On Voronoi Diagrams in the L_p -Metric in $\mathbb{R}^{D*}$

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Abstract. We prove upper bounds on the number of L_p -spheres passing through $D + 1$ points in general position in $\mathbb{R}^D$, and on the sum of the Betti numbers of the intersection of bisectors in the L_p -metric, where p is an even positive integer. The bounds found do not depend on p . Our result implies that the complexity of Voronoi diagrams (for point sites in general position) in the L_p -metric is bounded for increasing p . The proof for this upper bound involves the techniques of Milnor [12] and Thom [16] for finding a bound on the sum of the Betti numbers of algebraic varieties, but instead of the usual degree of polynomials we use their additive complexity, and apply results of Benedetti and Risler [2], [13]. Furthermore, we prove that in D dimensions and for even p the number of L_p -spheres passing through $D + 1$ points in general position is odd. In particular, combined with results of [8], [9], our results clarify the structure of Voronoi diagrams based on the L_p -metric (with even p) in three dimensions. For the proof we use the theory of degree of continuous mappings in $\mathbb{R}^D$, which is a tool widely applied in nonlinear analysis [14].

1. Introduction

We first recall the definition of the L_p -metric in D dimensions. Let p be a real number with $1 \leq p < \infty$, and let x and y be two points in $\mathbb{R}^D$ with coordinates $(x_1, \dots, x_D)$ and $(y_1, \dots, y_D)$, respectively. Then the distance between x and y in the L_p -metric, denoted by $d_p(x, y)$, is defined to be

$$d_p(x, y) = (|x_1 - y_1|^p + \dots + |x_D - y_D|^p)^{1/p}.$$

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The L_p -sphere with center y and radius $R > 0$ is the set of all $x \in \mathbb{R}^D$ such that $d_p(x, y) = R$.

Let $d(\cdot, \cdot)$ denote a smooth strictly convex distance function in D -space. The concept of the *Voronoi diagram* for convex distance functions in D dimensions can be defined analogously to the case of the L_2 -metric, see [4], [5]. For any two distinct points s and t in $\mathbb{R}^D$ we call the *bisector* of s and t , denoted by $B(s, t)$, the set of all $x \in \mathbb{R}^D$ that have equal distances to s and t :

$$B(s, t) = \{x \in \mathbb{R}^D \mid d(x, s) = d(x, t)\}.$$

Voronoi diagrams based on the Euclidean metric in any dimension are well understood (see, e.g., [5]; an extensive bibliography on this fundamental data structure can be found in [1]), but not much is known when more general distance functions, namely, convex distance functions, are considered. An exception are planar Voronoi diagrams: since in this case bisectors behave fully analogous to those in the Euclidean metric, they can still be computed with the same techniques [10], [4]. Recently, geometrical properties of bisectors based on smooth strictly convex distance functions, and their intersections, in 3-space have been clarified [8], [9].

The reason for the success in solving problems related to standard Voronoi diagrams in higher dimensions lies in their *linearity*: bisectors are linear spaces, and so are their intersections. The situation in smooth strictly convex distance functions is different: bisectors are no longer linear, and their intersections need not be connected [8], [9].

In this paper, besides the L_p -metrics we also consider smooth strictly convex distance functions, but we devote more attention to the L_p -metrics with even $p \geq 2$, since these are immediate generalizations of the usual Euclidean metric. For odd p , analogous results on upper bounds in the L_p -metric can also be obtained in a similar way.

First Problem. We consider the problem of estimating the complexity of Voronoi diagrams in the L_p -metric in D dimensions. To simplify the discussion, we take $D = 3$ and even p . Then the distance between two points x and y is given by

$$d_p(x, y) = ((x_1 - y_1)^p + \cdots + (x_3 - y_3)^p)^{1/p}.$$

Assume that the point sites are in general position. We must bound the number of vertices of the Voronoi diagram for four sites, or, equivalently, the number of L_p -spheres passing through the given points. By taking the p -powers of the distance functions above we can reduce the problem to that of bounding the number of real solutions of a system of three polynomials of order $p - 1$. Our goal is to find a bound that is more favorable than $(p - 1)^3$ derived from Bézout's theorem, see, e.g., p. 299 of [2].

Second Problem. Given $D + 1$ points in general position in D -space, it is known that there exists exactly one standard sphere passing through the given points. Does the analogy hold in the L_p -metric?

Our Results to the First Problem. We show that for point sites in general position there is an upper bound *independent of p* to the number of L_p -spheres passing through them.

In fact, we prove more general results in D -space by using another kind of measure for complexity, namely, the *Betti numbers*. We take a close look at them.

The Betti numbers have their origin in the so-called “connectivity number” for a compact surface. Consider a closed curve C in a compact surface M . If the curve C equals the boundary of a face contained in M , then we say that C *bounds* in M . Now let $C_1, \dots, C_k$ be closed curves in M . If the *sum*, i.e., the union, of $C_1, \dots, C_k$ is identical to the boundary of a face or a union of faces in M , then we say as above that the sum of $C_1, \dots, C_k$ *bounds* in M . Let n be the maximal number of closed curves $C_1, \dots, C_n$ such that no C_i bounds in M , but for any other curve C' either C' itself or the sum of C' and some of $C_1, \dots, C_n$ bounds in M . Then the number n is called the *connectivity number* of M . Examples: the connectivity number of a disk with h holes is h , and of a torus is 2.

For topological spaces or manifolds with dimension ≥ 2 the connectivity number is generalized to the k th Betti number by replacing curves by appropriate k -dimensional submanifolds. Of course, the first Betti number is just the connectivity number. See [7] for further details.

The k th Betti number is the number of appropriate k -dimensional submanifolds (“ k -cycles”) of M that do *not* bound in M , it is therefore a kind of measure for the geometric complexity of M .

We show that the Betti number of intersections of L_p -bisectors is bounded above by a constant that depends only on the space dimension. Our proof involves results of Milnor and Thom [12], [16] in algebraic geometry, and results of Benedetti and Risler [13], [2] on additive complexity of multivariate polynomials. The results of Benedetti and Risler substantially improve and generalize the results of Borodin and Cook [3].

Our results imply that in D dimensions the complexity of Voronoi diagrams of n point sites in general position in the L_p -metric (with even p) can be bounded by $O(n^c)$ for some constant c that does not depend on p .

Our Results to the Second Problem. We show that an analogy to the standard Euclidean metric holds, namely, the number of L_p -spheres (with even p) passing through $D + 1$ points in general position in D -space is always an *odd* number. To this end, we compute the *algebraic number* of such spheres. This number seems to have some impact on algorithms for computing Voronoi diagrams in the L_p -metric, because the algebraic number mentioned is a numerical *invariant*; see also Section 4.2 for a geometrical interpretation.

