Star-Shaped Sets in Normed Spaces

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Abstract. We prove a generalization of the Krasnosel'ski theorem on star-shaped sets. Using d -segments in n -dimensional Minkowski spaces instead of usual segments, the notions " d -visibility" and " d -star-shapedness" are introduced. Our main aim is to give necessary and sufficient conditions for d -star-shapedness in finite-dimensional normed spaces.

1. Introduction

Krasnosel'ski [7] proved a theorem on *star-shaped sets*. We recall its statement in a generalized form, as it is contained, e.g., in the book by Leichtweiss [8]:

Let $M \subset \mathbf{R}^n$ be a compact set. We say that M is *star-shaped* if a point $x_0 \in M$ exists such that for every point $x \in M$ the segment $[x_0, x]$ is contained in M . In other words, every point $x \in M$ is *visible in M* from the point x_0 . Krasnosel'ski's theorem affirms that, if every $n + 1$ points of M are visible in M from some point, then all the points of M are visible from some point of M , i.e., M is star-shaped.

The proof of this theorem, offered by Krasnosel'ski for closed regions in $\mathbf{R}^2$ which are bounded by a finite number of segments, was given in a simple and general form in [3], see also [8]. This proof is based on the classical Helly theorem. For further background, we refer to the survey articles [3] and [4].

In the present paper we prove a theorem (and several consequences thereof) which generalizes Krasnosel'ski's assertion. Namely, we consider *d -star-shapedness* in a

Minkowski space $\mathfrak{R}^n$, where d -star-shaped sets are obtained when we replace the usual segments by d -segments. Thus, we first recall briefly several notions connected with d -convexity in a Minkowski space $\mathfrak{R}^n$, see also [2].

2. Basic Notions

Let $\mathfrak{R}^n$ denote a (finite-dimensional) *normed space* or *Minkowski space*, whose unit ball is a compact, convex body symmetric about the origin and defining the distance between two points $x, y \in \mathfrak{R}^n$ in the usual way. We denote this distance by $d(x, y) = \|x - y\|$. For every two points $a, b \in \mathfrak{R}^n$ we denote by $[a, b]_d$ the d -segment with the endpoints a, b , i.e., the set of all points $x \in \mathfrak{R}^n$ satisfying the condition

$$\|a - x\| + \|x - b\| = \|a - b\|.$$

A set $M \subset \mathfrak{R}^n$ is said to be d -convex if for every $a, b \in M$ the inclusion $[a, b]_d \subset M$ holds. It is obvious that every d -convex set is *linearly convex* (i.e., it is convex in the usual sense, considering linear segments instead of d -segments). Since, evidently, the intersection of any family of d -convex sets is also d -convex, it is possible to define the d -convex hull $\text{conv}_d X$ for every set $X \subset \mathfrak{R}^n$ (as the intersection of all d -convex sets is d -convex).

It should be noticed that the notion of d -convexity was introduced by Menger [9], de Groot [5], Aleksandrov and Zalgaller [1], Soltan and Prisakaru [12], and other mathematicians. However, except for definitions, these papers contain almost no theorems on d -convex sets. The theory of d -convex sets was mainly developed by Soltan and his students, see, e.g., [10] and [11].

A set $M \subset \mathfrak{R}^n$ is said to be d -star-shaped if a point $x_0 \in M$ exists such that for every point $x \in M$ the d -segment $[x_0, x]_d$ is contained in M . (In other words, every point $x \in M$ is d -visible in M from x_0 .) In order to establish a necessary and sufficient condition for d -star-shapedness (which generalizes Krasnosel'ski's theorem), we introduce some auxiliary notions.

Let a be a point of a set $M \subset \mathfrak{R}^n$ (in particular, a boundary point). A ray l emanating from $a \in M$ is said to have a *free direction* (with respect to M) if a point $b \in l$ distinct from a exists, such that the interval $]a, b[$ does not contain any points of M , i.e., $]a, b[\cap M = \{a\}$. Otherwise, the ray l has a *nonfree direction*. The union of all rays emanating from $a \in M$ which have nonfree directions (with respect to M) is said to be the cone of nonfree directions at a and is denoted by $\text{nof}_a M$.

