Deterministic Rational Transducers and Random Sequences

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Abstract

This paper presents some results about transformations of infinite random sequences by letter to letter rational transducers. We show that it is possible by observing initial segments of a given random sequence to decide whether two given letter to letter rational transducers have the same output on that sequence. We use the characterization of random sequences by *Kolmogorov Complexity*. We also prove that the image of a random sequence is either random, or non-random and non-recursive, or periodic, depending on some transducer's structural properties that we give.

Introduction

This paper starts the study of the deterministic rational transducers behaviour on infinite random sequences. Firstly, we show that it is possible by observing initial

segments of a given random sequence to decide whether two given letter to letter rational transducers have the same output on that sequence (we call this problem *equality problem*). The analogous problem is undecidable on the class of all input sequences. Secondly, we prove that the image of a random sequence is either random or non-random and non-recursive or periodic, depending on the transducer's structural properties. Finally, we derive an arithmetical theorem from the previous result : the image of a 'random real' by a letter to letter rational transducer is either rational or transcendental.

This work is a part of a larger study on transformations processes of discrete curves. These processes are 'real time' transformations on random sequences. For us, 'real time' means that the computation time as well as the memory of the processes are bounded. Consequently, these processes are rational transducers [PD97].

The concept of random sequence has been well defined for the first time by Martin-Löf in 1966 [ML66]. A sequence is random if and only if it does not belong to any constructive null set. Theory of *Kolmogorov Complexity* provides a characterization of such sequences in terms of compressibility. Levin and Schnorr have shown that the notions of random sequences and incompressible sequences are equivalent [Lev73] [Sch73]. This theory has enabled to obtain results on regular languages [Sei86] [CL88] [LV93], but our aim is different. The originality of this paper lies in the study of the effect of algebraic objects (rational transducers) on random sequences. This is another approach to Markov chains, which are probabilistic finite automata : here we consider deterministic automata and it is the input sequence that chooses the transitions. This choice is 'typically random' when the sequence is random.

We briefly present in section 1.1 the definition of random sequences due to Martin-Löf, and an equivalent definition in terms of *Kolmogorov Complexity*. There are several variants of this theory (see the foreword of Chaitin in [Cal94]), but here we consider only the *Prefix Complexity*. This variant is more convenient for the study of infinite sequences.

We consider classical rational transducers [MS97], introduced by M. Nivat [Niv68], reading infinite binary sequences in input and, unlike Büchi automata e.g. [Tho90], without any particular acceptance conditions. We do not require that for each state and each input letter there exists the corresponding transition. We say that a transducer accepts a sequence if it can read all its letters. In section 1.2, we introduce a classification of transducers' states. This classification is drawn from the theory of Markov chains [MT96]. The main aim of this classification is Lemma 5. It gives a necessary and sufficient condition of the acceptance of a random sequence.

In part 2, we show that the equality problem of two transducers is decidable on the class of random sequence, when it is undecidable on the class of all sequences. In order to

accepts an infinite sequence if and only if both initial transducers accept this sequence and have the same output.

In part 3, Theorem 2 establishes a classification of the images of random sequences. This classification depends on the transducer's structural properties. On the one hand, the image of a random sequence is random if and only if there are not two paths with the same output linking two recurrent states. On the other hand, the image of a random sequence has finite complexity if and only if all paths starting from a recurrent state have the same output. Moreover, in this case, the image sequence is periodic.

Section 3.2 presents an application of the previous result within the frame of arithmetic. We call *random real* a real number of which the binary expansion is a random sequence. Theorem 3 claims that the image of any random real by a letter to letter rational transducer is either rational or transcendental.

1 Preliminaries

1.1 Random sequences and Kolmogorov Complexity

In this part we briefly present the definition of random sequences from Martin-Löf and the theory of *Kolmogorov Complexity*. In the following we call *word* (respectively *sequence*) any finite (respectively any infinite) string over the binary alphabet $\{0, 1\}$.