We briefly describe the algebraic number or *degree* associated to a continuous mapping $f: \Omega \rightarrow \mathbb{R}^D$. Assume that f is differentiable and y is a regular value of f , then the *degree of f (on Ω over y)* is defined to be the number of points in $f^{-1}(y)$ at which the linearization of f , i.e., the Jacobian matrix of f , preserves the orientation minus the number of points in $f^{-1}(y)$ at which the linearization reverses the orientation. The degree is therefore also called the *algebraic number* of solutions of the equation $f(x) = y$. One important property of degree is that whenever the degree of f (over y) differs from zero then the equation $f(x) = y$ has at least one solution. For this reason the degree is one important tool widely used in nonlinear analysis for proving the existence of solutions of the equation $f(x) = y$, see [14].

Our technique does not apply to the case of odd p . This is because we need the unique solvability of the equation $x^{p-1} = y$, for any real number y (see the proof of Lemma 14).

Related Works. Deep results of Sharir [15] imply that the combinatorial complexity of the Voronoi diagram in the L_p -metric (for even p) of a set of n points sites in D -space is $O(n^{D+\varepsilon})$, for any $\varepsilon > 0$. However, the “constant” factor in this bound tends to ∞ as $p \rightarrow \infty$. Thus, keeping n fixed our bound is more favorable for large p .

Makeev [11] has shown in a context quite different from ours that, given $D + 1$ affinely independent points and a smooth convex body in D -space, there always exists a body homothetic to the given one so that its boundary passes through the given points. His proof is geometrical and is based on a similar kind of degree of mapping.

Using a result of Goodey [6, Theorem 4], it can be shown that, for any convex distance function d in D -space whose unit sphere is not an ellipsoid, there exist $D + 1$ affinely independent points and at least two d -spheres (i.e., sets homothetic to the unit sphere induced by d) that pass through these points.

Implications. In particular, our results provide a clear picture of the geometrical structure of bisectors and their intersections based on the L_p -metric in 3-space:

- Bisectors are homeomorphic to a plane (indeed, every bisector is the graph of a smooth real-valued function defined on a plane). They intersect in a single curve homeomorphic to a line, see [8] and [9].
- The number N of L_p -spheres passing through four points in general position is odd and can exceed one, see [8]. There exists a constant C that does not depend on p such that $N \leq C$.

The rest of this paper is organized as follows. Section 2 introduces the basic definitions, among them the important notion of “in general position” for points sites with respect to a smooth convex distance function; there we state a key result needed in Sections 3 and 4. Section 3 is devoted to the upper bounds for the number of L_p -spheres passing through $D + 1$ points in general position, and on the Betti number of L_p -bisector intersections. In Section 4 we compute the algebraic number of L_p -spheres passing through $D + 1$ points in general position. Conclusions and open problems are discussed in Section 5.

2. Preliminaries

In the special case of the L_p -metric with even p , it can easily be proved that the bisector $B(s, t)$ of two points s and t in D -space is a $(D - 1)$ -dimensional submanifold of $\mathbb{R}^D$. In fact, the gradient (with respect to x) of the polynomial $d_p(x, s)^p - d_p(x, t)^p$, whose zero-set yields the bisector $B(s, t)$, does not vanish whenever $s \neq t$. Therefore the zero-set of $d_p(x, s)^p - d_p(x, t)^p$ is a $(D - 1)$ -dimensional manifold. More generally, it can be shown [9] that for smooth strictly convex distance functions in D dimensions bisectors are $(D - 1)$ -dimensional hypersurfaces homeomorphic to a hyperplane, and they intersect in a smooth surface homeomorphic to a $(D - 2)$ -dimensional linear space.

Let S be a finite set of points in D dimensions. For every point $x \in \mathbb{R}^D$, let $C(x)$ contain all sites $s \in S$ that minimize $d(x, s)$, and call two points x and y *equivalent* if $C(x) = C(y)$. As in the classical case of the Euclidean metric [5, p. 294], we call an equivalence class F a k -face if its dimension is k . However, note that in our context of general convex distance functions this definition allows a k -face to be disconnected,

which does not occur in the Euclidean metric. As we see later, allowing a face to be disconnected makes the discussion on Betti numbers of L_p -bisector intersections more uniform.

As in [5, p. 294], we define the *Voronoi diagram* $V(S)$ of S to be the cell complex whose faces are the equivalence classes of points. As usual, we use *vertex* and *edge* to denote a connected component of a 0-face (a single point) and a connected component of a 1-face, respectively. We call the set

$$VR(s) = \{x \in \mathbb{R}^D \mid d(x, s) < d(x, t) \text{ for each } t \in S - \{s\}\}$$

the *Voronoi region* of s —clearly, $VR(s)$ is a D -face. Now let $S = \{s^{(1)}, \dots, s^{(N)}\}$, then the tuple $(s^{(1)}, \dots, s^{(N)}) \in (\mathbb{R}^D)^N$ is called an N -*configuration*. Note that though another indexing of the N given points will yield another N -configuration, we regard all N -configurations having the same set of components as being equivalent. Let $T = \{t^{(1)}, t^{(2)}, \dots, t^{(M)}\}$ be a subset of S . By $B(T)$ or $B(t^{(1)}, t^{(2)}, \dots, t^{(M)})$ we denote the set of all $x \in \mathbb{R}^D$ having equal distances to the points of T , i.e.,

$$B(T) = \{x \in \mathbb{R}^D \mid d(x, t^{(1)}) = d(x, t^{(2)}) = \dots = d(x, t^{(M)})\}.$$

To introduce the notion of points being in general position with respect to a smooth convex distance function, we need the concept of transversality. We say that two manifolds M_1 and M_2 of $\mathbb{R}^D$ *intersect transversely at a point* $x \in M_1 \cap M_2$ if the tangent spaces $T_x M_1$ and $T_x M_2$ of M_1 and M_2 at x (respectively) intersect transversely, i.e., their vector sum is the whole $\mathbb{R}^D$:

$$T_x M_1 + T_x M_2 = \mathbb{R}^D.$$

(In particular, if $M_1 \cap M_2 = \emptyset$, then M_1 and M_2 intersect transversely.) See Fig. 1. We say that M_1 and M_2 *intersect transversely* (in $\mathbb{R}^D$) if they do so at every $x \in M_1 \cap M_2$.

Definition 1. We say that the points $s^{(1)}, \dots, s^{(N)}$ are *in general position* or *nondegenerate* (with respect to the smooth convex distance function d) if for any subsets T of S , any $q \in S - T$, and any $t \in T$ the manifolds $B(T)$ and $B(t, q)$ intersect transversely in $\mathbb{R}^D$, i.e., at every point of $B(T) \cap B(t, q)$ the tangents to $B(T)$ and $B(t, q)$ span the whole $\mathbb{R}^D$.

