Correction to the Paper "On the Theory of Statistical Decision Functions"

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The following correction should be made to the above-mentioned paper.

1 The proof of Corollary 4 should be replaced by the following:

For an arbitrary open set e in D , let $e = \bigcup e_i$, where e_i are the neighborhoods with their closures $\bar{e}_i$ contained in e . Then we have

$$\{x; \varphi(x) \in e\} = \bigcup_i \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} \{x; \varphi_n(x) \in \bar{e}_i\} \in \mathfrak{R}$$

This proves the corollary.

2 The passage from the line 3 to the line 6 on the page 20, "Thenis a closed set" should be omitted. Though it is not wrong, it is liable to raise misunderstanding.

3 The statement in Lemma 3, " $r(F, \varphi)$ is a continuous function on $\Omega \times \mathfrak{D}$ and satisfies", should be replaced by "When $F_i \rightarrow F$, $\varphi_i(x) \rightarrow \varphi(x)$ ($i \rightarrow \infty$), then we have $r(F_i, \varphi_i) \rightarrow r(F, \varphi)$ ($i \rightarrow \infty$) and". The proof remains available.

4 The passages from the line 9 to the line 29 on the page 21, "Lemma 4.....is a compact, metric and separable space", should be replaced by the following:

Lemma 4' As a function of F in Ω $r(F, \varphi)$ is uniformly continuous with respect to φ .

Proof As $W(F, d)$ is continuous on the compact space $\Omega \times D$, for any positive number ε there exists a positive number δ such that

$$\|F_1 - F_2\| < \delta \rightarrow |W(F_1, d) - W(F_2, d)| < \varepsilon \quad \text{for all } d.$$

Since $\|F_1 - F_2\| < \delta$ implies

$$|F_1(E) - F_2(E)| \leq 2 \|F_1 - F_2\| < 2\delta$$

for any set E of $\mathfrak{R}$, we have

$$|r(F_1, \varphi) - r(F_2, \varphi)| = \left| \int_{\mathfrak{R}} W(F_1, \varphi(x)) dF_1 - \int_{\mathfrak{R}} W(F_2, \varphi(x)) dF_2 \right|$$

$$\begin{aligned}
&\leq \left| \int_R \{W(F_1, \varphi(x)) - W(F_2, \varphi(x))\} dF_1 \right| \\
&+ \left| \int_R W(F_2, \varphi(x)) dF_2 - \int_R W(F_2, \varphi(x)) dF_1 \right| \\
&< \varepsilon + 2\delta
\end{aligned}$$

This proves the lemma.

Now we introduce a metric in $\mathfrak{D}$.

Definition Let φ_1, φ_2 be two elements of $\mathfrak{D}$. We define the distance $\rho_{\mathfrak{D}}$ between φ_1, φ_2 by

$$\rho_{\mathfrak{D}}(\varphi_1, \varphi_2) = \max_{F \in \Omega} |r(F, \varphi_1) - r(F, \varphi_2)|$$

Clearly we have $\rho_{\mathfrak{D}}(\varphi_1, \varphi_2) \leq 1$. Let $\mathfrak{D}^*$ denote the metric space obtained from $\mathfrak{D}$, in identifying the elements between which the $\rho_{\mathfrak{D}}$ distances are zero, and metrized by $\rho_{\mathfrak{D}}$.

Lemma 4'' $\mathfrak{D}^*$ is a conditionally compact space.

Proof Let ε be an arbitrarily chosen positive number. Then, according to Lemma 4', every F has a neighborhood, $U(F)$, such that

$$F' \in U(F) \rightarrow |r(F, \varphi) - r(F', \varphi')| < \varepsilon \quad \text{for all } \varphi.$$

As Ω is compact, Ω can be covered by a finite number of such neighborhoods, i.e.,

$$\Omega = U(F_1) \cup U(F_2) \cup \dots \cup U(F_s)$$

Now put

$$\rho'(\varphi_1, \varphi_2) = \max_{1 \leq i \leq s} |r(F_i, \varphi_1) - r(F_i, \varphi_2)|$$