1.1.1 Random sequences

The first satisfactory definition of a random sequence has been given by Martin-Löf in [ML66]. A sequence is random if and only if it does not belong to any constructive null set. More precisely, a sequence s is *non random* if and only if there is a recursive function $u(i, j)$, mapping pairs of integers to words, satisfying the following properties :

1. $\forall i \mu(\Omega_i) \leq 2^{-i}$;
2. $\forall i s \in \Omega_i$.

where $\Omega_i = \bigcup_j \Gamma_{u(i,j)}$, $\Gamma_{u(i,j)}$ being the set of all the sequences having the prefix $u(i, j)$, and μ denotes uniform measure on the set of all sequences.

1.1.2 Kolmogorov Complexity

Theory of *Kolmogorov Complexity* [LV97], also called *Algorithmic Information Theory*, gives rigorous mathematical foundations to the notion of information quantity contained in an object x . This quantity $K(x)$ is the length of a smallest program computing x without any input. The programming language must satisfy the following property : for any computable partial function ϕ mapping words to words there exists

a constant C such that for any p in the domain of ϕ there exists a program computing $\phi(p)$ of length at most $\text{length}(p) + C$. We will use the variant of *Kolmogorov Complexity* called *Prefix Complexity*, in which the set of programs must be a *prefix set*: none of the programs is the prefix of another program. The complexity varies no more than by an additive constant when we change the programming language. Indeed,

$$\exists C \forall x |K_1(x) - K_2(x)| < C$$

where $K_1(x)$ and $K_2(x)$ are Kolmogorov complexities defined for two different programming languages.

1.1.3 Randomness and incompressibility

Theory of *Kolmogorov Complexity* provides a characterization of randomness using incompressibility properties. Actually Levin and Schnorr [Lev73] [Sch73] have shown that random sequences are exactly those incompressible for some variants of *Kolmogorov Complexity*. Their proof can be easily translated in the frame of *Prefix Complexity* that we use here. More specifically, all prefixes of a random sequence are compressible for less than a constant. Formally, if $a_{1:n}$ is the prefix of length n of a sequence a , then this sequence is random if and only if :

$$\exists c \forall n K(a_{1:n}) \geq n - c$$

A random sequence is non-recursive, but non-random ones may be either recursive or non-recursive.

- Assume for example that the sequence 010011010110... is random. Then the sequence 001100001111..., where each letter of the previous sequence appears twice, is non-random and non-recursive.
- The sequence 0101010101... is non-random and recursive.

In several proofs we show that a sequence a is non-random building a program that computes a using another sequence a' as an input. This program uses at most the λn first letters of a' to compute the n first letters of a , with $\lambda < 1$. The following lemma expresses this idea.

Lemma 1 *Let a and a' be two sequences. If there exists a program P such that :*

$$\exists (u_n)_{n \in \mathbb{N}} \exists (v_n)_{n \in \mathbb{N}} \exists \lambda < 1 \text{ such that } \forall n P(a'_{1:u_n}) = a_{1:v_n} \text{ and } u_n \leq \lambda v_n$$

then a is non-random.

Proof We will give an upper bound of the complexity of the prefix words $a_{1:v_n}$ of a . The program $\langle P, a'_{1:u_n} \rangle$ computes $a_{1:v_n}$, but the set of these programs is not a prefix

set. In order to obtain such a set we have to use the prefix coding of $a'_{1:u_n}$, denoted by $\overline{a'_{1:u_n}}$. For all n we have $K(a_{1:v_n}) \leq |\overline{a'_{1:u_n}}| + |P| + O(1)$. A classical upper bound for the prefix coding of a word w is $|w| + 2 \log(|w|) + O(1)$. Hence we have :

$$\begin{aligned} \forall n \quad K(a_{1:v_n}) &\leq |a'_{1:u_n}| + 2 \log(|a'_{1:u_n}|) + O(1) \\ &\leq \lambda |a_{1:v_n}| + 2 \log(|a_{1:v_n}|) + O(1) \end{aligned}$$

