Polytopes and the Mean Value Property

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Abstract. Let P be any (not necessarily convex nor connected) solid polytope in the n -dimensional Euclidean space $\mathbb{R}^n$, and let $P(k)$ be the k -skeleton of P . Let $\mathcal{H}_{P(k)}$ be the set of all continuous functions satisfying the mean value property with respect to $P(k)$. For any $k = 0, 1, \dots, n$, we show that $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of polynomials. This settles an open problem posed by Friedman and Littman [37] in 1962. Moreover, we show that if P admits ample symmetry, then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of harmonic polynomials. Some interesting examples are also given.

1. Introduction

Let μ be a nonnegative Borel measure with compact support K and total mass $\mu(K) = 1$ in the n -dimensional Euclidean space $\mathbb{R}^n$, and let Ω be an open subset of $\mathbb{R}^n$. A $\mathbb{C}$ -valued continuous function f in Ω is said to satisfy the μ -mean value property if

$$f(x) = \int_K f(x + ry) d\mu(y) \quad (1.1)$$

holds for any $x \in \Omega$ and $0 < r < r_x$, where r_x is a sufficiently small positive constant depending on $x \in \Omega$ such that $\inf_{x \in C} r_x > 0$ for any compact subset C of Ω . Let $\mathcal{H}_\mu(\Omega)$ be the set of all continuous functions in Ω satisfying the μ -mean value property (1.1).

It is a classical problem to characterize the linear space $\mathcal{H}_\mu(\Omega)$. We briefly review its long history. In the case of two dimensions Walsh [75], Beckenbach and Reade [8], and Friedman [36] showed that if μ is the homogeneous distribution on the vertices, sides, or area of a regular convex N -gon, then $\mathcal{H}_\mu(\Omega)$ is precisely the harmonic polynomials of degree $\leq N$ having zero N th derivatives in each of the directions of the radii of the polygons. See also [60] and [52]. In the case of three dimensions Beckenbach and Reade [9] gave a similar characterization of $\mathcal{H}_\mu(\Omega)$ when μ is the homogeneous distribution on the vertices of the regular convex solids. However, their work was incomplete for the dodecahedron and icosahedron. This gap was filled by Flatto [28]. He also gave a

complete characterization of $\mathcal{H}_\mu(\Omega)$ when μ is the homogeneous distribution on the vertices, surface, or volume of an n -dimensional regular tetrahedron or octahedron. In these special cases, $\mathcal{H}_\mu(\Omega)$ becomes a finite-dimensional linear space.

Flatto [27] considered the problem in the general setting as formulated above and showed that $f \in \mathcal{H}_\mu(\Omega)$ if and only if f is a weak solution of a certain infinite system of linear partial differential equations. He also pointed out that $\mathcal{H}_\mu(\Omega)$ is finite dimensional if and only if the characteristic variety of the system consists of the singleton set $\{0\}$. This was still previously noticed by Aronszajn [6] in a different context.

Then Friedman and Littman [37] discovered another system of partial differential equations characterizing the μ -mean value property (1.1), which is more convenient in many respects. In terms of this system Friedman and Littman [37] and then Flatto [28] obtained sufficient conditions on μ under which $\mathcal{H}_\mu(\Omega)$ is finite dimensional. Unfortunately, except for very restricted cases, their conditions seem to be difficult to verify even for Euclidean measures of familiar figures. In fact, in the end of their paper Friedman and Littman [37] wrote: "The problem of proving the finite dimensionality of the space of functions satisfying the mean value property (1.1) for such simple bodies as convex polyhedra, ellipsoids, etc., with the ordinary Euclidean measure does not seem to be easy. . . . (one sentence omitted). . . . Both methods, however, lead to rather involved technical problems in trying to apply them even to very simple bodies."

Several authors gave partial answers to the finite-dimensionality problem. For example, Garsia and Rodemich [42] considered the case of product measures and, in particular, established the finite dimensionality for the homogeneous distribution on an n -dimensional parallelepipedon. The case of discrete measures was investigated in great detail. In this case, the finite dimensionality was established by Friedman and Littman [37] and Garsia [41] in a general setting. Then more detailed investigations were made concerning various properties of the space $\mathcal{H}_\mu(\Omega)$, e.g., $\dim \mathcal{H}_\mu(\Omega)$, $\deg \mathcal{H}_\mu(\Omega)$, etc., where $\deg \mathcal{H}_\mu(\Omega)$ is the highest degree of the polynomials in $\mathcal{H}_\mu(\Omega)$. See [29], [32], and [33]. In another direction many authors considered the problem from the viewpoint of functional equations. See [2], [15], [10], [51], [53]–[58], [65], [68], [72], [77], etc. There is a vast literature on the related topics (see the References).

The finite-dimensionality problem for *arbitrary* polytopes seems to have been open since Friedman and Littman [37] posed the question. (The author searched the *Mathematical Reviews* (1940–1995) for all papers in which the words *mean value property* appear at least once.) The purpose of this paper is to settle this problem affirmatively.

In this paper a *polytope* in $\mathbb{R}^n$ means a finite union of closed convex solid polytopes in $\mathbb{R}^n$. So a polytope is not necessarily convex nor connected. Given a polytope P in $\mathbb{R}^n$ and $k = 0, 1, \dots, n$, let $P(k)$ be the k -skeleton of P , i.e., the union of all faces of P of dimension $\leq k$. Let v_k be the k -dimensional Euclidean measure on $P(k)$, $|P(k)| = v_k(P(k))$, the k -dimensional Euclidean volume of $P(k)$. Then our problem is to describe those continuous functions f in Ω which satisfy the μ -mean value property (1.1) with respect to the measure $\mu = v_k/|P(k)|$. In this special case, the condition (1.1) is rewritten as

$$f(x) = \frac{1}{|P(k)|} \int_{P(k)} f(x + ry) dv_k(y) \quad (0 < r < r_x). \quad (1.2)$$

This condition is referred to as the $P(k)$ -mean value property. Let $\mathcal{H}_{P(k)}(\Omega)$ be the set

of all continuous functions in Ω satisfying (1.2). The problem of describing $\mathcal{H}_{P(k)}(\Omega)$ is referred to as the *k-skeleton problem*. For $k = 0, n - 1$, and n , the *k-skeleton problem* is also referred to as the *vertex*, *surface*, and *volume problem*, respectively (see [28]). Previously only these three problems (for only special figures) have been considered.

Let $x = (x_1, \dots, x_n)$ denote a standard coordinate system of $\mathbb{R}^n$. We set $\partial = (\partial_1, \dots, \partial_n)$, where $\partial_j = \partial/\partial x_j$. Let $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \dots, \partial_n]$ be the ring of partial differential operators with constant coefficients. The most fundamental result in this paper is the following:

Theorem 1.1. *Let P be any (not necessarily convex nor connected) polytope in $\mathbb{R}^n$, and let Ω be any open subset of $\mathbb{R}^n$. Then for each $k = 0, 1, \dots, n$, the following assertions hold:*

- (1) $\mathcal{H}_{P(k)}(\Omega)$ is a finite-dimensional linear space of polynomials.
- (2) A basis of $\mathcal{H}_{P(k)}(\Omega)$ can be taken from homogeneous polynomials.
- (3) $\mathcal{H}_{P(k)}(\Omega)$ admits a structure of $\mathbb{C}[\partial]$ -module.

For $k = 0$ the result is not new since the vertex problem is a special case of the discrete measure problem for which the finite dimensionality is well known as mentioned previously. Theorem 1.1 is established in Section 4.

Remark 1.2. It can be shown that the restriction map $\mathcal{H}_{P(k)}(\mathbb{R}^n) \rightarrow \mathcal{H}_{P(k)}(\Omega)$ is an isomorphism (see Proposition 3.9). So $\mathcal{H}_{P(k)}(\Omega)$ is independent of the domain Ω . This remark allows us to simplify the notation as $\mathcal{H}_{P(k)} = \mathcal{H}_{P(k)}(\Omega)$. Sometimes we use this simplified notation.

2. Main Results

We state the main results of this paper other than Theorem 1.1. They are either a main step for establishing Theorem 1.1 or an amplification of Theorem 1.1. First we introduce some notations: $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are vectors in $\mathbb{R}^n$ while $\xi = (\xi_1, \dots, \xi_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ are vectors in $\mathbb{C}^n$. We set $\langle \xi, \eta \rangle = \xi_1 \eta_1 + \dots + \xi_n \eta_n$. Let $\{P_{i_j}\}_{i_j \in I_j}$ be the set of j -dimensional faces of P , H_{i_j} the j -dimensional affine subspace of $\mathbb{R}^n$ containing P_{i_j} , $\pi_{i_j}: \mathbb{R}^n \rightarrow H_{i_j}$ the orthogonal projection from $\mathbb{R}^n$ down to the subspace H_{i_j} . Let $p_{i_j} \in \mathbb{R}^n$ be the vectors defined by

$$p_{i_j} = \pi_{i_j}(0) \in H_{i_j}. \quad (2.1)$$

We remark that $H_{i_n} = \mathbb{R}^n$ and $p_{i_n} = 0$ for any $i_n \in I_n$. For $i_j \in I_j$ and $i_{j+1} \in I_{j+1}$ we write $i_j < i_{j+1}$ if P_{i_j} is a face of $P_{i_{j+1}}$. For $i_j < i_{j+1}$ let $\mathbf{n}_{i_j i_{j+1}}$ be the outer unit normal vector of $\partial P_{i_{j+1}}$ in $H_{i_{j+1}}$ at the face P_{i_j} . The vector $p_{i_j} - p_{i_{j+1}}$ is parallel to $\mathbf{n}_{i_j i_{j+1}}$, so that we can define $[i_j : i_{j+1}] \in \mathbb{R}$ by the relation

$$p_{i_j} - p_{i_{j+1}} = [i_j : i_{j+1}] \mathbf{n}_{i_j i_{j+1}}. \quad (2.2)$$

Let $I(k)$ be the index set defined by

$$I(k) = \{i = (i_0, i_1, \dots, i_k); i_j \in I_j, i_0 < i_1 < \dots < i_k\}. \quad (2.3)$$

For $i = (i_0, i_1, \dots, i_k) \in I(k)$ we set

$$[i] = \begin{cases} 1 & (k = 0), \\ [i_0 : i_1][i_1 : i_2] \cdots [i_{k-1} : i_k] & (k = 1, 2, \dots, n). \end{cases} \quad (2.4)$$

Let $h_m^{(j)}(\xi)$ be the complete symmetric polynomial of degree m in j -variables:

$$h_m^{(j)}(\xi_1, \dots, \xi_j) = \sum_{m_1 + \dots + m_j = m} \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_j^{m_j}, \quad (2.5)$$

where the summation is taken over all j -tuples $(m_1, \dots, m_j)$ of nonnegative integers satisfying the indicated condition (see [64]).