Fig. 1. Illustrating the notion of transversality.

Note that if $B(T)$ is a finite set of points, then its tangents are zero-dimensional vector spaces, so the only way for $B(T)$ and $B(t, q)$ to “intersect” transversely is that $B(T) \cap B(t, q) = \emptyset$.

In the L_2 -metric in 3-space the usual definition of points being in general position is that no five points are cospherical and no four points are cocircular. Our definition of points being in general position generalizes this definition to smooth strictly convex distance functions. This claim follows by observing that in the L_2 -metric the bisector of two points p and q is a plane passing through the middle of the segment pq and perpendicular to pq . However, in the general case of convex distance functions different from the L_2 -metric, our definition of “in general position” still needs a formal justification: any set of configurations of N distinct points $\in \mathbb{R}^D$ being “in general position” should be very large. Indeed, the following result [9] justifies our definition:

Let d be a smooth strictly convex distance function in $\mathbb{R}^D$. Then the set $\mathcal{G} \subset (\mathbb{R}^D)^N$ of N -configurations being in general position (with respect to the metric d) is dense in $(\mathbb{R}^D)^N$. Moreover, the complementary set of $\mathcal{G}$ in $(\mathbb{R}^D)^N$ has measure zero, if it is measurable.

This result says that “almost all” sets of N distinct points in $\mathbb{R}^D$ have the property of being in general position. Note that we can in addition assume that any set of $D + 1$ points (among the N points in general position) are linearly independent.

Let S be a set of points in general position, and let $T \subset S$ with $|T| \geq 2$. The following claim follows directly from the definition of general position.

Lemma 2. *If $|T| \leq D + 1$, then $B(T)$ is either a manifold with the dimension $D + 1 - |T|$ or an empty set. If $|T| > D + 1$, then $B(T) = \emptyset$.*

Clearly, vertices of the Voronoi diagram $V(S)$ are contained in $B(T)$ for $|T| = D + 1$, edges of $V(S)$ are subsets of $B(T)$ for $|T| = D$. Thus, if we assume that the points of S are in general position, then, by the discussion above, in order to construct the faces of a Voronoi region we need to investigate only the sets $B(T)$ with $|T| \leq D + 1$.

For shorter notation, from now on we write d to denote the L_p -metric (instead of d_p). Let $S = \{s^{(1)}, \dots, s^{(K+1)}\}$ be a set of $K + 1$ points in $\mathbb{R}^D$. To avoid dealing with fractional powers we define

$$\begin{aligned} f_1(x) &= d(x, s^{(1)})^p - d(x, s^{(K+1)})^p, \\ &\vdots \\ f_K(x) &= d(x, s^{(K)})^p - d(x, s^{(K+1)})^p. \end{aligned}$$

Then clearly $B(s^{(1)}, \dots, s^{(K+1)})$ is the zero-set of the polynomial system $f_1 = 0, \dots, f_K = 0$. Using the definition of $d(\cdot, \cdot)$ we obtain

$$f_1(x) = (x_1 - s_1^{(1)})^p + \dots + (x_D - s_D^{(1)})^p - (x_1 - s_1^{(K+1)})^p - \dots - (x_D - s_D^{(K+1)})^p. \quad (1)$$

We obtain analogous expressions for $f_2, \dots, f_K$.

3. Bounding the Complexity of Voronoi Diagrams in the L_p -Metric

3.1. Additive Complexity of Multivariate Polynomials

We consider real polynomials in D real variables $x_1, \dots, x_D$. If α is a *multi-index*, i.e., $\alpha = (\alpha_1, \dots, \alpha_D) \in \mathbb{Z}^D$, we write x^α to denote the formal product $x_1^{\alpha_1} \cdots x_D^{\alpha_D}$, “formal” because if $\alpha_i < 0$, then $x_i^{\alpha_i}$ is not defined for $x_i = 0$. The notion of “additive complexity” of a real multivariate polynomial was generalized from the univariate case [3] by Risler [13] as follows (see also Definition 4.3.1 of [2]).

Let P be a real polynomial in D variables. If P is a monomial, then the *additive complexity* of P is defined to be zero. Otherwise we can find a positive integer k and formal expressions $S_1, \dots, S_{k+1}$ having the following form:

$$\left\{ \begin{array}{l} S_1 = c_1 + x^{\alpha_1}, \\ S_2 = c_2 + x^{\alpha_2} S_1^{a_{1,2}}, \\ \vdots \\ S_k = c_k + x^{\alpha_k} \prod_{i=1}^{k-1} S_i^{a_{i,k}}, \\ S_{k+1} = c_{k+1} \times x^{\alpha_{k+1}} \prod_{i=1}^k S_i^{a_{i,k+1}} \end{array} \right. \quad (2)$$

with real numbers c_i , integers $a_{i,j}$, and multi-indices α_i (for $i = 1, \dots, k$) such that

$$P = S_{k+1}$$

holds after successively expanding S_i for $i = 2, \dots, k+1$. The expressions S_i ’s are called formal because if an exponent therein is negative, then they are undefined for some specific value of the corresponding variable. We define the *additive complexity* of P to be the minimum of such k ’s, and denote it by $L_+(P)$. Informally, $L_+(P)$ is the minimal number of additions needed for computing P , ignoring the number of multiplications.

Example. Consider the polynomial $P = x_1^2 + 2x_1x_2 + x_2^2$. We set

$$S_1 = 2 + x_1^{-1}x_2, \quad (3)$$

$$S_2 = 1 + x_1^{-1}x_2S_1, \quad (4)$$

$$S_3 = 1 \times x_1^2S_2. \quad (5)$$

After replacing S_2 in (5) by the right-hand side of (4), and then replacing S_1 by the right-hand side of (3) we clearly obtain

$$P = S_3 = x_1^2 + 2x_1x_2 + x_2^2.$$

So the additive complexity of P is at most 2. However, observing that P also equals $(x_1 + x_2)^2$, we can alternatively write

$$S_1 = 1 + x_1^{-1}x_2, \quad (6)$$

$$S_2 = 1 \times x_1^2S_1^2 \quad (7)$$

so that $P = S_2$ after replacing S_1 in (7) by the right-hand side of (6). Thus $L_+(P) \leq 1$, which is better than the bound 2 found before.

The following result was proved by Benedetti and Risler, see [13] and Proposition 4.3.3 of [2]. It is a substantial improvement and generalization of the results of Borodin and Cook [3], and shows that the notion of additive complexity is as useful as the ordinary degree used in Bézout's theorem for bounding the number of solutions of polynomial systems.

