Maximum Packing for Biconnected Outerplanar Graphs

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Abstract

The problem of determining the maximum number of vertex-disjoint subgraphs of a biconnected outerplanar graph H on n_h vertices isomorphic to a "pattern" biconnected outerplanar graph G on n_g vertices is shown to be solvable in time $O((n_h n_g)^2)$.

1 Introduction

Following Hell and Kirkpatrick [12], a G -packing of a graph H is a set of vertex-disjoint subgraphs $H_1, H_2, \dots, H_l$ of H such that each H_i is isomorphic to G . In analogy to the problem of maximum matching in H , the problem of *maximum G -packing in H* is to determine the maximum cardinality of G -packing of H . Hell and Kirkpatrick proved this problem to be NP-complete in general [13], and Berman *et al.* showed it to be NP-complete even when H is a planar graph [5]. Recently, Lingas has shown the problem of maximum graph-packing restricted to trees, where the guest tree is of arbitrary size, to be solvable in time $O(n^{\frac{5}{2}})$ [15]. All previously known polynomial-time solutions to restrictions of the graph-packing problem relied on the constant size of the guest graph G .

In this paper we consider the restriction of the maximum graph-packing problem to biconnected outerplanar graphs. Both the guest graph G and the host graph H are assumed to be biconnected outerplanar graphs. In particular, the guest graph G is of arbitrary size. The assumption on biconnectivity of G

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is important since even the problem of *subgraph isomorphism* for a connected outerplanar graph G and biconnected outerplanar graph H (i.e., the problem of determining whether such a graph G is isomorphic to a subgraph of such a graph H) is known to be NP-complete [22].

Generally, the maximum graph-packing problem is not easier than the problem of subgraph isomorphism since the solution to an instance of the former immediately yields solution to the corresponding instance of the latter. The only known polynomial-time algorithms for subgraph isomorphism are those for k -connected partial k -trees (biconnected outerplanar graphs are special case of 2-connected partial 2-trees) and partial k -trees of bounded degree. Substantially weakening any of the two assumptions yields NP-completeness [19, 10]. Besides general polynomial-time algorithms for these two problems [5, 19, 9, 8], there are known several specific polynomial-time algorithms for special cases of these problems, e.g., for trees [18], biconnected outerplanar graphs [14], biconnected partial 2-trees [17], and k -connected partial k -paths [10].

Partial k -trees are known to admit efficient polynomial-time solutions for many problems that are unfeasible for general graphs (see [4]), especially for those that can be described in the language of extended Monadic Second Order Logic, [3, 5]. Since graph isomorphism is not one of the former problems, it is doubtful that the general techniques of [3] can be used to construct an efficient subgraph isomorphism (not to say about maximum graph-packing) algorithm for the feasible cases of partial k -trees.

Presently the best known upper time-bound for subgraph isomorphism restricted to biconnected outerplanar graphs is due to Lingas who presented a cubic-time dynamic programming algorithm for this problem [14]. In Lingas' algorithm, the recursive reduction of an imbedding subproblem to smaller ones is solved not by a matching technique, as it is in the case of all the other aforementioned polynomial-time solutions to restrictions of subgraph isomorphism, but by finding a simple path in an auxiliary graph.

In this paper, we generalize Lingas' algorithm for subgraph isomorphism restricted to biconnected outerplanar graphs to include the maximum packing problem for biconnected outerplanar graphs. The recursive reduction of a graph-packing subproblem to smaller ones is solved by finding shortest paths in auxiliary graphs. In effect, for biconnected outerplanar graphs G , H on respectively n_g and n_h vertices, we can solve the problem of maximum G -packing in H in

time $O((n_h n_g)^2)$. The increase in time complexity in comparison with the cubic-time algorithm for subgraph isomorphism from [14] is primarily caused by the necessity of considering a larger family of “rooted” subgraphs of the guest graph G .

Our result has potential applications in image processing, chemical structure analysis and computational biology. For instance, the related problem of maximum two-dimensional text packing occurs in the Comparison Algorithm for Navigating Digital Image Databases (CANDID) developed at the Los Alamos National Laboratory [6]. CANDID facilitates a query-by-example approach to image retrieval. A user poses queries such as, “Show me all or the maximum number of non-overlapping images in the database that contain textures similar to those in this example image”. Such queries are useful in a variety of settings such as analysis of the images sent by remote sensing satellites and medical diagnostics. Since the problem of maximum two-dimensional text packing is NP-complete in general [1], its restrictions to instances solvable in polynomial-time are of interest.

