .

Theorem 5. Assuming that $h : \{0, 1\}^* \mapsto \{0, 1\}^{k-1}$ is a random function, where $k = |n|$, the proposed scheme is secure if and only if the anomalous E_n -Diffie-Hellman problem is intractable.

See Appendix A for the proof.

4 Anomalous E_n -Diffie-Hellman Problem is Tractable

Now we show that the anomalous E_n -Diffie-Hellman problem is *tractable*. More precisely, given (n, E_n) , there is an efficient algorithm to factor n .

Let E_p and E_q be anomalous elliptic curves such that $E_n = [E_p, E_q]$, and $\overline{E_p}$ and $\overline{E_q}$ be the twists of E_p and E_q . That is, $\#E_p = p$, $\#E_q = q$, $\#\overline{E_p} = p + 2$, and $\#\overline{E_q} = q + 2$. (see [2] for the twist of an elliptic curve).

When X is a random integer in $\mathbf{Z}/n\mathbf{Z}$, there are four possibilities as follows:

1. X is the x -coordinate of a point on $[E_p, E_q]$. Then $n * X$ is $[\mathcal{O}_p, \mathcal{O}_q]$.

2. X is the x -coordinate of a point on $[\overline{E_p}, E_q]$. Then $n * X$ is [a finite value, $\mathcal{O}_q$].
3. X is the x -coordinate of a point on $[E_p, \overline{E_q}]$. Then $n * X$ is $[\mathcal{O}_p$, a finite value] .
4. X is the x -coordinate of a point on $[\overline{E_p}, \overline{E_q}]$. Then $n * X$ is a finite value (i.e., [a finite value , a finite value]).

In cases 2 and 3, $n * X$ can be used for factor n , while $n * X$ cannot be used for factor n in cases 1 and 4. That is, in cases 2 and 3, $\gcd(d, n)$ should be p or q , where d be the denominator of $n * X$.

Hence, given $(n, E(\mathbf{Z}/n\mathbf{Z}))$, if X is randomly selected from $\mathbf{Z}/n\mathbf{Z}$, n can be factored with probability $1/2$. By repeating this procedure many ($\text{poly}(|n|)$) times, the success probability of factoring n is overwhelming.

5 Another Trial and Failure

We can easily construct an elliptic curve version of a public-key cryptosystem [11]. Here, [11] is constructed on the multiplicative group over ring $\mathbf{Z}/n\mathbf{Z}$ ($n = p^2q$; p, q : primes), while the elliptic curve version is constructed on elliptic curves, E_n , over ring $\mathbf{Z}/n\mathbf{Z}$ ($n = pq$; p, q : primes), $E_n = [E_p, E_q]$, E_p is anomalous elliptic curve over $\mathbf{F}_p$ and E_q is not an anomalous elliptic curve over $\mathbf{F}_q$. The p -Sylow group in [11] corresponds to the anomalous curve, E_p .

Obviously n can be easily factored from the parameter of E_n in a manner similar to Section 4 (see Appendix B for the related reductions).

6 Reductions

Through the observation on our algorithm of breaking the anomalous E_n -Diffie-Hellman problem and on the reason why it is hard to utilize anomalous curves for constructing secure cryptographic schemes, we have realized a relationship between the breaking algorithm and some reductions.

For this purpose, this section will show two reductions (Reductions 1 and 2) of factoring n to computing the order of an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$.

We will also show other reductions which are related to the attack introduced in Section 5 in Appendix B.

The algorithms of our reductions are based on the algorithm of breaking our identity cryptosystem (i.e., the anomalous E_n -Diffie-Hellman problem). Hence this section follows the notations and techniques of Section 4 and the previous sections (e.g., $\overline{E_p}$ for the twist curve of elliptic curve E_p , and $*$ for the elliptic curve addition operation with only x -coordinates).

Definition 6. Let G_1 be an instance generator such that $G_1(1^k) \rightarrow n$, $n = pq$, $|n| = k$, (p, q : primes, $|p| = |q|$), where p and q are independently selected. The “factoring problem” is, given (n, k) by G_1 , to find (p, q) .

The factoring problem is *intractable*, if for any probabilistic polynomial time machine Adv , for any constant c , for sufficiently large k ,

$$\Pr[Adv(1^k, n) = (p, q)] < 1/k^c.$$

The probability is taken over the coin flips of G_1 and Adv .