Proposition 3. *Let $P_1, \dots, P_D$ be real polynomials in D variables $x_1, \dots, x_D$. Let N be the number of nondegenerate solutions of the system*

$$\begin{cases} P_1(x_1, \dots, x_D) = 0, \\ \vdots \\ P_D(x_1, \dots, x_D) = 0. \end{cases}$$

Then we have

$$N \leq 2^A (A + 1)^B 2^{B(B-1)/2},$$

where

$$A = D + \sum_{i=1}^D L_+(P_i) \quad \text{and} \quad B = D + 2 \sum_{i=1}^D L_+(P_i).$$

To apply Proposition 3, we have to estimate the additive complexity of polynomials defining bisectors in the L_p -metric. The following lemma provides useful rules for getting bounds on additive complexity of expressions that are composed of polynomials whose additive complexities are known.

Lemma 4. *Let $P, P_1, \dots, P_n$ and Q be real polynomials. Then:*

1. $L_+(P_1 + \dots + P_n) \leq n - 1 + L_+(P_1) + \dots + L_+(P_n)$.
2. $L_+(P^m) \leq L_+(P)$, for any integer $m \geq 0$.
3. $L_+(PQ) \leq L_+(P) + L_+(Q)$.

Proof. The proof is easy (though lengthy), and follows almost immediately from the definition of additive complexity. We prove only Claim 1.

We only show that $L_+(P + Q) \leq 1 + L_+(P) + L_+(Q)$. The general claim can then be proved using a simple induction on n . Informally, to compute $P + Q$, we need $L_+(P)$ additions for computing P , $L_+(Q)$ additions for computing Q , and one addition more to get their sum, ignoring the multiplications. Thus, in total we need $L_+(P) + L_+(Q) + 1$ additions to compute $P + Q$. The formal proof follows. Let $k = L_+(P)$ and $l = L_+(Q)$.

Then, for P , we have

$$\left\{ \begin{array}{l} S_1 = c_1 + x^{\alpha_1}, \\ S_2 = c_2 + x^{\alpha_2} S_1^{\alpha_{1,2}}, \\ \vdots \\ S_k = c_k + x^{\alpha_k} \prod_{i=1}^{k-1} S_i^{\alpha_{i,k}}, \\ S_{k+1} = c_{k+1} \times x^{\alpha_{k+1}} \prod_{i=1}^k S_i^{\alpha_{i,k+1}} \end{array} \right. \quad (8)$$

so that $P = S_{k+1}$ after expanding the S_i 's. Similarly, for Q , we have

$$\left\{ \begin{array}{l} T_1 = d_1 + x^{\beta_1}, \\ T_2 = d_2 + x^{\beta_2} T_1^{\beta_{1,2}}, \\ \vdots \\ T_l = d_l + x^{\beta_l} \prod_{i=1}^{l-1} T_i^{\beta_{i,l}}, \\ T_{l+1} = d_{l+1} \times x^{\beta_{l+1}} \prod_{i=1}^l T_i^{\beta_{i,l+1}} \end{array} \right. \quad (9)$$

so that $Q = T_{l+1}$ after expanding the T_i 's. To prove the required bound for $L_+(P + Q)$, we patch (8) and (9) together by setting

$$\begin{aligned} W_1 &= S_1 = c_1 + x^{\alpha_1}, \\ &\vdots \\ W_k &= S_k = c_k + x^{\alpha_k} \prod_{i=1}^{k-1} S_i^{\alpha_{i,k}} = c_k + x^{\alpha_k} \prod_{i=1}^{k-1} W_i^{\alpha_{i,k}}, \\ W_{k+1} &= T_1 = d_1 + x^{\beta_1}, \\ &\vdots \\ W_{k+l} &= T_l = d_l + x^{\beta_l} \prod_{i=1}^{l-1} T_i^{\beta_{i,l}} = d_l + x^{\beta_l} \prod_{i=1}^{l-1} W_{k+i}^{\beta_{i,l}}, \\ W_{k+l+1} &\stackrel{(*)}{=} \frac{c_{k+1}}{d_{l+1}} + c_{k+1} P^{-1} \frac{Q}{d_{l+1}} = \frac{c_{k+1}}{d_{l+1}} + x^{-\alpha_{k+1}} \left(\prod_{i=1}^k S_i^{-\alpha_{i,k+1}} \right) x^{\beta_{l+1}} \left(\prod_{i=1}^l T_i^{\beta_{i,l+1}} \right) \\ &= \frac{c_{k+1}}{d_{l+1}} + x^{-\alpha_{k+1}} \left(\prod_{i=1}^k W_i^{-\alpha_{i,k+1}} \right) x^{\beta_{l+1}} \left(\prod_{i=1}^l W_{k+i}^{\beta_{i,l+1}} \right), \\ W_{k+l+2} &\stackrel{(**)}{=} d_{l+1} \frac{P}{c_{k+1}} W_{k+l+1} = d_{l+1} x^{\alpha_{k+1}} \left(\prod_{i=1}^k S_i^{\alpha_{i,k+1}} \right) W_{k+l+1} \\ &= d_{l+1} x^{\alpha_{k+1}} \left(\prod_{i=1}^k W_i^{\alpha_{i,k+1}} \right) W_{k+l+1}. \end{aligned}$$

Using the equalities (*) and (**), which define W_{k+l+1} and W_{k+l+2} in terms of P and Q , it is easy to verify that $P + Q = W_{k+l+2}$ after expanding the W_i 's (without actually doing this). Therefore $L_+(P + Q) \leq 1 + L_+(P) + L_+(Q)$. $\square$

Given two distinct points u and v in $\mathbb{R}^D$, we know that the bisector $B(u, v)$ of u and v in the L_p -metric is the zero-set of the following polynomial having the same form as (1):

$$f = (x_1 - u_1)^p + \cdots + (x_D - u_D)^p - (x_1 - v_1)^p - \cdots - (x_D - v_D)^p. \quad (10)$$

Lemma 5. *The additive complexity of f is at most $4D - 1$.*

Proof. From Lemma 4 and the fact that the additive complexity of a constant or a monomial is zero, we have $L_+((x_i - u_i)^p) \leq 1$ and $L_+(-(x_i - v_i)^p) \leq 1$. Applying Rule 1 of Lemma 4 to f we obtain

$$\begin{aligned} L_+(f) &\leq 1 + \left(D - 1 + \sum_{i=1}^D L_+((x_i - u_i)^p) \right) + \left(D - 1 + \sum_{i=1}^D L_+(-(x_i - v_i)^p) \right) \\ &\leq 4D - 1. \end{aligned} \quad \square$$

3.2. L_p -Metric

We point out that the assumption of nondegeneracy in the following theorem—which is required in order to apply Proposition 3—is meaningful by the discussion in Section 2.

