Comments on Hadron Mass Dependence in Quark Models (*).

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Summary. — A modified harmonic-oscillator potential provides dynamics consistent with known empirical information in the nonrelativistic quark model. The composite mass depends linearly on the constituent mass for deviations about the constituent mass in the Dirac quark model.

In the nonrelativistic quark model the dynamical mechanism underlying quark bound states is described by a hierarchy of forces ⁽¹⁾. The Hamiltonian for the most dominant of these forces, the so-called «superstrong» ones, is

$$(1) \quad H = \frac{1}{2M} \sum_{i=1}^n \mathbf{p}_i^2 + nM + V^{(n', n-n')},$$

$$V^{(n', n-n')} \equiv V^{(n', n-n')}(x_1, \dots, x_n; \sigma_1, \dots, \sigma_n),$$

in which the σ_i are Pauli matrices for each particle, $\mathbf{p}_i \equiv -i\nabla_i$ and M is the mass of each constituent particle. We have set $\hbar = c = 1$.

For a potential homogeneous under scale transformations,

$$V^{(n', n-n')}(\lambda x_1, \dots, \lambda x_n; \sigma_1, \dots, \sigma_n) = \lambda^\gamma V^{(n', n-n')}(x_1, \dots, x_n; \sigma_1, \dots, \sigma_n)$$

for all real $\lambda > 0$, we can establish a mass relation. This is accomplished by introducing the dimensionless variables $\varepsilon \equiv (m - 3M)^{\gamma/(2+\gamma)}$ and $\xi_i \equiv x_i(2M)^{1/(2+\gamma)}$

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(1) J. J. J. KOKKEDEE: *The Quark Model* (New York and Amsterdam, 1969), p. 27.

in the equation $H\psi = m\psi$ from which it follows that

$$(2) \quad \varepsilon = \varepsilon_1^{(2)} + \varepsilon_2^{(-\gamma)},$$

where

$$\varepsilon_1^{(2)} \equiv \frac{\left(\psi(\xi_1, \dots, \xi_n), - \sum_{i=1}^n \nabla_{\xi_i}^2 \psi(\xi_1, \dots, \xi_n) \right)}{(\psi(\xi_1, \dots, \xi_n), \psi(\xi_1, \dots, \xi_n))}$$

and

$$\varepsilon_2^{(-\gamma)} \equiv \frac{(\psi(\xi_1, \dots, \xi_n) V^{(n', n-n')} \psi(\xi_1, \dots, \xi_n))}{(\psi(\xi_1, \dots, \xi_n), \psi(\xi_1, \dots, \xi_n))}.$$

The superscript on the energy terms refer to the degree of homogeneity of the functional under scale transformations. Using the pseudovirial theorem ⁽²⁾ on (2) it follows that

$$(3) \quad \varepsilon_1^{(2)} = \frac{\gamma}{2} \varepsilon_2^{(-\gamma)}$$

and

$$(4) \quad \varepsilon = \frac{2 + \gamma}{\gamma} \varepsilon_1^{(2)}.$$

Since $\varepsilon_1^{(2)} \geq 0$, we find that $\varepsilon < 0$ for $-2 < \gamma < 0$ and $\varepsilon \geq 0$ otherwise.

From the definition of ε , $m = \varepsilon(2M)^{-\gamma/(2+\gamma)} + 3M$ and the only integer value of γ which produces binding (that is $m < M$) is $\gamma = -1$, in which case

$$(5) \quad m = (2\varepsilon + 3)M.$$

Since it is known that $M \geq 10$ GeV, eq. (5) implies that the system is in the relativistic domain. Thus, $\gamma = -1$ is also excluded from consideration.

Several authors ^(3,4) have suggested a harmonic-oscillator model for baryons. Such a model is not compatible with $m/M \leq 1/10$. A more realistic model is one in which there is a minimum range of interaction. A potential which accomplishes this is

$$(6) \quad V = \frac{1}{2} K \sum_{i < j} (|\mathbf{x}_i - \mathbf{x}_j| - a)^2 - \frac{1}{2} K a^2 = \frac{1}{2} K \sum_{i < j} (|\mathbf{x}_i - \mathbf{x}_j|^2 - 2a|\mathbf{x}_i - \mathbf{x}_j|).$$

In the Appendix it is shown that if the potential is composed of two terms, one of homogeneity with respect to scale transformations of order 2, the other

⁽²⁾ G. ROSEN: *Journ. Math. Phys.*, **7**, 2066 (1966).

