Digital Topologies Revisited: An Approach Based on the Topological Point-Neighbourhood*

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Abstract. Adopting the point-neighbourhood definition of topology, which we think may in some cases help acquire a very good insight of digital topologies, we unify the proof technique of the results on 4-connectedness and on 8-connectedness in $\mathbb{Z}^2$. We also show that there is no topology compatible with 6-connectedness. We shortly comment on potential further use of this approach.

Keywords: Image processing, digital topology, adjacency, path-connectedness and topological connectedness in $\mathbb{Z}^2$.

1 Introduction and Basic Definitions

The domain of a *digital picture* can be viewed as a subset of $\mathbb{Z}^2$, where $\mathbb{Z}$ stands for the set of integers, together with some adjacency neighbourhood structure [2, 7, 6] assigned to each point. Thus, for instance, we may talk on 8-adjacency neighbourhood structure (in shorthand, 8-structure) if each point $(x, y) \in \mathbb{Z}^2$ is given the adjacency neighbourhood

$$\begin{array}{ccccc} (x-1, y-1) & (x, y-1) & (x+1, y+1) \\ (x, y-1) & (x, y) & (x, y+1) \\ (x-1, y+1) & (x+1, y) & (x+1, y+1) \end{array}$$

In the sequel, let us assume as plausible that *the adjacency neighbourhood structure is homogeneous* (i.e., for each (x_0, y_0) , the natural translation $(x, y) \rightarrow (x + x_0, y + y_0)$ is an adjacency isomorphism) and *symmetric* with respect to the point (x, y) .

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We would like to demonstrate that the point-neighbourhood definition of topology adopted here provides us with a good method for deciding when (if) there is a topology compatible with adjacency. We will present different proofs of previously known results (see [2, 6, 7, 9]) and slightly extend them.

We also assume that the adjacency neighbourhood never exceeds the 8-neighbourhood, but note the result holds also for any larger neighbourhood.

In the following we give some elementary definitions. In *section 2* we present the different *adjacency neighbourhood structures*. *Section 3* contains the main results. Finally some conclusions are given in *section 4*.

1.1 Path-Connectedness in $\mathcal{S}$

Suppose that $\mathbb{Z}^2$ is given a (homogeneous and symmetric) neighbourhood structure. In order to refer to this structure, let us call it $\mathcal{S}$. Suppose that p, q are points of $\mathbb{Z}^2$. By an $\mathcal{S}$ -path from p to q we understand a finite sequence of points $p = p_1, p_2, \dots, p_n = q$ such that p_i is a neighbour of p_{i-1} ($1 < i \leq n$) in the structure $\mathcal{S}$. Let us call the points p, q $\mathcal{S}$ -related if there is an $\mathcal{S}$ -path from p to q .

Let X be a subset of $\mathbb{Z}^2$. Since the relation of being $\mathcal{S}$ -related, for a given $\mathcal{S}$, is obviously an equivalence relation on X . This relation gives rise to a partition of X into classes of $\mathcal{S}$ -related elements. Let us call each class of this equivalence an $\mathcal{S}$ -component.

A natural question arises ([8, 2], etc.) if (when) the $\mathcal{S}$ -components can be obtained as the components of a connectedness relation of a topology. Let us view basic notions we need for pursuing this question. Out of several possible definitions of (classical) topology, the definition involving the point-neighbourhood structure may best serve the purpose.

1.2 Topological Connectedness

Definition (Topological space): Let P be a set. Let us assign to each $x \in P$ a set $\mathcal{U}(x)$ of subsets of P which is subject to the following conditions:

- (i) if $U \in \mathcal{U}(x)$, then $x \in U$,
- (ii) if $U \in \mathcal{U}(x)$ and $U \subset V$, then $V \in \mathcal{U}(x)$,
- (iii) if $U, V \in \mathcal{U}(x)$, then $U \cap V \in \mathcal{U}(x)$,
- (iv) if $U \in \mathcal{U}(x)$, then there exists $V \in \mathcal{U}(x)$ such that, for each $y \in V$, $U \in \mathcal{U}(y)$.