The first theorem gives a characterization of the $P(k)$ -mean value property (1.2) in terms of a system of partial differential equations.

Theorem 2.1. *Any $f \in \mathcal{H}_{P(k)}(\Omega)$ is real-analytic and satisfies the system of partial differential equations:*

$$\tau_m^{(k)}(\partial) f = 0 \quad (m = 1, 2, 3, \dots), \quad (2.6)$$

where $\tau_m^{(k)}(\xi)$ is the homogeneous polynomial of degree m defined by

$$\tau_m^{(k)}(\xi) = \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \dots, \langle p_{i_k}, \xi \rangle). \quad (2.7)$$

Conversely, any weak solution of (2.6) is real-analytic and belongs to $\mathcal{H}_{P(k)}(\Omega)$.

Theorems 2.1 and 2.3 below play a key role in establishing Theorem 1.1. Theorem 2.1 is also used to establish a relation between the volume problem and the surface problem.

Theorem 2.2. *For each $i_{n-1} \in I_{n-1}$ there exists a unique $i_n \in I_n$ such that $i_{n-1} \prec i_n$. We set $c_{i_{n-1}} = [i_{n-1} : i_n]$. Assume that $c_{i_{n-1}}$ is independent of $i_{n-1} \in I_{n-1}$. Then,*

$$\mathcal{H}_{P(n-1)}(\Omega) = \mathcal{H}_{P(n)}(\Omega). \quad (2.8)$$

Friedman and Littman [37] have already obtained a system of partial differential equations which characterizes the μ -mean value property (1.1) for a general measure μ (see (3.1) in Theorem 3.1). In fact, our system (2.6) turns out to be equivalent to their system (3.1) for the special measure $\mu = v_k/|P(k)|$ (see Theorem 3.3). The important point here is that their system admits such an explicit representation as (2.6)–(2.7). They gave no such explicit representation. Theorems 2.1 and 2.2 are established in Section 3.

We proceed to the finite-dimensionality problem: *Is $\mathcal{H}_{P(k)}(\Omega)$ finite dimensional?*

The following theorem plays a crucial role in solving this problem.

Theorem 2.3. *The system of algebraic equations:*

$$\tau_m^{(k)}(\xi) = 0 \quad (m = 1, 2, 3, \dots), \quad (2.9)$$

has $\xi = 0$ as its only common root in $\mathbb{C}^n$.

Theorem 2.3 and a standard technique used by Flatto [27], [28] and Friedman and Littman [37] (a method based on the Hilbert Nullstellensatz) lead to an affirmative answer to the finite-dimensionality problem (see Theorem 3.2). Theorem 2.3 is established in Section 4.

Another interesting problem is: *When is any function in $\mathcal{H}_{P(k)}(\Omega)$ harmonic?* This problem is closely related to the symmetry of $P(k)$.

Theorem 2.4. *Let $G \subset O(n)$ be a group of symmetries of $P(k)$. Assume that, for any nonzero vector $v \in \mathbb{R}^n$, the vectors belonging to the orbit Gv span the entire Euclidean space $\mathbb{R}^n$. Then $\mathcal{H}_{P(k)}(\Omega)$ is a finite-dimensional linear space of harmonic polynomials.*

We remark that Theorem 2.4 applies to each skeleton $P(k)$ of any (convex or nonconvex) regular polytopes P in $\mathbb{R}^n$ with center at the origin, $G = G_P$ being the complete symmetry group of P . Theorem 2.4 is established in Section 5.

Following Remark 1.2 we use the simplified notation $\mathcal{H}_{P(k)} = \mathcal{H}_{P(k)}(\Omega)$. Theorem 1.1(2) asserts that there exists a direct sum decomposition:

$$\mathcal{H}_{P(k)} = \bigoplus_{m \geq 0}^{\text{finite}} \mathcal{H}_{P(k)}(m), \quad (2.10)$$

where $\mathcal{H}_{P(k)}(m)$ is the linear space of all homogeneous polynomials of degree m satisfying the $P(k)$ -mean value property (1.2). Moreover Theorem 1.1(3) asserts that $\mathcal{H}_{P(k)}$ carries a structure of $\mathbb{C}[\partial]$ -module. Several problems naturally arise. For examples the following problems are interesting from a combinatorial point of view.

Problem 2.5.

- (1) Determine $\dim \mathcal{H}_{P(k)}$ and construct a basis of $\mathcal{H}_{P(k)}$.
- (2) Determine $\dim \mathcal{H}_{P(k)}(m)$ and construct a basis of $\mathcal{H}_{P(k)}(m)$.
- (3) Determine the structure of $\mathcal{H}_{P(k)}$ as a $\mathbb{C}[\partial]$ -module.

These problems are of particular interest when $P(k)$ admits ample symmetry. We consider the two-dimensional cases to which Theorem 2.4 applies effectively. Let P be a polygon. The k -skeleton $P(k)$ ($k = 0, 1, 2$) is said to admit a $\mathbb{Z}_N$ -symmetry if it admits a cyclic group of rotations of order N as a symmetry group. We identify $\mathbb{R}^2$ with the Gauss plane $\mathbb{C}$ with coordinate $z = x + \sqrt{-1}y$.

Theorem 2.6. *Let P be a polygon and assume that $P(k)$ admit a $\mathbb{Z}_N$ -symmetry for an integer $N \geq 3$. Let $a_{P(k)} \in \mathbb{C}$ be the constant defined by*

$$a_{P(k)} = \int_F \bar{z}^N dv_k(z), \quad (2.11)$$

where F is a fundamental domain of $P(k)$ with respect to the $\mathbb{Z}_N$ -action. Assume that

the constant $a_{P(k)}$ does not vanish. Then $\mathcal{H}_{P(k)}$ is of dimension $2N$ and

$$\mathcal{H}_{P(k)}(m) = \begin{cases} \mathbb{C} & (m = 0), \\ \mathbb{C} z^m \oplus \mathbb{C} \bar{z}^m & (1 \leq m \leq N-1), \\ \mathbb{C} \operatorname{Im}(a_{P(k)} z^N) & (m = N), \\ \{0\} & (m \geq N+1). \end{cases} \quad (2.12)$$

As a $\mathbb{C}[\partial]$ -module $\mathcal{H}_{P(k)}$ is generated by the single element $\operatorname{Im}(a_{P(k)} z^N)$.

Theorem 2.6 is established in Section 5. This theorem applies to any regular star-polygon $\{N/M\}$ with center at the origin. See p. 93 of [20] for the definition of regular star-polygons and the notation $\{N/M\}$. In case $M = 1$, $\{N\}$ is a regular convex N -gon.

Theorem 2.7. *Let P be a regular star-polygon $\{N/M\}$ with center at the origin, where N and M are coprime natural numbers such that $N > 2M$. Then for each $k = 0, 1, 2$, $\mathcal{H}_{P(k)}$ is of dimension $2N$. Let (x, y) be an orthonormal coordinate system of $\mathbb{R}^2$ such that P is symmetric with respect to the x -axis. Then $\mathcal{H}_{P(k)}(m)$ is given by (2.12) with $a_{P(k)}$ replaced by 1. As a $\mathbb{C}[\partial]$ -module $\mathcal{H}_{P(k)}$ is generated by the single element $\operatorname{Im}(z^N)$.*

Theorem 2.7 is established in Section 6. Another application of Theorem 2.6 is also given in Section 6 (see Theorem 6.1).