Let $c > 0$ be fixed. There exists n_c such that :

$$\forall n \geq n_c \quad \lambda |a_{1:v_n}| + 2 \log(|a_{1:v_n}|) + O(1) \leq |a_{1:v_n}| - c$$

Eventually we have :

$$\forall c \exists n_c \forall n \geq n_c \quad K(a_{1:v_n}) \leq |a_{1:v_n}| - c$$

Therefore a is non-random. □

1.2 Transducers

We consider classical transducers without final states since inputs are only infinite sequences. In this first approach we consider only letter to letter transducers : the output is a letter at each letter read in input. When the transducer reaches a state where there is no transition labelled by the input letter, it halts and we say that it rejects the input sequence. Otherwise it accepts the input sequence.

We give now some definitions and preliminary results considering only inputs on transducers.

Definition 1 Let T be a transducer and a be a sequence accepted by T . A state of T is recurrent for a if it is reached an infinity of times with a as an input. T_a denotes the set of recurrent states for a .

Definition 2 Let q be a state of T . Let $\text{occ}_n^a(q)$ denote the number of occurrences of the state q during the reading of $a_{1:n}$. The frequency $f^a(q)$ of q is defined by :

$$f^a(q) = \limsup_{n \rightarrow \infty} \frac{\text{occ}_n^a(q)}{n}$$

A state q is frequent if $f^a(q) > 0$.

Definition 3 Let q be a state of T and t be a transition from q . Let $\text{occ}_n^a(t)$ denote the number of occurrences of the transition t during the n first occurrences of the state q with a as an input. The frequency $f^a(t)$ of t is defined by :

$$f^a(t) = \limsup_{n \rightarrow \infty} \frac{\text{occ}_n^a(t)}{n}$$

Lemma 2 *Let a be a sequence accepted by T . T contains at least one frequent state.*

Proof Suppose that for each state q_i of T_a we have $f(q_i) = 0$. Let $\epsilon = \frac{1}{2Q}$ where Q is the number of states of T . Therefore we have :

$$\begin{aligned} \exists n \forall q_i \in T \quad \frac{\text{occ}_n^a(q_i)}{n} &\leq \epsilon \\ \sum_{i=1}^Q \frac{\text{occ}_n^a(q_i)}{n} &\leq \epsilon Q \\ 1 &\leq \frac{1}{2} \end{aligned}$$

□

Definition 4 *A complete state is a state out of which get two transitions labelled by 0 and 1. We say that a set of states is complete if each state of this set is complete.*

Claim 1 *Consider the preorder on the set of the states of a transducer defined by :*

$$q < q' \Leftrightarrow \text{we can reach } q' \text{ from } q$$

and the equivalence classes of states. We say that maximal classes are absorbing. There is an algorithm that splits a transducer in sub-automata $T_1, \dots, T_N$ and T_P (See Figure 1. $T_1, \dots, T_N$ are the complete absorbing classes and T_P is the union of the remaining classes.

Figure 1: Complete absorbing classes

Lemma 3 *Let a be a random sequence accepted by T . Any frequent state of T is complete.*

Proof Let q be a frequent state. Suppose that q is not complete. Thus, there is only one transition from this state, say the transition labelled by input letter 0 and leading to the state q' . We show this implies that a is non-random. Consider the sequence a' obtained as follows : in the sequence a we delete letters read at each occurrence of q . The following program computes a using T and a' .

```

Program  $P(a')$ 
  Repeat
    If  $T$  is in state  $q$  do
      Output 0 ; Place  $T$  in state  $q'$ 
    Else
      Read a letter  $b$  of  $a'$  and output  $b$ 
      Simulate  $T$  on input  $b$ 
    End if
  End repeat
End

