TOWARDS A HISTORY OF THE GEOMETRIC FOUNDATIONS OF MATHEMATICS

LATE XIXth CENTURY*

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RÉSUMÉ : Beaucoup de « géomètres » du XIX^e siècle – Bernhard Riemann, Hermann von Helmholtz, Felix Klein, Riccardo De Paolis, Mario Pieri, Henri Poincaré, Federigo Enriques, et autres – ont joué un rôle important dans la discussion sur les fondements des mathématiques. Mais, contrairement aux idées d'Euclide, ils n'ont pas identifié « l'espace physique » avec « l'espace de nos sens ». Partant de notre expérience dans l'espace, ils ont cherché à identifier les propriétés les plus importantes de l'espace et les ont posées à la base de la géométrie. C'est sur la connaissance active de l'espace que les axiomes de la géométrie ont été élaborés ; ils ne pouvaient donc pas être *a priori* comme ils le sont dans la philosophie kantienne. En outre, pendant la dernière décennie du siècle, certains mathématiciens italiens – De Paolis, Gino Fano, Pieri, et autres – ont fondé le concept de nombre sur la géométrie, en employant des résultats de la géométrie projective. Ainsi, on fondait l'arithmétique sur la géométrie et non l'inverse, comme David Hilbert a cherché à faire quelques années après, sans succès.

MOTS-CLÉS : Riemann, Helmholtz, fondements des mathématiques, géométrie, espace.

ABSTRACT : Many XIXth century « geometers » – such as Bernhard Riemann, Hermann von Helmholtz, Felix Klein, Riccardo De Paolis, Mario Pieri, Henri Poincaré, Federigo Enriques, and others – played an important role in the discussion about the foundations of mathematics. But in contrast to Euclid's ideas, they did not simply identify « physical space » with the « space of the senses ». On the basis of our experience in space, they intended to determine the main properties of space and put them at the very foundation of geometry. The axioms of geometry were hence based on active knowledge of space and were not *a priori*, as in the case according to kantian philosophy. Moreover, in the last decade of the century some Italian mathematicians – De Paolis, Gino Fano, Pieri, and others – founded the concept of number itself on geometry, by using results of projective geometry. Arithmetic was then founded on geometry and not reversely, as David Hilbert tried – without success – to do some years later.

KEYWORDS : Riemann, Helmholtz, foundations of mathematics, geometry, space.

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ZUSAMMENFASSUNG : Viele « Geometer » vom 19. Jahrhundert – Bernhard Riemann, Hermann von Helmholtz, Felix Klein, Riccardo De Paolis, Mario Pieri, Henri Poincaré, Federigo Enriques, und anderen – haben eine wichtige Rolle in der Debatte über die Grundlagen der Mathematik gespielt. Im Gegensatz zu den euklidischen Vorstellungen haben sie aber nicht den « physikalischen Raum » mit dem « Raum unserer Sinnen » identifiziert. Ausgehend von unseren Erfahrungen im Raum haben sie versucht, dessen wichtigsten Eigenschaften zu bestimmen um daraus die Grundlagen der Geometrie zu machen. Die Axiome der Geometrie waren also auf aktiver Kenntnis des Raumes gebaut und nicht a priori wie nach der kantianischen Philosophie. Ausserdem haben während des letzten Jahrzehnts des Jahrhunderts gewisse italienische Mathematiker – darunter De Paolis, Gino Fano und Pieri – unter Anwendung von Ergebnisse der projektiven Geometrie den Begriff selbst der Zahl auf der Geometrie gebaut. Sie bauten also die Arithmetik auf der Geometrie und nicht umgekehrt, wie es David Hilbert ohne Erfolg einige Jahre später versucht hat.

STICHWÖRTER : Riemann, Helmholtz, Grundlagen der Mathematik, Geometrie, Raum.

RIASSUNTO : Molti « geometri » del diciannovesimo secolo – Bernhard Riemann, Hermann von Helmholtz, Felix Klein, Riccardo De Paolis, Mario Pieri, Henri Poincaré, Federigo Enriques e altri ancora – hanno svolto un ruolo importante nella discussione sui fondamenti della matematica. Ma, contrariamente alle idee di Euclide, essi non hanno identificato lo « spazio fisico » con lo « spazio dei nostri sensi ». A partire dalla nostra esperienza nello spazio, essi hanno cercato di individuare le proprietà più importanti dello spazio e le hanno poste a fondamento della geometria. Gli assiomi della geometria risultavano così fondati sulla conoscenza attiva dello spazio e non erano a priori, come nella filosofia kantiana. Inoltre, durante l'ultima decade del secolo, alcuni matematici italiani (come De Paolis, Gino Fano, Pieri e altri) hanno fondato il concetto stesso di numero sulla geometria, impiegando risultati propri della geometria proiettiva. In tal modo, l'aritmetica si fondava sulla geometria e non viceversa, come David Hilbert avrebbe tentato di fare qualche anno più tardi ma senza successo.

PAROLE CHIAVE : Riemann, Helmholtz, fondamenti della matematica, geometria, spazio.

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This paper is not a complete *compendium* on the history of the foundations of mathematics during the late XIXth century. The author's aim is to show that in the XIXth century geometry played an important role in the discussion of the foundations of mathematics and that many mathematicians tried to found geometric axioms on our active organisation and experience of space.

Riemann's memoir on manifolds, Helmholtz's papers on rigid bodies, and the works of Klein and Sophus Lie on the theory of transformation groups can be regarded as elements of a great strategy for the foundations of mathematics, which aims at founding geometry on our perception and construction of space.

They did not simply identify « physical space » with the « space of the senses » ! From this identification Euclid wrongly extended our « medium sized » experience to any scale. According to Euclid, since similar figures exist by our experience, they also exist in any space – very small or very large. This property characterizes Euclidean geometry ; as a consequence, Euclidean geometry should be valid for any scale of space.

Many XIXth century mathematicians thought that geometry is an experimental science, but that it is not the simple image of the world. In other words, « physical space » and the « space of the senses » are not the same thing. They tried to realize which are the most relevant properties of space – the properties which *structure* our space – and posed them at the basis of geometry. Of course, their ideas were sometimes very different.

Some of them considered « measurement » as the most significant operation in space. This is possible when a « rigid meter » is preserved during any motion in space. In such a case we may measure, since we may compare lengths. Since spaces with constant curvature – positive, negative or zero – preserve rigid movements, as Riemann and Helmholtz pointed out, any of them can be considered like the space where we live and measure.

According to them, our active knowledge of space plays an essential role in the formulation of geometric axioms and foundations of geometry are based on our experience in space. But not only that. During the last decades of the century, some Italian mathematicians – such as De Paolis, Fano, Pieri, and others – succeeded in founding the theory of real numbers – and the concept of *number* itself – on geometry, by using some results of projective geometry. In such a way, arithmetic was based on geometry. Some years later, in his formalistic programme Hilbert aimed to do the opposite. Nevertheless Kurt Gödel proved that Hilbert's hope – to found geometry on arithmetic – had to remain just a hope.

RIEMANN'S « HYPOTHESES »

On June 10, 1854 Riemann held his *Habilitationsvortrag* on the principles of geometry¹ at the university of Göttingen. His lecture strongly contributed to differential geometry, topology, foundations of geometry, and to physics as well. Riemann tried to be as clear as possible, since many professors of philosophy with little mathematical knowledge attended his lecture. Therefore he did not use the typical language of professional mathematicians ; this choice made his arguments and results very difficult to understand.

Riemann thought that the difficulties regarding the foundations of geometry were mainly due to the following reasons : the confusion between topological concepts – depending on shape – and metrical concepts – depending on measure – and the fact that « the general concept of multiply extended quantities, which includes spatial quantities, remains completely unexplored² ». According to Riemann, the notion of « multiply extended quantity » must be deepened and posed as the foundation of geometry.

In the first part of Riemann's lecture a « multiply extended quantity » has the simple meaning of set of points, discrete or continuous, but in the following sections it assumes a geometric meaning and gets closer and closer to the modern definition of n -dimensional manifold. The spherical surface and the plane are examples of 2-dimensional manifolds ; generally speaking, a manifold is a smooth space, where a system of coordinates is defined in the neighbourhood of each point and all the systems of coordinates coincide on the overlapping regions.

Riemann posed a differential expression – the linear element or first fundamental form – as the basis of his theory, actually known as the metric tensor, which represents the distance between two infinitely close points :

$$(2.1) \, ds^2 = \sum_{ij} g_{ij} \, dx^i \, dx^j$$

where g_{ij} is a symmetrical and positively definite matrix, and $x^1 \dots x^n$ are coordinates on the manifold.

This hypothesis is not accidental. In fact, for a Euclidean 3-dimensional space the formula (2.1) becomes $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$, which is the usual distance, given by the theorem of Pythagoras, between the origin of a Cartesian coordinate system and the point (dx_1, dx_2, dx_3) infinitely close to the origin. Therefore, the linear element (2.1) extends the usual concept of distance between infinitely close points to an n -dimensional space, not necessarily *flat* such as the Euclidean space is.

A generalization of the Euclidean distance has already been defined by Carl Friedrich Gauss, who was Riemann's teacher at the university of Göttingen, in

1. RIEMANN, 1854.

2. RIEMANN, 1854, here 1975, p. 135.

his theory of curved surfaces³. The linear element (2.1) reduces to the Gaussian first fundamental form in the case that the manifold has 2 dimensions, that is to say if it is a surface.