The set P together with the assignment $\mathcal{U}(x)$, for each $x \in P$, is called a topological space. The sets $U \in \mathcal{U}(x)$ are called topological neighbourhoods of x . We denote the assignment $x \rightarrow \mathcal{U}(x)$ by t - the topology. So we can refer to the couple (P, t) , meaning the corresponding topological space.

It should be noted that this definition of topological space is equivalent to the “base-for-open-sets” definition or to the “closure” definition. This can be easily verified by a straightforward translation (see e.g. [1]). One should also observe that it is the axiom (iv) which is usually responsible for inconveniences when one looks for “suitable” topologies (this can be compensated by a possibility to apply the topological result back into the real model).

Let (P, t) be a topological space defined in the sense of previous definition (i. e. via point-neighbourhoods) and let X be a subset of P . We say that (X, t_1) is a *topological subspace* of (P, t) if, for each $x \in X$, the set $\mathcal{U}(x) \cap X$ is the set of all neighbourhoods of x in the topology t_1 . It can be seen easily that (X, t_1) is then a topological space in its own right. Moreover, if (P, t) is a topological space and Y, V are subsets of P with $Y \subset V$, then if $(Y, t_1), (V, t_2)$ are topological subspaces of (P, t) , then (Y, t_1) is a topological subspace of (V, t_2) .

Definition (Topological connectedness): Let (P, t) be a topological space. We say that (P, t) is *disconnected* if there are two disjoint sets R, S such that $R \cup S = P$ and, moreover, for each $r \in R$ and each $s \in S$ the set R is a neighbourhood of r and S is a neighbourhood of s . The space (P, t) is said to be *connected* if it is not disconnected. Finally, a subset X of (P, t) is said to be *connected* if the subspace (X, t_1) of (P, t) is connected.

Obviously, a connected topological space may have plenty of disconnected subspaces. For instance, the set $(-\infty, 0) \cup (0, +\infty)$ or the set $\{1, 2, 3, \dots\}$ are obviously disconnected subspaces of the space $\mathbb{R}$ of reals (resp. the topology induced by one of the equivalent n -norms $n = 1, 2, \infty$). Also, the set Q of all rational numbers is also disconnected – we can write $Q = Q_1 \cup Q_2$, where $Q_1 = \{q \in Q \mid q < \sqrt{2}\}$ and $Q_2 = \{q \in Q \mid q > \sqrt{2}\}$. The last example shows that the question on deciding about connectedness of a subspace may be sometimes nontrivial.

2 Neighbourhood Structures on $\mathbb{Z}^2$

Definition (Compatibility): Let S denote a given adjacency structure on $\mathbb{Z}^2$ and let t be a topology on $\mathbb{Z}^2$. We say that t is *compatible* with S if for each $X, X \subset \mathbb{Z}^2$, the set X is connected (with respect to t) if and only if X is S -connected.

Let us employ the following definition (see also [3]). Let (P, t) be a topological space. We say that (P, t) is *locally finite* if each point $x \in P$ possesses the finite neighbourhood $U \in \mathcal{U}(x)$, i.e. $|U| < \infty$. Obviously, if (P, t) is locally finite then *each point* $x \in P$ possesses a *smallest neighbourhood* in t and this neighbourhood is finite. This follows from the closedness of the neighbourhood under the formation of intersections. An important fact in our considerations is this: If $U(x)$ is the smallest (topological) neighbourhood of x , then $U(x)$ must also be a neighbourhood of all points in $U(x)$. This immediately follows from the point-neighbourhood definition of topology (the axiom (iv)).

We will now investigate all possible (homogeneous and symmetric) adjacency neighbourhood structures in $\mathbb{Z}^2$ (see the figure below) and formulate results on the compatible topologies. We will in turn take up the 0-adjacency, 2-adjacency, 4-adjacency, 6-adjacency and 8-adjacency.