Now let M be a set in a Minkowski space $\mathfrak{R}^n$. We say that M possesses the property of *internal-local d -conicity* if for every point $a \in M$ the set $\text{cl}(\text{conv}(\text{nof}_a M))$ is d -convex. We now describe one more condition for a set $M \subset \mathfrak{R}^n$. We say that M possesses the property of *external-local d -conicity* if, for every $a \in M$ and every ray l emanating from a and having a free direction, there is a positive number ε such that for each positive $\mu < \lambda < \varepsilon$ the relation $[a + \lambda v, a + \mu v]_d \cap M = \emptyset$ holds, where v is a vector directed along l .

By $\text{him } \mathfrak{R}^n$ we denote the *Helly dimension* of the Minkowski space $\mathfrak{R}^n$, i.e., the minimal integer h with the following property: For every finite family $\{M_1, \dots, M_s\}$ of d -convex sets in $\mathfrak{R}^n$ ($s > h + 1$) the relation $M_1 \cap \dots \cap M_s \neq \emptyset$ holds, if every $h + 1$ of them has a nonempty intersection. Note that by Helly's theorem, $\text{him } \mathfrak{R}^n \leq n$.

3. Results

The following theorem is our main assertion. It is an analogue of Krasnosel'ski's theorem, and its proof uses some ideas from Krasnosel'ski's original proof, adapted for d -convexity.

Theorem. *Let $M \subset \mathfrak{R}^n$ be a compact set that is internal- and external-local d -conic. The number $\text{him } \mathfrak{R}^n$ is denoted by h . The set M is d -star-shaped if and only if for every $h + 1$ points $x_1, \dots, x_{h+1} \in M$ there is a point $y \in M$ such that all points $x_1, \dots, x_{h+1}$ are d -visible in M from y , i.e., $[y, x_i]_d \subset M$ for $i = 1, \dots, h + 1$.*

Proof. For every point $a \in M$, we put

$$C(a) = \text{cl conv}_d(M \cap \text{nof}_d M). \quad (1)$$

We establish that the d -convex set

$$N = \bigcap_{a \in M} C(a)$$

is nonempty, and every point $x_0 \in N$ has the desired property, i.e., each point $x \in M$ is d -visible in M from x_0 :

$$[x, x_0]_d \subset M \quad \text{for each } x \in M.$$

Let $x_1, \dots, x_{h+1}$ be points of M . Then (according to the assumption of the theorem) there is a point $y \in M$ such that

$$[y, x_i]_d \subset M, \quad i = 1, \dots, h + 1.$$

Hence, $y \in [y, x_i]_d \subset \text{nof}_{x_i} M$, and $y \in M$, i.e., $y \in C(x_i)$, $i = 1, \dots, h + 1$. Thus, every $h + 1$ of the sets $C(a)$, $a \in M$, has a common point. Consequently, each *finite* family of sets $C(a)$ has a nonempty intersection.

Now let $Q \subset \mathfrak{R}^n$ be a Euclidean ball containing M in its interior. For every $x \in \text{bd } Q$ we consider the maximal closed Euclidean ball $U(z)$ centered at z , which contains no points of M in its interior. We take a point $a \in M \cap \text{bd } U(z)$ and denote by $P(a)$ the closed half-space satisfying

$$P(a) \cap U(z) = \{a\}. \quad (2)$$

It follows from (2) that $\text{nof}_a M \subset P(a)$, and consequently

$$C(a) = \text{cl conv}_d(M \cap \text{nof}_a M) \subset \text{cl conv}_d(\text{nof}_a M) = \text{cl conv}(\text{nof}_a M) \subset P(a) \quad (3)$$

(by virtue of the internal-local d -conicity of M). We conclude that the set $C(a)$ has no common point with the ray r_z which emanates from z and has the direction of the outward normal vector to the ball Q . This means that $N \cap r_z = \emptyset$. Since this is true for all points $z \in \text{bd } Q$, the inclusion $N \subset Q$ holds. Furthermore, since B is compact and $C(a)$ is closed, every finite family of sets $C(a)$ has a nonempty intersection. Thus, the whole family $C(a)$, $a \in M$, has a nonempty intersection, i.e., $N \neq \emptyset$.