In his memoir Riemann follows Gauss's fundamental steps, by extending Gaussian concepts and results for surfaces to n -dimensional manifolds, such as the measure of curvature and some properties of geodesic lines. Riemann's approach, as that of Gauss, is metric ; in fact, the concept of distance plays the fundamental role whether in the theory of curved surfaces or in Riemannian theory of manifolds ; moreover, the essential properties of manifolds are expressed by means of the linear element.

In his lecture Riemann refers to the « Privy Councillor Gauss », to the works of Euclid and Adrien-Marie Legendre, and to « some philosophical researches of Herbart ». As one can read in some notes found in his personal papers posthumously – the *Fragmente philosophischen Inhalts*, written about 1853⁴ – Riemann appreciated highly Johann Friedrich Herbart's psychology as contained in his *Promotionsschrift* and *Habilitationsschrift*. By applying Herbartian psychological theory Riemann answered the question⁵ : « When is our knowledge of the world true ? » as follows : « When connections between our [mental] representations correspond to connections of things », and « connections of things » are deduced from « connections of phenomena ». This approach is typical of all Riemann's work and leads him to investigate links between different laws of nature, knowledge of which is based on the exactness of our description of phenomena in infinitesimal regions. As Hermann Weyl wrote in *Space, time, matter*, « the principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry⁶ ». And his deep conviction is in line with Riemann's ideas : to found the concept of « multiply extended quantity » (n -dimensional manifold) on the notion of distance between infinitely close points. This point of view is the core of Riemann's metric approach to the study of geometry and its foundations.

As Riemann notices in his lecture, « the basis for the metric relations must be sought outside it [*i. e.* space], in binding forces acting upon it⁷ », in its own structure. In his *Fragmente philosophischen Inhalts*⁸, Riemann elaborated a model of the ether by analysing interactions between two infinitely close particles ; according to his model, the ether was able to propagate magnetic and

3. GAUSS, 1827.

4. RIEMANN, 1876.

5. The references are in RIEMANN, 1876, here 1990, p. 523. Our translation.

6. WEYL, 1952, p. 92.

7. RIEMANN, 1854, here 1975, p. 152.

8. RIEMANN, 1876.

electromagnetic forces, light, heat and gravity. Therefore, local properties of infinitesimal regions of space force the global geometry of space⁹.

The definition of a manifold by means of its linear element (2.1) can also be interpreted as independent of our perception of space. In some unpublished notes concerning his lecture, Riemann claimed : « The concept of an n -dimensional manifold exists independently of our intuition of space. Space, surface, line are only intuitive examples of manifold with 1, 2, or 3 dimensions respectively. *Without any spatial intuition we could build all geometry*¹⁰. » From the concept of temperature, Riemann added, the idea of *continuum* can be gained, by considering all possible temperatures between $-\infty$ and $+\infty$. One may then look at a linear equation $ax + by + c = 0$, and consider all the cases when x and y satisfy this equation. Then one obtains a straight line and « from this definition of the straight line one can deduce all the geometric properties of straight lines. It is evident that it is possible to go on this way, without any appeals to spatial intuition¹¹ ». Therefore, all the Euclidean axioms appealing to intuition are not needed any more, such as the axiom that only one straight line passes through two given points. Riemann noticed : « However, even if this treatment of geometry is very interesting, the pursuit of this approach would be extremely unfruitful, since no new principle could be found : that which is simple and clear in the representation of space would become complicated and difficult¹². » So analysis and intuition of space are essential in order to develop a geometric research really « fruitful » and meaningful. In the « Plan of the investigation » at the beginning of his lecture, Riemann wrote : « As is well known, geometry presupposes as something given both the concept of space and the basic principles for constructions in space¹³ », and hence the relations between points in space.

In his lecture Riemann posed another essential « assumption » based on our perception of space, particularly on the fact that our body is an instrument of measure. Riemann writes :

« Measurement involves the superposition of the quantities to be compared ; it therefore requires a means of transporting one quantity to be used as a standard for the others. Otherwise, one can compare two quantities only when one is part of the other, and then only as to *more* or *less*, not as *how much*¹⁴. »

9. BOTTAZZINI and TAZZIOLI, 1995.

10. RIEMANN, 1994, p. 118. Our emphasis. Our translation.

11. RIEMANN, 1994, p. 118. Our translation.

12. RIEMANN, 1994, p. 119. Our translation.

13. RIEMANN, 1854, here 1975, p. 135.

14. RIEMANN, 1854, here 1975, p. 137.

In order to help intuition, one can image a 3-dimensional space where the quantity to be measured is placed. Measurement is then possible if one considers a segment with standard length (for example 1 metre long) and superposes it on the quantity ; then the dimensions of the quantity are evaluated with respect to the unit of length. This procedure is allowed only on the assumption that « lines have a length independent of their configuration, so that every line can be measured by every other¹⁵ ». In fact, if this hypothesis is not valid, the unit of length will become shorter or longer during its motion – that is what happens over an *egg*. In that case, measurement of quantities in space would not be possible ; that is to say, a « rigid body » – a body in which distances between its points are constant – would change its shape during the motion.

Therefore such a space can not be homogeneous, Bertrand Russell significantly remarked in his book *An essay on the foundations of geometry*¹⁶, and added that space would not be passive, but it would exert a well defined effect on things. Therefore, since the unit of length must be preserved, then space has necessary constant curvature – positive, negative or zero. Plane (curvature equal to zero) and spherical surface (positive constant curvature) are examples of 2-dimensional manifolds. Each manifold of constant curvature is such that any portion of it can be continuously superposed anywhere on the manifold. In his memoir, Riemann considers only manifolds with constant curvature, and compares their measure of curvature to the Gaussian definition of curvature given for curved surfaces.

The definition of the linear element (2.1) is very general and includes also manifolds whose curvature is variable from one point to another. Such manifolds are the most interesting ones from the theoretical point of view and constitute the main subject for modern differential geometry. Riemann studied manifolds with variable curvature in a memoir, published only posthumously, which answered a question posed by the Academy of Sciences in Paris about the distribution of heat in a body¹⁷. Riemann faced the following problem : « *Trouver quel doit être l'état calorifique d'un corps solide homogène indéfini pour qu'un système de couches isothermes, à un instant donné, restent isothermes après un temps quelconque, de telle sorte que la température d'un point puisse s'exprimer en fonction du temps et de deux autres variables indépendantes*¹⁸. » In the second part of the work, he shows that the original question can be reduced to establishing the necessary and sufficient conditions that allow the two manifolds to transform continuously one into the other. Then he was led to introduce the curvature tensor of a manifold, as it is actually called. Riemann proved that the problem proposed by the Academy of

15. RIEMANN, 1854, here 1975, p. 140.

16. RUSSELL, 1897.

17. RIEMANN, 1861.

18. RIEMANN, 1990, p. 391.

Sciences has a solution if and only if the curvature tensor is the same for the two manifolds ; such a statement is the generalization of the Gaussian *therorema egregium*, which is valid for curved surfaces, to the case of manifolds.

But why does Riemann consider only manifolds with constant curvature in his lecture, if he knows the mathematical tools necessary for treating manifolds with any curvature ? Riemann could indeed elaborate a systematic theory including manifolds with variable curvature as well by introducing new and fruitful concepts, such as the tensor of curvature. « The restriction to the case of constant curvature » – Jeremy Gray remarked as a comment to a Riemann's memoir – « is artificial for a mathematician¹⁹ », who always tries to extend theories from particular to general cases. In my opinion, the reasons of Riemann's attitude are in his purpose : he indeed intends to contribute to foundations of geometry more than to theoretic geometry, as he claims in his lecture on many occasions. Riemann's aim is to pose the *real* « hypotheses » as foundation of geometry. « Thus arises the problem of seeking out the simplest data from which the metric relations of space can be determined²⁰. » One of these « simplest data » concerns the measurement of quantities in space ; as a consequence, space must have constant curvature.

The last part of his memoir is entitled « Applications to space » and develops a deep analysis of Euclidean space, which Riemann characterizes as a 3-dimensional flat space (with curvature equal to zero). The results and theorems of Euclidean geometry can be deduced from these hypotheses. But other geometries – the geometries on manifolds with positive or negative constant curvature – can be included as well, since they satisfy Riemann's assumptions of rigid motion. Incidentally, they are the non-Euclidean geometries which Riemann never mentioned in his lecture : the hyperbolic geometry (also called Lobachevskij-Bolyai's geometry), the elliptic geometry, and the spherical geometry (or Riemannian geometry). Non-Euclidean and Euclidean geometries are then deduced as particular cases from Riemann's theory. Therefore, after the publication of Riemann's lecture non-Euclidean geometries gained a relevant theoretic status, since they belonged to differential geometry, which was considered an important and fruitful theory by nineteenth-century mathematicians. That was important for the diffusion of non-Euclidean geometries in the academic world.