- (i) 0-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the only point (x, y) is adjacent)
- (ii) 2-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x - 1, y)$ and $(x + 1, y)$ are adjacent)
- (iii) 4-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, $(x, y + 1)$ are adjacent)
- (iv) 6-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, $(x, y + 1)$, $(x - 1, y - 1)$, $(x + 1, y + 1)$ are adjacent)
- (v) 8-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, $(x, y + 1)$, $(x - 1, y - 1)$, $(x + 1, y + 1)$, $(x - 1, y + 1)$, $(x + 1, y - 1)$ are adjacent).

Remark: The n -adjacencies, n is odd, are not homogeneous and 2-point connections (1-adjacency) can be compared only with a trivial topology like the discrete topology.

3 Compatible Topologies on $\mathbb{Z}^2$

Theorem 1: *There is exactly one topology which is compatible with the 0-adjacency. This topology is the discrete topology on $\mathbb{Z}^2$.*

Proof: The discrete topology making each point $(x, y) \in \mathbb{Z}^2$ a neighbourhood of (x, y) is clearly compatible with 0-adjacency.

Theorem 2: *There are infinitely many topologies which are compatible with the 2-adjacency in $\mathbb{Z}^2$. Among those there are infinitely many, which are locally finite.*

Proof: For a fixed $x \in \mathbb{Z}$, we obviously have a topology, t_x , on the subspace $T_x = \{(x, y) \mid y \in \mathbb{Z}\}$ which is compatible with the 2-adjacency on the set T_x . Indeed, it suffices to take $\mathcal{U}(x) = \{U \subset T_x \mid \{(x, y)\} \subset U\}$ provided y is odd, $\mathcal{U}(x) = \{U \subset T_x \mid \{(x, y-1), (x, y), (x, y+1)\} \subset U\}$ provided y is even. Note that the above neighbourhoods contain two types as smallest and finite neighbourhood (i.e. the 1-dimensional Marcus-Wyse topology).

Since the roles of odd and even numbers are obviously interchangeable and since the adjacency connectedness (2-connectedness) of “different levels” do not affect each other, we can take, for each $x \in \mathbb{Z}$, one of the two topologies on T_x , obtaining a topology that is compatible with 2-adjacency. Since we have infinitely many combinations at our disposal, the result is proved.

Remark (not locally finite topologies): A question of separate purely topological curiosity may arise whether we can construct, for the 2-adjacency, a topology which is compatible with the adjacency and which is not locally finite. This seems to be possible – the standard ultrafilter construction can be applied in this case or more easily the Frechet-filter of the infinite intervals $\{x \in \mathbb{Z} \mid x \geq z\}$, $z \in \mathbb{Z}$. These topologies are, however, hardly relevant to digital pictures.

Let us now consider the 4-adjacency. The following result, which is due to [9] and [6] is in force. We will show how one obtains this result in the point-neighbourhood formalism. The little auxiliary results stated as Observations 3.1, 3.2 may be of certain value in their own right.

Theorem 3: *There are 2 topologies which are compatible with the 4-adjacency — the 2-dimensional Marcus-Wyse topologies τ [9]:*

$$U \in \tau \quad \equiv \quad \begin{cases} U(x, y) & : \text{ if } x + y \text{ is even (resp. odd),} \\ \{(x, y)\} & : \text{ else,} \end{cases}$$

with $U(x, y) = \{(x, y), (x, y-1), (x, y+1), (x+1, y), (x-1, y)\}$.

Proof: The result easily follows from the following two observations.

Observation 3.1: Each topology which is compatible with 4-adjacency is locally finite. Moreover, if $U(x, y)$ is the smallest neighbourhood of (x, y) in a topology compatible with 4-adjacency, then

$$U(x, y) \subseteq \{(x, y), (x, y-1), (x, y+1), (x+1, y), (x-1, y)\}.$$

Proof: Let $(x, y) \in \mathbb{Z}^2$ and let us show that the point (x, y) must have a finite neighbourhood.

Let $X_1 = \{(x, y)\}$ and $X_2 = \{(u, v) \in \mathbb{Z}^2 \mid |x - u| + |y - v| \geq 2\}$, see Fig. 1.

