On the Quantitative Steinitz Theorem in the Plane

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Dedicated to Professor Dr. Jürgen Flachsmeier on the occasion of his sixtieth birthday

Abstract. We prove that for any $k \geq 4$, any set X of points in the plane, and any point $P \in \text{interior conv}(X)$, there is a subset $Y \subset X$ of at most k points such that if $\text{conv}(X)$ contains a disk with radius r around P , then $\text{conv}(Y)$ contains a disk with radius $[\cos(2/(k+1))\pi]/[\cos(1/(k+1))\pi]r$ around P . This generalizes the quantitative Steinitz theorem in the plane and proves a conjecture of Bárány and Heppes.

1. Introduction

It is a well-known theorem in convexity theory that for any set $X \subset \mathbb{R}^d$ and any point P in the interior of $\text{conv}(X)$ there is a subset $Y \subset X$ of at most $2d$ points such that P is in the interior of $\text{conv}(Y)$. This number of $2d$ points can be necessary, for if $(e_i)_{i=1}^d$ is a basis of $\mathbb{R}^d$, then $X = \{e_1, -e_1, \dots, e_d, -e_d\}$ (the vertexset of a cross-polytope) contains 0 in the interior of its convex hull, but no proper subset of X does.

This theorem of Steinitz was made quantitative by Bárány *et al.* [2], who proved the existence of constants $c_d > 0$ such that if $X \subset \mathbb{R}^d$ contains a ball with center P and radius r in its convex hull, then there is a subset $Y \subset X$ of at most $2d$ points whose convex hull contains a ball with center P and radius $c_d r$. Without loss of generality we may always assume that $r = 1$ and $P = 0$. Bárány *et al.* gave a lower bound for c_d which is of order d^{-2d} , sketched an improvement to $d^{-(1/2)d-2}$, but conjectured the right order of magnitude to be $d^{-1/2}$.

Bárány and Heppes [1] determined the constant c_2 of the quantitative Steinitz theorem in the plane. They proved that any set $X \subset \mathbb{R}^2$ whose convex hull contains a disk with center 0 and radius 1 has a subset $Y \subset X$ of at most four points whose convex hull contains a disk with center 0 and radius

$$c_2 = \frac{\cos \frac{2}{5}\pi}{\cos \frac{1}{5}\pi} = \frac{3 - \sqrt{5}}{2} = 0.38 \dots =: r_4.$$

Fig. 1. Extremal situation for four- and five-point subsets.

They also showed the existence of a subset $Y \subset X$ of at most five points whose convex hull contains a disk with center 0 and radius

$$\frac{\cos \frac{2}{6}\pi}{\cos \frac{1}{6}\pi} = \frac{1}{\sqrt{3}} = 0.57 \dots =: r_5,$$

and asked for the largest numbers r_k such that X always has a subset $Y \subset X$ of at most k points which contains a disk with center 0 and radius r_k in its convex hull $\text{conv}(Y)$.

They conjectured that the worst case is always a regular $(k+1)$ -gon which contains the unit circle as incircle (as they proved in the cases $k = 4, 5$, see Fig. 1), so

$$r_k = \frac{\cos(2/(k+1))\pi}{\cos(1/(k+1))\pi}.$$

It is the aim of this paper to prove that conjecture.

Theorem. *If $X \subset \mathbb{R}^2$ contains a disk with center 0 and radius 1 in its convex hull $\text{conv}(X)$, then for each $k \geq 4$ there is a subset $Y \subset X$ of at most k points which contains a disk with center 0 and radius $[\cos(2/(k+1))\pi]/[\cos(1/(k+1))\pi]$ in the convex hull $\text{conv}(Y)$.*

These numbers were also investigated by Kirkpatrick *et al.* [3] for large subset-sizes k and higher dimensions d in connection with a problem in robotics. In the two-dimensional case they had already proved the inequalities

$$1 - 2 \frac{\pi^2}{k^2} < r_k \leq \frac{\cos(2/(k+1))\pi}{\cos(1/(k+1))\pi} < 1 - \frac{3}{2} \frac{\pi^2}{(k+1)^2}.$$