Definition 7. Let G_2 be an instance generator such that $G_2(1^k) \rightarrow (E_n, n)$, $n = pq$, $|n| = k$ (p, q : primes, $|p| = |q|$), $E_n = [E_p, E_q]$ (E_p and E_q are elliptic curves over $\mathbf{F}_p$ and $\mathbf{F}_q$, respectively). Here p and q are independently selected.

The “*elliptic curve factoring problem*” is, given (E_n, n, k) by G_2 , to find (p, q) .

The elliptic curve factoring problem is *intractable*, if for any probabilistic polynomial time machine Adv , for any constant c , for sufficiently large k ,

$$\Pr[Adv(1^k, E_n, n) = (p, q)] < 1/k^c.$$

The probability is taken over the coin flips of G_2 and Adv .

Definition 8. Let G_3 be an instance generator such that $G_3(1^k) \rightarrow (E_n, n)$, $n = pq$, $|n| = k$ (p, q : primes, $|p| = |q|$), $E_n = [E_p, E_q]$. Here, p and q are independently selected.

The “*elliptic curve order problem*” is, given (E_n, n, k) , to find $\Phi(E_n)$, where $\Phi(E_n) = \text{lcm}(\#E_p, \#E_q)$.

The elliptic curve order problem is *intractable*, if for any probabilistic polynomial time machine Adv , for any constant c , for sufficiently large k ,

$$\Pr[Adv(1^k, E_n, n) = \Phi(E_n)] < 1/k^c.$$

The probability is taken over the coin flips of G_3 and Adv .

Theorem 9. [Reduction 1] Let G_3 for the elliptic curve order problem be equivalent to G_2 for the elliptic curve factoring problem. Assume G_2 and G_3 output E_n such that $\#E_p$ and $\#E_q$ have large prime factors, p' and q' , respectively, i.e., $\#E_p = p'u$, $\#E_q = q'v$, $1 \leq u < q^{\frac{1}{4}}$, $0 \leq v < q^{\frac{1}{4}}$.

Then, the elliptic curve order problem is intractable if and only if the elliptic curve factoring problem is intractable.

Proof. (Only if:)

Assume that the elliptic curve factoring problem is not intractable. That is, an adversary, Adv , given (E_n, n, k) , can factor n with non-negligible probability. We will then construct a polynomial time machine M with help of Adv (with knowing (p, q)), which, given (E_n, n, k) , computes $\Phi(E_n)$ with non-negligible probability. M obtains elliptic curves E_p and E_q such that $E_n = [E_p, E_q]$. M computes $\#E_p$, $\#E_q$ with using Schoof’s algorithm [13], whose running time is of the order of polynomial. So, M^{Adv} can compute $\Phi(E_n) = \text{lcm}(\#E_p, \#E_q)$ with non-negligible probability.

(If:)

Let assume that the elliptic curve order problem is not intractable. That is, an adversary, Adv , can compute $\Phi(E_n)$ from (E_n, n, k) with non-negligible probability. We will then construct a probabilistic polynomial time algorithm M , with help of Adv as an oracle, which, given (E_n, n, k) , can factor n with non-negligible probability.

When M randomly selects $z \in \mathbf{Z}/n\mathbf{Z}$, $\left(\frac{f(z)}{n}\right) = -1$ with probability $1/2$, where f denotes a cubic polynomial which defines the elliptic curve $E_n = [E_p, E_q]$ and $\left(\frac{f(z)}{n}\right)$ the Jacobi symbol.

Then we have two possibilities: z is the x -coordinate of a point on $[E_p, \overline{E}_q]$ and is the x -coordinate of a point on $[\overline{E}_p, E_q]$, where $\overline{E}_q$ (and $\overline{E}_p$) is the twist of

E_q (and E_p) [2], and $\# \overline{E}_q = q+1+s$ (and $\# \overline{E}_p = p+1+t$) when $\# E_q = q+1-s$ (and $\# E_p = p+1-t$). Without loss of generality, we assume that the first case occurs.

Since the distribution of (E_n, n, k) is given by $G_3 = G_2$, Adv can output a correct $\Phi(E_n)$ with non-negligible probability.

Let $a = \gcd(q+1-s, q+1+s)$, $ab = q+1+s$, then, it is easy to see that $q' \nmid q+1+s$, $a \leq 4\sqrt{q}$. On the other hand, since $p' \nmid q+1+s$ with overwhelming probability over the distribution of p' , a is a divisor of v .

Hence, we have $a < q^{\frac{1}{4}}$ and $\gcd(p+1-t, b) = \gcd(u, b) < q^{\frac{1}{4}}$.