Fig. 1. The two sets $X_1 = \{\blacksquare\}$ and $X_2 = \{\bullet\}$

Then both X_1 and X_2 are 4-connected but $X_1 \cup X_2$ is not. It follows that there is a topological neighbourhood of (x, y) in any topology compatible with 4-adjacency which is disjoint with X_2 . In other words, there is a topological neighbourhood, $U(x, y)$, such that $U(x, y) \subseteq \mathbb{Z}^2 - X_2$. This is what we wanted to show.

Observation 3.2: Let $(x, y) \in \mathbb{Z}^2$ and let t be a topology compatible with 4-adjacency. Let $U(x, y)$ be the smallest neighbourhood of (x, y) in t . Then either $U(x, y) = \{(x, y)\}$ or $U(x, y) = \{(x, y), (x, y - 1), (x, y + 1), (x + 1, y), (x - 1, y)\}$.

Proof: By the previous observation, $U(x, y) \subseteq \{(x, y), (x, y - 1), (x, y + 1), (x + 1, y), (x - 1, y)\}$. Suppose that $U(x, y) \neq \{(x, y)\}$. Then, without a loss of generality, $(x - 1, y) \in U(x, y)$. Suppose now that $(x, y + 1) \notin U(x, y)$ (again, one argues analogously if there is another edge than $(x, y + 1)$ outside of $U(x, y)$). Since the set $\{(x - 1, y), (x, y + 1)\}$ is not 4-connected and the set $\{(x, y), (x, y + 1)\}$ is 4-connected, we infer that the smallest neighbourhood of $(x, y + 1)$, some set $U(x, y + 1)$, must not contain the point $(x - 1, y)$ and must contain the point (x, y) . Consequently considering other edges analogously,

$$U(x, y) \cap U(x, y + 1) = \{(x, y)\}.$$

But since $U(x, y + 1)$ is a topological neighbourhood, it must be also a topological neighbourhood of the point (x, y) . It follows that $U(x, y) \cap U(x, y + 1)$ must be a topological neighbourhood of the point (x, y) .

But $U(x, y) \cap U(x, y + 1) = \{(x, y)\}$ which is a contradiction. This completes the proof of Observation 3.2.

Let us return to the proof of Theorem 3. Let t be a topology which is compatible with the 4-adjacency. It is obvious that the singleton sets $\{(x, y)\}$ cannot

constitute the neighbourhoods for all $(x, y) \in \mathbb{Z}^2$ (we would obtain the discrete topology; the discrete topology is obviously not compatible with 4-adjacency). It is also obvious that the sets

$$\{(x, y), (x-1, y), (x+1, y), (x, y-1), (x, y+1)\}$$

cannot be the smallest neighbourhoods for all $(x, y) \in \mathbb{Z}^2$ (we would not have a topology at all). Thus, in every topology compatible with 4-adjacency we must have, for some points, both the singleton smallest neighbourhoods and the “star-like” neighbourhoods. Having found the necessary conditions for a compatible topology, the rest consists of an easy inductive argument already presented in [9].

Choose, for instance, the point $(0, 0)$. Then either the set $U_1(0, 0) = \{(0, 0), (-1, 0), (1, 0), (0, 1), (0, -1)\}$ or the set $U_2(0, 0) = \{(0, 0)\}$ must be the smallest neighbourhood of $(0, 0)$.

In the former case, the smallest neighbourhoods of the points $(-1, 0), (1, 0), (0, -1), (0, 1)$ must necessarily be the singleton sets, the smallest neighbourhoods of the points $(-1, -1), (-1, 1), (1, -1), (1, 1)$ must be the “star” sets, and so on. In the latter case, the smallest neighbourhoods of the points $(-1, 0), (1, 0), (0, -1), (0, 1)$ must be the “star” sets, the smallest neighbourhoods of the points $(-1, -1), (-1, 1), (1, -1), (1, 1)$ must be the singleton sets, and so on.

Consequently, there are only two topologies on $\mathbb{Z}^2$ which are compatible with the 4-adjacency – either the Marcus-Wyse topology or the topology obtained from it by the shift $(x, y) \rightarrow (x, y+1)$. The Marcus-Wyse topology allows for a simple description as also the present consideration shows: the smallest neighbourhood of (x, y) is a singleton set provided $x+y$ is even, and the smallest neighbourhood of (x, y) is a star set provided $x+y$ is odd.

