Absolute Moments in 2-dimensional Normal Distribution

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Let x and y be distributed according to the following 2-dimensional normal distribution,

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_1^2} - \frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right) \right\} dx dy.$$

It is our purpose to express absolute moments in terms of elementary functions. Putting $E(|x^m y^n|) = (m, n)$ for simplicity, we have

$$\begin{aligned} (m, n) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x^m y^n| \exp \left\{ -\frac{1}{2(1-\rho^2)} \right. \\ &\quad \left. \times \left(\frac{x^2}{\sigma_1^2} - 2\rho \frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right) \right\} dx dy \\ &= \frac{2^{\frac{m+n}{2}} \sigma_1^m \sigma_2^n}{\pi} (1-\rho^2)^{\frac{m+n+1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x^m y^n| \exp(-x^2 + 2\rho xy - y^2) dx dy \\ &= \frac{2^{\frac{m+n}{2}} \sigma_1^m \sigma_2^n}{\pi} (1-\rho^2)^{\frac{m+n+1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x^m y^n| e^{-x^2-y^2} \sum_{k=0}^{\infty} \frac{(2\rho xy)^k}{k!} dx dy \\ &= \frac{2^{\frac{m+n}{2}} \sigma_1^m \sigma_2^n}{\pi} (1-\rho^2)^{\frac{m+n+1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{m+1}{2} + k\right) \Gamma\left(\frac{n+1}{2} + k\right)}{(2k)!} (2\rho)^{2k} \\ &= \frac{2^{\frac{m+n}{2}} \sigma_1^m \sigma_2^n}{\pi} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right) (1-\rho^2)^{\frac{m+n+1}{2}} \\ &\quad \times F\left(\frac{m+1}{2}, \frac{n+1}{2}; \frac{1}{2}; \rho^2\right) \\ &= \frac{2^{\frac{m+n}{2}} \sigma_1^m \sigma_2^n}{\pi} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \rho^2\right). \end{aligned}$$

Here

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{1!\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} z^2 + \dots$$

is the hypergeometric function, which reduces to the polynomial of z if α or β is a non-positive integer and γ is positive. Thus, when at least one of the integers m, n is an even number, (m, n) reduces to the polynomial of ρ^2 multiplied by $\sigma_1^m \sigma_2^n$.