```

Since q is frequent we have $\limsup_{n \rightarrow \infty} \frac{\text{occ}_n^a(q)}{n} = \alpha$ with $\alpha > 0$. Thus there is a series $(v_n)_{n \in \mathbb{N}}$ and n_0 such that for all $n > n_0$ we have $\frac{\text{occ}_{v_n}^a(q)}{v_n} \geq \frac{\alpha}{2}$. Let $u_n = v_n - \text{occ}_{v_n}^a(q)$. For all $n > n_0$ we have $u_n \leq \lambda v_n$ with $\lambda = 1 - \frac{\alpha}{2} < 1$. The program P is such that for all n we have $P(a'_{1:u_n}) = a_{1:v_n}$. Thus hypothesis of Lemma 1 are satisfied, what enables us to conclude that a is non-random. □

Lemma 4 *Let a be a random sequence accepted by T . Then all states of T_a are frequent and T_a is a complete absorbing class.*

Proof According to Lemma 2, T contains at least one frequent state q . This state belongs to T_a and, according to lemma 3, is complete. Let t_0 (respectively t_1) be the transition accepting 0 (respectively 1) and leading to q_0 (respectively q_1) from q (q_0 and q_1 can be the same).

Step 1 : We show that both transitions t_0 and t_1 are frequent. Suppose for example that t_0 is not frequent, i.e. $f^a(t_0) = 0$. Obviously we have $f^a(t_1) = 1$. Consider the sequence a' defined as follows : in the sequence a we delete each 1 read at each occurrence of t_1 and we replace each 0 read at each occurrence of t_0 with a prefix code of the number of occurrences of t_1 before the next occurrence of t_0 . The following program computes a using a' .

```

Program  $P(a')$ 
  Let  $n$  be the number of occurrences of  $t_1$  before next occurrence of  $t_0$ 
  Repeat
    If  $T$  is in state  $q$  then
      If  $n = 0$  then
        Output 0 ; Let  $n$  be the prefix coded number appearing
          from current position in  $a'$ 
        Place  $T$  in state  $q_0$ 
      Else
        Output 1 ; Let  $n = n - 1$ 
        Place  $T$  in state  $q_1$ 
      End if
    Else
      Read a letter  $b$  of  $a'$  and output  $b$ 
      Simulate  $T$  on input  $b$ 
    End if
  End repeat
End

```

The same arguments as in the proof of Lemma 3 enable us to conclude that a is non-random.

Conclusion : Since $f^a(q) > 0$, $f^a(t_0) > 0$ and $f^a(t_1) > 0$ both states q_0 and q_2 are frequent.

Step by step we show that any state reachable from q is frequent and thus, according to Lemma 3, is complete. Such a state is recurrent. Therefore, the class of q is absorbing and complete.

□

The following lemma is an immediate corollary of Lemma 4 and of the fact that a complete absorbing class accepts any sequence.

Lemma 5 *For all random sequence a , T accepts a if and only if T reaches on a one of its complete absorbing classes.*

2 Decidability of the equality of letter to letter rational transducers on random sequences

We say that two transducers T_1 and T_2 are equal on a sequence s if $T_1(s) = T_2(s)$. Note the difference between this definition and the definition of the equivalence of transducers : T_1 and T_2 are equivalent if $T_1(w) = T_2(w)$ for all word w . The equality of a class Θ of rational transducers on a class S of sequences is decidable if there is an algorithm that decides, for any transducers T_1 and T_2 in Θ , and any sequence s in S spelled letter by letter, whether $T_1(s) = T_2(s)$, using a finite prefix (of unknown length) of s .

$$\exists A \forall (T_1, T_2) \in \Theta^2 \forall s \in S \begin{cases} A(T_1, T_2, s) = \text{yes} & \text{if } T_1(s) = T_2(s) \\ A(T_1, T_2, s) = \text{no} & \text{otherwise} \end{cases}$$

If Θ is the set of letter to letter rational transducers and S is the set of all sequences, the equality is not decidable. Consider transducers T_1 and T_2 defined in Figure 2. Suppose

Figure 2: Two transducers equal on 0^∞ and not equal on $0^n 10^\infty$

there is an algorithm A deciding, for any sequence s , whether both transducers T_1 and T_2 are equal on s . Let $s_1 = 0^\infty$. Since $T_1(s_1) = T_2(s_1)$, $A(T_1, T_2, s_1)$ halts having read a prefix of length n of s_1 and outputs *yes*. Now, let $s_2 = 0^n 10^\infty$. Since the n

first letters of s_1 and s_2 are the same, $A(T_1, T_2, s_2)$ halts and outputs *yes*, although we obviously have $T_1(s_2) \neq T_2(s_2)$.