We now prove that $N \subset M$. We suppose the contrary, i.e., a point $z \in N$ not belonging to M exists. Then, denoting the maximal closed Euclidean ball centered at z which contains no point of M in its interior by $U(z)$, we take a point $a \in M \cap \text{bd } U(z)$. As above, we consider the half-space $P(a)$ satisfying (2). We conclude that $\text{nof}_a M \subset P(a)$, and consequently relation (3) holds. This means that $z \in N \subset P(a)$, contradicting the inclusion $z \in \text{int } U(z)$. This contradiction shows that $N \subset M$.

Furthermore, we establish that for all points $x_0 \in N$ and $x \in M$ the linear segment $[x_0, x]$ is contained in M , i.e., M is star-shaped in the linear sense. We suppose the contrary, i.e., there is a point $w \in [x_0, x]$ which does not belong to M . Since M is compact, there is a closed Euclidean ball B centered at w , which has no common point with M . Now we translate the ball B in the direction $x - x_0$ until it has a point a in

common with M . We denote the translated ball by U . Then $(\text{int } U) \cap M = \emptyset$, and $a \in (\text{bd } U) \cap M$. Let $P(a)$ be the closed half-space such that $P(a) \cap U = \{a\}$ and $a \in \text{bd } P(a)$. We conclude that $\text{nof}_a M \subset P(a)$, and hence (see (3)) $C(a) \subset P(a)$, contradicting the relations $x_0 \notin P(a)$, $x_0 \in N \subset P(a)$. This contradiction shows that M is star-shaped in the linear sense.

Thus, from the assumption of the internal-local d -conicity of M we have deduced that the whole set M is visible (in the linear sense) from every point $x_0 \in N$. Now, with the help of the property of external-local d -conicity, we prove that M is d -visible (from each point $x_0 \in N$), i.e., for all points $x_0 \in N$, $x \in M$ the d -segment $[x_0, x]_d$ is contained in M .

We assume that a point $p \in [x_0, x]_d$ exists which does not belong to M . Then all the points close enough to p do not belong to M . Let ρ be a positive number, such that the point $q = p + \rho(x_0 - p)$ is not contained in M . We denote by a that point from $[q, x]$ which is nearest to q and belongs to M , and v denotes the vector $q - a$. By virtue of the external-local d -conicity, an $\varepsilon > 0$ exists such that $[a + \lambda v, a + \mu v]_d \cap M = \emptyset$ for every positive $\mu < \lambda < \varepsilon$. We notice that $[x_0, a] \subset M$ (according to the linear star-shapedness of M which was established above).

Furthermore, since $p \in [x_0, x]_d$ and therefore $q \in [x_0, x]_d$, the following relations hold:

$$\|x - x_0\| = \|x - p\| + \|p - x_0\| = \|x - p\| + \|p - q\| + \|q - x_0\|,$$

$$\|x - x_0\| = \|x - q\| + \|q - x_0\|.$$

Consequently,

$$\|x - p\| + \|p - q\| = \|x - q\|, \quad (4)$$

i.e., $p \in [q, x]_d$. We now denote by k the number $\|q - a\|/\|q - x\|$, and by g the homothety with center q and ratio k . Then $g(x) = a$, and the point $m = g(p)$ belongs to the segment $[q, p]$. Equation (4) implies (by virtue of the relations $g(x) = a$, $g(p) = m$, $g(q) = q$) that

$$\|a - m\| + \|m - q\| = \|a - q\|.$$

As a second consequence, we consider the case of a compact, polyhedral region $M \subset \mathbb{R}^n$, i.e., $M = \text{cl}(\text{int}(M))$, and the boundary of M is the union of a finite number of $(n - 1)$ -dimensional convex polytopes which are called the *facets* of the region.

Corollary 2. *Let $M \subset \mathbb{R}^n$ be a compact, polyhedral region, each facet of which has a d -convex affine hull. The number $\text{him } \mathbb{R}^n$ is denoted by h . The set M is d -star-shaped if and only if each $h + 1$ points of M are d -visible in M from some point of the set M .*

Indeed, let $a \in M$. Then for a sufficiently small ball B , centered at a , the relation $M \cap B = (\text{nof}_a M) \cap M$ holds. Moreover, $\text{nof}_a M$ is a polyhedral cone (in general not convex) such that each of its $(n - 1)$ -dimensional faces is situated in a d -convex hyperplane. It follows that for every $x \in \mathbb{R}^n$ the d -segment $[a, x]_d$ is either contained in $\text{nof}_a M$, or its relative interior $\text{ri}([a, x]_d)$ has no common point with $\text{nof}_a M$. This means that the properties of internal- and external-local d -conicity are satisfied for M . It remains to apply our theorem.