2. Proof of the Theorem

Let D be the disk with center 0 and radius 1. Let a set X in the plane have property $P(r, k)$ if $D \subset \text{conv}(X)$ but for each subset $Y \subset X$ of at most k points $rD \not\subset \text{interior conv}(Y)$. Then

$$r_k = \inf \{r \mid \text{there is a set } X \text{ with } P(r, k)\}.$$

If X has $P(r, k)$ and is an infinite set, then for each $\delta > 0$ there is a finite subset $X_\delta \subset X$ such that $(1 - \delta)D \subset \text{conv}(X_\delta)$. Then $(1 - \delta)^{-1}X_\delta$ has property $P((1 - \delta)^{-1}r, k)$, therefore

$$r_k = \inf \{r \mid \text{there is a finite set } X \text{ with } P(r, k)\}. \quad (1)$$

The vertices of the regular $(k + 1)$ -gon which is circumscribed by the disk D give an upper bound

$$r_k \leq r_k^* := \frac{\cos(2/(k + 1))\pi}{\cos(1/(k + 1))\pi}$$

for each diagonal from one vertex to the next but one touches the disk r_k^*D as a tangent, so this set has $P(r_k^*, k)$.

We show that $r_k = r_k^*$, i.e., each set X with $D \subset \text{conv}(X)$ has for each $\varepsilon > 0$ a subset $Y_\varepsilon \subset X$ of at most k points such that $(r_k^* - \varepsilon)D \subset \text{interior conv}(Y_\varepsilon)$. This implies the theorem; for if X is finite, then there are only finitely many possible choices for Y_ε , so there is a Y with $r_k^*D \subset \text{conv}(Y)$.

Let $r := \frac{1}{2}(r_k + r_k^*)$ and let X be a minimum-cardinality set with $P(r, k)$. By (1) X is a finite set. X is the set of vertices of some convex n -gon; for if there were a point $x \in X$ with $\text{conv}(X \setminus \{x\}) = \text{conv}(X)$, then x could be removed from X without losing $P(r, k)$, a contradiction to the minimum cardinality of X . Let $X = \{x_1, \dots, x_n\}$ clockwise, continued mod n .

Let $\tau: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $\tau(i) := \min\{j \mid j > i \text{ and } \overline{x_i x_j} \cap rD \neq \emptyset\}$. If rD lies in the interior of the angle $\angle x_j x_i x_{j+1}$ for some j , then $rD \subset \text{interior conv}(\{x_i, x_j, x_{j+1}\})$, so in that case X would not even have $P(r, 3)$. Therefore there is always such a j .

τ is a monotone increasing mapping, for $i_1 < i_2 < \tau(i_2) \leq \tau(i_1)$ implies that $\overline{x_{i_1} x_{\tau(i_2)}}$ also intersects rD , so $\tau(i_1) = \tau(i_2)$. Since $r < 1$ we have $rD \subset \text{interior}(D) \subset \text{interior conv}(X)$, so $\tau(i) \geq i + 2$ for each i . Also since $x_{i+n} = x_i$ we have $\tau(i + n) = \tau(i) + n$.

Case 1: τ is injective. Then τ is strictly monotone increasing. This implies $\tau(i) = i + a$ for $a := \tau(1) - 1$ and all i , since

$$\tau(i) + n \leq \tau(i + 1) + (n - 1) \leq \dots \leq \tau(i + n - 1) + 1 \leq \tau(i + n) = \tau(i) + n.$$

Let $l := \lceil n/a \rceil$, $n =: a(l - 1) + b$, so $1 \leq b \leq a$. Consider the l -gon $x_i, x_{i+a}, x_{i+2a}, \dots, x_{i+(l-1)a}$. Since $i + (l - 1)a < i + n$ we have

$$\begin{aligned} \sum_{\lambda=1}^{l-1} |\angle x_{i+(l-1)a} 0 x_{i+\lambda a}| &= 2\pi - |\angle x_{i+(l-1)a} 0 x_{i+n}| \\ &= 2\pi - \sum_{\beta=1}^b |\angle x_{i+(l-1)a+\beta-1} 0 x_{i+(l-1)a+\beta}| \\ &< 2\pi. \end{aligned}$$