Theorem 6. *For any $D + 1$ points in $\mathbb{R}^D$ in general position, the number of L_p -spheres passing through the given points is bounded by a constant depending only on the dimension D but not on p .*

Proof. Let $s^{(1)}, \dots, s^{(D+1)}$ be the given points. Then the centers of the L_p -spheres passing through them are the solutions of the following system of polynomial equations:

$$\begin{cases} f_1(x_1, \dots, x_D) = 0, \\ \vdots \\ f_D(x_1, \dots, x_D) = 0, \end{cases} \quad (11)$$

where each f_i , for $i = 1, \dots, D$, has the same form as the polynomial (1). Let N_1 be the number of solutions of (11), then by Proposition 3 we have

$$N_1 \leq 2^A (A + 1)^B 2^{B(B-1)/2},$$

where

$$A = D + \sum_{i=1}^D L_+(P_i) \leq 4D^2,$$

and

$$B = D + 2 \sum_{i=1}^D L_+(P_i) \leq D(8D - 1). \quad \square$$

From the result above, combined with results in [9], it follows that in D -space the complexity of the Voronoi diagram of n points in general position in the L_p -metric is bounded by $O(n^c)$ for some c independent of p . However, the constant factor is huge so that the bound has little practical implications.

In the following theorem, the general-position assumption is not required. However, then the set $B(s^{(1)}, \dots, s^{(K+1)})$ need not be finite when $K = D$, nor empty when $K > D$.

Theorem 7. *For any set of $K + 1$ points $s^{(1)}, \dots, s^{(K+1)}$ in $\mathbb{R}^D$ the sum of the Betti numbers of $B(s^{(1)}, \dots, s^{(K+1)})$ is bounded by a constant depending only on K and the space dimension D but not on p .*

Proof. In deriving the equation system whose number of zeros determine the upper bound to the sum of the Betti numbers to the algebraic variety $B(s^{(1)}, \dots, s^{(K+1)})$, we essentially follow [12] and [2]. For $i = 1, \dots, K$ let f_i be the polynomials defining the bisectors $B(s^{(i)}, s^{(K+1)})$, see (1). Since $B(s^{(1)}, \dots, s^{(K+1)})$, which is the zero-set of

$$F_B = f_1^2 + \dots + f_K^2,$$

is not necessarily a compact and nonsingular hypersurface in $\mathbb{R}^D$, we cannot apply Morse theory directly to F_B to obtain the bound desired. In place of F_B , following Milnor, we consider the function

$$F = F_B + \alpha(x_1^2 + \dots + x_D^2) - \beta$$

that depends on the positive parameters α and β . For any line L let π_L denote the orthogonal projection from $\mathbb{R}^D$ onto L (after choosing a coordinate on L , π_L maps each point of $\mathbb{R}^D$ to a real number, its “height”). Then (see [12]) there exist $\alpha, \beta > 0$ and a line L such that the zero-set $F^{-1}(0)$ of F is a compact hypersurface in $\mathbb{R}^D$, and such that the restriction of π_L to $F^{-1}(0)$ is a Morse function, i.e., all its critical points are nondegenerate. (A critical point p of a function φ is *nondegenerate* if there is a local coordinate system $(y_1, \dots, y_D)$ in a neighborhood U of p with $y_i(p) = 0$ for all i and so that $\varphi = \varphi(p) - y_1^2 - \dots - y_k^2 + y_{k+1}^2 + \dots + y_D^2$ holds on U for some k .)

Let N_2 denote the sum of the Betti numbers of the zero-set of F_B (thus, of $B(s^{(1)}, \dots, s^{(K+1)})$), and let N_3 denote the sum of the Betti numbers of the zero-set of F . Then, using results from algebraic topology, Milnor was able to relate N_2 to N_3 by

$$N_2 \leq \frac{1}{2}N_3.$$

Moreover, Morse theory says that

$$N_3 \leq \# \text{ critical points of } \pi_L.$$

Fig. 2. Here, $F^{-1}(0)$ is a torus. The critical points of π_L are the points on the torus at which $\text{grad } F$ is orthogonal to the plane spanned by $\lambda^{(1)}$ and $\lambda^{(2)}$.

So we have

$$N_2 \leq \frac{1}{2}(\# \text{ critical points of } \pi_L).$$

We now estimate the right-hand side of the inequality above. Let $\lambda^{(1)}, \dots, \lambda^{(D-1)}$ denote linearly independent vectors spanning a hyperplane that is orthogonal to the line L , see Fig. 2. Then clearly the critical points of π_L are the solutions of the system

$$\begin{cases} F = 0, \\ G_1 := \lambda^{(1)} \text{grad } F = 0, \\ \vdots \\ G_{D-1} := \lambda^{(D-1)} \text{grad } F = 0, \end{cases}$$

where $\text{grad } F$ denote the gradient of F .

This system is equivalent to

$$\begin{cases} F = f_1^2 + \dots + f_K^2 + \alpha(x_1^2 + \dots + x_D^2) - \beta = 0, \\ G_1 = \lambda_1^{(1)} \partial F / \partial x_1 + \dots + \lambda_D^{(1)} \partial F / \partial x_D = 0, \\ \vdots \\ G_{D-1} = \lambda_1^{(D-1)} \partial F / \partial x_1 + \dots + \lambda_D^{(D-1)} \partial F / \partial x_D = 0. \end{cases} \quad (12)$$

Since π_L is a Morse function, solutions of (12) are nondegenerate. Thus we can apply Proposition 3 to get an explicit upper bound for the number of solutions of (12). For this, we need estimates for the additive complexity of F and $G_1, \dots, G_{D-1}$. Using the rules of Lemma 4, we come after some simple calculations to the estimates that are given below for completeness:

$$\begin{aligned} L_+(F) &\leq 4KD + D, \\ L_+(G_i) &\leq KD(4D + 3) + (D - 1) \quad \text{for } i = 1, \dots, D - 1. \end{aligned}$$

By Proposition 3 we obtain

$$N_2 \leq \frac{1}{2} 2^A (A+1)^B 2^{B(B-1)/2},$$

where

$$A = D + L_+(F) + \sum_{i=1}^{D-1} L_+(G_i) \leq K D(4D^2 - D + 1) + D^2 + 1$$

and

$$B = D + 2L_+(F) + 2 \sum_{i=1}^{D-1} L_+(G_i) \leq K D(8D^2 - 2D + 2) + 2D^2 - D + 2. \quad \square$$

The constant mentioned in the theorem above, which is the upper bound on the number N_2 in the proof, is very large. So it might have no immediate practical applications.