In order to evaluate the probability of the case when $\Phi(E_n) * z$ is the x -coordinate of $[\mathcal{O}_p, \mathcal{O}_q]$, it is sufficient to consider the point of $a \gcd(u, b) * z$.

Let the maximum order of a point on $\overline{E}_q(\mathbf{F}_q) \setminus H_a$ be o_{max} . Then, o_{max} is at least $\sqrt{(q+1+s) - \#H_a}$ due to the group structure of elliptic curve, where H_a denotes the set of a -torsion points of $\overline{E}_q(\mathbf{F}_q)$. Since $\#H_a \leq a^2$, we obtain $o_{max} \geq \sqrt{(q+1+s) - \sqrt{q}}$.

Obviously, $\sqrt{(q+1+s) - \sqrt{q}} \geq \sqrt{q+1-3\sqrt{q}} > q^{\frac{1}{4}}$, so we have $o_{max} > q^{\frac{1}{4}}$.

Now, if we choose z randomly such that $z_q = z \bmod q$ is not the x -coordinate of point H_a and has order o_{max} , then $a \gcd(b, u) * z$ is the x -coordinate of $[\mathcal{O}_p, \text{a finite value}]$. We will then evaluate the probability that a randomly selected point, z_q , is not the x -coordinate of a point H_a and has order o_{max} .

It is easy to see that there are $\varphi(o_{max})$ many points of order o_{max} . Hence the probability that a randomly selected point, z_q , is not the x -coordinate of a point H_a and has order o_{max} is at least $\frac{1}{6 \log \log(q^{1/4})} = O(\frac{1}{\log k})$.

Hence, M selects polynomially many z randomly, for at least one of them, $\Phi(E_n) * z$ is the x -coordinate of a point, $[\mathcal{O}_p, \text{a finite value}]$, with overwhelming probability.

Thus, M can factor n with help of Adv with non-negligible probability.

Theorem 10. [Reduction 2] *Let the distribution of n given by G_1 for the factoring problem be equivalent to that given by G_3 for the elliptic curve order problem. Let E_n be uniformly selected by G_3 (i.e., the parameters of E_n are uniformly and independently selected over $\mathbf{Z}/n\mathbf{Z}$). Assume that the distribution of $\#E_p$ (and $\#E_q$) is statistically close to the uniform distribution over $\{0, 1\}^{k/2}$, where the distribution is taken over the coin flips of G_3 (i.e., random selection of p, q, E_p and E_q).*

Then the elliptic curve order problem is intractable if and only if the factoring problem is intractable.

Proof. Since the proof of the *only if part* is almost trivial, we only give the proof of the *if part*.

Let assume that the elliptic curve order problem is not intractable, so, there exists an adversary, Adv , which can compute $\Phi(E_n)$ from (E_n, n, k) with non-negligible probability.

Then we will construct a probabilistic polynomial time algorithm M , with help of Adv as an oracle, which, given n by G_1 , can factor n with non-negligible probability.

M uniformly selects an elliptic curve mod n , E_n , and randomly chooses $z \in \mathbf{Z}/n\mathbf{Z}$, then $\left(\frac{f(z)}{n}\right) = -1$ with probability $1/2$ (see the proof of Theorem 9 for the notations of f and $\left(\frac{f(z)}{n}\right)$).

Then we have two possibilities: z is the x -coordinate of a point on $[E_p, \overline{E}_q]$ and is the x -coordinate of a point on $[\overline{E}_p, E_q]$. Without loss of generality, we can assume that the first case occurs.

Since M chooses E_n uniformly as G_3 does, Adv outputs correct $\Phi(E_n)$ with non-negligible probability.

Let $\#E_p = p + 1 - t$ and $\#E_q = q + 1 - s$, therefore $\#\overline{E}_p = p + 1 + t$ and $\#\overline{E}_q = q + 1 + s$. We now evaluate the probability of the case when $\Phi(E_n) * z$ is the x -coordinate of $[\mathcal{O}_p, \mathcal{O}_q]$ (only in this case M fails in factoring n).

Let $a = \gcd(q + 1 - s, q + 1 + s)$ and $q + 1 + s = ab$, then it is obvious that $1 \leq a \leq 4\sqrt{q}$. Hence $b \geq \frac{q+1+s}{4\sqrt{q}} = O(2^{k/2})$.