However, the equality becomes decidable if we restrict ourselves to random sequences. In order to show this result, we need to define the notion of product transducer.

Definition 5 Let T_1 and T_2 be two transducers. Their product transducer $T = T_1 \times T_2$ is defined as follows : if we can reach q'_1 from q_1 in the transducer T_1 and q'_2 from q_2 in the transducer T_2 , with the same letter b_{in} in input and the same letter b_{out} in output, then T contains a transition between (q_1, q_2) and (q'_1, q'_2) reading b_{in} and outputting b_{out} .

The product transducer $T = T_1 \times T_2$ accepts a sequence a if and only if T_1 and T_2 accept a and output the same sequence.

Theorem 1 The equality of letter to letter rational transducers on random sequences is decidable.

Proof Let a be a random sequence. T_1 and T_2 accept a and output the same sequence if and only if their product transducer T accepts a or, according to Lemma 5, if and only if T reaches on a one of its complete absorbing classes. If T halts, we distinguish there are different cases :

- T_1 and T_2 accept a and output different sequences and thus T_1 and T_2 are not equal on a ;
- one transducer accepts a and the other rejects a : T_1 and T_2 are not equal on a ;
- T_1 and T_2 rejects a : if the output words are the same T_1 and T_2 are equal on a , otherwise they are not.

Finally, T_1 and T_2 are equal on a if and only if either T reaches on a one of its complete absorbing classes or T, T_1, T_2 halt simultaneously with the same output words. These observations lead to the following decision algorithm :

```

Program  $A(T_1, T_2, a)$ 
  Build product transducer  $T = T_1 \times T_2$ 
  Split  $T$  in complete absorbing classes as in claim 1
  For each letter of  $a$  repeat
    If  $T$  reaches one of its complete absorbing classes then
      Output yes ; Halt
    Elseif  $T$  halts then
      If  $T_1$  and  $T_2$  halt at same place and output same word then
        Output yes ; Halt
      Else
        Output no ; Halt
      End if
    End if
  End for
End

```

□

3 Classification of images of random sequences

In this part we are interested in the images of random sequences by letter to letter rational transducers. Theorem 2 shows the influence of the transducer's structure on the sequence we obtain.

Definition 6 A state q is partially indifferent if there are two different input words having the same length, leading to a same state from q and having the same image.

Definition 7 A state q is discriminating if there exists $l > 0$ satisfying the following property : from q , the images of two words w_1 and w_2 of length l end with same letter if and only if w_1 and w_2 start with same letter.

Theorem 2 Let T be a letter to letter rational transducer, let a be a random sequence accepted by T and let s denote $T(a)$.

1. if T_a does not contain any partially indifferent state then s is random ;
2. (a) if T_a contains at least one partially indifferent state then s is non-random.

Moreover :

- (b) if T_a contains a discriminating state then s is non-recursive ;
- (c) if T_a does not contain any discriminating state, then s is recursive and periodic : $s = uv^*$.

Remark 1 Let Q denote the number of states of a transducer T . We can show that the existence of a partially indifferent state in T is decidable in a time $\mathcal{O}(Q^2)$ and the existence of a discriminating state in T is decidable in a time $\mathcal{O}(2^{2Q})$.

Example 1 Call a transducer one-to-one if it defines an injective mapping on (infinite) sequences. A one-to-one transducer does not contain any partially indifferent state. The transducer of Figure 3 is a non-trivial example of a one-to-one transducer, that transforms a sequence a in $0a$. It is non-trivial because it is not just a injective morphism of words.