Taking the average over all $i \in \{1, \dots, n\}$ and using

$$\sum_{i=1}^n \sum_{\beta=1}^b |\angle x_{i+(l-1)a+\beta-1} 0 x_{i+(l-1)a+\beta}| = 2\pi b$$

we find that there is an $\hat{i}$ with $\sum_{\beta=1}^b |\angle x_{\hat{i}+(l-1)a+\beta-1} 0x_{\hat{i}+(l-1)a+\beta}| \geq 2\pi(b/n)$, so

$$\sum_{\lambda=1}^{l-1} |\angle x_{\hat{i}+(\lambda-1)a} 0x_{\hat{i}+\lambda a}| \leq 2\pi \left(1 - \frac{b}{n}\right).$$

For each $m \in \mathbb{Z}$ we have $d(x_m, 0) \geq 1$, $d(x_{m+a}, 0) \geq 1$ but $d(\overline{x_m x_{m+a}}, 0) \leq r \leq r_k^*$, therefore $|\angle x_m 0x_{m+a}| \geq 2 \arccos(r) \geq 2 \arccos(r_k^*)$ holds for each angle $\angle x_m 0x_{m+a}$. Together this implies $2(l-1) \arccos(r_k^*) \leq 2\pi(1 - b/n)$ or

$$r_k^* \geq \cos \left(\frac{\pi}{l-1} \left(1 - \frac{b}{n}\right) \right) = \cos \left(\pi \frac{a}{a(l-1) + b} \right). \quad (2)$$

To get a bound for k in terms of a, l, n we note that

$$rD \subset \text{interior conv}(\{x_{a-1}, x_{2(a-1)}, \dots, x_{\lceil n/(a-1) \rceil (a-1)}\}),$$

for by the definition of τ and a none of the edges of this $\lceil n/(a-1) \rceil$ -gon has a nonempty intersection with rD . So X does not have property $P(r, \lceil n/(a-1) \rceil)$, therefore

$$k+1 \leq \left\lceil \frac{n}{a-1} \right\rceil = \left\lceil \frac{a}{a-1}(l-1) + \frac{b}{a-1} \right\rceil. \quad (3)$$

Using (2) and (3) and the definition and monotonicity of r_k^* we obtain

$$\frac{\cos \frac{2\pi}{\lceil (a/(a-1))(l-1) + b/(a-1) \rceil}}{\cos \frac{\pi}{\lceil (a/(a-1))(l-1) + b/(a-1) \rceil}} \geq \cos \left(\pi \frac{a}{a(l-1) + b} \right).$$

Since $(\cos 2x)/(\cos x) < 1 - \frac{3}{2}x^2$ for $x \in]0, \pi/2[$ and $\cos x > 1 - \frac{1}{2}x^2$ for $x \in [0, 1]$ we have

$$1 - \frac{3}{2} \left(\frac{\pi}{\lceil (a/(a-1))(l-1) + b/(a-1) \rceil} \right)^2 > 1 - \frac{1}{2} \left(\pi \frac{a}{a(l-1) + b} \right)^2.$$

This implies (using $\lceil (a/(a-1))(l-1) + b/(a-1) \rceil \leq (a/(a-1))(l-1) + (a-2)/(a-1) + b/(a-1)$)

$$\sqrt{3} < \frac{a}{a-1} \left(1 + \frac{a-2}{a(l-1) + b} \right);$$

since $l \geq 3$ and $b \geq 1$ this inequality has no solution with $a \geq 3$. Therefore $a = 2$. In this case X has property $P(r, n-1)$. Theorem 3 of [1] states that if a set of n points has $P(r, n-1)$, then $r \geq r_{n-1}^*$. Since $(r_\nu^*)_{\nu=4}^\infty$ is a monotonic increasing sequence and $n > k$

Fig. 2. Situation of Case 2a.

we have $r_k^* \geq r = \frac{1}{2}(r_k + r_k^*) \geq r_{n-1}^* \geq r_k^*$. So $n - 1 = k$ and $r_k^* = r_k$, as the theorem claims.