Corollary 8. *For any set of $K+1$ points $s^{(1)}, \dots, s^{(K+1)}$ in $\mathbb{R}^D$ the number of connected components of $B(s^{(1)}, \dots, s^{(K+1)})$ is bounded by a constant depending only on K and D but not on p .*

Proof. The claim follows from Theorem 7 and the fact that the 0th Betti number of a topological space is the number of its connected components [7]. $\square$

Remark. We have restricted our investigation to the L_p -metrics with even p , because for odd $p > 2$ analogous results on upper bounds can be derived in a similar manner. In fact, in that case—although we must deal with *semialgebraic* sets—the additional number of polynomials to be considered depends only on D but not on p .

4. The Algebraic Number of L_p -Spheres Passing Through $D+1$ Points in D -Space

4.1. Degree of Continuous Mappings

The precise definition of degree makes use of approximation theory and results on properties of critical values of a map. Throughout this section, let Ω be an open and bounded subset of $\mathbb{R}^D$. We denote by $C(\bar{\Omega})$ the set of all continuous functions $f: \bar{\Omega} \rightarrow \mathbb{R}^D$, and by $C^k(\Omega)$ the set of all k -times continuous differentiable functions $f: \Omega \rightarrow \mathbb{R}^D$. If $f \in C^k(\Omega)$ we also say that f is C^k on Ω . If g is C^1 on Ω , then $g'(x)$ denotes the Jacobian matrix of g at $x \in \Omega$:

$$g'(x) = \left(\frac{\partial g_i}{\partial x_j}(x) \right)_{i,j=1,\dots,D}.$$

The determinant of g' —the Jacobian determinant of g —is denoted by $J_g(x)$. Let N_g denote the set of critical points of g , i.e., points at which the Jacobian of g vanishes.

Suppose that $g \in C(\bar{\Omega})$ is C^1 on Ω and that $y \in \mathbb{R}^D$ is a point not belonging to $g(\partial\Omega \cup N_g)$. Then at each $x \in \mathbb{R}^D$ with $g(x) = y$ the Jacobian $J_g(x)$ does not vanish, so the sign of $J_g(x)$ is unambiguously defined. We define the *degree of g on Ω over y* to be the integer

$$\deg(g, \Omega, y) = \sum_{x \in g^{-1}(y)} \text{sign } J_g(x), \quad \text{with the convention } \sum_{\emptyset} = 0. \quad (13)$$

Suppose now that $y \in g(N_g)$. Then it can be shown that there is a sequence of points $\{y_n\}$ with $y_n \notin g(\partial\Omega \cup N_g)$ so that $\lim_{n \rightarrow \infty} y_n = y$. Clearly, $\deg(g, \Omega, y_n)$ is defined by (13). Moreover, it can be shown that $\lim_{n \rightarrow \infty} \deg(g, \Omega, y_n)$ exists and the limit does not depend on the choice of the sequence $\{y_n\}$. We define the degree of g on Ω over y by

$$\deg(g, \Omega, y) = \lim_{n \rightarrow \infty} \deg(g, \Omega, y_n).$$

However, an arbitrary function $f \in C(\bar{\Omega})$ need not be C^1 on Ω . In this situation, we define the *degree of f on Ω over y* as follows [14, Definition 3.14].

Definition 9. Let $f \in C(\bar{\Omega})$ and let $\{g_n\}$ be a sequence of functions in C^1 that converge uniformly to f . Then for every $y \notin f(\partial\Omega)$ a sequence $\{\deg(g_n, \Omega, y)\}$, with $n \geq N$ for some N large enough, is defined, the limit

$$\lim_{n \rightarrow \infty} \deg(g_n, \Omega, y)$$

exists and does not depend on the special choice of $\{g_n\}$, and we then define

$$\deg(f, \Omega, y) = \lim_{n \rightarrow \infty} \deg(g_n, \Omega, y).$$

A justification of the definition above can be found on p. 71 of [14].

Lemma 10. Suppose the mapping $\Phi: [0, 1] \times \bar{\Omega} \rightarrow \mathbb{R}^D$ is continuous and $y \notin \Phi(t, \partial\Omega)$ for every $t \in [0, 1]$. Then $\deg(\Phi(t, \cdot), \Omega, y)$ is constant on $[0, 1]$.

Proof. See, e.g., Theorem 3.16(1) of [14]. □

Let x be a point in $\mathbb{R}^D$ with coordinates $(x_1, \dots, x_D)$. Then the *length* of x or more usually the *norm* of x is defined by

$$\|x\| = \sqrt{x_1^2 + \dots + x_D^2}.$$

For the proofs of the main results of the next subsection we need the following lemmas.

Lemma 11. Let h and $\delta \in C(\bar{\Omega})$ such that $\|\delta(x)\| < \|h(x)\|$ holds for all $x \in \partial\Omega$. Then $\deg(h + \delta, \Omega, 0) = \deg(h, \Omega, 0)$.

Proof. We set $\Phi(t, x) = h(x) + t\delta(x)$. Then for every $t \in [0, 1]$ and $x \in \partial\Omega$ we have

$$\begin{aligned}\|\Phi(t, x)\| &\geq \|h(x)\| - t\|\delta(x)\| \\ &> 0.\end{aligned}$$

This implies that $0 \notin \Phi(t, \partial\Omega)$, for every $t \in [0, 1]$. Applying Lemma 10, we obtain $\deg(\Phi(1, \cdot), \Omega, 0) = \deg(\Phi(0, \cdot), \Omega, 0)$, which clearly implies the claim. $\square$

Lemma 12. *Let $f \in C(\bar{\Omega})$. If y and z belong to the same connected component of $\mathbb{R}^D - f(\partial\Omega)$, then $\deg(f, \Omega, y) = \deg(f, \Omega, z)$.*

Proof. See Theorem 3.16(4) of [14]. $\square$

4.2. Computing the Algebraic Number of L_p -Spheres Passing Through $D + 1$ Points in D -Space

For convenience, we fix the dimension by taking $D = 3$. The proofs for the general case are similar to those for the special case. Let $S = \{s, t, u, v\}$ be a set of points in $\mathbb{R}^3$ in general position. The bisectors $B(s, v)$, $B(t, v)$, and $B(u, v)$ are the zero-sets of the following polynomials, which are analogous to the polynomial (1), respectively:

$$\begin{aligned}f_1(x) &= d(x, s)^p - d(x, v)^p, \\ f_2(x) &= d(x, t)^p - d(x, v)^p, \\ f_3(x) &= d(x, u)^p - d(x, v)^p.\end{aligned}$$

We discuss how the degree of $f = (f_1, f_2, f_3)$ over zero can be given a geometrical interpretation. Consider the polynomial system $f_1 = 0$, $f_2 = 0$, and $f_3 = 0$ whose zeros yield the set $B(s, t, u, v)$. Observe that the bisector $B(s, v)$ is the zero-set of $f_1 = 0$, and that $B(t, u, v)$ is the zero-set of the system $f_2 = 0$, $f_3 = 0$. Hence $B(s, t, u, v)$ is the intersection of the bisector surface $B(s, v)$ and the curve $B(t, u, v)$, see Fig. 3. Let us see how the orientation of $B(s, v)$ and $B(t, u, v)$ are related to the Jacobian determinant used in the definition of degree. At each point in $B(s, v) \cap B(t, u, v)$ the Jacobian determinant of f is the scalar product of the tangent vector to $B(t, u, v)$ and the normal to $B(s, v)$. By the general-position assumption these scalar products do not vanish. Hence the degree is the algebraic number of points in $B(s, v) \cap B(t, u, v)$, where each point is counted with appropriate signs determined by the orientations of $B(s, v)$ and $B(t, u, v)$.