Let us consider our space, the space where we live : which kind of manifold is it ? From Riemann's hypotheses it has constant curvature. But is its curvature positive, negative or zero ? In particular, is space Euclidean or not ? Riemann tries to answer this crucial question by appealing to physics : « If one

19. GRAY, 1992, p. 38.

20. RIEMANN, 1854, here 1975, p. 135.

assumes that bodies exist independently of position, then the curvature is everywhere constant, and it then follows from astronomical measurements that it cannot be different from zero²¹. » As a consequence, space is Euclidean. However, if the region of space in « the range of our telescopes » can be neglected in comparison with all space, the previous statement can not be considered as definitive. And what about infinitely small areas of space ? Riemann writes : « It is therefore quite definitely conceivable that the metric relations of space in the infinitely small do not conform to the hypotheses of [Euclidean] geometry²². » Other geometries could be valid in different regions of space, which are infinitely small or infinitely large in comparison with those usually measured by our instruments. According to his pluralistic approach, Riemann would leave Euclidean geometry if another geometry helped him more in understanding phenomena.

Riemann's ideas are very similar to those expressed by Nicolai Lobachevskij, the founder together with Janos Bolyai of non-Euclidean geometry, in conclusion of his astronomical calculations on the triangle Earth-Sun-Sirius in 1835-1838²³. Euclidean geometry is certainly a good approximation of physical space, such as it is perceived by our senses ; however, Lobachevskij noticed that Euclidean geometry could be false even in the perceptible world. Lobachevskij also showed that in hyperbolic geometry the angular defect of a triangle, which is equal to the difference between 2π and the sum of its angles, is proportional to the surface of the triangle. Such a theorem was rediscovered by Gauss²⁴ in a more general formulation, for any curved space, by using methods of differential geometry.

According to Lobachevskij, Euclidean geometry could give a false description of space ; moreover, the *real* geometry should be found from empirical data. This idea, shared by Riemann too, is strongly in contrast with Kantian philosophy. As Kant wrote in his *Transcendental aesthetics*, space is not an empirical concept, deduced by external experiences but a representation, which is necessary *a priori*. In fact, if the representation of space was a concept *a posteriori*, deduced by external experiences, the first principles of mathematics would be accidental and it would not be necessary that only one straight line passes through two points, but experience would show it every-time.

If it is true that Riemann is anti-Kantian²⁵, it is also evident that he is not a simple empiricist. His « hypothesis » (2.1) is valid for infinitely small regions ; so it is not empirical in character, since it is impossible to test it experimentally

21. RIEMANN, 1854, here 1975, p. 151.

22. RIEMANN, 1854, here 1975, p. 152.

23. LOBACHEVSKIJ, 1835-1838.

24. GAUSS, 1827.

25. NOWAK, 1989.

in physical space. Erhard Scholz²⁶ defines Riemann's epistemological attitude as « dialectic » : in his elaboration of concepts and hypotheses Riemann indeed develops arguments either theoretically or empirically, and integrates them one with the other in order to give the correct conceptual ground for a real geometry of space.

Moreover, Riemann's attitude can be defined neo-Kantian in his approach to the foundations of geometry : in fact, if it is true that axioms of geometry are not *a priori*, however, the spatial structure which Riemann posed at the basis of his geometric theory by means of suitable « hypotheses » is not arbitrary and is generated from our organized experiences in space. An essential innovation in Riemann's thought, as Luciano Boi remarks, is the intrinsic conception of space, which :

« [...] consiste plutôt à comprendre les propriétés qui peuvent caractériser la structure propre à l'espace, et à les conceptualiser mathématiquement. [...] Ajoutons que la géométrie n'est plus conçue par Riemann comme la science des figures qui peuvent être construites dans l'espace, mais comme la science des différentes formes spatiales – le concept de variété exprime en fait ce caractère qualitatif de forme spatiale, et une variété est un exemple parmi d'autres de "forme spatiale" – mathématiquement concevables, et en même temps, comme la science de l'espace lui-même, c'est-à-dire de l'espace physique²⁷. »

Such a conception of space will be taken over and enriched by Edmund Husserl and Hermann Weyl, who identify regularities of space in the properties – isotropy, homogeneity, symmetry – which are posed as foundation of geometry. However, « *ces régularités, c'est nous qui choisissons de les voir* ». To identify such regularities as foundational characteristics of geometry is the result of an activity « *de notre communauté communicante* », which aims to make understandable the world where we live ; an activity which can not forget neither our perception of space nor our experience in it²⁸.

HELMHOLTZ'S « FACTS » AND TRANSFORMATION GROUPS

Hermann von Helmholtz, who was a mathematician and a physicist as well as a physician, stated ideas about geometry and its foundations in contrast with Kantian philosophy and more in accordance with Riemann's philosophy of space. His research on physiology led Helmholtz to claim that our perception

26. SCHOLZ, 1992.

27. BOI, 1992, p. 79-80.

28. BAILLY and LONGO, 2003a and 2003b.

of space has a fundamental role in the formulation of geometric axioms. « My research on the spatial intuition of visual space allowed me to investigate the origin and essence of our spatial intuitions²⁹ », Helmholtz remarked. In another work published in 1879, Helmholtz stressed the relevance of our intuition for the foundations of geometry³⁰. Here he criticized Kant's argument, according to which spatial intuition had to contain « before each possible experience » the principles expressed by the axioms of geometry. This point of view was claimed in many popular lectures, which Helmholtz held from the 70's on.

In one of them, Helmholtz³¹ shows the non-intuitive character of spatial conceptions, by imaging that on a sphere there are living beings with our logic abilities ; however, they would formulate a system of geometric axioms very different from the axioms, which could be stated by the same living beings moving on a plane or by ourselves, who are living in a 3-dimensional Euclidean space. In some subsequent writings Helmholtz is less critical towards Kantian ideas and tries to put in evidence that from the sentence « space is not a pure form of intuition », one does not necessarily deduce the statement : « Euclidean geometry is true *a priori*³². »

The empirical approach to the study of geometry and its foundations, usually called geometric empiricism, characterizes Helmholtz's celebrated paper « Über die Thatsachen, die der Geometrie zum Grunde liegen », published in the same year as Riemann's lecture³³. In his paper Helmholtz tried to prove rigorously some theorems which had been only stated in a previous paper without any proof³⁴. Helmholtz aimed to establish « which statements in geometry express factual truth and which ones are definitions or consequences of definitions³⁵ » with no empirical bases ; finally, he purposed to deduce the real foundations of geometry.

Helmholtz received the text of Riemann's lecture – not yet published – only in 1868, some months before the publication of his own paper on the foundation of geometry. Anyway, Helmholtz could directly refer to Riemann's ideas and wrote that he substantially made the same way as Riemann did in his *Habilitationsvortrag*. In reality, their points of view are very different. Riemann started from the fundamental « hypotheses » that the distance between two infinitely close points is expressed by the linear element (2.1), and that bodies can move preserving their shape. These assumptions led him to

29. HELMHOLTZ, 1868, here 1882-1895, t. II, p. 618. Our translation.

30. HELMHOLTZ, 1879.

31. HELMHOLTZ, 1870.

32. BOI, 1996 ; KRAUSE, 1878 ; RIEHL, 1904.

33. HELMHOLTZ, 1868.

34. HELMHOLTZ, 1866.

35. HELMHOLTZ, 1868, here 1882-1895, t. II, p. 618. Our translation.

consider only the manifolds with constant curvature. On the contrary Helmholtz just observes the « facts » in our physical world ; here « rigid bodies can move unlimitedly [...] without changing their shape³⁶ ». Helmholtz's axioms take in account not only spatial relations, but also the « mechanic behaviour » of a rigid body during its motion. They are the following :

(1) Space has n dimensions.

(2) A system of points moving in space exists – the « rigid body » – such that the $2n$ coordinates of each couple of points are linked by an equation independent of its motion – the distance. Such an equation must be the same for all the couples of points, which can be superposed by a movement in space.

(3) Rigid bodies can move freely in space, that is to say, their points can be placed anywhere in space ; moreover, $\frac{n(n+1)}{2}$ conditions are generally requested in order to establish the position of a rigid body.

(4) If a rigid body turns about $(n-1)$ of its points, then after a complete rotation of 2π the body takes the initial position again (axiom of monodromy).

From these « facts » Helmholtz aimed to deduce the Riemann's original hypothesis on the linear element ; in particular, he showed that the distance between two infinitely close points – expressed by (2.1) – does not change for any movements of the body in space. Riemann's assumption appeared to Helmholtz both essential and arbitrary, since it lies at the basis of geometry and is too far from the empirical spirit of his research. From his « factual » axioms Helmholtz tried to deduce the validity of Riemann's assumption ; such a success would have shown that his theory was relevant and fruitful.

Helmholtz's paper, where only a space with three dimensions is considered, is very long and hard to read. His mathematical procedures and methods are indeed too weak and unsuitable to the study of « rigid displacements » in space. From his axioms Helmholtz proved the existence of an expression, homogeneous of the second degree in its differentials – Riemann's linear element – which « is preserved by any movement of two infinitely close and rigidly related points³⁷ ». Although his axioms (1)-(4) are claimed for finite regions and bodies, they are assumed valid in infinitesimal regions as well, without any words of explication, as Lie would have remarked. Helmholtz then proved a result valid in infinitesimal regions of space, whose consequence allowed him to « gain the starting point of Riemann's research³⁸ ». However, his deduction contains a mistake, as Lie noticed some years later³⁹.