The remainder of the paper is divided into two sections. In Section 2, we introduce basic notions, definitions, facts and lemmata necessary to design the maximum packing algorithm for biconnected outerplanar graphs. In Section 3, we present the algorithm, prove its correctness and analyze its time complexity. In Section 4, we discuss possible extensions of our algorithm, e.g., to include maximum topological packing and maximum homeomorphic packing for biconnected outerplanar graphs.

2 Preliminaries

We shall adhere to standard set and graph theoretic notation and definitions (for instance, see [7, 11]). Specifically, we assume the following set and graph conventions:

- 1) For a graph G , $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G .
- 2) A *path* in a graph G is a sequence $v_0, v_1, \dots, v_k$ of its vertices such that for $i = 0, \dots, k-1$, v_i is adjacent to v_{i+1} . If v_k is also adjacent to v_0 then the sequence can be also called a *cycle*. A path or a cycle is *simple* if all vertices that occur in

it are distinct. G is *connected* (*biconnected*, respectively) if for any two vertices v, w in G there is a simple path (simple cycle, respectively) including v and w .

3) A *directed path* in a directed graph D is a sequence $v_0, v_1, \dots, v_k$ of its vertices such that for $i = 0, \dots, k-1$, there is an edge of D leaving v_i and entering v_{i+1} . As in the undirected case, a directed path is simple if all vertices that occur in it are distinct.

An alternative way of defining the concept of subgraph isomorphism discussed in the introduction is as follows.

Let G, H be two graphs. A one-to-one mapping ϕ of $V(G)$ to $V(H)$ is called an *imbedding* of G in H if for any adjacent vertices $v, w \in V(G)$, $\phi(v)$ is adjacent to $\phi(w)$. G can be *imbedded* in H if there exists an imbedding of G in H .

We shall consider the graph-packing problem for the so called outerplanar graphs. An *outerplanar* graph is a graph which can be embedded in the plane in such a way that all vertices lie on the exterior face [20] (note that if the graph is biconnected, the exterior face is bounded by a simple cycle). We shall call such an embedding of a graph in the plane, an *outerplanar embedding*. By adapting the algorithm of Mitchell for recognition of outerplanar graphs in linear time [20], the following fact has been deduced in [14].

Fact 1: *Given a biconnected outerplanar graph, we can find the simple cycle bounding the exterior face of its outerplanar embedding in linear time.*

Using the standard geometric definitions of a straight-line segment, a simple polygon and convex polygon [21] and the following specific ones, we will be able to specify outerplanar embeddings of biconnected outerplanar graphs more precisely.

- a) A *diagonal* of a simple polygon P is a straight-line segment connecting two vertices of P and having its interior inside P .
- b) A *partial triangulation* of a simple polygon is a set of non-intersecting diagonals of the polygon. A *partially triangulated* polygon Q is a union of a simple polygon and a partial triangulation of the simple polygon. The vertices of the simple polygon are vertices of Q , whereas the edges of the simple polygon and the diagonals from the partial triangulation of the simple polygon are edges of Q . The former edges of Q are called *boundary edges* of Q , the latter edges of

Q are called *diagonal edges* of Q . We shall also consider directed (boundary or diagonal) edges of Q . Clearly, each edge of Q induces two directed edges of Q .

Mitchell notes in [20] that an outerplanar biconnected graph is in fact a partially triangulated polygon ("polygon with chords"). The following fact has been also observed in [14].

Fact 2: *Given a biconnected outerplanar graph, we can find its outerplanar embedding in the form of a partially triangulated (convex) polygon in linear time.*

It follows that a biconnected outerplanar graph has a unique outerplanar embedding (in the topological sense) up to the mirror image (see also [20]).

In the next section, to design our algorithm, we shall use Fact 2 and Fact 3 ([14]) which is an easy consequence of Fact 2.

Fact 3: *Let F be an outerplanar graph with vertices identified with numbers 0 through n . If the vertex sequence $0, 1, \dots, n$ forms the cycle bounding the exterior face of an outerplanar embedding of F , then any simple path in F that begins with 0 and ends with n is an increasing number sequence.*

3 The maximum packing algorithm for biconnected outerplanar graphs

Let G, H be biconnected outerplanar graphs on n_g and n_h vertices, respectively. To determine the cardinality of maximum G -packing in H , we proceed as follows.

