The Kissing Numbers of Tetrahedra*

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Abstract. We determine the lattice kissing numbers of tetrahedra, by which we disprove a conjecture by Grünbaum. At the same time, we present a strange phenomenon concerning kissing numbers and packing densities of tetrahedra.

1. Introduction

In 1694 David Gregory and Isaac Newton in a famous conversation discussed the following problem:

Can a ball be brought into contact with 13 nonoverlapping balls of the same size?

Gregory believed “yes” while Newton thought “no.” One hundred and eighty years later, this problem was first solved by Hoppe [7] in favor of Newton.

Let K be a convex body in R^d . Denote by $N(K)$ the translative kissing number of K and by $N^*(K)$ the lattice kissing number of K . In other words, $N(K)$ is the maximum number of nonoverlapping translates of K which can be brought into contact with K and $N^*(K)$ is the maximum number of translates of K in a lattice packing which can be brought into contact with K .

In 1896 Minkowski [11] first proved

$$N^*(K) \leq 3^d - 1.$$

* This article is based upon part of the author's Ph.D. thesis which was supported by the Austrian Academic Exchange Service.

Seventy years later Groemer [3] improved Minkowski's result to

$$N^*(K) \leq N(K) \leq 3^d - 1$$

and pointed out that the upper bound $3^d - 1$ can be attained if and only if K is a parallelepiped. With regard to the lower bounds of $N^*(K)$ and $N(K)$, in 1953 Swinnerton-Dyer [14] proved a theorem which implies

$$N^*(K) \geq d(d + 1).$$

In 1961, after sketching a proof of

$$N(K) \geq d(d + 1),$$

Grünbaum [5] made a conjecture which said: *If S is a d -dimensional simplex, then $N(S) = d(d + 1)$.*

Evidently, if this conjecture were true the lower bound $d(d + 1)$ of $N(K)$ would be attained by simplices in every dimension. Unfortunately, as will be seen, this conjecture is not true. In fact, Hoylman [8] already provides a counterexample for the conjecture of Grünbaum even if he did not consider the kissing number problem.

Our knowledge about the exact values $N^*(K)$ and $N(K)$ for a special convex body K is very limited. Let B be the d -dimensional unit sphere. In 1971, by studying quadratic forms, Watson [15] found

$$N^*(B) = 24, 40, 72, 126, 240, \text{ and } 272$$

when the dimension number is 4, 5, 6, 7, 8, and 9, respectively. Surprisingly, in 1979 Levenstein [10] and Odlyzko and Sloane [12] found

$$N^*(B) = N(B) = 240 \text{ and } 196,560$$

when the dimension number is 8 and 24, respectively, by applying code theory. For further references, refer to [1] and [2].

In this article we mainly prove

Theorem 1. *Let S be a tetrahedron, then*

$$N^*(S) = 18.$$

and verify

Phenomenon 1. *Every tetrahedron is in contact with 14 others in its densest lattice packings. However, in a lattice packing with a much smaller density every one is in contact with 18 others.*

2. Proof of Theorem 1

Let $abcd$ be the quadrilateral with vertices a, b, c , and d , let abc be the triangle with vertices a, b , and c , let ab be the closed linear segment from a to b , let $[ab)$ be the half-open linear segment from a to b containing a , and let (ab) be the open linear segment from a to b .

In this section S indicates the tetrahedron with vertices $p_1 = o = (0, 0, 0)$, $p_2 = (1, 0, 0)$, $p_3 = (0, 1, 0)$, and $p_4 = (0, 0, -1)$, and $S + T$ indicates a translative packing of S in R^3 (T is a suitable set of points $t_i \in R^3$, $t_0 = o$). We write S_i for the translated tetrahedra $S + t_i$ and $p_{j,i}$ for the translated vertices $p_j + t_i$, in particular, $S_0 = S$, $p_{j,0} = p_j$. Denote by $d(p, \overline{xy})$ the distance from p to the straight line determined by x and y , by $M(S)$ the number of translates S_i ($i \neq 0$) in $S + T$ which are in contact with S , and by $I(p_1 p_2 p_3)$ the number of indices i ($i \neq 0$) such that $S_i \cap p_1 p_2 p_3 \in \text{int}(p_1 p_2 p_3)$ (when we address the interior points of a facet or an edge, we always mean the relative interior in the correspondent two- or one-dimensional spaces, respectively). With these preparations, we define a valuation function

$$Z(S_i, p_1 p_2 p_3) = \begin{cases} \frac{1}{3} & \text{if } S_i \cap p_1 p_2 p_3 = p \text{ is an interior point of a facet of } S_i, \\ \frac{1}{2} & \text{if } S_i \cap p_1 p_2 p_3 = p \text{ is the midpoint of } p_{4,i} p_{j,i} \text{ for some } j \neq 4, \\ 1 & \text{if } S_i \cap p_1 p_2 p_3 = p \in p_{4,i} p_{j,i} \text{ for some } j \text{ and } \|p_{j,i} p\| > \|p_{4,i} p\|, \\ 0 & \text{in all other cases (especially when } i = 0) \end{cases}$$

(where $\|ab\|$ indicates the Euclidean distance from a to b) and set

$$G(p_1 p_2 p_3) = \sum_{S_i \cap S \neq \emptyset} Z(S_i, p_1 p_2 p_3).$$

For convenience, we abbreviate $\sum Z(S_i, p_1 p_2 p_3)$ to $\sum$.