⁽³⁾ D. FAIMAN and A. W. HENDRY: *Phys. Rev.*, **173**, 1720 (1968).

⁽⁴⁾ V. C. NAVARRO, M. MOSHINSKY and W. W. YEH: *Revista Mexicana de Fisica*, **17.4**, 241 (1968).

of order γ , for binding to occur consistent with $m/M \lesssim 1/10$ the potential must be similar to the above one. For the type of potential given by (6) one expects $nM = \frac{1}{2}Ka^2$ consistent with $\langle \text{kinetic energy} \rangle < m$.

For the case $n = 2$ (mesons) it is straightforward to establish the mass formula

$$(7) \quad m = CM^{-\frac{1}{2}} + 2M - \frac{1}{2}Ka^2,$$

where $C \equiv (l + 2k + \frac{3}{2})K^{\frac{1}{2}}$, l is the orbital angular-momentum quantum number, and the parameter k is associated with the number of nodes of the wave function. For the $J^P = 1^-$ mesons of mass $m = 760$ MeV the requirement $\langle \text{kinetic energy} \rangle = \frac{1}{2}CM^{-\frac{1}{2}} < m$ along with assumptions $M \approx 10$ GeV and $2M \approx \frac{1}{2}Ka^2$ yields the inequality $C < 1.4 \cdot 10^5 (\text{MeV})^{\frac{3}{2}}$. For a meson in the ground state our model predicts $a > 2.1 \cdot 10^{-3} (\text{MeV})^{-1}$.

The dynamics of this particular model deserves further study. It has been shown⁽³⁾ that a paraquark harmonic-oscillator shell model can explain the band structure of the known N^* resonances and $N^*\pi$ and $\Delta\pi$ partial decay widths.

Let us now consider the Dirac quark-model Hamiltonian for a system of n' quarks and $n - n'$ antiquarks,

$$(8) \quad \begin{cases} H = \sum_{i=1}^n (\alpha_i \cdot \mathbf{p}_i + \beta_i M_i) + V^{(n', n-n')}, \\ V^{(n', n-n')} \equiv V(\mathbf{x}_1, \dots, \mathbf{x}_n; \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n), \end{cases}$$

in which the α 's and β 's are Dirac matrices, $\mathbf{p}_i = -i\nabla_i$, and M_i is the mass of the i -th particle. For a potential homogeneous under scale transformations, we have

$$\begin{aligned} V^{(n', n-n')}(\lambda \mathbf{x}_1, \dots, \lambda \mathbf{x}_n; \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \\ = \lambda^\gamma V^{(n', n-n')}(\mathbf{x}_1, \dots, \mathbf{x}_n; \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n). \end{aligned}$$

From the pseudovirial theorem⁽²⁾ it follows that

$$(9) \quad m = \frac{1 + \gamma}{\gamma} E_1^{(1)} + E_2^{(0)},$$

where

$$m = \frac{(\psi, H\psi)}{(\psi, \psi)},$$

$$E_1^{(1)} \equiv \frac{(\psi, \sum_{i=1}^n \alpha_i \cdot \mathbf{p}_i \psi)}{(\psi, \psi)},$$

$$E_2^{(0)} \equiv \sum_{i=1}^n M_i \frac{(\psi, \beta_i \psi)}{(\psi, \psi)},$$

$$E_3^{(-\gamma)} = \frac{(\psi, V^{(n, n-n')}\psi)}{(\psi, \psi)}.$$

We have employed similar notation to that previously used. If the system is in the ground state, then the pseudovirial theorem may be extended⁽⁵⁾ to establish the inequality

$$(10) \quad \gamma(\gamma + 1) E_3^{(-\gamma)} \geq 0.$$

Consider the following cases:

$$\begin{aligned} \text{I) } & \gamma < -1, & E_3^{(-\gamma)} & \geq 0, & E_1^{(1)} & < 0, & E_2^{(0)} & \geq 0, \\ \text{II) } & 0 > \gamma > -1, & E_3^{(-\gamma)} & < 0, & E_1^{(1)} & \geq 0, & E_2^{(0)} & \geq 0, \\ \text{III) } & \gamma > 0, & E_3^{(-\gamma)} & \geq 0, & E_1^{(1)} & \geq 0. \end{aligned}$$