First, we construct outerplanar embeddings G', H' of G and H respectively, in the form of partially triangulated convex polygons. Such embeddings can be produced in time $O(n_g + n_h)$ by Fact 2. We shall identify the vertices on the boundary of H' with numbers $0, 1, \dots, n-1$, in counterclockwise order.

Then, for $F \in \{G, H\}$, we consider the class $PE(F')$ of pairs composed of parts of F' and directed edges of F' , defined as follows. (I, e) is in $PE(F')$ if and only if I equals F' and e is a directed boundary edge of F' or e is a directed diagonal edge of F' splitting F' into two partially triangulated polygons such that, one of them, including the splitting edge, equals I . Given $M = (I, e)$ in $PE(F')$, we denote by $S(I)$ the subgraph of F represented by I and call e the *root edge* of M . If $I = F'$ then (I, e) is called *complete*. Next, given an edge d

of F' and a directed diagonal edge e of F' different from d , $p(e, d)$ is the unique member (I, e) in $PE(F')$ where d is not an edge of I , see Fig. 1.

Figure 1: The embedding F' , edges e , d , and the figure $p(e, d)$ (thick marked).

By a *root-imbedding* of $(I, (v_0, v_1)) \in PE(G')$ in $(J, (w_0, w_1)) \in PE(H')$, we shall mean an imbedding of $S(I)$ in $S(J)$ such that v_0, v_1 are respectively mapped onto w_0, w_1 .

Note that G can be imbedded in H if and only if for an arbitrary directed boundary edge b of G' there exists (J, d) in $PE(H')$ such that the imbedding problem for (G', b) and (J, d) can be answered positively.

For $K = (J, d)$ in $PE(H')$, let $N(K)$ be the cardinality of maximum packing in the graph $(V(S(J)), E(S(J)) \setminus \{d\})$, i.e., the maximum number of vertex-disjoint subgraphs of $S(J)$ isomorphic to G and disjoint from d . Next, let $L(K)$ be a list of all pairs M in $PE(G')$ that can be root-imbedded in K so the part of $S(J)$ unmapped in the root-imbedding still contains $N(K)$ vertex-disjoint subgraphs isomorphic to G . Finally, let $MP(K)$ be the cardinality of maximum G -packing in $S(J)$.

Clearly, we have $N(K) \leq MP(K) \leq \lceil \frac{n_h}{n_g} \rceil$.

Remark 1. If $L(K)$ contains a complete pair from $PE(G')$ then $MP(K)$ is equal to $N(K) + 1$ else it is equal to $N(K)$.

Our algorithm will recursively determine $N(K)$ and $L(K)$ for all pairs $K = (J, d)$ in $PE(H')$ in the partial order induced by the size of $S(J)$.

Let us inductively assume that for any pair $Q \in PE(H')$, where the partially triangulated polygon in Q is a proper subfigure of J , we already know $N(Q)$ and $L(Q)$. Then, we can determine $N(K)$ and $L(K)$ as follows.

Let $z_0, z_1, \dots, z_l$ be the vertices of the inner face of J adjacent to d in the order induced by the direction of d such that $d = (z_l, z_0)$. Consider the class C of colorings of the edges (z_i, z_{i+1}) , $i = 0, \dots, l-1$, with green and red such that no two incident edges are red.

Remark 2. $N(K)$ is equal to the maximum of $\sum_{c \text{ is green}} N(p(d, c)) + \sum_{c \text{ is red}} MP(p(d, c))$ over all colorings in the class C .

In order to determine $N(K)$ using the expression of Remark 2, we construct the weighted auxiliary directed acyclic graph (dag) Z as follows:

1. $V(Z) = \{z_0, \dots, z_l\} \times \{1, 2\} \times \{\text{green}, \text{red}\}$;
2. $((z_i, q, a), (z_j, r, b)) \in E(Z)$ iff either $j = i+1, r = q+1$ and $a = b$ or $j = i, r = q-1$ and $\text{green} \in \{a, b\}$;
3. the edges of the form $((z_i, 1, \text{green}), (z_{i+1}, 2, \text{green}))$ have weight $\lceil \frac{n_h}{n_g} \rceil - N(p(d, (z_i, z_{i+1})))$ whereas the edges of the form $((z_i, 1, \text{red}), (z_{i+1}, 2, \text{red}))$ have weight $\lceil \frac{n_h}{n_g} \rceil - MP(p(d, (z_i, z_{i+1})))$;
4. all the remaining edges have weight 0.