Remark 1. Since $Z(S_i, p_1 p_2 p_3)$ and $G(p_1 p_2 p_3)$ keep invariant under linear transformations, we can define $Z(S_i, p_1 p_2 p_4)$, $Z(S_i, p_1 p_3 p_4)$, $Z(S_i, p_2 p_3 p_4)$, $G(p_1 p_2 p_4)$, $G(p_1 p_3 p_4)$, and $G(p_2 p_3 p_4)$ in a similar way.

Remark 2. From the definition of $Z(S_i, p_1 p_2 p_3)$, we see that:

- (a) If $Z(S_i, p_1 p_2 p_3) = \frac{1}{2}$ and $S_i \cap p_1 p_2 p_3 = p$, then we have $p + (0, 0, \frac{1}{2}) + \frac{1}{2}S \subset S_i$.
- (b) If $Z(S_i, p_1 p_2 p_3) = 1$ and $S_i \cap p_1 p_2 p_3 = p$, then a positive number η exists such that $p + (0, 0, \frac{1}{2} + \eta) + (\frac{1}{2} + \eta)S \subset S_i$.

Let a , b , and c be the midpoints of $p_1 p_2$, $p_2 p_3$, and $p_3 p_1$, respectively, let $q_1, q_2 \in p_1 p_2$, $q_3, q_4 \in p_2 p_3$, and $q_5, q_6 \in p_3 p_1$ be the points such that

$$\frac{p_1 q_1}{p_1 p_2} = \frac{q_2 p_2}{p_1 p_2} = \frac{p_2 q_3}{p_2 p_3} = \frac{q_4 p_3}{p_2 p_3} = \frac{p_3 q_5}{p_3 p_1} = \frac{q_6 p_1}{p_3 p_1} = \frac{1}{4}$$

and let $q_7 = q_1 q_4 \cap q_3 q_6$, $q_8 = q_2 q_5 \cap q_3 q_6$ and $q_9 = q_2 q_5 \cap q_1 q_4$. Then we define

$$\begin{aligned} ap_2 b^- &= ap_2 b \setminus \{b\}, \quad bp_3 c^- = bp_3 c \setminus \{c\}, \quad cp_1 a^- = cp_1 a \setminus \{a\}, \\ H_1(p_1 p_2 p_3) &= \{p_1 q_1 q_7 q_6 \setminus (q_1 q_7 \cup q_6 q_7)\} \\ &\quad \cup \{q_2 p_2 q_3 q_8 \setminus (q_3 q_8 \cup q_2 q_8)\} \cup \{p_3 q_5 q_9 q_4 \setminus (q_5 q_9 \cup q_4 q_9)\}, \\ H_2(p_1 p_2 p_3) &= \{q_1 q_2 q_8 q_7 \setminus q_7 q_8\} \cup \{q_3 q_4 q_9 q_8 \setminus q_8 q_9\} \cup \{q_5 q_6 q_7 q_9 \setminus q_7 q_9\}, \\ H_3(p_1 p_2 p_3) &= q_7 q_8 q_9. \end{aligned}$$

Remark 3. $H_k(p_1 p_2 p_3)$ have the following properties:

- (a) $p_1 p_2 p_3 = H_1(p_1 p_2 p_3) \cup H_2(p_1 p_2 p_3) \cup H_3(p_1 p_2 p_3)$.
- (b) $H_i(p_1 p_2 p_3) \cap H_j(p_1 p_2 p_3) = \emptyset, i \neq j$.
- (c) For every point $x \in H_k(p_1 p_2 p_3)$, there are k edges $p_i p_j$ of $p_1 p_2 p_3$ such that $d(x, \overline{p_i p_j}) \geq \frac{1}{4}d(p_l, \overline{p_i p_j})$ holds with $\{l, i, j\} = \{1, 2, 3\}$.

Before turning to the complicated estimation, we quote two intersection properties of tetrahedra which are frequently demanded.

Assertion 1. If $\text{card}\{S \cap (S + p)\} \geq 2$, then $\text{int}(S) \cap (\text{int}(S) + p) \neq \emptyset$.