For any finite subgroup G of $GL(n, \mathbb{R})$ there exists the notion of G -harmonic functions due to Steinberg [74]. Let $\mathcal{H}_G$ be the linear space of all G -harmonic functions. Steinberg [74] established the inequality:

$$\dim \mathcal{H}_G \geq |G|. \quad (2.13)$$

We consider the $G_{P(k)}$ -harmonic functions, where $G_{P(k)} \subset O(n)$ is the complete symmetry group of $P(k)$. For any $m = 1, 2, 3, \dots$, $\tau_m^{(k)}(\xi)$ is a $G_{P(k)}$ -invariant polynomial. Hence Theorem 2.1 leads to the inclusion:

$$\mathcal{H}_{G_{P(k)}} \subset \mathcal{H}_{P(k)}. \quad (2.14)$$

So (2.13) and (2.14) yield a lower bound for the dimension of $\mathcal{H}_{P(k)}$:

$$\dim \mathcal{H}_{P(k)} \geq |G_{P(k)}|. \quad (2.15)$$

Steinberg [74] showed that if G is a finite reflection group, then the inequality in (2.13) is just an equality and as a $\mathbb{C}[\partial]$ -module $\mathcal{H}_G$ is generated by Δ_G , where Δ_G is the fundamental alternating polynomial for G . We notice that the dimension of $\mathcal{H}_{P(k)}$ in Theorem 2.7 coincides with the order of the dihedral group D_{2N} (a finite reflection group) to which the complete symmetry group of the regular star-polygon $\{N/M\}$ is isomorphic. Theorem 2.7 asserts that the inclusion (2.14) is, in fact, an equality for any regular star-polygon $\{N/M\}$. This observation leads to the following:

Problem 2.8. Find (necessary and) sufficient conditions for the equality

$$\mathcal{H}_{G_{P(k)}} = \mathcal{H}_{P(k)}. \quad (2.16)$$

For any regular convex polyhedra P in $\mathbb{R}^3$ with center at the origin we have (2.16) for every $k = 0, 1, 2, 3$ (see [59]). Flatto [28], and Flatto and Wiener [35] established (2.16) for the regular tetrahedron and octahedron in $\mathbb{R}^n$ (for $k = 0, n - 1, n$), for the dodecahedron and icosahedron in $\mathbb{R}^3$, and for all regular convex solids in $\mathbb{R}^4$ (for $k = 0$). See also [47]. Even in dimension 1 there exists an example for which the inclusion (2.14) is strict (see (6.10) and (6.11)). Also in dimension 2 if a polygon P satisfies the assumption of Theorem 2.6 and has $\mathbb{Z}_N$ as the complete symmetry group, then the inclusion (2.14) is strict. There are uncountably many such polygons.

We conclude this section by giving two variations of our problem.

Problem 2.9. In the definition of the $P(k)$ -mean value property we assumed (1.2) for all radii r smaller than r_x . What happens if we assume (1.2) for just one radius, or two or a fixed finite number? This problem is clearly related to the Pompeiu problem in integral geometry (see [11]–[13] and [79]).

Problem 2.10. Our problem makes sense for “polytopes” on any complete Riemannian manifold of constant curvature, and even more generally on symmetric spaces. What happens in such general cases?

Problems 2.9 and 2.10 were suggested by one of the referees.

3. Differential Equations

Flatto [27] first discovered an infinite system of partial differential equations which characterizes the μ -mean value property (1.1). Then Friedman and Littman [37] discovered another system, which is more convenient in many respects. See also [78]. Let us recall some of their results needed in the sequel. We use the usual multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n)$ we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$. The following theorems are fundamental in studying the μ -mean value property (1.1).

Theorem 3.1 [37, Theorem 1]. *Let μ be a nonnegative Borel measure in $\mathbb{R}^n$ with compact support K and total mass $\mu(K) = 1$. Assume that K is not contained in any $(n - 1)$ -dimensional hyperplane. Then any $f \in \mathcal{H}_\mu(\Omega)$ is real-analytic and satisfies the infinite system of partial differential equations:*

$$\sigma_m(\partial)f = 0 \quad (m = 1, 2, 3, \dots), \quad (3.1)$$

where $\sigma_m(\xi)$ is the homogeneous polynomial of degree m defined by

$$\sigma_m(\xi) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} s_\alpha \xi^\alpha, \quad (3.2)$$

where the coefficient s_α is the α -moment of the measure μ :

$$s_\alpha = \int_K y^\alpha d\mu(y). \quad (3.3)$$

Conversely, any weak solution f of (3.1) in Ω is real-analytic and belongs to $\mathcal{H}_\mu(\Omega)$.

Our multi-index notation is slightly different from that in [37]. So (3.1)–(3.3) are slightly different from (1.2)–(1.3) in [37], but equivalent to them.

Theorem 3.2 [37, Theorem 2]. *The linear space $\mathcal{H}_\mu(\Omega)$ is finite dimensional if and only if the system of algebraic equations:*

$$\sigma_m(\xi) = 0 \quad (m = 1, 2, 3, \dots), \quad (3.4)$$

has $\xi = 0$ as its only common root in $\mathbb{C}^n$. If this is the case then any $f \in \mathcal{H}_\mu(\Omega)$ is a polynomial.

We would like to find an explicit representation of the polynomials $\sigma_m(\xi)$ in our situation. The following theorem gives an answer to this problem.

Theorem 3.3. *For the measure $\mu = v_k/|P(k)|$ the polynomials $\sigma_m(\xi)$ in (3.2) are represented as*

$$\sigma_m(\xi) = \frac{\tau_m^{(k)}(\xi)}{|P(k)| \prod_{j=1}^k (m+j)} \quad (m = 1, 2, 3, \dots), \quad (3.5)$$

where $\tau_m^{(k)}(\xi)$ are the polynomials defined by (2.7).

In our situation the coefficients s_α of $\sigma_m(\xi)$, and hence the polynomials $\sigma_m(\xi)$ themselves, may be computed rather explicitly by the repeated use of integration by parts or the Stokes theorem. But so far we have not succeeded along this line. A difficulty lies in the fact that the crucial role played by the vectors p_{i_j} is far from clear from this point of view. So we take another approach to derive the formula (3.5). Our approach is based on the successive applications of the radial differential operator $\delta_r = r \partial_r$ to the $P(k)$ -mean value property (1.2).

In addition to those in Section 2 we use the following notations. For $t = 0, 1, \dots, k+1$, let $I(t, k)$ be the index set defined by

$$I(t, k) = \begin{cases} \{i = (i_t, i_{t+1}, \dots, i_k); i_j \in I_j, i_t < i_{t+1} < \dots < i_k\} & (t = 0, \dots, k), \\ \{\emptyset\} & (t = k+1). \end{cases} \quad (3.6)$$

We remark that $I(0, k) = I(k)$, where $I(k)$ is defined by (2.3). For $i = (i_t, \dots, i_k) \in I(t, k)$ we set

$$[i] = \begin{cases} [i_t : i_{t+1}][i_{t+1} : i_{t+2}] \cdots [i_{k-1} : i_k] & (t = 0, \dots, k-1), \\ 1 & (t = k). \end{cases} \quad (3.7)$$

Let D_{i_j} be the first-order partial differential operator defined by

$$(D_{i_j} f)(x) = \langle p_{i_j}, \partial \rangle f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{f(x + \varepsilon p_{i_j}) - f(x)\}. \quad (3.8)$$

Let v_{i_j} be the Euclidean measure on the affine subspace H_{i_j} . It also denotes the Euclidean measure on $x + rH_{i_j}$ for every $x \in \mathbb{R}^n$ and $r > 0$. We remark that for $i_0 \in I_0$, H_{i_0} is the singleton set $\{p_{i_0}\}$ and v_{i_0} is the delta measure at the point p_{i_0} . We recall that v_k is the Euclidean measure on $P(k)$. It also denotes the Euclidean measure on $x + rP(k)$. We remark that $v_k = v_{i_k}$ on $x + rP_{i_k}$ for $i_k \in I_k$.

Lemma 3.4. *Let f be a C^1 -function in $\mathbb{R}^n$. Then for each $j = 1, 2, \dots, n$ and $i_j \in I_j$ we have*

$$\begin{aligned} \delta_r \int_{x+rP_{i_j}} f dv_{i_j} \\ = r \int_{x+rP_{i_j}} D_{i_j} f dv_{i_j} + r \sum_{i_{j-1} < i_j} [i_{j-1} : i_j] \int_{x+rP_{i_{j-1}}} f dv_{i_{j-1}}. \end{aligned} \quad (3.9)$$

Proof. For any fixed $x \in \mathbb{R}^n$, $r > 0$, and $i_j \in I_j$, we set

$$u(\varepsilon) = \int_{x+e^\varepsilon r P_{i_j}} f dv_{i_j} \quad (\varepsilon \in \mathbb{R}).$$

Let $\pi_\varepsilon: \mathbb{R}^n \rightarrow x + e^\varepsilon r H_{i_j}$ be the orthogonal projection from the Euclidean space $\mathbb{R}^n$ down to the affine subspace $x + e^\varepsilon r H_{i_j}$. We write $u(\varepsilon) = u_1(\varepsilon) + u_2(\varepsilon)$, where

$$u_1(\varepsilon) = \int_{\pi_\varepsilon(x+rP_{i_j})} f dv_{i_j}, \quad u_2(\varepsilon) = \int_{x+e^\varepsilon r P_{i_j}} f dv_{i_j} - \int_{\pi_\varepsilon(x+rP_{i_j})} f dv_{i_j}.$$

A geometric consideration based on Figs. 3.1 and 3.2 yields

$$\begin{aligned} u_1(\varepsilon) &= u(0) + \varepsilon r \int_{x+rP_{i_j}} D_{i_j} f dv_{i_j} + O(\varepsilon^2), \\ u_2(\varepsilon) &= \varepsilon r \sum_{i_{j-1} < i_j} [i_{j-1} : i_j] \int_{x+rP_{i_{j-1}}} f dv_{i_{j-1}} + O(\varepsilon^2). \end{aligned} \quad (3.10)$$