Figure 3: A non trivial one-to-one transducer

Example 2 Injectivity is not a necessary condition to map random sequences on random sequences. Consider the transducer T of Figure 4. Since sequences 0^∞ and 10^∞ have the same image 01^∞ , T is not a one-to-one transducer. However, T does not contain any partially indifferent state since there is no state reachable from two transitions having same output letter.

Figure 4: A many-to-one transducer without partially indifferent states

3.1 Proof

Proof of 1. and 2.(a) We have to show that s is random if and only if T_a does not contain any partially indifferent state.

Only if : We prove that if T_a contains at least one partially indifferent state q then s is non-random.

Since q is a partially indifferent state, there are two words w_1 and w_2 leading to the same state q' from q and outputting the same word u . Let l be the length of w_1 and w_2 . Let f be a coding function from $\{0, 1\}^l$ to $\{0, 1\}^l$ such that $f(w_1)$ and $f(w_2)$ differ only in their last letter. For example let $f(w_1) = v0$ and $f(w_2) = v1$, where v is a word of length $l - 1$. Let a' be defined as follows : in the sequence a we replace each word w of length l read from state q with v if $w = w_1$ or $w = w_2$, with $f(w)$ otherwise. The following program computes s with a' as an input.

```

Program  $P(a')$  Simulate  $T$  on input sequence  $a'$ 
  Each time  $T$  reaches state  $q$  do
    Let  $w$  be word of length  $l - 1$  from current position in  $a'$ 
    If  $w = v$  then
      Output  $u$ 
      Place  $T$  in state  $q'$ 
    Else
      Let  $w$  be word of length  $l$  from current position in  $a'$ 
      Let  $w'$  be the word such that  $f(w') = w$ 
      Replace  $w$  with  $w'$  in  $a'$ 
      Resume simulation of  $T$ 
    End if
  End do
End

```

The same arguments as in the proof of Lemma 3 enable us to conclude that s is non-random.

If : We prove that if T_a does not have any partially indifferent state then s is random. Suppose that s is non-random. We show that this implies that a is non-random too.

Let $u(i, j)$ denote a recursive function mapping pairs of integers to words. Let $\Gamma_{u(i, j)}$ denote the set of all sequences having the prefix $u(i, j)$, let Ω_i denote $\bigcup_j \Gamma_{u(i, j)}$ and let μ denotes uniform measure on the set of all sequences. Since s is non-random, s belongs to a constructive null set, i.e. there is a recursive function $u(i, j)$ such that :

1. $\forall i \mu(\Omega_i) \leq 2^{-i}$
2. $\forall i s \in \Omega_i$

Note that we can suppose that, given i , all sets $\Gamma_{u(i, j)}, j \in \mathbb{N}$ are disjoint. We will define a function $\hat{u}(i, j)$ mapping pairs of integers to words such that for all i we have $\{T(\hat{u}(i, j)) \mid j \in \mathbb{N}\} = \{u(i, j) \mid j \in \mathbb{N}\}$. The following program $P(i, j)$ computes $\hat{u}(i, j)$ using the recursive function $u(i, j)$ and the transducer T .

```

Program  $P(i, j)$ 
   $count = 0 ; k = 0$ 
  Repeat
     $k = k + 1 ; w = u(i, k)$ 
    For all words  $\hat{w}$  of length  $|w|$  do
      If  $T(\hat{w}) = w$  then
         $count = count + 1$ 
        If  $count = j$  then
          Output  $\hat{w} ;$  Halt
        End if
      End if
    End for
  End repeat
End