Fig. 3. Illustrating the geometrical interpretation of the degree.

Without loss of generality assume that $v = 0$ (which means that we translate the origin of the coordinate system to v). Expanding, for example, f_1 and since the monomials with the exponent p cancel out, we obtain

$$f_1(x) = \sum_{v=1}^p b_v (s_1^v x_1^{p-v} + s_2^v x_2^{p-v} + s_3^v x_3^{p-v}),$$

where we put the binomial coefficient $(-1)^v \binom{p}{v} = b_v$. We have analogous expressions for f_2 and f_3 .

Theorem 13. *The algebraic number of L_p -spheres passing through $D + 1$ points in general position is either 1 or -1 .*

The idea of the proof is as follows. We decompose each polynomial f_i (with $i = 1, 2, 3$) into the sum of a term h_i consisting of monomials with the exponent $p - 1$ and a remainder δ_i containing monomials having exponents strictly less than $p - 1$. The polynomials h_i are given by

$$\begin{aligned} h_1(x) &= b_1 (s_1 x_1^{p-1} + s_2 x_2^{p-1} + s_3 x_3^{p-1}), \\ h_2(x) &= b_1 (t_1 x_1^{p-1} + t_2 x_2^{p-1} + t_3 x_3^{p-1}), \\ h_3(x) &= b_1 (u_1 x_1^{p-1} + u_2 x_2^{p-1} + u_3 x_3^{p-1}), \end{aligned} \quad (14)$$

and the polynomials δ_i are given by

$$\begin{aligned} \delta_1(x) &= \sum_{v=2}^p b_v (s_1^v x_1^{p-v} + s_2^v x_2^{p-v} + s_3^v x_3^{p-v}), \\ \delta_2(x) &= \sum_{v=2}^p b_v (t_1^v x_1^{p-v} + t_2^v x_2^{p-v} + t_3^v x_3^{p-v}), \\ \delta_3(x) &= \sum_{v=2}^p b_v (u_1^v x_1^{p-v} + u_2^v x_2^{p-v} + u_3^v x_3^{p-v}). \end{aligned} \quad (15)$$

Now the mapping f on $\mathbb{R}^3$ can be regarded as the perturbation of a principal term $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $h(x) = (h_1(x), h_2(x), h_3(x))$ by a “small” term $\delta: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $\delta(x) = (\delta_1(x), \delta_2(x), \delta_3(x))$. Here “small” is understood in a sense such that Lemma 11 is applicable. We need two lemmas for proving the theorem above.

Let U_ρ denote the open ball of radius $\rho > 0$, i.e., $U_\rho = \{x \in \mathbb{R}^D \mid \|x\| < \rho\}$.

Lemma 14. *$\deg(h, U_\rho, 0)$ is independent of ρ . Moreover, $\deg(h, U_\rho, 0)$ is 1 or -1 .*

Proof. We compute $\deg(h, U_\rho, 0)$ by using (13) and Lemma 12. To this end, we prove first that the mapping h is a homeomorphism on $\mathbb{R}^D$.

Clearly, h is continuous. We prove that h is bijective. For each $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ the equation $h(x) = y$ can be solved uniquely as follows. We first consider this equation

as a linear equation system for the unknowns x_1^{p-1} , x_2^{p-1} , and x_3^{p-1} . Now the matrix corresponding to this linear equation system is

$$\mathcal{D} = b_1 \begin{pmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \end{pmatrix}. \quad (16)$$

By the general-position assumption we have $\det \mathcal{D} \neq 0$. Therefore this linear system is uniquely solvable. Moreover, since x_1^{p-1} , x_2^{p-1} , and x_3^{p-1} have odd exponents, we can uniquely compute x_1 , x_2 , and x_3 . Thus the mapping h is bijective. Applying the Domain Invariance Theorem [14, Corollary 3.22], the mapping h is a homeomorphism.

We apply Lemma 12. Take y with $\|y\|$ sufficiently small such that y and zero are still contained in the same connected component of $\mathbb{R}^D - h(\partial U_\rho)$ and such that none of the coordinates of $x = h^{-1}(y)$ vanishes; since h is a homeomorphism, such a y exists. By Lemma 12 we have

$$\deg(h, U_\rho, 0) = \deg(h, U_\rho, y). \quad (17)$$

From (14) we see that the Jacobian matrix of h at x is given by

$$h'(x) = (p-1) \mathcal{D} \begin{pmatrix} x_1^{p-2} & 0 & 0 \\ 0 & x_2^{p-2} & 0 \\ 0 & 0 & x_3^{p-2} \end{pmatrix},$$

whose determinant is clearly

$$J_h(x) = (p-1)^3 x_1^{p-2} x_2^{p-2} x_3^{p-2} \det \mathcal{D}.$$

Because of the special choice of y , we have $x_1 x_2 x_3 \neq 0$. So the Jacobian determinant $J_h(x)$ does not vanish. Thus the right-hand side of the equality (17) can be computed by using (13). It follows that

$$\deg(h, U_\rho, y) = \text{sign } J_h(x). \quad \square \quad (18)$$

Remark. Since U_ρ is symmetric and h is an odd mapping, i.e., $h(x) = -h(-x)$, Borsuk's theorem (see Theorem 3.23 of [14]) implies that $\deg(h, U_\rho, 0)$ is an odd number. However, we would like to know the exact value of $\deg(h, U_\rho, 0)$. Indeed, if $|\deg(h, U_\rho, 0)| \geq 3$ holds, then we could prove that, for any $D+1$ points in general position, there exist at least three L_p -spheres passing through them. This result would be an interesting one.