By Remark 2 and the definition of the graph Z , we obtain immediately:

Remark 3. $N(K)$ is equal to $l \lceil \frac{n_h}{n_g} \rceil$ decreased by the weight of a shortest path from $(z_0, 2, \text{green})$ to $(z_l, 1, \text{green})$ in the dag Z .

Given $N(p(d, (z_i, z_{i+1})))$ and $L(p(d, (z_i, z_{i+1})))$ for $i = 0, \dots, l-1$, the dag Z can be constructed in time $O(l)$ by Remark 1. Hence, combining Remark 3 with the linear-time algorithm for the single-source shortest-paths in directed acyclic graphs (see [7], p. 536), we obtain the following lemma.

Lemma 3.1 $N(K)$ can be determined on the basis of $N(p(d, (z_i, z_{i+1})))$ and $L(p(d, (z_i, z_{i+1})))$, $i = 0, \dots, l-1$, in time $O(l)$.

In order to determine $L(K)$, consider an arbitrary $M = (I, e) \in PE(G')$. Let $v_0, v_1, \dots, v_k$ be the vertices of the inner face f of I adjacent to e in the order induced by the direction of e such that $e = (v_k, v_0)$.

Remark 4. $S(I)$ can be imbedded in $S(J)$ such that v_0, v_k are respectively mapped onto z_0, z_l and the part of $S(J)$ unmapped in the imbedding still contains $N(K)$ vertex-disjoint subgraphs isomorphic to G if and only if there is a simple cycle $w_0, w_1, \dots, w_k$ in $S(J)$ such that

(*) $w_0 = z_0, w_k = z_l, \sum_{i=0}^{k-1} N(p(d, (w_i, w_{i+1}))) = N(K)$, and for $i = 0, 1, \dots, k-1$, if (v_i, v_{i+1}) is a diagonal edge of I then (w_i, w_{i+1}) is a diagonal edge of J and $p(e, (v_i, v_{i+1})) \in L(p(d, (w_i, w_{i+1})))$.

Note that the vertices of J occur on the outer boundary of J as the sequence $z_0, z_0 + 1 \bmod(n), \dots, z_l$, or as the sequence $z_0, z_0 - 1 \bmod(n), \dots, z_l$, in counter-clockwise order. Further, we may assume without loss of generality the former numbering and $z_0 < z_l$. See Fig. 2.

Figure 2: An example of the partially triangulated polygons I and J .

It follows from Fact 3 that $w_0, w_1, \dots, w_k$, where $w_0 = z_0, w_k = z_l$, is a simple cycle in $S(J)$ if and only if $w_0 - z_0, w_1 - z_0, \dots, w_k - z_0$ is an increasing sequence. By the above observation, to determine whether there exists a simple cycle in $S(J)$ satisfying (*), we may proceed as follows.

First, we construct a weighted dag D with the vertices $(z_0, 0), (z_l, k)$, and all vertices (z, j) , where z is any vertex of $S(J)$ different from z_0, z_l , at a distance $\leq k$ from $z_0, 1 \leq j < k$, and with edges specified as follows: there is an edge leaving (u, r) and entering (w, m) of weight $N(p(d, (u, w)))$ if and only if (u, w) is an edge of $S(J)$, $u < w, m = r + 1$ and if (v_r, v_{r+1}) is a diagonal edge of I , then (u, w) is a diagonal edge of J and $p(e, (v_r, v_{r+1})) \in L(p(d, (u, w)))$.

Lemma 3.2 *There is a one-to-one correspondence between directed simple paths of weight $N(K)$ from $(z_0, 0)$ to (z_l, k) in D and simple cycles C in $S(J)$ satisfying (*).*

Proof: Let C be the cycle in $S(J)$ induced by such a directed path of weight $N(K)$ in D . Clearly, C is of weight $N(K)$. The simplicity of C follows from the monotonicity of the path. The satisfiability of (*) by C immediately follows from the construction of D . Conversely, any simple cycle in $S(J)$ satisfying (*) easily induces a simple directed path of weight $N(K)$ from $(z_0, 0)$ to (z_k, k) in D , by the construction of D . $\square$

By Remark 4 and Lemma 3.2, it is sufficient to determine whether there exists a simple directed path of weight not less than $N(K)$ in D from $(z_0, 0)$ to (z_l, k) . Again, it can be done by employing the algorithm for the single-source shortest-paths in dags [7], in time linear in the size of D . The construction of D also can be done by k -phase breadth-first search in H , in time proportional to the size of D . Now, it is sufficient to observe that D has $O(n_h k)$ vertices and edges to obtain the following lemma.