```

Since $u(i, j)$ is recursive, $\hat{u}(i, j)$ is recursive too. Let $\hat{\Omega}_i = \bigcup_j \Gamma_{\hat{u}(i, j)}$. Since $\hat{u}(i, j)$ is the preimage of a $u(i, k)$, a belongs to $\hat{\Omega}_i$ for all i . Moreover we have :

$$\forall i \mu(\hat{\Omega}_i) \leq \sum_j \mu(\Gamma_{\hat{u}(i, j)})$$

Let Q be the number of states of T . Since T_a does not contain any partially indifferent state there are at most Q different words having the same image. Thus we have :

$$\begin{aligned}
 \forall i \mu(\hat{\Omega}_i) &\leq \sum_j Q \mu(\Gamma_{u(i, j)}) \\
 &= Q \mu(\Omega_i) \\
 &\leq Q 2^{-i}
 \end{aligned}$$

Hence a belongs to a constructive null set defined by $\hat{u}(i, j) : a$ is non-random.

□

Proof of 2.(b) Let q be a discriminating state. There exists $l > 0$ such that, observing the last letter of the image of an input word of length l from q , we can retrieve the first letter of this input word. Assume for example that all words of length l starting with 0 (respectively with 1) have their images ending with 0 (respectively with 1).

Suppose s is recursive. Let P be a program that computes s . Consider the sequence a' defined as follows : in the sequence a we delete letters read at each occurrence of q . The following program computes a using P , T and a' . Let q_0 (respectively q_1) denote the state reached when 0 (respectively 1) is read at state q .

```

Program  $P(a')$ 
  Repeat
    If  $T$  is in state  $q$  then
      Compute next  $n$ -th letter of  $s$  using program  $P$ 
      If this letter is 0 then
        Output 0 ; Place  $T$  in state  $q_0$ 
      Else
        Output 1 ; Place  $T$  in state  $q_1$ 
      End if
    Else
      Read a letter  $b$  of  $a'$  and output  $b$ 
      Simulate  $T$  on input  $b$ 
    End if
  End repeat
End

```

The same arguments as in the proof of Lemma 3 enable us to conclude that a is non-random.

□

Proof of 2.(c) Firstly we prove the following property : for all state q of T_a , for all l , for all words w_1 and w_2 of length l , the images of w_1 and w_2 from q are the same. We use a recurrent reasoning on l to show this. $W(q, l)$ denotes the set of image words of length l from q .

Step 1 : The property is true for $l = 1$. Indeed each state outputs the same letter whatever the input letter is, since each state of T_a is not discriminating.

Step 2 : Let q be a state of T_a . Let q_0 (respectively q_1) be the state reached from q when 0 (respectively 1) is read. Suppose the property is true for all $l \leq L$. Thus all words of $W(q_0, L)$ (respectively $W(q_1, L)$) end with the same letter b_0 (respectively b_1). Since q is not discriminating, we necessarily have $b_0 = b_1$. Thus all input words of length $L + 1$ have images ending with same letter from q .

We prove now that s is periodic, i.e. there are two words u and v such that $s = uv^*$. Let a' be the suffix sequence of a read from the first occurrence of a recurrent state q . According to the previous property, we have $q(a') = q(0^*)$. Since the automaton is finite and deterministic, there is necessary a loop.

□

3.2 From an arithmetical point of view

The following theorem is an application of Theorem 2 within the frame of arithmetic. We call ‘random real’ a real of which the binary expansion is a random sequence. We remind of the reader that a real is *algebraic* if it is the root of a polynomial equation having integer coefficients. A real is *transcendental* if it is not algebraic.

Theorem 3 *The image of a random real by a letter to letter rational transducer is either rational or transcendental.*

Proof Let a be the binary expansion of the image of a random real. If a has a finite complexity then it is periodic and then the real number is rational. If a has an infinite complexity then the real number is transcendental. Indeed, since an algebraic number is a root of a polynomial equation with integer coefficients, it has a finite complexity : we can give more and more accurate approximations of this root using an algorithm.

□

Open problems

In this first approach we have only considered letter to letter rational transducers. We should study whether the results we have obtained remain true in the general case. Moreover we think that links between simple and ‘pure’ algebraic objects (rational transducers) and ‘pure’ complex objects (random sequences), and also the links with Markov chains, should be study thoroughly.

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