Lemma 15. *For all sufficient large $\rho > 0$ we have $\deg(h + \delta, U_\rho, 0) = \deg(h, U_\rho, 0)$.*

Proof. For any nonzero $x \in \mathbb{R}^3$ we can write uniquely $x = \|x\|(x/\|x\|)$, thus we may define the variable transformation $x = \rho n$ where $\rho > 0$ and $n = (n_1, n_2, n_3) \in \mathbb{R}^3$ with $\|n\| = 1$. In (14) after replacing the variables x_1 by ρn_1 , x_2 by ρn_2 , and x_3 by ρn_3 the

components of h become

$$\begin{aligned} h_1(x) &= b_1 \rho^{p-1} (s_1 n_1^{p-1} + s_2 n_2^{p-1} + s_3 n_3^{p-1}), \\ h_2(x) &= b_1 \rho^{p-1} (t_1 n_1^{p-1} + t_2 n_2^{p-1} + t_3 n_3^{p-1}), \\ h_3(x) &= b_1 \rho^{p-1} (u_1 n_1^{p-1} + u_2 n_2^{p-1} + u_3 n_3^{p-1}). \end{aligned} \quad (19)$$

Analogously, after applying the transformation $x = \rho n$ to the components of δ given by (15) we obtain

$$\begin{aligned} \delta_1(x) &= \sum_{v=2}^p b_v \rho^{p-v} (s_1^v n_1^{p-v} + s_2^v n_2^{p-v} + s_3^v n_3^{p-v}), \\ \delta_2(x) &= \sum_{v=2}^p b_v \rho^{p-v} (t_1^v n_1^{p-v} + t_2^v n_2^{p-v} + t_3^v n_3^{p-v}), \\ \delta_3(x) &= \sum_{v=2}^p b_v \rho^{p-v} (u_1^v n_1^{p-v} + u_2^v n_2^{p-v} + u_3^v n_3^{p-v}). \end{aligned} \quad (20)$$

In order to apply Lemma 11, we show that $\|\delta\| < \|h\|$ or equivalently that $\|h\|^2 - \|\delta\|^2 > 0$ for all points on the sphere with center zero having a sufficient large radius ρ . First we compute $\|h\|^2$ from (19) and get

$$\|h\|^2(x) = b_1^2 \rho^{2(p-1)} \left(\left(\sum_{1 \leq i \leq 3} s_i n_i^{p-1} \right)^2 + \left(\sum_{1 \leq i \leq 3} t_i n_i^{p-1} \right)^2 + \left(\sum_{1 \leq i \leq 3} u_i n_i^{p-1} \right)^2 \right). \quad (21)$$

Now observe that $\|h\|^2 - \|\delta\|^2$ is a polynomial in ρ . Furthermore, the monomial in $\|h\|^2 - \|\delta\|^2$ having the highest exponent, which is $2(p-1)$, is exactly $\|h\|^2$. Hence, the coefficient for $\rho^{2(p-1)}$ can be extracted from formula (21):

$$b_1^2 \left(\left(\sum_{1 \leq i \leq 3} s_i n_i^{p-1} \right)^2 + \left(\sum_{1 \leq i \leq 3} t_i n_i^{p-1} \right)^2 + \left(\sum_{1 \leq i \leq 3} u_i n_i^{p-1} \right)^2 \right). \quad (22)$$

We show that it does not vanish. Suppose the contrary. Then each term of the sum in the expression above would vanish, i.e.,

$$\begin{aligned} s_1 n_1^{p-1} + s_2 n_2^{p-1} + s_3 n_3^{p-1} &= 0, \\ t_1 n_1^{p-1} + t_2 n_2^{p-1} + t_3 n_3^{p-1} &= 0, \\ u_1 n_1^{p-1} + u_2 n_2^{p-1} + u_3 n_3^{p-1} &= 0. \end{aligned}$$

Since $\det \mathcal{D} \neq 0$ we obtain $n_1^{p-1} = n_2^{p-1} = n_3^{p-1} = 0$. Thus we have $n_1 = n_2 = n_3 = 0$, which contradicts the equality $\|n\| = 1$.

By the result above, we conclude that for any sufficiently large ρ , say $\rho > \rho_0$, the inequality $\|\delta(x)\| < \|h(x)\|$ holds for all x with $\|x\| = \rho$. Applying Lemma 11 we obtain

$$\deg(h + \delta, U_\rho, 0) = \deg(h, U_\rho, 0),$$

where $\rho > \rho_0$. □

We are now ready to prove Theorem 13.

Proof of Theorem 13. From Lemma 15 we have $\deg(f, U_\rho, 0) = \deg(h + \delta, U_\rho, 0) = \deg(h, U_\rho, 0)$. Using Lemma 14 we finally obtain $\deg(f, U_\rho, 0) = 1$ or -1 . $\square$

Corollary 16. *Let K be a natural number $\leq D$. Then for any set of $K + 1$ points $s^{(1)}, \dots, s^{(K+1)} \in \mathbb{R}^D$ in general position the set $B(s^{(1)}, \dots, s^{(K+1)})$ is nonempty. In particular, for $K = D$ the set $B(s^{(1)}, \dots, s^{(D+1)})$ consists of an odd number of points that is bounded by a constant depending only on the dimension D but not on p .*

Proof. From $\deg(f, U_\rho, 0) \neq 0$ we conclude that $B(s^{(1)}, \dots, s^{(D+1)})$ is not empty. Therefore $B(s^{(1)}, \dots, s^{(K+1)})$ must also be nonempty.

Since $|\deg(f, U_\rho, 0)| = 1$, using the definition of degree we see that the number of points in $B(s^{(1)}, \dots, s^{(D+1)})$ must be odd. This fact together with Theorem 6 imply the claim. $\square$

Remark. From Corollary 16 and Theorem 4 of [6] it follows that there actually exist $D + 1$ points in general position so that the number of L_p -spheres, with even $p \geq 4$, passing through these points is an odd number ≥ 3 .

5. Concluding Remarks

In this paper we have investigated some structural and combinatorial properties of Voronoi diagrams in the L_p -metric by following the paradigm of expressing geometrical objects (e.g., intersections of bisectors) as zero-sets of systems of polynomials. In this way, we were able to apply methods from the theory of additive complexity, algebraic topology, and the theory of degree to our problem. Our main results are as follows. We have shown that there exists an upper bound for the number of L_p -spheres passing through $D + 1$ points in general position in $\mathbb{R}^D$, and for the sum of the Betti numbers of $B(s^{(1)}, \dots, s^{(K+1)})$. These bounds do not depend on p . We have determined the algebraic number of L_p -spheres passing through $D + 1$ points in general position in D -space.

Our results raise some open problems:

- (1) Determine the exact value of the maximum number of L_p -spheres passing through $D + 1$ points (in general position) in D -space.
- (2) Compute the algebraic number of spheres (induced by a smooth strictly convex distance function) passing through $D + 1$ points in general position in D -space.

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