We note that

$$(11) \quad \frac{\partial m}{\partial M_i} = \frac{(\psi, (\partial H / \partial M_i) \psi)}{(\psi, \psi)} = \frac{(\psi, B_i \psi)}{(\psi, \psi)} \equiv C_i.$$

Differentiating (9), and using the fact that $E_2^{(0)} = \sum_{j=1}^n M_j C_j$ leads to

$$(12) \quad \frac{1 + \gamma}{\gamma} \frac{\partial E_1^{(1)}}{\partial M_i} = - \sum_{j=1}^n M_j \frac{\partial C_j}{\partial M_i}.$$

Suppose $C_1 = C_2 = \dots = C_n$ and $\partial C / \partial M_1 = \partial C / \partial M_2 = \dots = \partial C / \partial M_n$ (as in the case of mutually aligned spins).

Then, (12) reduces to

$$(13) \quad \frac{1 + \gamma}{\gamma} \frac{\partial E_1^{(1)}}{\partial M_i} = -M \frac{\partial C}{\partial M_i} = M \frac{\partial^2 m}{\partial M_i^2},$$

in which M is defined by

$$M \equiv \sum_{i=1}^n M_i.$$

From Schwartz's inequality⁽⁶⁾ we establish the inequality

$$(14) \quad \frac{\partial m}{\partial M_i} = \frac{(\psi, B_i \psi)}{(\psi, \psi)} \leq \frac{(\psi, B_i^2 \psi)(\psi, \psi)^{\frac{1}{2}}}{(\psi, \psi)} = 1.$$

⁽⁵⁾ For a variation around the ground state $(d^2/d\lambda^2)m[\psi(\lambda \mathbf{x}i)]|_{\lambda=1} \geq 0$ since $(\psi, H\psi)/(\psi, \psi) \geq E_0$.

⁽⁶⁾ G. ROSEN: *Mass dependence in the Dirac quark model* (preprint).

If $\partial^2 m / \partial M_i < \infty$ it follows from (13) that for $M \sim 0$, $E_1^{(1)} \sim A M^n$, $n \geq 2$, A a constant independent of M_i , and hence that $C = E_2^{(0)} / M = \partial m / \partial M_i \sim B M^{n-1}$ with $B = -((1 + \gamma) / \gamma)(n / (n - 1)) A$. Substituting both of these limiting values for $E_1^{(1)}$ and $E_2^{(0)}$ into (9) implies that, as $M \rightarrow 0$, $m \rightarrow 0$. This cannot be immediately deduced from (8). Case III) ($\gamma > 0$, $E_2^{(0)} \geq 0$) can be eliminated, since near $M = 0$ we have $A > 0$ which implies that $\beta > 0$ and hence that $\partial m / \partial M_i < 0$. Thus we conclude $m(0) = 0^-$.

For cases I) and II) note that $\partial m / \partial M_i > 0$ for $M_i > 0$. From (13) we see that large inflections are damped out by the M^{-1} factor. This suggests the mass relation shown in Fig. 1. Such a relationship lends credence to the use of (11) to find the mass dependence of the composite system about the constituent mass point $M_i = M_q$. An attempt has been made to employ the quasi-independent model to obtain such an approximation of (11) (6).

Fig. 1. — A plot of composite mass (m) vs. constituent mass (M_i) in units of the quark mass, $M_q \approx 10$ GeV.

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APPENDIX

Consider the nonrelativistic Hamiltonian

$$(A.1) \quad H = \sum_{i=1}^n \frac{P_i^2}{2M} + nM + \frac{1}{2} K \sum_{i < j} |\mathbf{x}_i - \mathbf{x}_j|^2 + V',$$

in which V' is defined by

$$V' \equiv V'(\mathbf{x}_1, \dots, \mathbf{x}_n; \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n)$$

and

$$V'(\lambda \mathbf{x}_1, \dots, \lambda \mathbf{x}_n; \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n) = \lambda^\gamma V'(\mathbf{x}_1, \dots, \mathbf{x}_n; \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n).$$