We use the fact that the radial differential operator δ_r is the infinitesimal generator of the similarity transformations with base point at the origin. This implies that the left-hand side of (3.9) equals $(du/d\varepsilon)_{\varepsilon=0}$. But (3.10) shows that $(du/d\varepsilon)_{\varepsilon=0}$ equals the right-hand side of (3.9). The proof is complete. $\square$

For $i = (i_t, i_{t+1}, \dots, i_k) \in I(t, k)$ let $A_m^{(i)}$ be the m th-order homogeneous partial differential operator defined by

$$A_m^{(i)} = \begin{cases} h_m^{(k-t+1)}(D_{i_t}, D_{i_{t+1}}, \dots, D_{i_k}) & (t = 0, \dots, k), \\ 0 & (t = k+1), \end{cases} \quad (3.11)$$

Fig. 3.1

where $h_m^{(j)}(\xi)$ is defined by (2.5). At first glance the right-hand side of (3.11) depends on t and k while the left-hand side does not. We remark that, however, the index i on the left-hand side depends implicitly on t and k . For $i = (i_t, i_{t+1}, \dots, i_k) \in I(t, k)$ we define $i' \in I(t+1, k)$ by

$$i' = \begin{cases} (i_{t+1}, \dots, i_k) & (t = 0, \dots, k-1), \\ \emptyset & (t = k). \end{cases} \quad (3.12)$$

The definition (2.5) of the complete symmetric polynomials $h_m^{(j)}(\xi)$ and the definition (3.11) of $A_m^{(i)}$ readily yield the following:

Lemma 3.5. For $i \in I(t, k)$ ($t = 0, \dots, k$) the following recurrence relation holds:

$$A_m^{(i)} = \begin{cases} 1 & (m = 0), \\ D_{i_t} A_{m-1}^{(i)} + A_m^{(i')} & (m \geq 1). \end{cases} \quad (3.13)$$

Fig. 3.2

Lemma 3.6. For any C^∞ -function f let $F(x, r)$ be the function defined by

$$F(x, r) = \int_{P(k)} f(x + ry) dv_k(y) = \frac{1}{r^k} \int_{x+rP(k)} f dv_k, \quad (3.14)$$

Then the following equality holds for $t = 0, 1, \dots, k$:

$$r^t \prod_{j=t+1}^k (\delta_r + j) F(x, r) = \sum_{j=t}^k \sum_{i \in I(j, k)} [i] \int_{x+rP_{ij}} A_{j-t}^{(i)} f dv_{ij}. \quad (3.15.t)$$

Proof. The definition (3.14) of $F(x, r)$ is rewritten as

$$r^k F(x, r) = \sum_{i_k \in I_k} \int_{x+rP_{i_k}} f dv_{i_k}.$$

This is nothing but (3.15.k). The proof goes by downward induction on t . If (3.15.t) holds then an application of δ_r to (3.15.t) yields

$$\begin{aligned} r^t \prod_{j=t}^n (\delta_r + j) F(x, r) &= \sum_{j=t}^k \sum_{i \in I(j, k)} [i] \int_{x+rP_{ij}} r D_{ij} A_{j-t}^{(i)} f dv_{ij} \\ &\quad + \sum_{j=t}^k \sum_{i \in I(j, k)} \sum_{i_{j-1} < i_j} [i] r [i_{j-1} : i_j] \int_{x+rP_{i_{j-1}}} A_{j-t}^{(i)} f dv_{i_{j-1}} \end{aligned}$$

$$\begin{aligned}
&= r \sum_{j=t}^m \sum_{i \in I(j,k)} [i] \int_{x+rP_{ij}} D_{ij} A_{j-t}^{(i)} f \, dv_{ij} \\
&\quad + r \sum_{j=t}^k \sum_{i \in I(j-1,k)} [i] \int_{x+rP_{ij-1}} A_{j-t}^{(i')} f \, dv_{ij-1} \\
&= r \sum_{j=t}^k \sum_{i \in I(j,k)} [i] \int_{x+rP_{ij}} \{D_{ij} A_{j-t}^{(i)} + A_{j-t+1}^{(i')}\} f \, dv_{ij} \\
&\quad + r \sum_{i \in I(t-1,k)} [i] \int_{x+rP_{it-1}} A_0^{(i')} f \, dv_{it-1} \\
&= r \sum_{j=t-1}^k \sum_{i \in I(j,k)} [i] \int_{x+rP_{ij}} A_{j-i+1}^{(i)} f \, dv_{ij},
\end{aligned}$$

where the first equality follows from (3.9), the second from (3.7), and the fourth from (3.13), respectively. Dividing these equalities by r we establish (3.15. $t-1$). Hence the proof is complete. $\square$

We remark that $P_{i_0} = \{p_{i_0}\}$ is a singleton set for any $i_0 \in I_0$. Hence (3.15.0) is rewritten as

$$\begin{aligned}
&\prod_{j=1}^k (\delta_r + j) F(x, r) \\
&= \sum_{i \in I(k)} [i] f(x + r p_{i_0}) + \sum_{j=1}^k \sum_{i \in I(j,k)} [i] \int_{x+rP_{ij}} A_j^{(i)} f \, dv_{ij}. \quad (3.16)
\end{aligned}$$

Lemma 3.7. *Let $F(x, r)$ be defined by (3.14). Then the following formula holds for every nonnegative integer m :*

$$\begin{aligned}
&\prod_{j=1}^k (\delta_r + m + j) \partial_r^m F(x, r) \\
&= \sum_{i \in I(k)} [i] (A_m^{(i)} f)(x + r p_{i_0}) + \sum_{j=1}^k \sum_{i \in I(j,k)} [i] \int_{x+rP_{ij}} A_{m+j}^{(i)} f \, dv_{ij}. \quad (3.17.m)
\end{aligned}$$

Proof. Since (3.17.0) is just the same as (3.16), (3.17.0) is already established in Lemma 3.6. The proof goes by induction. If (3.17. m) holds then an application of the radial differential operator δ_r to (3.17. m) yields

$$\begin{aligned}
&r \prod_{k=j}^k (\delta_r + m + 1 + j) \partial_r^{m+1} F(x, r) \\
&= r \sum_{i \in I(k)} [i] (D_{i_0} A_m^{(i)})(x + r p_{i_0}) + \sum_{j=1}^k \sum_{i \in I(j,k)} [i] \int_{x+rP_{ij}} r D_{ij} A_{m+j}^{(i)} f \, dv_{ij}
\end{aligned}$$

$$+ \sum_{j=1}^k \sum_{i \in I(j,k)} \sum_{i_{j-1} < i_j} [i] r [i_{j-1} : i_j] \int_{x+rp_{i_{j-1}}} A_{m+j}^{(i)} f \, dv_{i_{j-1}},$$

where Lemma 3.4 is used. Manipulating the second and third terms in the right-hand side in a similar manner as in the proof of Lemma 3.6, we obtain

$$\begin{aligned} & r \prod_{j=1}^k (\delta_r + m + 1 + j) \partial_r^{m+1} F(x, r) \\ &= r \sum_{i \in I(k)} [i] (\{D_{i_0} A_m^{(i)} + A_{m+1}^{(i')}\} f)(x + rp_{i_0}) \\ &+ r \sum_{j=1}^k \sum_{i \in I(j,k)} [i] \int_{x+rp_{i_j}} \{D_{i_j} A_{m+j}^{(i)} + A_{m+1+j}^{(i')}\} f \, dv_{i_j}. \end{aligned}$$

Dividing this equality by r and using (3.13) we establish (3.17. $m+1$). The proof is complete. $\square$

Lemma 3.8. Let $F(x, r)$ and $\tau_m^{(k)}(\xi)$ be defined by (3.14) and (2.7), respectively. Then we have

$$\prod_{j=1}^k (m+j) (\partial_r^m F)_{r=0} = \tau_m^{(k)}(\partial) f \quad (m = 0, 1, 2, \dots). \quad (3.18)$$

Proof. The formula (3.17. m) in Lemma 3.7 is rewritten as

$$\begin{aligned} \prod_{j=1}^k (\delta_r + m + j) \partial_r^m F &= \sum_{i \in I(k)} [i] (A_m^{(i)} f)(x + rp_{i_0}) \\ &+ \sum_{j=1}^k r^j \sum_{i \in I(j,k)} [i] \int_{P_{i_j}} (A_{m+j}^{(i)} f)(x + ry) \, dv_{i_j}(y). \end{aligned}$$

As $r \rightarrow 0$ the left-hand side tends to $\prod_{j=1}^k (m+j) (\partial_r^m F)_{r=0}$ and the second term in the right-hand side vanishes. Hence we obtain

$$\begin{aligned} \prod_{j=1}^k (m+j) (\partial_r^m F)_{r=0} &= \sum_{i \in I(k)} [i] A_m^{(i)} f \\ &= \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \partial \rangle, \dots, \langle p_{i_k}, \partial \rangle) f \\ &= \tau_m^{(k)}(\partial) f. \end{aligned} \quad \square$$

Now we are in a position to establish Theorem 3.3.