The rest would consist in checking that, indeed, the Marcus-Wyse topology is compatible with 4-adjacency. This is not difficult and has been done in detail in [9]. The proof is complete.

Let us now consider the 6-adjacency (see the schema in the figure (iv)). In this case the search for compatible topology would be in vain.

Theorem 4: *There is no topology on $\mathbb{Z}^2$ which is compatible with the 6-adjacency.*

Proof: Suppose that t is a topology compatible with the 6-adjacency. By the very same reasoning we employed in Observations 3.1, 3.2 we can derive the following results:

- (i) The topology t is locally finite,
- (ii) If $(x, y) \in \mathbb{Z}^2$, then the smallest neighbourhood of (x, y) in t is either the singleton set $\{(x, y)\}$ or the whole 6-star set

$$\{(x, y), (x-1, y), (x+1, y), (x, y-1), (x, y+1), (x-1, y-1), (x+1, y+1)\}.$$

There must be a point in $\mathbb{Z}^2$ with the proper 6-adjacency neighbourhood. Let us denote it again by (x, y) . It follows that the points $(x + 1, y)$ and $(x + 1, y + 1)$ must have the singleton set neighbourhoods. This is absurd since the set $\{(x + 1, y), (x + 1, y + 1)\}$ is 6-connected. The proof is complete.

The following corollary to the previous result can be viewed as another proof of the result by Chassery [2] and L. Latecki [7].

Theorem 5: *There is no topology on $\mathbb{Z}^2$ which is compatible with the 8-adjacency.*

Proof: It is easily seen that if S_1 and S_2 are two adjacency relations on $\mathbb{Z}^2$ and S_2 is finer than S_1 , then the absence of a locally finite compatible topology for S_1 implies the absence of a locally finite compatible topology for S_2 . Since there is no topology compatible with 6-adjacency, there is no topology compatible with the 8-adjacency.

4 Conclusion

We have completed the tour over all possible “nice” adjacencies in $\mathbb{Z}^2$. Presumably the next step is testing the suitability of the point-neighbourhood approach is the examination of concrete (finite) configurations of points in $\mathbb{Z}^2$ and, of course, the digital topologies in $\mathbb{Z}^n$ for $n \geq 3$. We intend to pursue this elsewhere. It should be observed, however, that it does not seem possible to analyze $\mathbb{Z}^n$ with the help of viewing $\mathbb{Z}^n$ as a Cartesian product of $\mathbb{Z}^m$ for m smaller than n . This can be graphically seen even for $\mathbb{Z}^2$. Indeed, none of the adjacencies on $\mathbb{Z}^2$ with the exception of the discrete one is obtained as a “Cartesian product” of adjacencies on $\mathbb{Z}$. We may however obtain nontrivial (non-homogeneous) adjacencies this way or, in other words, homogeneity may be a too restricting property for digital topologies. In particular if the data stem from a projection of a higher dimensional space (e.g. 3-dim) onto a lower dimension (e.g. 2-dim). In the case of digital images, we may seek the topological properties of the 3-dim objects in the 2-dim image rather than establishing adjacency across object boundaries. Such occluding boundaries represent discontinuities of the image function and adjacent pixels may correspond to 3-dim points of different objects which are far apart in reality.

If, for instance, we take the 2-adjacency on $\mathbb{Z}$ and multiply it with each other, we obtain an adjacency on $\mathbb{Z}^2$ which is the 4-adjacency on the points of the type *(even, even)*, the 2-adjacency on the points of the type *(even, odd)* and *(odd, even)* – in the former case vertically and in the latter case horizontally, and the 1-adjacency on the points of the type *(odd, odd)*. Since the 2-adjacency on $\mathbb{Z}$ allows for a compatible topology, so does our “mixed” adjacency on $\mathbb{Z}^2$, which directly leads to the abstract cellular complexes of Kovalevsky [3, 4].

This survey is part of an ongoing research project with two primary goals to extend the results to *higher dimensions* and to *irregular adjacency neighbourhood structures* [5].

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