First we investigate the maximum size of a square factor of $\#\overline{E}_q$. Let $b = cd_1d_2^2$, where c is square free and $d_1 = \gcd(a, b)$. Let $d = d_1d_2$, $l_1 = |\#\overline{E}_q|$, $l_2 = |d|$ and $l_1 - 2l_2 = |ac/d_1|$. Since $\#\overline{E}_q$ is assumed to be uniformly distributed over $\{0, 1\}^{l_1}$, the probability, T , that $\#\overline{E}_q$ has a square factor with size of at least l_2 bits is

$$T = \sum_{l_2 \leq l \leq l_1/2} \frac{2^{l+(l_1-2l)}}{2^{l_1}} = \sum_{l_2 \leq l \leq l_1/2} \frac{1}{2^l}.$$

Therefore, if $l_2 > e \log k$ for any constant e and for sufficiently large k (i.e., 2^{l_2} is super-polynomial), then T is negligible in k . Therefore, $|d| = e \log k$ for constant e with overwhelming probability.

Let the maximum order of a point on $\overline{E}_q$ be $\sigma_{\overline{E}_q}$. Then, $\sigma_{\overline{E}_q}/a$ is at least b/d due to the group structure of elliptic curve.

We will then evaluate the probability that b/d divides $\#E_p = (p + 1 - t)$. In this case, $\Phi(E_n) * z$ is the x -coordinate of $[\mathcal{O}_p, \mathcal{O}_q]$, when $\sigma_{\overline{E}_q}/a = b/d$. Since we assume $\#E_p$ is distributed statistically uniformly and independently from $\#E_q$ (i.e., b/d), the probability that b/d divides $\#E_p = (p + 1 - t)$ is $\frac{d}{b} = O(\frac{k^e}{2^{k/2}})$, which is negligible in k .

If b/d does not divide $\#E_p = (p + 1 - t)$, then $\Phi(E_n) * z$ is the x -coordinate of a point, $[\mathcal{O}_p, \text{a finite value}]$, for a point, z , such that $z_q = z \bmod q$ has the maximum order on $\overline{E}_q$. The probability that a randomly selected point, z_q , has the the maximum order is $\frac{1}{\phi \ln \ln q + 1 + s} = O(\frac{1}{\log k})$.

Hence, when M selects polynomially many z randomly, for at least one of them, $\Phi(E_n) * z$ is the x -coordinate of a point, $[\mathcal{O}_p, \text{a finite value}]$, with overwhelming probability.

Thus, M can factor n with help of the oracle, Adv , with non-negligible probability.

7 Conclusion

This paper has introduced our trial of realizing an identity-based cryptosystem based on the SSSA algorithm, and then showed that our scheme is not secure. We have generalized the observation of our breaking algorithm and presented reductions of factoring n to computing the order of an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$. Through these reductions, we can clarify the essential reason why our system was broken: E_n in our system is a very specific curve such that $\Phi(E_n)$ is known ($\Phi(E_n) = n$).

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Appendix A: Proof of Theorem 5

Since the only-if part is trivial, we will prove the if part. To prove it, assume that the proposed scheme is not secure, i.e., there exists Adv who can break the proposed scheme with non-negligible probability. We will then construct a probabilistic polynomial time algorithm M , with the help of Adv as an oracle, which can break the anomalous E_n -Diffie-Hellman problem with non-negligible probability.

First, M generates $s_i \in \mathbf{Z}/n\mathbf{Z}$ uniformly ($i = 1, \dots, l$), and calculates $X_i = s_i * X_{G_n}$. If $X_i \geq 2^k$, then discard it and select s_i again in the same way. Repeat the procedure until all of X_i ($i = 1, \dots, l$) are within the range of $\{0, 1\}^k$. Since $2^k/n > 1/2$, the expected number of the repetitions for each X_i is less than 2, and in total, the expected number of the repetitions is less than $2l$.

M then runs Adv and gives $X_1, \dots, X_l$ to Adv along with $(r_1, \dots, r_l)$ as the reply to Adv 's queries to random oracle h . Namely, after Adv gives $(ID_1, \dots, ID_l)$ to M as h , M randomly selects r_i such that its distribution is the same as that of

the real system, and M returns $X_1, \dots, X_l$ as $h(ID_1||r_1), \dots, h(ID_l||r_l)$ to Adv along with $(r_1, \dots, r_l)$. Clearly, the distribution of M 's reply is the same as that of the reply by random oracle.

M then gives $(s_1, \dots, s_l)$ to Adv as the authority's reply to Adv .