Proof of Theorem 3.3. We differentiate (3.14) m -times with respect to r . Differentiations under the integral sign yield

$$(\partial_r^m F)(x, r) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{P(k)} y^\alpha (\partial^\alpha f)(x + ry) \, dv_k(y).$$

Putting $r = 0$ we obtain

$$(\partial_r^m F)_{r=0} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} u_\alpha \partial^\alpha f, \quad (3.19)$$

where the coefficient u_α is the α -moment of the measure $v_k = |P(k)|\mu$:

$$u_\alpha = \int_{P(k)} y^\alpha dv_k(y) = |P(k)|s_\alpha.$$

Comparing (3.2)–(3.3) with (3.18)–(3.19) we establish the theorem. $\square$

Proof of Theorem 2.1. Theorem 2.1 is an immediate consequence of Theorems 3.1 and 3.3. $\square$

Proof of Theorem 2.2. Given any $i_{n-1} \in I_{n-1}$, let $V_{i_{n-1}}^\pm$ be the (closed) n -dimensional half-spaces having $H_{i_{n-1}}$ as their boundary, p an interior point of $P_{i_{n-1}}$ in $H_{i_{n-1}}$, and U a sufficiently small open neighborhood of p in $\mathbb{R}^n$. Then for each $i_n \in I_n$ such that $i_{n-1} < i_n$, $P_{i_n} \cap U$ is contained in just one of the half-spaces $V_{i_{n-1}}^\pm$. We denote it by $V_{i_{n-1}}^{\varepsilon_{i_n}}$. For any $i_n, i'_n \in I_n$ such that $i_{n-1} < i_n, i'_n$, we cannot have $\varepsilon_{i_n} \neq \varepsilon_{i'_n}$; otherwise, $P_{i_n} \cup P_{i'_n}$ would be an n -dimensional cell of P , and hence $P_{i_{n-1}}$ could not be an $(n-1)$ -dimensional face of P (contradiction). But $\varepsilon_{i_n} = \varepsilon_{i'_n}$ implies $P_{i_n} = P_{i'_n}$, i.e., $i_n = i'_n$. Hence there exists precisely one $i_n \in I_n$ such that $i_{n-1} < i_n$. In particular, there exists a one-to-one correspondence between the index sets $I(n-1)$ and $I(n)$. Now we recall that $p_{i_n} = 0$ for any $i_n \in I_n$. Hence for $k = n$, (2.7) is rewritten as

$$\begin{aligned} \tau_m^{(n)}(\xi) &= \sum_{i \in I(n)} [i] h_m^{(n+1)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \dots, \langle p_{i_{n-1}}, \xi \rangle, 0) \\ &= \sum_{i \in I(n-1)} c_{i_{n-1}} [i] h_m^{(n)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \dots, \langle p_{i_{n-1}}, \xi \rangle). \end{aligned} \quad (3.20)$$

We remark that $c_{i_{n-1}} = [i_{n-1} : i_n]$ is nonzero for at least one $i_{n-1} \in I_{n-1}$. This follows from Lemma 4.1(3) in Section 4. So if $c_{i_{n-1}}$ is independent of $i_{n-1} \in I_{n-1}$ then this common number c does not vanish. Now (2.7) for $k = n-1$ and (3.20) imply that $\tau_m^{(n)}(\xi) = c \tau_m^{(n-1)}(\xi)$ for every m . Hence it follows from Theorem 2.1 that $\mathcal{H}_{P(n-1)}(\Omega) = \mathcal{H}_{P(n)}(\Omega)$. The proof is complete. $\square$

Proposition 3.9. *Let μ be as in Theorem 3.1. Assume that $\mathcal{H}_\mu(\Omega)$ is finite dimensional for an open subset Ω of $\mathbb{R}^n$.*

- (1) *The restriction map $\mathcal{H}_\mu(\mathbb{R}^n) \rightarrow \mathcal{H}_\mu(\Omega)$ is an isomorphism.*
- (2) *Any function $f \in \mathcal{H}_\mu(\mathbb{R}^n)$ satisfies the global μ -mean value property, i.e., f satisfies (1.1) for every $x \in \mathbb{R}^n$ and $r > 0$.*

Proof. Assertion (1): Theorem 3.1 implies that any $f \in \mathcal{H}_\mu(\mathbb{R}^n)$ is real-analytic in $\mathbb{R}^n$. Hence the injectivity of the restriction map $\mathcal{H}_\mu(\mathbb{R}^n) \rightarrow \mathcal{H}_\mu(\Omega)$ follows from the unicity theorem for real-analytic functions. We proceed to the surjectivity. Theorem 3.2

implies that any $f \in \mathcal{H}_\mu(\Omega)$ is a polynomial. In particular, f extends to an entire function in $\mathbb{C}^n$. The unicity theorem for holomorphic functions implies that f satisfies the system (3.1) in $\mathbb{C}^n$. Theorem 3.1 implies that $f \in \mathcal{H}_\mu(\mathbb{R}^n)$. Hence the restriction map $\mathcal{H}_\mu(\mathbb{R}^n) \rightarrow \mathcal{H}_\mu(\Omega)$ is surjective. Assertion (2): By Assertion (1), $\mathcal{H}_\mu(\mathbb{R}^n)$ is finite dimensional. Hence Theorem 3.2 implies that any $f \in \mathcal{H}_\mu(\mathbb{R}^n)$ is a polynomial. We expand $f(x + ry)$ as $f(x + ry) = \sum_{j=0}^m f_j(x, y)r^j$ where $f_j(x, y)$ are polynomials in (x, y) . The μ -mean value property (1.1) states that

$$f(x) = \sum_{j=0}^m r^j \int_K f_j(x, y) d\mu(y) \quad (3.21)$$

holds for every $0 < r < r_x$, where r_x is a sufficiently small positive constant depending on $x \in \mathbb{R}^n$. Since the both sides of (3.21) are polynomials in r , it is clear that f satisfies (3.21) for every $r > 0$. $\square$

4. Characteristic Variety

We are interested in the characteristic variety of the system (2.6) of partial differential equations, i.e., the common roots of the system (2.9) of algebraic equations. In addition to those in the previous sections we use the following notations. Let $I^*(k)$ be the subset of $I(k)$ defined by

$$I^*(k) = \{i \in I(k); [i] \neq 0\}, \quad (4.1)$$

where $I(k)$ is given by (2.3). Given any $i_k \in I_k$, let $I(i_k)$ be the index set defined by

$$I(i_k) = \{i = (i_0, \dots, i_{k-1}, i_k); i_j \in I_j, i_0 < \dots < i_{k-1} < i_k\}. \quad (4.2)$$

We remark that (4.2) is different from (2.3) since the index i_k is now fixed; $I(k)$ is the union of $I(i_k)$ over all $i_k \in I_k$. Moreover, we set

$$I^*(i_k) = I^*(k) \cap I(i_k). \quad (4.3)$$

Finally, let $H_{i_k}^*$ be the k -dimensional linear subspace of $\mathbb{R}^n$ parallel to the affine subspace H_{i_k} .

Lemma 4.1.

- (1) For any $i \in I(i_k)$, i belongs to $I^*(i_k)$ if and only if the $(k+1)$ -points $p_{i_0}, p_{i_1}, \dots, p_{i_k}$ are in a general position in the affine space H_{i_k} .
- (2) $\sum_{i \in I^*(i_k)} [i] = k! |P_{i_k}|$ for any $i_k \in I_k$.
- (3) $\sum_{i \in I^*(k)} [i] = k! |P(k)|$ for any $k = 0, \dots, n$.

Proof. For any $i = (i_0, \dots, i_t, \dots, i_k) \in I(k)$, the definition (2.1) of the vectors p_{i_j} implies

$$p_{i_t} - p_{i_k} = \sum_{j=t}^{k-1} [i_j : i_{j+1}] \mathbf{n}_{i_j i_{j+1}} \quad (t = 0, \dots, k-1). \quad (4.4)$$

The vectors $(\mathbf{n}_{i_0 i_1}, \dots, \mathbf{n}_{i_{k-1} i_k})$ form an orthonormal basis of $H_{i_k}^*$. Let Δ_i be the k -dimensional simplex in H_{i_k} defined as the convex hull of the points $p_{i_0}, p_{i_1}, \dots, p_{i_k}$. Then (4.4) implies that $[i] = [i_0 : i_1][i_1 : i_2] \cdots [i_{k-1} : i_k]$ counts $k!$ -times the *signed* volume of the simplex Δ_i . Three assertions of the lemma easily follow from this fact. Assertion (1): For any $i \in I(i_k)$, $[i] \neq 0$ if and only if Δ_i has a nonzero volume. This is equivalent to the condition that $p_0, \dots, p_k$ are in a general position in H_{i_k} . Assertion (2): The volume of P_{i_k} equals the sum of the signed volumes of Δ_i for $i \in I(i_k)$. Clearly the sum can be taken only over $i \in I^*(i_k)$. This establishes Assertion (2). Assertion (3): Summing up the formula in Assertion (2) over $i_k \in I_k$, we obtain Assertion (3) immediately. $\square$

Lemma 4.2. *For any $\xi = (\xi_0, \dots, \xi_k) \in \mathbb{C}^{k+1}$, the following formula holds as an identity among holomorphic functions in a neighborhood of $z = 0$:*

$$\prod_{j=0}^k (1 - \xi_j z)^{-1} = \sum_{m=0}^{\infty} h_m^{(k+1)}(\xi_0, \dots, \xi_k) z^m. \quad (4.5)$$