Lemma 3.3 *We can determine whether $M = (I, e)$ is in $L(K)$ on the basis of $L((I', e'))$'s, where I' is a subfigure of I , in time $O(n_h k)$.*

Note that for the face f , $M = (I, e)$ is one of the at most $2k + 2$ members of $PE(G')$ where I includes f and the root edge e of M is adjacent to f . Let $size(f)$ mean the number of vertices adjacent to f . It follows that we can determine $L(K)$ in time $O(n_h \cdot \sum_{f \text{ is an inner face of } G'} (size(f))^2)$ which is $O(n_h(n_g)^2)$. By Euler's formula for planar graphs [3], H' has $O(n_h)$ edges. Thus, there are $O(n_h)$ pairs K in $PE(H')$. Hence, determining $L(K)$ for all $K \in PE(H')$ takes $O((n_h n_g)^2)$ time. Similarly, by Lemma 3.1, we conclude that $N(K)$ for all $K \in PE(H')$ can be determined in time $O(\sum_{f \text{ is an inner face of } H'} (size(f))^2)$ which is $O((n_h)^2)$. Summarizing, by Remark 1, we obtain our main result:

Theorem 3.4 *The problem of determining the maximum number of vertex-disjoint subgraphs of a biconnected outerplanar graph H on n_h vertices isomorphic to a biconnected outerplanar graph G on n_g vertices is solvable in time $O((n_h n_g)^2)$.*

4 Extensions

1) We can easily extend our algorithm for maximum packing for biconnected outerplanar graphs by replacing the concept of subgraph isomorphism with the more general concepts of topological imbedding [9] and subgraph homeomorphism [11].

A graph G can be *topologically imbedded* in a graph H if the latter contains a subgraph which after contracting some paths with internal vertices of degree two becomes isomorphic to G . A *topological G -packing* of a graph H is a set of vertex-disjoint subgraphs $H_1, H_2, \dots, H_l$ of H such that G can be topologically imbedded in each H_i . The problem of *maximum topological G -packing* in H is to determine the cardinality of maximum topological G -packing in H .

To solve the problem of maximum topological G -packing in H for biconnected outerplanar graphs G and H , we need only to slightly modify our algorithm for maximum G -packing in H . Firstly, we replace the notion of root-imbedding with that corresponding of *topological root-imbedding*. Secondly, we extend $PE(H')$ by the set of pairs (J, c) where c is a directed diagonal of an internal face of H' . Next, if v is a vertex of J of degree two adjacent to s (or t , respectively), different from s , t , and s (or t , respectively) has also degree two in J then let J_s (or J_t , respectively) denote the partially triangulated polygon obtained from J by deleting s (or t , respectively) with the incident edges and adding the diagonal edge (v, t) (or (s, v) , respectively). For such a v , we augment $L(J, (s, t))$ by the pairs in $L(J_s, (v, t))$ (or $L(J_t, (s, v))$, respectively), and can directly set $N(J, (s, t))$ to $N(J_s, (v, t))$ (or $N(J_t, (s, v))$, respectively).

We leave to the reader the details of these slight modifications.

The corresponding problem of *maximum homeomorphic G -packing* in H for biconnected outerplanar graphs G and H can be simply solved by contracting all maximal paths in G with internal vertices of degree two and then running the algorithm for maximum topological packing.

2) Our algorithm for maximum (or maximum homeomorphic, respectively) G -packing in H , where G, H are biconnected outerplanar graphs, can be also extended to produce the maximum number of vertex-disjoint subgraphs of H isomorphic (homeomorphic, respectively) to G by retracing appropriate shortest paths in top-down fashion.

3) The problem of maximum packing for biconnected outerplanar graphs and its extensions seem to admit NC algorithms like subgraph isomorphism for biconnected outerplanar graphs [16].

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