36. HELMHOLTZ, 1868, here 1882-1895, t. II, p. 621. Our translation.

37. HELMHOLTZ, 1868, here 1882-1895, t. II, p. 637. Our translation.

38. HELMHOLTZ, 1868, here 1882-1895, t. II, p. 637. Our translation.

39. LIE, 1892a.

Lie, together with Klein, were going to create the theory *ad hoc* which make Helmholtz's formalism and calculations simpler and more understandable : the theory of transformation groups. The fundamental principles and aims of this theory were expounded in the lecture which Klein held at the beginning of the academic year 1872-1873 when he was named full professor at the university of Erlangen – it is known as *Erlanger Antrittsrede*⁴⁰. In order to apply group theory to geometry, Klein and Lie started from the empirical observation that in space there are transformations preserving the geometric properties of bodies ; for example, they preserve the properties which are independent of the position of the body. The main role is played by the transformation groups of space, which Klein introduced in a non-rigorous way : he simply requested that by composing any transformations of space, another transformation is obtained. The following conditions must be added in order to obtain a *real* transformation group :

- a) the identity transformation I exists – it is such that $I \cdot T = T \cdot I = T$ for any transformation T ;
- b) for any transformation T the inverse transformation S exists – it is such that $S \cdot T = T \cdot S = I$.

Relevant examples of transformation groups of space are : all motions of space (if a motion is a transformation acting on all space) and all rotations about a fixed point, which is a group contained in the former one. According to Klein⁴¹, this new point of view is very fruitful in geometry, since any geometric problem can be reduced to the following general question : if a manifold and a transformation group of it are given, develop the invariant theory with respect to the group.

These ideas can be successfully applied to Helmholtz's theory developed in the paper on the « factual » principles of geometry⁴². A rigid body is indeed submitted to different transformation groups – translations, rotations about a fixed point or about an axis – which preserve its essential properties. Lie reformulated Helmholtz's axioms by employing transformation groups and, in the same time, gave a clear mechanical interpretation of Helmholtz's arguments by characterizing « rigid motions » of space. In a paper entitled « Bemerkungen zu V. Helmholtz's Arbeit. Über die Thatsachen, die der Geometrie zum Grunde liegen »⁴³, Lie assumed that in a 3-dimensional space each couple of points P, Q determine an invariant of motion : their distance $d(P, Q)$. Moreover he posed the following axiom of « free mobility » : if the point P is fixed, then Q can be placed in any position X of space satisfying the condition : $d(P, X) = d(P, Q)$ – X belongs to a spherical surface with centre P and radius

40. KLEIN, 1872.

41. KLEIN, 1872.

42. HELMHOLTZ, 1868.

43. LIE, 1886.

$d(P, Q)$; if two points P, R are fixed, then Q can be placed in any position X such that : $d(P, X) = d(P, Q)$, $d(R, X) = d(R, Q)$ – X belongs to a circumference ; if three points are fixed, then all points of space are still as well. Lie was able to notice a « gap » in Helmholtz's paper, which however had no consequence on the final result. Such a gap was explained by Lie in terms of transformation groups : Helmholtz identified a finite transformation groups with the correspondent group of infinitesimal transformations. This statement is generally false, as Lie himself proved in some subsequent papers⁴⁴ by showing counterexamples. Lie pointed out : « My remarks conclusively show that Helmholtz's research contains irreparable mistakes, even if it is very interesting⁴⁵. » And in his lectures on the theory of transformation groups⁴⁶, he added that Helmholtz « did not prove » that his axioms can really characterize all rigid movements of space.

Lie's attitude is maybe too hard towards Helmholtz, who surely erred but was the first to attribute to the « rigid movement » the main role in geometry. This point of view on the foundations of geometry is shared by Klein and Lie himself. In his papers « Über die Grundlagen der Geometrie⁴⁷ », Lie characterizes the groups of rigid movements by using the factual axioms posed by Helmholtz, which he reformulates in the language of his theory of transformation groups. He proves rigorously that Helmholtz's axioms characterize Euclidean and non-Euclidean movements in space ; in other words, he finds that rigid motions can only take place in a space with constant curvature (zero, negative, or positive). In terms of transformation groups, the group of isometries (an isometry is a transformation preserving the distance between points) is the most extensive group which preserves the property of free mobility. The group of isometries selects spaces with constant curvature as the natural spaces, where the properties of « rigidity » are preserved.

In 1882 Lie visited Paris, met Poincaré, and discussed with him about the main ideas expounded in the *Programme of Erlangen*. Since 1880 Poincaré had been interested in transformation groups and related them to his studies on the theory of automorphic functions ; from 1883 to 1892 he published many works containing applications of the theory of transformation groups to functions of a complex variable, differential equations, and algebraic geometry.

In 1887 Poincaré published the paper « Les hypothèses fondamentales de la géométrie⁴⁸ », where his aim was to « énoncer toutes les hypothèses nécessaires [de la géométrie] et n'énoncer que celles-là ». And he added : « Je crois que ce problème n'est pas encore résolu et je cherche à contribuer à sa

44. LIE, 1892a, here 1927-1934, t. II, part. I, p. 478-479 ; LIE, 1893, p. 454-456.

45. LIE, 1892a, here 1927-1934, t. II, part. I, p. 479. Our translation.

46. LIE, 1893, p. 460.

47. LIE, 1892a and 1892b.

48. POINCARÉ, 1887.

*solution*⁴⁹. » By employing the theory of transformation groups and some results of projective geometry, Poincaré found the lowest number of independent hypotheses, which are at the basis of Euclidean and non-Euclidean geometries over a plane :

- (1) The plane has 2-dimensions.
- (2) The position of a figure depends on 3 conditions.
- (3) A figure can not move if two of its points are fixed.
- (4) The distance between two different points can not be zero.

The geometry deduced from the axioms (1), (2), (3) is coherent and absolutely new, Poincaré remarked ; however, some consequences of this geometry « *sont trop contraires aux habitudes de notre esprit*⁵⁰ » and that is the reason why mathematicians have never considered it.

If the following axiom is posed in addition to the previous ones (1)-(4) :

- (5) The sum of the angles of any triangle is constant,

then only Euclidean geometry is selected.

The latter axiom has no empirical origin ; it is equivalent to the request that « similar figures exist, which are not equal ». All the results and theorems of Euclidean geometry follow from this statement, which can be judged more intuitive than the axiom (5) or more in accordance with what happens in space. However, as Lobachevskij and Riemann had already noticed, we are able to check only bounded spatial regions (and figures), while we can not say whether the criteria for similitude are valid – or not – for infinitesimal or infinitely large figures.

Therefore, if « *faire de la géométrie, c'est étudier les propriétés de nos instruments, c'est-à-dire du corps solide*⁵¹ », the allowable spaces are all manifolds with constant curvature, where rigid movement is possible. And in order to deduce the Euclidean geometry alone, another axiom is necessary which can not be checked generally by experience.

But which is the nature of geometric axioms ? On many occasions, Poincaré expressed his opinion on this point : he was contrary to Kantian judgments *a priori* (« [les jugements synthétiques] s'imposeraient alors à nous avec une telle force, que nous ne pourrions concevoir la proposition contraire [...]. Il n'y aurait pas la géométrie non euclidienne⁵² »), but he could not be in favour of empiricism (« *Si la géométrie était une science exacte, elle serait soumise à une continuelle révision*⁵³. »).

According to Poincaré, geometric axioms are conventions. But one must be careful, since not all conventions can be posed as foundations of geometry :

49. POINCARÉ, 1887, here 1916-1956, t. XI, p. 80.

50. POINCARÉ, 1887, here 1916-1956, t. XI, p. 82.

51. POINCARÉ, 1913, p. 40.

52. POINCARÉ, 1925, p. 65.

53. POINCARÉ, 1925, p. 66.

« [...] *notre choix, parmi toutes les conventions possibles, est guidé par des faits expérimentaux*⁵⁴. » Then experience has the fundamental role and shows us which are the most suitable conventions to be assumed⁵⁵. Therefore, it makes no sense to speak of truth or falsity of Euclidean geometry. It is simply the most « convenient » geometry for the representation of our physical world for two fundamental reasons : it is the simplest geometry and is in accordance with the experimental facts.

MANIFOLDS WITH CONSTANT CURVATURE AS REAL SPACES

At the beginning of the XIXth century, mathematicians thought that geometry describes the physical world and considered it as the first chapter of physics. It was a common opinion that Euclidean geometry is the real geometry of space ; that was another reason why non-Euclidean geometry was completely accepted by the academic world only in the last decades of the century.

In the second half of the XIXth century, non-Euclidean geometry was included in other geometric theories, which were considered relevant and fruitful, such as differential geometry, theory of transformation groups, and projective geometry. Moreover, in his « *Saggio di interpretazione della geometria non euclidea*⁵⁶ », Eugenio Beltrami proved the coherence of Lobachevskij-Bolyai geometry, by assuming the coherence of Euclidean geometry. In spite of this success, mathematicians seldom devoted themselves to studying non-Euclidean geometry. In their opinion, in fact, non-Euclidean geometry could not be applied to our physical space, whose geometric description seemed in total accordance with Euclidean geometry. Moreover, they did not know any helpful application of non-Euclidean geometry to the solutions of relevant mathematical problems. As it is shown below, however, not all mathematicians shared these ideas.

Anyway, non-Euclidean geometry posed new epistemological questions : What do geometric theories speak about ? Space where we live is one, and then just a single geometry can be regarded as its natural geometry ? Therefore, what do other geometries apply to ? Which is their deep meaning ?