Proof. If $\xi \neq 0$, then we put $R = \min |\xi_j|^{-1}$, where the minimum is taken over all $j = 0, \dots, k$ such that $\xi_j \neq 0$. If $\xi = 0$, then we put $R = \infty$. For $|z| < R$, we have

$$\begin{aligned} \prod_{j=0}^k (1 - \xi_j z)^{-1} &= \prod_{j=0}^k \sum_{m_j=0}^{\infty} (\xi_j z)^{m_j} \\ &= \sum_{m_0=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \xi_0^{m_0} \cdots \xi_k^{m_k} z^{m_0 + \cdots + m_k} \\ &= \sum_{m=0}^{\infty} \left\{ \sum_{m_0 + \cdots + m_k = m} \xi_0^{m_0} \cdots \xi_k^{m_k} \right\} z^m \\ &= \sum_{m=0}^{\infty} h_m^{(k+1)}(\xi_0, \dots, \xi_k) z^m. \end{aligned} \quad \square$$

Lemma 4.3. *For any $\xi \in \mathbb{C}^n$, the following formula holds as an identity among holomorphic functions in a neighborhood of $z = 0$:*

$$\sum_{i \in I^*(k)} [i] \prod_{j=0}^k (1 - \langle p_{i_j}, \xi \rangle z)^{-1} = k! |P(k)| + \sum_{m=1}^{\infty} \tau_m^{(k)}(\xi) z^m. \quad (4.6)$$

Proof. Lemma 4.1(3) and Lemma 4.2 imply

$$\sum_{i \in I(k)} [i] \prod_{j=0}^k (1 - \langle p_{i_j}, \xi \rangle z)^{-1} = \sum_{i \in I(k)} [i] \sum_{m=0}^{\infty} h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \dots, \langle p_{i_k}, \xi \rangle) z^m$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} z^m \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \dots, \langle p_{i_k}, \xi \rangle) \\
&= k! |P(k)| + \sum_{m=1}^{\infty} \tau_m^{(k)}(\xi) z^m.
\end{aligned}$$

Clearly the sum in the left-hand side can be taken only over $i \in I^*(k)$. This establishes the lemma. $\square$

Lemma 4.4. For any $i_k \in I_k$, let $\theta_{i_k}: \mathbb{C}^n \rightarrow \{0, 1\}$ be the function defined by

$$\theta_{i_k}(\xi) = \begin{cases} 1 & (\langle v, \xi \rangle = 0 \text{ for all } v \in H_{i_k}^* \oplus \mathbb{R}p_{i_k}), \\ 0 & (\text{otherwise}). \end{cases} \quad (4.7)$$

If $\xi \in \mathbb{C}^n$ is a common root of the system (2.9) of algebraic equations, then

$$\sum_{i_k \in I_k} \theta_{i_k}(\xi) |P_{i_k}| = |P(k)|. \quad (4.8)$$

Proof. Since $\xi \in \mathbb{C}^n$ is a common root of the system (2.9), Lemma 4.3 implies that

$$\sum_{i \in I^*(k)} [i] \prod_{j=0}^k (1 - \langle p_{i_j}, \xi \rangle z)^{-1} = k! |P(k)| \quad (4.9)$$

holds in a neighborhood of $z = 0$. Since both sides are rational in z this equality holds as an identity among rational functions in z . Hence, we can let z tend to ∞ in (4.9). We have

$$\lim_{z \rightarrow \infty} \prod_{j=0}^k (1 - \langle p_{i_j}, \xi \rangle z)^{-1} = \hat{\theta}_i(\xi) \quad (i \in I^*(k)), \quad (4.10)$$

where $\hat{\theta}_i(\xi)$ is given by

$$\hat{\theta}_i(\xi) = \begin{cases} 1 & (\langle p_{i_j}, \xi \rangle = 0 \text{ for all } j = 0, \dots, k), \\ 0 & (\text{otherwise}). \end{cases} \quad (4.11)$$

For any $i \in I^*(i_k)$ Lemma 4.1(1) implies that the $(k+1)$ -points $p_{i_0}, \dots, p_{i_k}$ are in a general position in H_{i_k} . Hence the vectors $p_{i_0}, \dots, p_{i_k}$ span the linear subspace $H_{i_k}^* \oplus p_{i_k} \mathbb{R}$. Hence (4.7) and (4.11) imply that for any $i \in I^*(k)$,

$$\hat{\theta}_i(\xi) = \theta_{i_k}(\xi) \quad \text{if } i \in I^*(i_k). \quad (4.12)$$

Applying (4.10) and (4.12) to (4.9) we obtain

$$\begin{aligned}
k! |P(k)| &= \sum_{i \in I^*(k)} \hat{\theta}_i(\xi) [i] = \sum_{i_k \in I_k} \sum_{i \in I^*(i_k)} \theta_{i_k}(\xi) [i] \\
&= \sum_{i_k \in I_k} \theta_{i_k}(\xi) \sum_{i \in I^*(i_k)} [i] = k! \sum_{i_k \in I_k} \theta_{i_k}(\xi) |P_{i_k}|,
\end{aligned}$$

where Lemma 4.1(2) is used to derive the last equality. The proof is complete. $\square$

Lemma 4.5. *Let $\xi \in \mathbb{C}^n$ be any common root of the system (2.9) of algebraic equations. We write $\xi = \xi' + \sqrt{-1}\xi''$, where ξ' and ξ'' are real vectors in $\mathbb{R}^n$. Then ξ' and ξ'' are orthogonal to the subspace $H_{i_k}^* \oplus \mathbb{R}p_{i_k}$ for any $i_k \in I_k$.*

Proof. The identity $\sum_{i_k \in I_k} |P_{i_k}| = |P(k)|$ and (4.8) yields

$$\sum_{i_k \in I_k} \{1 - \theta_{i_k}(\xi)\} |P_{i_k}| = 0. \quad (4.13)$$

Since $\theta_{i_k}(\xi) = 0$ or 1 and $|P_{i_k}| > 0$ for any $i_k \in I_k$, we must have $\theta_{i_k}(\xi) = 1$ for any $i_k \in I_k$. Now (4.7) implies that $\langle v, \xi \rangle = 0$ for all $v \in H_{i_k}^* \oplus \mathbb{R}p_{i_k}$. Hence the real and imaginary parts of ξ are orthogonal to the space $H_{i_k}^* \oplus \mathbb{R}p_{i_k}$ for any $i_k \in I_k$. The proof is complete. $\square$

Lemma 4.6. *For any $j = 1, 2, \dots, n-1$ and $i_{j+1} \in I_{j+1}$ we have*

$$H_{i_{j+1}}^* = \sum_{i_j < i_{j+1}} H_{i_j}^*, \quad (4.14)$$

where the right-hand side is the sum of linear spaces.

Proof. For any $i_j \in I_j$ such that $i_j < i_{j+1}$, P_{i_j} is a j -dimensional face of the $(j+1)$ -dimensional polytope $P_{i_{j+1}}$. Now we recall that H_{i_j} (resp. $H_{i_{j+1}}$) is the j - (resp. $(j+1)$ -) dimensional affine subspace containing P_{i_j} (resp. $P_{i_{j+1}}$) and $H_{i_j}^*$ (resp. $H_{i_{j+1}}^*$) is the j - (resp. $(j+1)$ -) dimensional linear subspace parallel to H_{i_j} (resp. $H_{i_{j+1}}$). Hence $H_{i_j}^*$ is a linear subspace of $H_{i_{j+1}}^*$ of codimension one. In order for $P_{i_{j+1}}$ to be a polytope there exist at least two indices $i_j, i'_j \in I_j$ such that $i_j, i'_j < i_{j+1}$ and $H_{i_j}^* \neq H_{i'_j}^*$. Since $H_{i_j}^*$ and $H_{i'_j}^*$ are of dimension $j \geq 1$ and of codimension one in $H_{i_{j+1}}^*$, we obtain $H_{i_j}^* + H_{i'_j}^* = H_{i_{j+1}}^*$. This establishes the lemma. $\square$

Now we are in a position to establish Theorems 2.3 and 1.1.

Proof of Theorem 2.3. Let $\xi \in \mathbb{C}^n$ be any common root of the system (2.9). As in Lemma 4.5, we write $\xi = \xi' + \sqrt{-1}\xi''$. Lemma 4.5 implies that ξ' and ξ'' are orthogonal to $H_{i_k}^* \oplus \mathbb{R}p_{i_k}$ for any $i_k \in I_k$. We show that ξ' and ξ'' are orthogonal to $H_{i_{k+1}}^*$ for any $i_{k+1} \in I_{k+1}$. First we consider the case $k = 0$. For any $i_1 \in I_1$ there exist two indices $i_0, i'_0 \in I_0$ such that $i_0, i'_0 < i_1$ and the vector $p_{i_0} - p_{i'_0}$ spans $H_{i_1}^*$. Hence ξ' and ξ'' are orthogonal to $H_{i_1}^*$ for any $i_1 \in I_1$. In case $k \geq 1$, we apply Lemma 4.6 for $j = k$ to show that ξ' and ξ'' are orthogonal to $H_{i_{k+1}}^*$ for any $i_{k+1} \in I_{k+1}$. Using Lemma 4.6 we repeat the same argument for $j = k+1, \dots, n-1$. Then we eventually conclude that ξ' and ξ'' are orthogonal to $H_{i_n}^* = \mathbb{R}^n$ (for any $i_n \in I_n$). Hence we obtain $\xi' = \xi'' = 0$, i.e., $\xi = 0$. The proof is complete. $\square$

Proof of Theorem 1.1. Assertion (1) follows immediately from Theorems 2.3 and 3.2. Assertion (2) follows from the fact that the polynomials $\tau_m^{(k)}(\xi)$ in (2.7) are homogeneous.