This $\rho'(\varphi_1, \varphi_2)$ gives also a metric in $\mathfrak{D}$ and satisfies $0 \leq \rho'(\varphi_1, \varphi_2) \leq 1$. We denote by $\mathfrak{D}'$ the metric space obtained from $\mathfrak{D}$ by this ρ' . Then, an element of $\mathfrak{D}'$, say φ , is represented by a system of numbers $\{r(F_1, \varphi), r(F_2, \varphi), \dots, r(F_s, \varphi)\}$. Taking into account of the fact $0 \leq r(F, \varphi) \leq 1$, we see immediately that $\mathfrak{D}'$ is conditionally compact. Now it holds

$$0 \leq \rho_{\mathfrak{D}}(\varphi_1, \varphi_2) - \rho'(\varphi_1, \varphi_2) < 2\varepsilon$$

from which follows the lemma.

From Lemma 3 follows immediately

Lemma 5 When $\varphi_n(x) \rightarrow \varphi(x)$ ($n \rightarrow \infty$), $r(F, \varphi_n) \rightarrow r(F, \varphi)$ uniformly in F .

Now, let $\gamma(\mathfrak{D}^*)$ be the metric space of all fundamental sequences in $\mathfrak{D}^*$. The distance between two fundamental sequences $\varphi_1^* = \{\varphi_{1n}\}$, $\varphi_2^* = \{\varphi_{2n}\}$ is, as usual, given by

$$\rho(\varphi_1^*, \varphi_2^*) = \lim_{n \rightarrow \infty} \rho(\varphi_{1n}, \varphi_{2n})$$

Then, $\gamma(\mathfrak{D}^*)$ is complete and totally bounded, consequently compact. $\mathfrak{D}^*$ is clearly embedded in $\gamma(\mathfrak{D}^*)$, the closure of $\mathfrak{D}^*$, $\mathfrak{D}^*$ is $\gamma(\mathfrak{D}^*)$ and $r(F, \varphi)$ defined on $\Omega \times \mathfrak{D}^*$ is continuously extended on $\Omega \times \gamma(\mathfrak{D}^*)$. Therefore, we can extend the notion of decision function to the whole space $\gamma(\mathfrak{D}^*)$. The real meaning of an element of $\gamma(\mathfrak{D}^*)$ not belonging to $\mathfrak{D}^*$, say φ^* , consists in the fact, that for any positive number ε there exists a decision function φ in $\mathfrak{D}^*$ such that

$$|r(F, \varphi^*) - r(F, \varphi)| < \varepsilon \quad \text{for all } F$$

When $\mathfrak{D}^*$ is compact from the beginning, as in the case where the sample space R is a countable set, we have, of course, $\mathfrak{D}^* = \gamma(\mathfrak{D}^*)$. In the following we denote $\gamma(\mathfrak{D}^*)$ merely by $\mathfrak{D}^*$. $\mathfrak{D}^*$ is, thus, a compact metric space.

5 The passage from the line 6 from the bottom to the line 5 from the bottom on the page 22, "and are.....from Lemma 7 the", should be replaced by the following:

"We define the distances in M and Δ , respectively, as follows:

$$\rho(\mu_1, \mu_2) = \max_{\delta} |r(\mu_1, \delta) - r(\mu_2, \delta)|$$

$$\rho(\delta_1, \delta_2) = \max_{\mu} |r(\mu, \delta_1) - r(\mu, \delta_2)|$$

Then we have for $n \rightarrow \infty$

$$\mu_n \rightarrow \mu \text{ (in the ordinary sense)} \rightarrow \rho(\mu_n, \mu) \rightarrow 0$$

$$\delta_n \rightarrow \delta \text{ (in the ordinary sense)} \rightarrow \rho(\delta_n, \delta) \rightarrow 0$$

Thus, M and Δ are compact metric spaces. We obtain immediately the following "

6 In Theorem IV $r(\mu, \varphi)$ means $\int_{\Omega} r(E, \varphi) d\mu$.

7 In the line 2 from the bottom on the page 30 the sign " $\geq$ " should be replaced by " $>$ "

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