In the third note to his *Programme of Erlangen*, Klein noticed the relevance of spatial intuition in the different phases of the geometric thought – from the axioms posed at the basis of geometry to geometric deduction. According to Klein, intuition is a fundamental element of geometry, and geometry is a perceptible form of abstract considerations starting from postulates of the spatial

54. POINCARÉ, 1925, p. 66.

55. NABONNAND, 2000a, 2000b.

56. BELTRAMI, 1868a.

intuition. Klein clearly explained his ideas about the choice and origin of geometric axioms in some subsequent works⁵⁷. Klein remarked that axioms appeal to our intuition, which is « something *essentially* inexact ». An axiom constitutes « the request » by means of which « exact statements » are established in « inexact intuition ». In this sense, axioms are a theoretic substratum for intuitive considerations and show a starting choice characterized by the priority of spatial intuition.

By requesting free mobility of rigid bodies, Riemann and Helmholtz had developed their geometry from the metric properties of space. Other properties of space can also be posed as foundations of geometry – for example optical properties. This is Klein's point of view, which led him to pose « projective intuition » at the basis of his geometry. According to Klein, projective geometry is the pre-eminent geometry, since it includes metric geometry ; he indeed spent many years of his life to study projective geometry and formulated a correct system of axioms for it.

In order to create different systems of axioms and, as a consequence, different geometries, mathematicians often distinguished the space of our experience from the conceptual space, where any coherent geometry can be developed, without any care about its truth or falsity. This point of view was shared by many mathematicians and philosophers of mathematics. Husserl⁵⁸ expresses it very clearly, when he emphasizes that geometric space can be built conceptually by means of « ideality », that is to say by employing mental processes from particular to general cases ; these processes are based on experience, but then overcome it and arrive at conceiving geometric space – a space only « thinkable », as Husserl⁵⁹ wrote – as a space where any coherent mathematical theory can be developed. On the contrary, intuitive space is based on our perception of space, which is common to all people. According to Husserl, the idealization processes do not occur *on* the intuitive world but *in* it ; however, the representation of the geometric space is « only conceptual, and can not be transformed into the corresponding intuition ». The intuitive and geometric space are then completely distinguished, even if the geometric space is deeply rooted in intuition.

In an analogous way, Poincaré distinguishes the representative or sensible space, which is the description of our representations and of our sensations, from the geometric space, as Poincaré called it. Poincaré wondered : « *Quelles sont d'abord les propriétés de l'espace proprement dit ? je veux dire de celui qui fait l'objet de la géométrie et que j'appellerai l'espace géométrique*⁶⁰. »

57. BOTTAZZINI, 2000 and 2002 ; AVELLONE, BRIGAGLIA and ZAPPULLA, 2002.

58. HUSSERL, 1962.

59. HUSSERL, 1983.

60. POINCARÉ, 1925, p. 69.

According to Poincaré, the geometric space is a physical space with « idealized » properties, which extend the properties of the real space ; the geometric space is then continuous, unbounded, homogeneous and isotropic. Since space itself is « amorphous » (« *il est une forme flasque, sans rigidité, qui peut s'appliquer à tout*⁶¹ »), it has no particular property. Properties are then attributed to space by our active knowledge and practice in space – we can consider metric relations (metric geometry), or visual properties (projective geometry), or the shape of objects (topology) as the invariants of space. As Poincaré noticed, geometry is the study of properties attributed to instruments of measurement. But instruments are not perfect ; so geometry is the study of properties, which instruments of measurement would have, if they were perfect. He wrote :

« La géométrie est l'étude d'un ensemble de lois peu différentes de celles auxquelles obéissent réellement nos instruments, mais beaucoup plus simples, de lois qui ne régissent effectivement aucun objet naturel, mais qui sont concevables pour l'esprit. En ce sens la géométrie est une convention »⁶². »

In line with Klein's ideas, Poincaré thinks that geometry must study a particular transformation group and that the concept of group « pre-exists » in our mind « at least in power ». But there are many groups, which we can conceive of ; then, which group do we choose as the foundation of geometry ? and why ? Poincaré writes : « *L'expérience nous guide dans ce choix qu'elle ne nous impose pas ; elle nous fait reconnaître non quelle est la géométrie la plus vraie, mais quelle est la plus commode*⁶³. »

Poincaré's words are in contrast with the ideas of Riemann, Helmholtz, Klein – and others – about the role of experience in the formulation of geometric axioms and, more generally, in cognitive processes. These three had attributed to our experience the fundamental role in knowing the world, while Poincaré seems to reduce it to a simple « convention ». In reality, on many occasions Poincaré attributes great relevance to our experience and to our interactions with the surrounding space when searching for the axioms that may serve as the foundation of geometry. Poincaré remarks : « *Le sous-groupe rotatif attire d'abord notre attention parce qu'il conserve certaines sensations*⁶⁴. » As Philippe Nabonnand points out in his paper on the psychophysiology of geometry, Poincaré thinks that experience has a double role : at the beginning, a « psycho-physiologic » phase occurs – when experience originates « *notre capacité innée à former des groupes* » – and

61. POINCARÉ, 1913, p. 40.

62. POINCARÉ, 1913, p. 40-41.

63. POINCARÉ, 1925, p. 93-94.

64. POINCARÉ, 1899, p. 38.

successively experience has « *un rôle fondamental de guide pour le choix des conventions les plus commodes*⁶⁵ ».

According to Poincaré, the « psycho-physiologic » experience forces one to pose axioms as the foundation of geometry, which lead to consider the geometries on Riemannian manifolds with constant curvature – that is to say, the Euclidean and non-Euclidean geometries. One needs to introduce new hypotheses deduced from experience in order to select the unique group, and thereby the unique geometry of space, thus choosing between Euclidean geometry, hyperbolic geometry, elliptic geometry or spherical geometry. Poincaré formulates the request which the group should satisfy in these words : « *La proposition qui en tiendra lieu sera l'affirmation de l'existence d'un sous-groupe invariant dont tous les déplacements sont échangeables et qui est formé de toutes les translations*⁶⁶. » In such a way, he found the Euclidean group.

This approach aims at defining an objective property as an invariant of a certain transformation group, which on its part is determined by our sensations due to observations and measurements *about* and *in* the world. It is shared by many mathematicians. Among them Enriques, who criticizes Poincaré's ideas on conventionalism of science, and holds that scientific concepts give an approximate description of reality in which it is necessary to distinguish the invariants from the many other variables. Enriques⁶⁷ does not describe reality as a simple suite of sensations but as a set of sensations connecting with « willingly organized » conditions. Then reality is not a pure empirical factor, but something due to the rational coordinative activity : an invariant of the correspondence between « volitions » and « sensations ». The role of experience is fundamental in the description of reality. Enriques wrote that the reality of objects and empirical factors consist in the fact that they do not change if submitted to tests which organize them in different ways with respect to subject's will ; in other words, they are mathematical and psychological invariants⁶⁸.

The relevance of the theory of transformation groups in geometry will be pointed out even more powerfully by Weyl, who considers different invariant groups according to the structure of space that is to be preserved. Weyl writes : « *Nous avons constaté qu'objectivité signifie invariance pour le groupe des automorphismes*⁶⁹. » Anyway, reality does not clearly show which group of automorphism has to be considered for the investigation at issue. As Weyl and Enriques claim, objectivity is a cultural fact and changes in time : it is a historical datum.

65. NABONNAND, 2000a, p. 14.

66. POINCARÉ, 1899, p. 34.

67. ENRIQUES, 1938.

68. ENRIQUES, 1938, p. 87.

69. WEYL, 1964, p. 130.

Automorphisms preserve the structure of the empty space (the geometric space) as a vectorial space. But if *tests* are made, then instruments of measurement – which can be considered as rigid bodies – are requested to preserve their shape during their motion in the physical space : therefore, rigid movement selects the group of isometries. For example, affinities – automorphisms of the plane – are constituted by rotations, similitudes and translations. Weyl claims that the group of affinities describes the geometric plane ; the description of the physical plane is given by the proper subgroup of isometries (or rigid displacements). Weyl points out : « *Il n'y a pas lieu d'être surpris, comme le fut Kant, que ce groupe d'automorphismes géométriques soit plus large que celui des automorphismes physiques et comprenne les homothéties*⁷⁰. »

Let us notice that Weyl's geometric space is deeply different from that of Husserl, where all conceivable theories live. In fact Weyl assumes that the geometric structure of space is preserved. In the physical space – which in Weyl's opinion is the space of our experience, the phenomenological space – free mobility of figures is requested ; as a consequence, only the Euclidean and non-Euclidean geometries are possible.

This point of view, which is already contained in some ideas of the XIXth century mathematicians, even if it was expressed by using different words and formalism, is expounded very clearly by Weyl. Riemann, Helmholtz, Poincaré, and others seemed to realize that the conceptual space – where the manifolds with any curvature can be conceived abstractly – is different from the physical space – where the instruments of measurement select the manifolds with constant curvature. Then all the spaces with constant curvature become possible *real* spaces. But how can one choose among them ? on which bases ? The choice must be based on empirical facts.