Finally M gives X_{sG} and X_{tG} as $(h(ID_A||r_A))$ and $(h(ID_B||r_B))$ to Adv , and Adv replies $K_{AB} = s_A * (s_B * X_{G_n}) = X_{(st)G}$ to M with non-negligible probability, since the distribution of the messages given to Adv by M is exactly the same as that given by h and the authority.

Thus, M with the help of Adv can break the the anomalous E_n -Diffie-Hellman problem with non-negligible probability.

Appendix B: Another Type of Reductions

Here we will show another type of reductions, which are related to the attack algorithm shown in Section 5. That is, the attack algorithm is a special case of the reductions in this section.

Definition 11. Let G_4 be an instance generator such that $G_4(1^k) \rightarrow (E_n, n)$, $n = pq$, $|n| = k$ (p, q : primes, $|p| = |q|$), $E_n = [E_p, E_q]$. Here p and q are independently selected.

Let $\Phi(E_p) = m \cdot \#E_p$, where m is an integer such that $\frac{\#E_q}{\gcd(\#E_p, \#E_q)} \nmid m$ (i.e., $\Phi(E_n) \nmid \Phi(E_p)$).

The *modified elliptic curve order* problem is, given (E_n, n, k) , to find $\Phi(E_p)$.

The modified elliptic curve order problem is *intractable*, if for any probabilistic polynomial time machine Adv , for any constant c , for sufficiently large k ,

$$\Pr[Adv(1^k, E_n, n) = \Phi(E_p)] < 1/k^c.$$

The probability is taken over the coin flips of G_4 and Adv .

Theorem 12. [Reduction 3] Let G_4 for the modified elliptic curve order problem be equivalent to G_2 for the elliptic curve factoring problem.

The modified elliptic curve order problem is intractable if and only if the elliptic curve factoring problem is intractable.

Proof. Since the proof of the *only if* part is almost trivial, we only give the proof of the *if* part.

Assume that the modified elliptic curve order problem is intractable. That is, there exists an adversary, Adv , which can compute $\Phi(E_p)$ from (E_n, n, k) with non-negligible probability. We will then construct a probabilistic polynomial time machine M , with help of Adv as an oracle, which given (E_n, n, k) , can factor n with non-negligible probability.

Since the distribution of (E_n, n, k) given by G_2 is equivalent to that by G_4 , Adv outputs a correct $\Phi(E_p)$ with non-negligible probability.

M randomly selects $z \in \mathbf{Z}/n\mathbf{Z}$ such that $\left(\frac{f(z)}{n}\right) = 1$ with probability $1/2$ (see the proof of Theorem 9 for the notations of f and $(\frac{f(z)}{n})$).

Then we have two possibilities: z is the x -coordinate of a point on $[E_p, E_q]$ and is the x -coordinate of a point on $[\overline{E_p}, \overline{E_q}]$. In the former case, if z is a point such that $z_q = z \bmod q$ has the maximum order on $\overline{E_q}$, then $\Phi(E_p) * z$ is the x -coordinate of $[\mathcal{O}_p, \text{a finite value}]$.

Hence, when M selects polynomially many z randomly, for at least one of them, $\Phi(E_n) * z$ is the x -coordinate of a point, $[\mathcal{O}_p, \text{a finite value}]$, with overwhelming probability.

Thus, M can factor n with help of the oracle, Adv , with non-negligible probability.

Theorem 13. [Reduction 4] *Let the distribution of n given by G_1 for the factoring problem be equivalent to that given by G_4 for the modified elliptic curve order problem. Let E_n be uniformly selected by G_4 (i.e., the parameters of E_n are uniformly and independently selected over $\mathbf{Z}/n\mathbf{Z}$).*

Then the modified elliptic curve order problem is intractable if and only if the factoring problem is intractable.

Proof. Since the proof of the *only if* part is almost trivial, we only give the proof of the *if* part.

Let assume that the modified elliptic curve order problem is not intractable, i.e., there exists an adversary, Adv , which computes $\Phi(E_p)$ from (E_n, n, k) with non-negligible probability.

Then we will construct a probabilistic polynomial time algorithm M , with help of Adv as an oracle, which, given n by G_1 , can factor n with non-negligible probability.

M uniformly selects an elliptic curve mod n , E_n . Here note that the distribution of E_n given by M is the same as that given by G_4 .

The remaining algorithm for this reduction is the same as that of Reduction 3. Thus, M can factor n with help of the oracle, Adv , with non-negligible probability.