Indeed, let f be any element in $\mathcal{H}_{P(k)}(\Omega)$ and write $f = \sum_{j=0}^N f_j$ for some natural number N , where f_j is a homogeneous polynomial of degree j . Let $E = x_1 \partial_1 + \cdots + x_n \partial_n$ be the Euler operator. Since $\tau_m^{(k)}(\xi)$ is homogeneous of degree m , we have $[E, \tau_m^{(k)}(\partial)] = m \tau_m^{(k)}(\partial)$. Hence the Euler operator E keeps the solution space of (2.6) invariant. Hence Theorem 2.1 implies

$$E^s f = \sum_{j=0}^N j^s f_j \in \mathcal{H}_{P(k)}(\Omega) \quad (s = 0, 1, \dots, N).$$

Now the Vandermonde determinant formula implies $f_j \in \mathcal{H}_{P(k)}(\Omega)$ for any $j = 0, 1, \dots, N$. Assertion (2) is established. Assertion (3) follows easily from Theorem 2.1: If $f \in \mathcal{H}_{P(k)}(\Omega)$, then Theorem 2.1 implies that f satisfies the system (2.6) of partial differential equations. Since (2.6) is a system with constant coefficients, for any $L \in \mathbb{C}[\partial]$, Lf also satisfies the system (2.6). Hence Theorem 2.1 again implies that $Lf \in \mathcal{H}_{P(k)}(\Omega)$. The theorem is established $\square$

5. Polytopes with Symmetry

We show that if $P(k)$ admits ample symmetry, then any function in $\mathcal{H}_{P(k)}(\Omega)$ is harmonic in the usual sense.

Lemma 5.1. *Let $G \subset O(n)$ be a group of symmetries of $P(k)$. Assume that, for any nonzero vector $v \in \mathbb{R}^n$, the vectors belonging to the orbit Gv span the entire Euclidean space $\mathbb{R}^n$. Then $\tau_1^{(k)}(\xi) = 0$ and $\tau_2^{(k)}(\xi) = \lambda(\xi, \xi)$ for some constant $\lambda \in \mathbb{R}$.*

Proof. Since $G \subset O(n)$ is a group of symmetries of $P(k)$, the linear form $\tau_1^{(k)}(\xi)$ and the quadratic form $\tau_2^{(k)}(\xi)$ are G -invariant. There exists a nonzero vector $v \in \mathbb{R}^n$ such that $\tau_1^{(k)}(v) = 0$. Hence $\tau_1^{(k)}(gv) = 0$ for every $g \in G$. By assumption the vectors gv ($g \in G$) span the entire Euclidean space $\mathbb{R}^n$. It follows that $\tau_1^{(k)}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$. Hence we obtain $\tau_1^{(k)}(\xi) = 0$. Let $Q = (q_{st})$ be the symmetric matrix associated with the quadratic form $\tau_2^{(k)}(\xi)$. Since $\tau_2^{(k)}(\xi) = \sum_{s,t=1}^n q_{st} \xi_s \xi_t$, it suffices to show that Q is a scalar matrix. Let λ be an eigenvalue of Q , and v the corresponding eigenvector. The G -invariance of $\tau_2^{(k)}(\xi)$ means that the matrix Q commutes with every element $g \in G$. Hence all gv ($g \in G$) are eigenvectors of Q corresponding to the eigenvalue λ . Since these vectors span $\mathbb{R}^n$, the entire Euclidean space $\mathbb{R}^n$ is the eigenspace of Q corresponding to the eigenvalue λ . Hence Q is the scalar matrix λI . The proof is complete. $\square$

Lemma 5.2. *The constant $\lambda = \lambda_{P(k)}$ in Lemma 5.1 is explicitly given by*

$$\lambda_{P(k)} = \frac{1}{n} \sum_{i \in I(k)} [i] \sum_{j=0}^k (j+1) |p_{ij}|^2. \quad (5.1)$$

This constant is positive.

Proof. Let Δ be the Laplacian with respect to $\xi = (\xi_1, \dots, \xi_n)$. We use the formula $\Delta(\langle v, \xi \rangle \langle w, \xi \rangle) = 2\langle v, w \rangle$ for two vectors v, w . Lemma 5.1 implies

$$\tau_2^{(k)}(\xi) = \sum_{i \in I(k)} [i] \sum_{t=0}^k \sum_{s=0}^t \langle p_{i_s}, \xi \rangle \langle p_{i_t}, \xi \rangle = \lambda \sum_{j=1}^n \xi_j^2.$$

Applying the Laplacian Δ to this we obtain

$$\sum_{i \in I(k)} [i] \sum_{t=0}^k \sum_{s=0}^t \langle p_{i_s}, p_{i_t} \rangle = n\lambda. \quad (5.2)$$

For $s = 0, 1, \dots, t$, the vectors $(\mathbf{n}_{i_s i_{s+1}}, \dots, \mathbf{n}_{i_{t-1} i_t}, p_{i_t})$ form an orthogonal system. Hence (4.4) yields

$$\langle p_{i_s}, p_{i_t} \rangle = |p_{i_t}|^2 \quad (s = 0, 1, \dots, t). \quad (5.3)$$

Substituting (5.3) into (5.2) we establish (5.1). We show that the constant λ is positive. Theorem 3.3 and Lemma 5.1 yield

$$\sigma_2(\xi) = \frac{\lambda \langle \xi, \xi \rangle}{|P(k)| \prod_{j=1}^k (j+2)}. \quad (5.4)$$

On the other hand, (3.2) and (3.3) imply

$$\sigma_2(\xi) = \int_{P(k)} \langle \xi, y \rangle^2 d\mu(y). \quad (5.5)$$

(See also (1.12) in [37].) If $\sigma_2(\xi) = 0$ for some $\xi \in \mathbb{R}^n$, then (5.5) implies that $\langle \xi, y \rangle = 0$ for any $y \in P(k)$. Since $P(k)$ is the k -skeleton of a polytope in $\mathbb{R}^n$, the vectors $y \in P(k)$ span the entire Euclidean space $\mathbb{R}^n$. Hence we have $\xi = 0$. So the real quadratic form $\sigma_2(\xi)$ is positive definite. Therefore (5.4) implies $\lambda > 0$. $\square$

Proof of Theorem 2.4. Lemmas 5.1 and 5.2 imply $\tau_2^{(k)}(\partial) = \lambda_{P(k)} \Delta$, where $\lambda_{P(k)}$ is a positive constant and Δ is the Laplacian in $\mathbb{R}^n$. Hence Theorem 2.4 is an immediate consequence of Theorem 2.1. $\square$

Proof of Theorem 2.6. Let G be a cyclic group of rotations of order N and assume that G is a symmetry group of $P(k)$. It is well known that the homogeneous polynomials $1, z^m, \bar{z}^m$ ($m = 1, 2, \dots$) give a basis of the harmonic polynomials in two variables. Since $N \geq 3$, for any nonzero vector $v \in \mathbb{R}^2$, the vectors belonging to the orbit Gv span the entire Euclidean plane $\mathbb{R}^2$. It follows from Theorem 2.4 that $\mathcal{H}_P(m) \subset \mathbb{C}z^m \oplus \mathbb{C}\bar{z}^m$ for $m \geq 1$. For $m = 1, \dots, N-1$, we have

$$\int_{P(k)} z^m dv_k(z) = \sum_{j=0}^{N-1} \int_{\zeta^j F} z^m dv_k(z) = \sum_{j=0}^{N-1} \zeta^{mj} \int_F z^m dv_k(z) = 0,$$

where ζ is a primitive N th root of unity. This means that z^m and $\bar{z}^m$ ($m = 1, \dots, N-1$) satisfy the $P(k)$ -mean value property (1.2) at the origin. The binomial theorem

$$z^m = \sum_{j=0}^m \binom{m}{j} w^{m-j} (z-w)^j,$$

implies that they satisfy the $P(k)$ -mean value property (1.2) at any point $w \in \mathbb{C}$. Hence $1, z^m, \bar{z}^m \in \mathcal{H}_{P(k)}$ for $m = 1, \dots, N-1$. For simplicity, we put $a = a_{P(k)}$, where $a_{P(k)}$ is defined by (2.11). For $m = N$ we have

$$\begin{aligned} \int_P \operatorname{Im}(az^N) dv_k(z) &= \frac{N}{2\sqrt{-1}} \int_F (az^N - \bar{a}\bar{z}^N) dv_k(z) \\ &= \frac{N}{2\sqrt{-1}} (a\bar{a} - \bar{a}a) = 0. \end{aligned} \quad (5.6)$$

The binomial theorem and (5.6) implies that $\operatorname{Im}(az^N) \in \mathcal{H}_{P(k)}$. By using the assumption $a \neq 0$ it is easy to see that any $f \in \mathcal{H}_{P(k)}(N)$ is a constant multiple of $\operatorname{Im}(az^N)$. Finally, we show that $\mathcal{H}_{P(k)}(m) = \{0\}$ for any $m \geq N+1$. Since $\mathcal{H}_{P(k)}(m) \subset \mathbb{C}z^m \oplus \mathbb{C}\bar{z}^m$, any element $f \in \mathcal{H}_{P(k)}(m)$ is expressed as $f = \alpha z^m + \beta \bar{z}^m$ for some $\alpha, \beta \in \mathbb{C}$. Consider the differential operators:

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - \sqrt{-1}\partial_y), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + \sqrt{-1}\partial_y).$$

Since $\mathcal{H}_{P(k)}$ is a $\mathbb{C}[\partial]$ -module we apply these operators repeatedly to f . Then we obtain

$$\begin{aligned} (\partial_z)^{m-N} f &= \alpha m(m-1) \cdots (N+1) z^N \in \mathcal{H}_{P(k)}, \\ (\partial_{\bar{z}})^{m-N} f &= \beta m(m-1) \cdots (N+1) \bar{z}^N \in \mathcal{H}_{P(k)}. \end{aligned}$$

But the assumption $a \neq 0$ implies that neither z^N nor $\bar{z}^N$ belongs to $\mathcal{H}_{P(k)}$. Hence we must have $\alpha = \beta = 0$. The proof is complete. $\square$

6. Examples

The first example is regular star-polygons $P = \{N/M\}$ with center at the origin. A complete characterization of $\mathcal{H}_{P(k)}$ is given in Theorem 2.7, which we establish below.