It is no accident that in the second half of the XIXth century, just after the publication of Riemann's lecture, many mathematicians tried to extend classical theories of mathematical physics – such as mechanics, potential theory, and theory of elasticity – to manifolds with constant curvature. In Germany Rudolph Lipschitz, Ernst Schering and Wilhelm Killing published papers on potential theory, whose results and fundamental relations are extended to curved spaces, such as Kepler's laws, Hamilton-Jacobi method, and so on⁷¹. In Italy Eugenio Beltrami⁷² deduced some relevant theorems in potential theory, one of them concerning the extension of the Laplace equation to a Riemannian manifold, actually known as the Laplace-Beltrami equation. Alberto Tonelli, Ernesto Cesàro, Ernesto Padova, and Carlo Somigliana pursued Beltrami's

70. WEYL, 1964, p. 128.

71. TAZZIOLI, 1994a.

72. BELTRAMI, 1868b.

research and published interesting papers on mathematical physics in curved spaces⁷³.

As a natural development of such studies, presupposing only existence in an n -dimensional space with constant curvature, Killing established the transformation between a coordinate system moving with a body and another coordinate system thought to be fixed in space. The transformation laws found by Killing are surprising : they indeed generalize some relations, deduced by Hendrik Antoon Lorentz in 1904⁷⁴ and usually known as Lorentz transformations, which play a fundamental role in the theory of special relativity. Woldemar Voigt⁷⁵ was the first physicist to deduce the Lorentz transformations by imposing the invariance of Maxwell's equations. It is then relevant that in a purely geometric context, with no physical hypothesis, Killing established for a mechanical system the transformations of coordinates which later became the basis of the special theory of relativity⁷⁶.

Beltrami's contribution to mathematical physics and, particularly, to the theory of elasticity in curved spaces, is also very important. As many other mathematicians and physicists of the XIXth century, Beltrami imaged that an ether filled the whole universe and was able to propagate physical phenomena in space by means of its strains. If a physical force exists, then particles of the ether are forced to transmit the force from one point of space to the other by contact – by following the Newtonian principle of action and reaction. In such a way, physical forces are included in a mechanistic schema which allows to overcome the concept of action at-a-distance. Since the ethereal medium is supposed to be elastic, homogeneous and isotropic, then the usual laws of the theory of elasticity can be applied to it.

But is our universe, filled with ether, flat ? or can it be curved ? Beltrami does not answer this question and shows a pluralistic attitude, such as Riemann and Lobachevskij had done. In a paper from 1882⁷⁷ Beltrami extends the equations of elastic equilibrium to spaces with constant curvature : he shows that in such spaces the ether propagates physical phenomena by means of its strains, which « can not occur in Euclidean space ». However, Beltrami does not choose the curvature of space or the ether model in a definitive way : he tries to explain electric and magnetic forces as well as spherical waves mechanically by means of an ether filling a space with positive, negative or null curvature according to the phenomenon at issue.

Also Kingdon Clifford – according to Weyl « a solitary echo to Riemann's ideas » – asserts very clearly that space is able to transmit physical forces by

73. TAZZIOLI, 1993.

74. LORENTZ, 1904.

75. VOIGT, 1887.

76. TAZZIOLI, 1994a and 2000.

77. BELTRAMI, 1882.

changing its curvature. In Clifford's opinion, physical forces and geometry of space are very closely connected :

« There are three kinds of variation in the curvature of space which we ought to consider as within the range of possibilities :

« (1) Our space is perhaps really possessed of a curvature varying from point to point, which we fail to appreciate because we are acquainted with only a small portion of space, or because we disguise its small variations under changes in our physical condition which we do not connect with our change of position [...].

« (2) Our space may be really same (of equal curvature), but its degree of curvature may change as a whole with the time. In this way our geometry based on the sameness of space would still holds good for all parts of space, but the change of curvature might produce in space a succession of apparent physical changes.

« (3) We may conceive our space to have everywhere a nearly uniform curvature, but that slight variations of the curvature may occur from point to point, and themselves vary with the time⁷⁸. »

These words seem to point out the possibility that our universe may follow different geometries. Our experience is indeed confined to small portions of space, whose curvature is constant (or nearly constant) since free mobility of rigid bodies is experienced. But in infinitesimal or infinitely large regions of space different geometries could occur, as Riemann noticed in his lecture. The general theory of relativity would have shown that in space propagation of phenomena is linked to a variation of curvature, as Riemann, Clifford and Beltrami suggested from geometric considerations alone. In fact, they predicted that space is the real responsible in propagating forces and its curvature depends on the physical phenomenon that takes place.

CONCLUSIONS

In the second half of the XIXth century, as we have seen, Riemann, Beltrami, Clifford and others believed that physical phenomena structure the geometry of space – its metric and its curvature. Physics and geometry are then closely linked and the space of our active experience has a fundamental role in the foundations of mathematics.

During the last decades of the XIXth century, some Italian mathematicians tried to found the proper concept of number on geometry, by using results and methods of projective geometry. In his paper « Sui fondamenti della geometria

78. CLIFFORD, 1885, p. 202.

proiettiva », published in 1881, De Paolis made rigorous some ideas due to Christian von Staudt, who had defined the coordinates of a point on a straight line (and then the real numbers), by using only elementary geometrical constructions – such a process is known as coordinatization.

De Paolis⁷⁹ shows that rational and irrational numbers can be geometrically constructed by starting from a straight line if the principle of completeness is satisfied – that is to say, the straight line has no hole. Moreover, in order to define the concept of limit of segments, in a similar way as Georg Cantor did for the limit of rational numbers, he requested that any segment is preserved during its motion⁸⁰. For justifying this axiom, De Paolis reported the following words of the French mathematician Jules Hoüel : « *Il serait difficile de trouver une seule démonstration d'une proposition fondamentale de géométrie, dans laquelle n'entre pas l'idée de mouvement géométrique plus ou moins déguisée*⁸¹. » By supposing that any segment moves rigidly in space, De Paolis deduced the existence of irrational segments⁸² ; as a consequence, he could introduce irrational numbers in a purely geometric way.

De Paolis's paper is fairly rigorous, even if the treatment of the subject is sometimes unclear and the axioms are hidden in the text. About ten years later Gino Fano, one of Corrado Segre's students, developed a more abstract approach to geometry. In his work on the foundations of projective geometry, Fano⁸³ followed De Paolis's paper step by step but made it more rigorous and axiomatic. Fano, who was also influenced by Giuseppe Peano, introduced geometric quantities but did not define any of them. For example, a manifold is made by points, he wrote ; nevertheless, in his opinion « point » is just a name, which is « independent of its proper nature ». Moreover, his geometric postulates never refer to psychological or perceptive origins. Fano also proved the independence of his system of axioms by creating geometric models, in a similar way as Hilbert would do in his *Grundlagen der Geometrie*⁸⁴.

Other Italian mathematicians, in particular Mario Pieri, published a sequence of works on the foundations of geometry which inspired Hilbert even if their starting points were very different. Pieri, who was a student of De Paolis in Pisa and of Segre and Peano in Turin, wrote one the deepest works of the Italian geometric school⁸⁵. He introduced his axioms and methods of projective geometry with no reference to intuition or psychology ; he seems to come

79. DE PAOLIS, 1881.

80. AVELLONE, BRIGAGLIA and ZAPPULLA, 2002.

81. DE PAOLIS, 1881, p. 490.

82. More details about De Paolis's proof – and about his work in general – can be found in AVELLONE, BRIGAGLIA and ZAPPULLA, 2002.

83. FANO, 1892.

84. HILBERT, 1899.

85. PIERI, 1898.

very near to cutting the strong link between geometry and empiricism. This attitude would lead geometry to its complete formalization in Hilbert's work⁸⁶.

Another Italian mathematician who worked intensely on the foundations of geometry was Giuseppe Veronese. He thought that geometry is the science of space and that it is a « mixed science », both « empirical » and « formal ». In fact, experience and pure intuition are both fundamental for posing the correct axioms at the basis of geometry and to deduce geometrical concepts from our active observation of space.

His celebrated treatise on the foundations of geometry⁸⁷ is confused and very difficult to read. Here Veronese introduced a long series of « empirical remarks », in order to answer the question : « What is the *continuum* ? » The author claimed :

« This is a word whose meaning we realize, without any mathematical definition [...]. We know *by intuition* the continuum in its simplest form, since it is the common characteristic of many concrete things, such as for example the *time* and the *position* occupied by an object [in space]⁸⁸. »

Such an attitude was very different from the point of view expressed by Georg Cantor and Richard Dedekind on the same subject. « The continuity of space is an arbitrary assumption », Cantor wrote to Dedekind in April 15th, 1882 ; then it must be considered an axiom. By means of some assumptions deduced from empirical observations, Veronese succeeded in defining the real numbers from the points of a segment in a purely abstract way.

Veronese also introduced infinite and infinitesimal segments by considering a segment AB – A being the origin, and AB the unit of length – and a sequence of consecutive segments with length AB – the totality is called a field. By referring to a metaphysical, rather than a mathematical, « principle of the continuity of thought », Veronese supposed that an element outside the field always exists, say C. Then he defined AC an actual infinite with respect to AB and AB an actual infinitesimal with respect to AC. It is important to notice that in this geometry the principle of Archimedes – if $AB > CD$, then an integer n exists, such that $n \cdot CD > AB$ – is not valid ; in fact the unit AB can never overcome the actual infinite AC.

Veronese justified his infinitesimal and infinite on the basis of experience. He noticed that it is our experience, which leads us to search the indivisible segment on a straight line. An atomistic view which is, in Veronese's opinion, in accordance with our empirical knowledge of space. In fact we cannot divide

86. More details about the Italian geometric school can be found in AVELLONE, BRIGAGLIA and ZAPPULLA, 2002, and in BOTTAZZINI, 2000 and 2001.