Case 2: τ is not injective. Then there is an i with $\tau(i) = \tau(i + 1) =: j \geq i + 3$. We lead this and the assumption of minimal cardinality of X to a contradiction.

Let $x_i(t) := x_i + t(x_i - x_{i-1})$ and $X(t) := (X \setminus \{x_i\}) \cup \{x_i(t)\}$, so we move x_i along the line $x_{i-1}x_i$ away from x_{i-1} . Then for $t \geq 0$ we have $\text{conv}(X(t)) \supseteq \text{conv}(X) \supset D$.

Case 2a: There is a $t > 0$ such that $X(t)$ does not have $P(r, k)$. Then there is a k -gon $x_{j_1} := x_i(t), x_{j_2}, \dots, x_{j_k}$ (clockwise) which completely surrounds rD without touching it, so $rD \subset \text{interior conv}(\{x_{j_1}, x_{j_2}, \dots, x_{j_k}\})$. For each of the finitely many possible $(k-1)$ -tuples of indices $\{j_2, \dots, j_k\} \subset \{i+1, \dots, i+n-1\}$ the set $\{t > 0 \mid rD \subset \text{interior conv}(\{x_i(t), x_{j_2}, \dots, x_{j_k}\})\}$ is open. So if there is a $t > 0$ such that $X(t)$ does not have $P(r, k)$, then there is a $t_0 \geq 0$ such that $X(s)$ has $P(r, k)$ for $s \in [0, t_0]$ but for some $\varepsilon > 0$ and each $s \in]t_0, t_0 + \varepsilon[$ the set $X(s)$ does not have $P(r, k)$. Since only the edges $\overline{x_i(t)x_{j_2}}$ and $\overline{x_i(t)x_{j_k}}$ of the k -gon $x_i(t), x_{j_2}, \dots, x_{j_k}$ depend on t , this implies that all other edges have empty intersection with rD , but at least one of $\overline{x_i(t_0)x_{j_2}}$ and $\overline{x_i(t_0)x_{j_k}}$ is a tangent to rD which moves away from rD for $t > t_0$ (Fig. 2). rD lies in the angle $\angle x_{j_2}x_i(t_0)x_{j_k}$. With increasing t the segment $\overline{x_i(t)x_{j_k}}$ moves toward 0 but the segment $\overline{x_i(t)x_{j_2}}$ moves away from 0. This implies that $\overline{x_i(t_0)x_{j_2}}$ must be the tangent to rD and $\overline{x_i(t_0)x_{j_k}} \cap rD = \emptyset$. Since $\overline{x_i(t_0)x_{j_2}} \cap rD \neq \emptyset$ we have $\tau(i) \leq j_2$. Furthermore, the segment $\overline{x_{i-1}x_i(t_0)}$ does not intersect the line $x_{i+1}x_{j_2}$ (otherwise $x_{i+1} \in \text{conv}(\{x_{i-1}, x_i(t_0), x_{j_2}\})$, so $X(t_0) \setminus \{x_{i+1}\}$ was a set with $P(r, k)$, contradicting the minimal cardinality of X). Therefore the line $x_i(t_0)x_{j_2}$ separates rD and x_{i+1} , so it must also separate x_{i+1} and $x_{\tau(i+1)}$. This implies $\tau(i+1) > j_2 \geq \tau(i)$, a contradiction to the assumption of Case 2.

Case 2b: For each $t \geq 0$ the set $X(t)$ has property $P(r, k)$. Since X is of minimum cardinality, we have for each $t \geq 0$ that $D \not\subset \text{conv}(X(t) \setminus \{x_{i+1}\})$, so for each t the segment $\overline{x_i(t)x_{i+2}}$ intersects D . Therefore the half-line starting at x_{i+2} in direction $x_i - x_{i-1}$ intersects D . So the line g_1 through 0, perpendicular to $x_i x_{i-1}$, separates x_i and x_{i+2} .

Fig. 3. Situation of Case 2b.

Since $\tau(i) = \tau(i+1) \geq i+3$ by the assumption of Case 2, the segment $\overline{x_i x_{i+2}}$ does not intersect rD .