Proof of Theorem 2.7. Any regular star-polygon $\{N/M\}$ with center at the origin admits a $\mathbb{Z}_N$ -symmetry. To apply Theorem 2.6 we have only to show that the constant $a_{P(k)}$ in (2.11) is a nonzero real number for $k = 0, 1, 2$. We remark that Theorem 2.2 applies to any regular star-polygon $\{N/M\}$. So the area problem ($k = 2$) and the edge problem ($k = 1$) have the same solutions. Accordingly, we have only to consider the cases $k = 0, 1$. We set

$$z_j = \exp\left(\frac{2\pi\sqrt{-1}j}{N}\right), \quad w_j = wz_j \quad (j = 0, 1, \dots, N-1), \quad (6.1)$$

where w is the complex number defined by

$$w = \frac{\cos(\pi M/N)}{\cos\{\pi(M-1)/N\}} \exp\left(\frac{\pi\sqrt{-1}}{N}\right). \quad (6.2)$$

The coordinate system (x, y) is taken so that P is symmetric with respect to the x -axis. In case $M \geq 2$ we may assume that the vertices of P are given by $z_0, \dots, z_{N-1}$ and $w_0, \dots, w_{N-1}$. In case $M = 1$ the vertices of P are only $z_0, \dots, z_{N-1}$. (The point w_j is the midpoint of the edge $[z_j, z_{j+1}]$ in P .)

The vertex problem ($k = 0$): In case $M = 1$, we take the singleton set $\{1\}$ as a fundamental domain F . Then we have $a_{P(0)} = 1$. In case $M \geq 2$ we take $F = \{1, w\}$. Then we have

$$a_{P(0)} = 1 - \frac{\cos(\pi M/N)}{\cos\{\pi(M-1)/N\}} > 0. \quad (6.3)$$

In any case $a_{P(0)}$ is a positive real number.

The edge problem ($k = 1$): A fundamental domain F of $P(1)$ is given by the union of two line segments $[1, w_0]$ and $[1, w_{N-1}]$. We compute the constant $a_{P(1)}$ in (2.11) with respect to the fundamental domain F . In terms of the polar coordinates $a_{P(1)}$ is computed as

$$a_{P(1)} = \frac{2\pi}{N} \left(\cos \frac{\pi M}{N} \right)^{N+1} \int_0^{1/2} u(\theta) \sin \pi \theta d\theta, \quad (6.4)$$

where $u(\theta)$ is the function defined by

$$u(\theta) = \frac{1}{[\cos\{\pi(M - \frac{1}{2} + \theta)/N\}]^{N+1}} - \frac{1}{[\cos\{\pi(M - \frac{1}{2} - \theta)/N\}]^{N+1}}. \quad (6.5)$$

Since $u(\theta)$ is positive for $0 < \theta \leq \frac{1}{2}$, the constant $a_{P(1)}$ is a positive real number. This establishes the theorem. $\square$

As the second example we consider another type of star-polygons. Let N be any integer ≥ 3 . By definition a star-polygon $[2N]$ is the union $P = P_0 \cup P_1$ of two congruent regular convex N -gons P_0 and P_1 with center at the origin such that $P_0 \neq P_1$. A star-polygon $[2N]$ is said to be *regular* if the rotation of angle π/N around the origin maps P_0 onto P_1 ; otherwise, it is said to be *irregular*. We denote by $[2N]_r$ (resp. $[2N]_i$) a regular (resp. an irregular) star-polygon $[2N]$. A regular star-polygon $[2N]_r$ admits a $\mathbb{Z}_{2N}$ -symmetry while an irregular star-polygon $[2N]_i$ admits a $\mathbb{Z}_N$ -symmetry. The case $N = 3$ is illustrated by Figs. 6.1 and 6.2. We remark that Theorem 2.2 applies to any star-polygon $[2N]$. So we have only to consider the vertex ($k = 0$) and edge ($k = 1$) problems. We compute the constant $a_{P(k)}$ in (2.11) and apply Theorem 2.6 to establish the following:

Fig. 6.1 $[6]_r$.

Theorem 6.1. *Let N be any integer ≥ 3 , P any star-polygon $[2N]$, and $k = 0, 1, 2$. We take an orthonormal coordinate system (x, y) of $\mathbb{R}^2$ such that P is symmetric with respect to the x -axis. Then:*

- (1) *If P is a regular star-polygon $[2N]_r$, then $\dim \mathcal{H}_{P(k)} = 4N$ and $\mathcal{H}_{P(k)}(m)$ is given by (2.12) with N and $a_{P(k)}$ replaced by $2N$ and 1, respectively.*
- (2) *If P is an irregular star-polygon $[2N]_i$, then $\dim \mathcal{H}_{P(k)} = 2N$ and $\mathcal{H}_{P(k)}(m)$ is given by (2.12) with $a_{P(k)}$ replaced by 1.*

Fig. 6.2 $[6]_i$.

Theorem 2.6 has many other applications of the same type, which are omitted here. Polytopes with symmetry will be discussed more thoroughly elsewhere.

Finally, we give a complete description of $\mathcal{H}_{P(k)}$ for any one-dimensional polytope. In this case the problem is quite simple but still of some interest. Any one-dimensional polytope P is a finite union of closed bounded intervals

$$P = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{2N-1}, a_{2N}], \quad (6.6)$$

in the real line $\mathbb{R}$, where $a_1 < a_2 < \cdots < a_{2N}$.

Theorem 6.2. *Let P be any one-dimensional polytope given by (6.6). Then*

$$\mathcal{H}_{P(0)} = \begin{cases} \mathbb{C} & (\sum_{j=1}^{2N} a_j \neq 0), \\ \mathbb{C} \oplus \mathbb{C}x & (\sum_{j=1}^{2N} a_j = 0), \end{cases} \quad (6.7)$$

$$\mathcal{H}_{P(1)} = \begin{cases} \mathbb{C} & (\sum_{j=1}^{2N} (-1)^j |a_j| \neq 0), \\ \mathbb{C} \oplus \mathbb{C}x & (\sum_{j=1}^{2N} (-1)^j |a_j| = 0). \end{cases} \quad (6.8)$$

Proof. The polynomials $\tau_m^{(k)}(\xi)$ in (2.7) are computed as

$$\begin{aligned} \tau_m^{(0)}(\xi) &= A_m \xi^m, & \text{where } A_m &= \sum_{j=1}^{2N} a_j^m, \\ \tau_m^{(1)}(\xi) &= B_m \xi^m, & \text{where } B_m &= \sum_{j=1}^{2N} (-1)^j (\operatorname{sgn} a_j) a_j^m, \end{aligned}$$

where $\operatorname{sgn} a_j$ is the signature of a_j . We have $A_2 = \sum_{j=1}^{2N} a_j^2 > 0$. So the system (2.6) is equivalent to $A_1 \partial f = \partial^2 f = 0$. Hence Theorem 2.1 implies (6.7). Similarly, we claim $B_2 > 0$. Indeed, if $a_{2s} \leq 0 \leq a_{2s+1}$, then

$$B_2 = \sum_{j=1}^s (a_{2j-1}^2 - a_{2j}^2) + \sum_{j=s+1}^N (a_{2j}^2 - a_{2j-1}^2) > 0,$$

and if $a_{2s-1} \leq 0 \leq a_{2s}$, then

$$B_2 = \sum_{j=1}^{s-1} (a_{2j-1}^2 - a_{2j}^2) + (a_{2s-1}^2 + a_{2s}^2) + \sum_{j=s+1}^N (a_{2j}^2 - a_{2j-1}^2) > 0.$$

In any case, we have $B_2 > 0$. So the system (2.6) is equivalent to $B_1 \partial f = \partial^2 f = 0$. Hence Theorem 2.1 implies (6.8). The proof is complete. $\square$

For $k = 0, 1$, the complete symmetry group $G_{P(k)}$ of $P(k)$ is given by

$$G_{P(k)} = \begin{cases} \{1\} & (a_j + a_{2N+1-j} \neq 0 \text{ for some } j = 1, \dots, N), \\ \{\pm 1\} & (a_j + a_{2N+1-j} = 0 \text{ for all } j = 1, \dots, N), \end{cases} \quad (6.9)$$

Hence the inclusion (2.14) is strict if and only if

$$N \geq 2, \quad a_j + a_{2N+1-j} \neq 0 \quad \text{for some } j = 1, \dots, N, \quad (6.10)$$

and

$$\begin{cases} \sum_{j=1}^{2N} a_j = 0 & (k = 0), \\ \sum_{j=1}^{2N} (-1)^j |a_j| = 0 & (k = 1). \end{cases} \quad (6.11)$$

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