87. VERONESE, 1891.

88. VERONESE, 1891, p. 45. Our translation.

a segment into an infinite number of parts empirically, but must stop after many – but not infinite – steps. When we stop, we do not get a point but an actual infinitesimal. For a point, Veronese remarked, the usual properties of a segment are not valid any more, while any actual infinitesimal preserves such properties. As a consequence, the intuitive continuum is made by many actual infinitesimals joining two points, and not simply by points.

Infinite and infinitesimal segments introduced by Veronese find an immediate arithmetical interpretation ; anyway, Veronese's numbers differ a lot from the transfinite and infinitesimal numbers defined by Cantor, as Veronese himself remarked. He was proud to point out that his numbers satisfy the commutative property while Cantor's numbers do not⁸⁹.

In line with Cantor and Dedekind, Hilbert thought that continuity must be a postulate, on a par with the other properties of space, and posed the axiom of continuity at the basis of geometry. In his *Grundlagen der Geometrie*, Hilbert considered different systems of axioms leading to many geometries (Euclidean, non-Euclidean, Archimedean, and so on).

In his review of Hilbert's *Grundlagen* Poincaré⁹⁰ judged the formalistic approach interesting and fruitful, since it includes so many geometries and numbers. (According to Poincaré, who considered arithmetic as a consequence of geometry and not the contrary, many geometries lead to the introduction of many numbers ; therefore, Hilbert's system is very rich of numbers.) Anyway he could not share Hilbert's point of view *tout court*. Poincaré pointed out :

« Les raisonnements doivent pouvoir, d'après lui [Hilbert], se ramener à des règles purement mécaniques, et il suffit, pour faire la Géométrie, d'appliquer servilement ces règles aux axiomes, sans savoir ce qu'ils veulent dire. On pourra ainsi construire toute la Géométrie, je ne dirai pas précisément sans y rien comprendre, puisqu'on saisira l'enchaînement logique des propositions, mais tout au moins sans y rien voir. On pourrait confier les axiomes à une machine à raisonner, par exemple au piano raisonneur de Stanley Jevons, et l'on en verrait sortir toute la Géométrie⁹¹. »

Intuition and experience seemed to have no role in Hilbert's programme and, Poincaré went on, *« le point de vue logique paraît seul l'intéresser⁹² »*.

« Son œuvre est donc incomplète », Poincaré commented severely, even if his approach surely represents a *« progrès considérable, comparable à ceux que l'on devait à Lobatchevski, à Riemann, à Helmholtz et à Lie⁹³ »*. Hilbert's

89. More details about Veronese and his work are in CANTÙ, 1999.

90. POINCARÉ, 1902.

91. POINCARÉ, 1902, here 1916-1956, t. XI, p. 95.

92. POINCARÉ, 1902, here 1916-1956, t. XI, p. 112.

93. POINCARÉ, 1902, here 1916-1956, t. XI, p. 113.

geometries are simply deduced by replacing some axioms with others ; so, are they only logical games ? Poincaré could not accept that geometry was reduced to a sequence of logical inferences, of axioms and theorems with no meaning. Therefore, he included Hilbert's geometries in a meaningful theory – the theory of transformation groups – and showed that Hilbert's theory was not lacking of sense. « *On verrait que chacune de ses géométries est encore l'étude d'un groupe*⁹⁴ », Poincaré was proud to remark. His re-interpretation of Hilbert's geometry was then successfully concluded.

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94. POINCARÉ, 1902, here 1916-1956, t. XI, p. 112.

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LIST OF REFERENCES

- AVELLONE (Maurizio), BRIGAGLIA (Aldo) and ZAPPULLA (Carmela), 2002, « The foundations of projective geometry in Italy from De Paolis to Pieri », *Archive for History of Exact Sciences*, t. LVI, p. 363-425.
- BAILLY (François) and LONGO (Giuseppe), 2003a, « Espace, temps et cognition. À partir des mathématiques et des sciences de la nature », *Revue de synthèse*, 5^e sér., t. CXXXIV, année 2003, p. 61-118.
- BAILLY (F.) and LONGO (G.), 2003b, « Incomplétude et incertitude en mathématiques et en physique », in a volume in memory of Gilles Châtelet, Paris, to appear.
- BELTRAMI (Eugenio), 1868a, « Saggio di interpretazione della geometria non euclidea », *Giornale di Matematiche*, t. VI, p. 284-312, repr. in BELTRAMI, 1902-1920, t. I, p. 374-405.
- BELTRAMI (E.), 1868b, « Sulla teorica generale dei parametri differenziali », *Memorie dell'Accademia delle Scienze dell'Istituto di Bologna*, ser. 2, t. VIII, p. 551-590, repr. in BELTRAMI, 1902-1920, t. II, p. 74-118.
- BELTRAMI (E.), 1882, « Sulle equazioni generali dell'elasticità », *Annali di Matematica Pura ed Applicata*, ser. 2, t. X, p. 188-211, repr. in BELTRAMI, 1902-1920, t. III, p. 383-407.
- BELTRAMI (E.), 1902-1920, *Opere matematiche*, ed. Facoltà di scienze della Reale Università di Roma, Milano, Hoepli, 4 t.
- BOI (Luciano), 1992, « L'espace : concept abstrait et/ou physique. La géométrie entre formalisation mathématique et étude de la nature », in BOI, FLAMENT and SALANSKIS, ed., 1992, p. 65-90.
- BOI (L.), 1996, « Les géométries non euclidiennes, le problème philosophique de l'espace et la conception transcendantale. Helmholtz et Kant, les néo-kantiens, Einstein, Poincaré et Mach », *Kant Studien*, t. LXXXVII, p. 257-289.
- BOI (Luciano), FLAMENT (Dominique) and SALANSKIS (Jean-Michel), ed., 1992, *1830-1930. A century of geometry*, Berlin, Springer.
- BOTTAZZINI (Umberto), 2000, « I geometri italiani e i *Grundlagen der Geometrie* di Hilbert », in *Atti del XVI congresso dell'Unione matematica italiana*, Napoli, 13-18 settembre 1999, Bologna.
- BOTTAZZINI (U.), 2001, « I geometri italiani e il problema dei fondamenti (1889-1899) », *Bollettino Unione Matematica Italiana*, spec. issue « La Matematica nella società e nella cultura », ser. 7, t. IV-A, p. 281-329.
- BOTTAZZINI (Umberto) and TAZZIOLI (Rossana), 1995, « *Naturphilosophie* and its role in Riemann's mathematics », *Revue d'histoire des mathématiques*, t. I, p. 3-38.
- CANTÙ (Paola), 1999, *Giuseppe Veronese e i fondamenti della geometria*, Milano, Unicopli.

- CLIFFORD (William Kingdon), 1885, *The Common Sense of the exact sciences*, ed. Karl PEARSON, London.
- DE PAOLIS (Riccardo), 1881, « Sui fondamenti della geometria proiettiva », in *Atti della Reale Accademia dei Lincei*, ser. 3, t. IX, p. 489-503.
- ENRIQUES (Federigo), 1938, *La Théorie de la connaissance scientifique de Kant à nos jours*, Paris, Hermann.
- FANO (Gino), 1892, « Sui postulati fondamentali della geometria proiettiva in uno spazio lineare a un numero qualunque di dimensioni », *Giornale di Matematiche*, t. XXX, p. 106-132.
- GAUSS (Carl Friedrich), 1827, « Disquisitiones generales circa superficies curvas », *Commentationes Societatis Regiae Scientiarum Gottingensis recentiores*, t. VI, repr. in GAUSS, 1863-1933, t. IV, p. 217-258.
- GAUSS (C. F.) 1863-1933, *Werke*, Königliche Gesellschaft der Wissenschaften Göttingen ed., Göttingen, 12 t.
- GRAY (Jeremy), 1992, « Poincaré and Klein. Groups and geometries », in BOI, FLAMENT and SALANSKIS, ed., 1992, p. 35-44.
- HELMHOLTZ (Hermann von), 1866, « Über die thatsächlichen Grundlagen der Geometrie », *Verhandlungen des Naturhistorisch-Medicinischen Vereins zu Heidelberg*, t. IV, p. 197-202, repr. in HELMHOLTZ, 1882-1895, t. II, p. 610-616.
- HELMHOLTZ (H. von), 1868, « Über die Thatsachen, die der Geometrie zum Grunde liegen », *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen*, t. XIV, p. 193-221, repr. in HELMHOLTZ, 1882-1895, t. II, p. 618-639.
- HELMHOLTZ (H. von), 1870, « Über den Ursprung und die Bedeutung der geometrischen Axiome », in HELMHOLTZ, 1884, t. II, p. 1-31.
- HELMHOLTZ (H. von), 1879, *Die Thatsachen in der Wahrnehmung*, Berlin, Hirschwald, repr. in HELMHOLTZ, 1884, t. II, p. 217-251.
- HELMHOLTZ (H. von), 1882-1895, *Wissenschaftliche Abhandlungen*, Leipzig, Barth, 3 t.
- HELMHOLTZ (H. von), 1884, *Vorträge und Reden*, Braunschweig, Vieweg, 2 t.
- HILBERT (David), 1899, *Grundlagen der Geometrie*, Göttingen, Teubner.
- HUSSERL (Edmund), 1962, *L'Origine de la géométrie*, Paris, Presses universitaires de France.
- HUSSERL (E.), 1983, « Philosophische Versuche über den Raum », *Husserliana*, spec. issue « Studien zur Arithmetik und Geometrie », t. XXI, p. 262-310.
- KLEIN (Felix), 1871, « Über die sogenannte nicht-euklidische Geometrie », *Mathematische Annalen*, t. IV, p. 573-625, repr. in KLEIN, 1921-1923, t. I, p. 245-305.
- KLEIN (Felix), 1872, *Vergleichende Betrachtungen über neuere geometrische Forschungen. Programm zum Eintritt in die philosophische Fakultät und den Senat der K. Friedrich-Alexanders-Universität zu Erlangen*, Erlangen, Deichert.
- KLEIN (F.), 1873, « Über die sogenannte nicht-euklidische Geometrie (Zweite Aufsatz) », *Mathematische Annalen*, t. VI, p. 112-145, repr. in KLEIN, 1921-1923, t. I, p. 311-343.
- KLEIN (F.), 1921-1923, *Gesammelte Mathematische Abhandlungen*, ed. R. FRICKE, Berlin, Springer, 3 t.
- KRAUSE (Albrecht), 1878, *Kant und Helmholtz über den Ursprung und die Bedeutung der Raumanschauung und der geometrischen Axiome*, Lahr, Moritz Schauenburg.
- LIE (Sophus), 1886, « Bemerkungen zu v. Helmholtzs Arbeit. Über die Tatsachen, die der Geometrie zu Grunde liegen », *Berichte über die Verhandlungen der*