Let g_2 be the parallel to $x_i x_{i-1}$ which is a tangent to rD and which separates 0 and x_i . Any point x_m with $i+2 \leq m \leq i+n$ and $|\angle x_{i+2} 0 x_m| \geq \pi/2 + \arcsin(r)$ lies on the same side of g_2 as x_i , so for these points $\overline{x_m x_i}$ does not intersect rD (Fig. 3).

Let $m_1 = i+2$, $m_{v+1} := \tau(m_v)$ for $v \geq 1$ and let l be the largest l such that $\sum_{\lambda=2}^l |\angle x_{m_{\lambda-1}} 0 x_{m_\lambda}| < \pi/2 + \arcsin(r)$. As in Case 1, each of these angles is at least $2 \arccos(r)$, so we have

$$2(l-1) \arccos(r) < \frac{\pi}{2} + \arcsin(r). \quad (4)$$

If we use now $\arcsin(r) \leq \pi/2$ we obtain, from (4),

$$r > \cos\left(\frac{\pi}{2(l-1)}\right). \quad (5)$$

By the definition of τ we have

$$rD \subset \text{interior conv}(\{x_i, x_{m_1}, x_{m_2-1}, x_{m_2}, \dots, x_{m_l-1}, x_{m_l}, x_{m_{l+1}-1}, x_{m_{l+1}}\}),$$

so $k+1 \leq 2(l+1)$. As in Case 1 we now use

$$r \leq r_k^* \leq \frac{\cos(2\pi/2(l+1))}{\cos(\pi/2(l+1))}$$

together with (5) and $\cos(2x)/\cos(x) < 1 - \frac{3}{2}x^2$ and $\cos(x) > 1 - \frac{1}{2}x^2$ to obtain

$$\begin{aligned} 1 - \frac{3}{2} \left(\frac{\pi}{2(l+1)} \right)^2 &> \frac{\cos(2\pi/2(l+1))}{\cos(\pi/2(l+1))} \geq r_k^* \geq r \\ &> \cos\left(\frac{\pi}{2(l-1)}\right) > 1 - \frac{1}{2} \left(\frac{\pi}{2(l-1)} \right)^2. \end{aligned}$$

Since the only solutions $l \geq 2$ of $3(1/(l+1))^2 < (1/(l-1))^2$ are $l = 2, 3$, only the cases $l = 2, k = 4, 5$ and $l = 3, k = 7$ remain. Of these, only the case $l = 2, k = 5$

Fig. 4. Situation of Case 2b with $l = 2, k = 5$.

satisfies inequality (4) with $\arccos(r) > \arccos(r_k^*)$, $\arcsin(r) < \arcsin(r_k^*)$. To exclude this last subcase let m_1, m_2 be as before and let $\mu \geq i + 2$ be the first index such that g_2 separates 0 and x_μ (Fig. 4). We claim that

$$rD \subset \text{interior conv}(\{x_i, x_{m_1}, x_{m_2-1}, x_{m_2}, x_\mu\}).$$

As before none of the edges $\overline{x_i x_{m_1}}$, $\overline{x_{m_1} x_{m_2-1}}$, $\overline{x_{m_2-1} x_{m_2}}$, and $\overline{x_\mu x_i}$ touches rD . If $\mu \leq m_2$, this is already sufficient, so we may suppose $\mu - 1 \geq m_2$. Then $x_{\mu-1}$ and x_{m_2} are both in the region bounded by g_2 , the boundary of D , and the line g_3 which is a tangent to $r_5^* D \supset rD$ through the point of D which is farthest from $x_i x_{i-1}$. Since $x_\mu x_{\mu-1}$ does not intersect D , the tangent g_4 to D through the point of intersection of g_2 and g_3 separates 0 and x_μ . Then the parallel g_5 to g_4 through the point of intersection of the boundary of D and g_3 separates $r_5^* D \supset rD$ and $\overline{x_\mu x_{m_2}}$. So the set does not have $P(r_5^*, 5)$, and Case 2b is shown to be impossible.

This completes the proof of the theorem. $\square$

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