Corollary 3. *Let $M_1, \dots, M_s \subset \mathbb{R}^n$ (with $s > h + 1$ and $h = \text{him } \mathbb{R}^n$) be sets such that:*

- (i) *The union of every $h + 1$ of them is d -star-shaped.*
- (ii) *The set $M = M_1 \cup \dots \cup M_s$ is compact and possesses the properties of external- and internal-local d -conicity.*

Then M is d -star-shaped.

Indeed, let $x_1, \dots, x_{h+1} \in M$. For every $i \in \{1, \dots, h + 1\}$ we choose an index q_i such that $x_i \in M_{q_i}$. Then

$$x_1, \dots, x_{h+1} \in M_{q_1} \cup \dots \cup M_{q_{h+1}} \quad (5)$$

(if some of the indices $q_1, \dots, q_{h+1}$ coincide, then we add some other sets M_i , in order to have a union of $h + 1$ sets on the right-hand side of (5)). By assumption, the set $N = M_{q_1} \cup \dots \cup M_{q_{h+1}}$ is d -star-shaped, i.e., all the points $x_1, \dots, x_{h+1}$ are d -visible in N from a point $y \in N$. Thus, every $h + 1$ points of M are d -visible in M from a point $y \in M$. It remains to apply the theorem.

If, in particular, we consider the Euclidean space $\mathbb{R}^n$ (replacing d -convexity by linear convexity), then we obtain the following consequence.

Corollary 4. *Let $M_1, \dots, M_s \subset \mathbb{R}^n$, $s > n + 1$. If the union of every $n + 1$ of the sets $M_1, \dots, M_s$ is star-shaped and $M_1 \cup \dots \cup M_s$ is compact, then $M_1 \cup \dots \cup M_s$ is also star-shaped.*

This result was obtained by Kołodziejczyk [6].

4. Examples

Finally, we describe two examples which demonstrate that the notions of internal- and external-local d -conicity introduced above are independent from each other.

Example 1. In the space $\mathbf{R}^3$ with a fixed coordinate system (x_1, x_2, x_3) , we consider the set $\Sigma = \text{conv}(P_1 \cup P_2 \cup Q)$, where Q is described by the conditions $x_3 = 0$ and $x_1^2 + x_2^2 = 3$, whereas P_1 and P_2 are balls of radius 1 centered at $(0, 0, 1)$ and $(0, 0, -1)$, respectively. We denote by $\mathfrak{R}^3$ the normed space obtained from $\mathbf{R}^3$ by taking Σ as the unit ball. If now $a, b \in \mathfrak{R}^3$ are points such that the vector $a - b$ is parallel to the plane $\text{aff } Q$ or forms with this plane an angle not smaller than $\pi/3$, then the d -segment $[a, b]_d$ coincides with the linear segment $[a, b]$. If this angle is smaller than $\pi/3$, then the corresponding d -segment $[a, b]_d$ is a parallelogram. This parallelogram is situated in a plane parallel to the x_3 -axis, has $[a, b]$ as its longer diagonal and, moreover, two of its sides are parallel to the plane $x_3 = 0$, and its angles are given by $\pi/3$ and $2\pi/3$. It follows that the set Σ is external-local d -conical. At the same time, for the point $a = (-2/\sqrt{3}, 0, 1) \in \text{bd } \Sigma$ the corresponding cone $\text{cl}(\text{conv}(\text{nof}_a \Sigma))$ coincides with the half-space $x_3 \leq 3 + x_1\sqrt{3}$. Since this half-space is not d -convex, the body Σ is not internal-local d -conical.

Example 2. Let $\mathfrak{R}^2$ be the Minkowski plane in which the unit ball is the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$. We denote by Q_0 the first quadrant of the plane, and by Q_k its translate $a_k + Q_0$, where $a_k = (1/k^2, 1/k)$. Finally, we put

$$P = Q_0 \cup \left(\bigcup_{k=1}^{\infty} Q_k \right)$$

and denote by M the intersection of the set P with the square $|x_1| \leq 2$, $|x_2| \leq 2$. It is easily shown that M is internal-local d -conic, but not external-local conical.

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