- Sächsischen Akademie der Wissenschaften zu Leipzig*, 1887, p. 337-342, repr. in LIE, 1927-1934, t. II, part. I, p. 374-379.
- LIE (S.), 1890a, « Über die Grundlagen der Geometrie. I. – Abhandlung », *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig*, p. 284-321, repr. in LIE, 1927-1934, t. II, part. I, p. 380-413.
- LIE (S.), 1890b, « Über die Grundlagen der Geometrie. II. – Abhandlung », *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig*, 1891, p. 355-418, repr. in LIE, 1927-1934, t. II, part. I, p. 414-468.
- LIE (S.), 1892a, « Sur les fondements de la géométrie », *Comptes rendus hebdomadaires de l'Académie des sciences de Paris*, 114, p. 461-463, repr. in LIE, 1927-1934, t. II, part. I, p. 477-479.
- LIE (S.), 1892b, « Bemerkungen zu neueren Untersuchungen über die Grundlagen der Geometrie », *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig*, p. 106-114, repr. in LIE, 1927-1934, t. II, part. I, p. 469-476.
- LIE (S.), 1893, *Theorie der Transformationsgruppe*, ed. Friedrich ENGEL, Leipzig, Teubner, t. III.
- LIE (S.), 1927-1934, *Gesammelte Abhandlungen*, Leipzig, Teubner, 6 vol.
- LOBACHEVSKIJ (Nikolaij), 1835-1838, *Novye naala geometrij s polnoj teoriej parallel'nyh* ; here repr. *New principles of geometry with complete theory of parallels*, Austin, 1897, t. X.
- LONGO (Giuseppe), 2001a, « The reasonable effectiveness of mathematics and its cognitive roots », in BOI (Luciano), ed., *New interactions of mathematics with natural sciences and humanities*, Berlin-New York, Springer, to appear.
- LONGO (G.), 2001b, « Space and time in the foundations of mathematics, or some challenges in the interactions with other sciences », invited lecture, *First AMS/SMF meeting*, Lyon, July 2001 ; see : <<http://www.di.ens.fr/users/longo/download.html>>
- LORENTZ (Hendrik Antoon), 1904, « Electromagnetic phenomena in a system moving with any velocity smaller than that of the light », *Koninklijke Akademie van Wetenschappen te Amsterdam. Section of Sciences. Proceedings*, t. VI, p. 809-831.
- NABONNAND (Philippe), 2000a, « La genèse psychophysiologique de la géométrie selon Poincaré », in FLAMENT (Dominique), *Histoires de géométries*. Textes du séminaire de l'année 2000, Paris, Édition de la Maison des sciences de l'homme, p. 12-28.
- NABONNAND (P.), 2000b, « La polémique entre Poincaré et Russell au sujet du statut des axiomes de la géométrie », *Revue d'histoire des mathématiques*, t. VI, p. 219-269.
- NOWAK (Gregory), 1989, « Riemann's *Habilitationsvortrag* and the Synthetic *A Priori* Status », in ROWE (David) and MC CLEARY (John), eds., *The History of Modern Mathematics*, New York, NY, Academic Press, t. II, p. 17-46.
- PIERI (Mario), 1898, « I principii della geometria di posizione composti in sistema logico-deduttivo », *Memorie della Reale Accademia delle Scienze di Torino*, ser. 2, t. XLVIII, p. 1-62.
- POINCARÉ (Henri), 1887, « Les hypothèses fondamentales de la géométrie », *Bulletin de la Société mathématique de France*, t. XV, p. 203-216, repr. in POINCARÉ, 1916-1956, t. XI, p. 79-91.
- POINCARÉ (H.), 1899, « On the foundation of geometry », *Monist*, ser. 9, t. I, p. 1-43.

- POINCARÉ (H.), 1902, « Les fondements de la géométrie », *Bulletin des sciences mathématiques*, ser. 2, t. XXVI, p. 249-272 repr. in POINCARÉ, 1916-1956, t. XI, p. 92-113.
- POINCARÉ (H.), 1913, *Dernières pensées*, Paris, Flammarion.
- POINCARÉ (H.), 1916-1956, *Œuvres*, Paris, Gauthier-Villars, 11 t.
- POINCARÉ (H.), 1925, *La Science et l'hypothèse*, Paris, Flammarion.
- RIEHL (Alois), 1904, *Hermann von Helmholtz in seinem Verhältnis zu Kant*, Berlin, Reuther & Reichard.
- RIEMANN (Bernhard), 1854, *Über die Hypothesen, welche der Geometrie zu Grunde liegen, Habilitationsvortrag*, Göttingen, repr. in RIEMANN, 1990, p. 304-319. All page references are to the english translation, *A comprehensive introduction to differential geometry*, by Michael SPIVAK, Berkeley, Publish or perish, 1975, t. III, p. 135-153.
- RIEMANN (B.), 1861, « Commentatio mathematica, qua respondere tentatur quaestioni ab Ill.ma Academia Parisiense propositae : “Trouver quel doit être l'état calorifique d'un corps solide homogène indéfini pour qu'un système de courbes isothermes, à un instant donné, restent isothermes après un temps quelconque, de telle sorte que la température d'un point puisse s'exprimer en fonction du temps et de deux autre variables indépendantes” », in RIEMANN, 1990, p. 423-436.
- RIEMANN (B.), 1876, « Fragmente philosophischen Inhalts », in RIEMANN, 1990, p. 507-538.
- RIEMANN (B.), 1990, *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge*, 1st ed. Leipzig, Teubner, 1876, here ed. by Raghavan NARASIMHAN, Leipzig/Berlin, Teubner/Springer.
- RIEMANN (B.), 1994, *Sulle ipotesi che stanno alla base della geometria e altri scritti scientifici e filosofici*, ed. by R. PETTOELLO, Torino, Bollati Boringhieri.
- RUSSELL (Bertrand), 1897, *An essay on the foundations of geometry*, Cambridge, Cambridge University Press, repr. New York, NY, Dover, 1956.
- SCHOLZ (Erhard), 1992, « Riemann's vision of a new approach to geometry », in BOI, FLAMENT and SALANSKIS, ed., 1992, p. 22-34.
- TAZZIOLI (Rossana), 1993, « Ether and theory of elasticity in Beltrami's work », *Archive for History of Exact Sciences*, t. XLVI, p. 1-37.
- TAZZIOLI (R.), 1994a, « Rudolf Lipschitz's work on differential geometry and mechanics », in KNOBLOCH (Eberhard) and ROWE (David), eds., *The History of Modern Mathematics*, New York, NY, Academic Press, t. III, p. 113-137.
- TAZZIOLI (R.), 1994b, « Il problema di Riemann-Helmholtz-Lie. “Ipotesi”, “fatti” e “gruppi di trasformazione” », in *Seminari di geometria dell'Università di Bologna* (1991-1993), Bologna, p. 249-270.
- TAZZIOLI (R.), 2000, *Beltrami e i matematici « relativisti ». La meccanica in spazi curvi nella seconda metà dell'Ottocento*, Bologna, Pitagora (Quaderni dell'Unione Matematica Italiana, 47).
- TAZZIOLI (R.), 2002, « Riemann. Le géomètre de la nature », *Pour la science* (Les Génies de la science), Paris, août-nov.
- VERONESE (Giuseppe), 1891, *Fondamenti di geometria a più dimensioni e a più specie di unità rettilinee, esposti in forma elementare*, Padova, Tipografia del Seminario.
- VOIGT (Woldemar), 1887, « Über das Doppler'sche Princip », *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, p. 44-51.

WEYL (Hermann), 1952, *Space, time, matter*, 1st ed. Berlin, 1918, here transl. by Henry L. BROSE, New York, NY, Dover.

WEYL (H.), 1964, *Symétrie et mathématique moderne*, 1st ed. Princeton, Princeton University Press, 1952, here Paris, Flammarion.