We establish the mass relation

$$(A.2) \quad m = E_1^{(2)} + E_2^{(0)} + E_3^{(-2)} + E_4^{(-\gamma)},$$

m = mass of the composite system, $E_1^{(2)}, E_2^{(0)}, E_3^{(-2)}, E_4^{(-\gamma)}$ defined by

$$E_1^{(2)} = \frac{\left(\psi, \sum_{i=1}^n (\mathbf{p}_i^2/2M) \psi \right)}{(\psi, \psi)}, \quad E_2^{(0)} = nM,$$

$$E_3^{(-2)} = \frac{\left(\psi, \frac{1}{2} K \sum_{i < j} |\mathbf{x}_i - \mathbf{x}_j|^2 \psi \right)}{(\psi, \psi)}, \quad E_4^{(-\gamma)} = \frac{(\psi, V' \psi)}{(\psi, \psi)}.$$

From the pseudovirial theorem ⁽²⁾ it follows from (A.2) that

$$(A.3) \quad 2E_1^{(2)} - 2E_3^{(-2)} - \gamma E_4^{(-\gamma)} = 0.$$

Assuming the system to be in the ground state ⁽⁵⁾ a second variation of (A.2) is used to yield the inequality

$$(A.4) \quad 2E_1^{(2)} + 6E_3^{(-2)} + \gamma(\gamma + 1)E_4^{(-\gamma)} \geq 0.$$

Combining (A.2) and (A.3) with (A.4) we find

$$(A.5) \quad (\gamma + 2)E_1^{(2)} \geq (\gamma - 2)E_3^{(-2)}.$$

For $\gamma < 2$ we have

$$(A.6a) \quad E_3^{(-2)} \geq \frac{\gamma + 2}{\gamma - 2} E_1^{(2)}$$

and for $\gamma > 2$ we have

$$(A.6b) \quad E_3^{(-2)} \leq \frac{\gamma + 2}{\gamma - 2} E_1^{(2)}.$$

Substituting (A.3) into (A.2) produces

$$(A.7) \quad m = \frac{\gamma + 2}{\gamma} E_1^{(2)} + \frac{\gamma - 2}{\gamma} E_3^{(-2)} + nM.$$

From (A.2) we note that no binding can occur for $\gamma \geq 2$ and $\gamma < -2$ since

$$\frac{\gamma + 2}{2} E_1^{(2)} \geq 0 \quad \text{and} \quad \frac{\gamma - 2}{2} E_3^{(-2)} \geq 0.$$

Consider the following three cases.

Case I) $\gamma = -1$ ⁽⁷⁾. Equation (A.7) reduces to

$$(A.8) \quad m = -E_1^{(-2)} + 3E_3^{(-2)} + nM$$

and from (A.6a) it follows that $E_3^{(-2)} \geq -\frac{1}{3}E_1^{(-2)}$ and hence $m \geq nM$.

Case II) $\gamma = -2$. Equation (A.7) reduces to

$$(A.9) \quad m = 2E_3^{(-2)} + nM$$

and since $E_3^{(-2)} \geq 0$ then again we have $m \geq nM$.

Case III) $\gamma = 1$. From (A.6a), (A.2) and (A.3) we find

$$(A.10) \quad m = 2E_1^{(-2)} - E_3^{(-2)} + nM = \frac{2}{3}E_4^{(-1)} + 2E_3^{(-2)} + nM.$$

Substituting (A.3) into (A.9) yields

$$(A.11a) \quad E_3^{(-2)} + 3E_1^{(-2)} \geq 0,$$

or

$$(A.11b) \quad E_4^{(-1)} + \frac{8}{3}E_3^{(-2)} \geq 0.$$

We can have $m < nM$ if the inequality

$$(A.12) \quad -\frac{8}{3}E_3^{(-2)} \leq E_4^{(-1)} \leq -\frac{4}{3}E_3^{(-2)}$$

holds.

We note that $E_4^{(-1)}$ cannot be found by a perturbation calculation.

⁽⁷⁾ The case $\gamma = -1$ is considered in ref. ⁽⁶⁾.

RIASSUNTO (*)

Un potenziale modificato dell'oscillatore armonico fornisce una dinamica consistente con note informazioni empiriche sul modello a quark non relativistico. La massa composta dipende linearmente dalla massa costituente per deviazioni intorno alla massa costituente nel modello a quark di Dirac.

(*) Traduzione a cura della Redazione.

Замечания о зависимости от адронных масс в моделях кварков.

Резюме (*). — Модифицированный потенциал гармонического осциллятора обеспечивает динамику, согласующуюся с известной эмпирической информацией в нерелятивистской модели кварков. В дираковской модели кварков составная масса линейно зависит от массы составляющих частиц для отклонений, порядка массы составляющих частиц.

(*) Переведено редакцией.