Assertion 2. If $x, y \in z + \eta\{p_1 p_2 p_3\}$ and $\rho > \eta$, then

$$(x + \text{int}(\rho S)) \cap (y + \text{int}(\rho S)) \neq \emptyset.$$

Remark 4. From Assertion 2 we have $I(p_1 p_2 p_3) \leq 1$.

Lemma 1.

$$M(S) = G(p_1 p_2 p_3) + G(p_1 p_2 p_4) + G(p_1 p_3 p_4) + G(p_2 p_3 p_4).$$

Proof. Apparently, when $i \neq 0$, S_i intersects S in one of the following ways:

Case 1: $S_i \cap S = \emptyset$. Evidently, then we have

$$\begin{cases} Z(S_i, p_1 p_2 p_3) = 0, \\ Z(S_i, p_1 p_2 p_4) = 0, \\ Z(S_i, p_1 p_3 p_4) = 0, \\ Z(S_i, p_2 p_3 p_4) = 0. \end{cases} \quad (1)$$

Case 2: $S_i \cap S = p$ is an interior point of a facet of S_i . Without loss of generality, we take $p_{2,i} p_{3,i} p_{4,i}$ to be the mentioned facet. Then, since $S + T$ is a translative packing, we get $p = p_1$ and

$$\begin{cases} Z(S_i, p_1 p_2 p_3) = \frac{1}{3}, \\ Z(S_i, p_1 p_2 p_4) = \frac{1}{3}, \\ Z(S_i, p_1 p_3 p_4) = \frac{1}{3}, \\ Z(S_i, p_2 p_3 p_4) = 0. \end{cases} \quad (2)$$

Case 3: $S_i \cap S = p$ is an interior point of an edge of S_i . Without loss of generality, we take $p_{2,i} p_{3,i}$ to be the mentioned edge. Keeping the intersection properties of the

translative tetrahedra and the fact that $S + T$ is a packing in mind, it is easy to see that $p \in p_1 p_4$. So, if $\|pp_{2,i}\| = \|pp_{3,i}\|$ we have

$$\begin{cases} Z(S_i, p_1 p_2 p_3) = 0, \\ Z(S_i, p_1 p_2 p_4) = \frac{1}{2}, \\ Z(S_i, p_1 p_3 p_4) = \frac{1}{2}, \\ Z(S_i, p_2 p_3 p_4) = 0, \end{cases} \quad (3)$$

and if $\|pp_{2,i}\| > \|pp_{3,i}\|$ we have

$$\begin{cases} Z(S_i, p_1 p_2 p_3) = 0, \\ Z(S_i, p_1 p_2 p_4) = 1, \\ Z(S_i, p_1 p_3 p_4) = 0, \\ Z(S_i, p_2 p_3 p_4) = 0. \end{cases} \quad (4)$$

Case 4: $S_i \cap S = p$ is a vertex of S_i . Without loss of generality, we take $p = p_{4,i}$ as an example. Then we see that p is a point of $p_1 p_2 p_3$ and

$$\begin{cases} Z(S_i, p_1 p_2 p_3) = 1, \\ Z(S_i, p_1 p_2 p_4) = 0, \\ Z(S_i, p_1 p_3 p_4) = 0, \\ Z(S_i, p_2 p_3 p_4) = 0. \end{cases} \quad (5)$$

Thus, concluding from (1)–(5), we get

$$\begin{aligned} & Z(S_i, p_1 p_2 p_3) + Z(S_i, p_1 p_2 p_4) + Z(S_i, p_1 p_3 p_4) + Z(S_i, p_2 p_3 p_4) \\ &= \begin{cases} 0 & \text{if } S \cap S_i = \emptyset, \\ 0 & \text{if } i = 0, \\ 1 & \text{if } S \cap S_i \neq \emptyset, \quad i \neq 0, \end{cases} \end{aligned}$$

which implies

$$\begin{aligned} M(S) &= \sum_{S_i \cap S \neq \emptyset, i \neq 0} 1 \\ &= \sum_{S_i \cap S \neq \emptyset} Z(S_i, p_1 p_2 p_3) + \sum_{S_i \cap S \neq \emptyset} Z(S_i, p_1 p_2 p_4) \\ &\quad + \sum_{S_i \cap S \neq \emptyset} Z(S_i, p_1 p_3 p_4) + \sum_{S_i \cap S \neq \emptyset} Z(S_i, p_2 p_3 p_4) \\ &= G(p_1 p_2 p_3) + G(p_1 p_2 p_4) + G(p_1 p_3 p_4) + G(p_2 p_3 p_4). \end{aligned}$$

Lemma 1 is proved. $\square$

Lemma 2. *If $I(p_1 p_2 p_3) = 0$, then we have*

$$G(p_1 p_2 p_3) \leq 4\frac{2}{3}.$$

Proof. Based upon the previous definitions, we study the three numbers

$$\sum_{S_i \cap a p_2 b^- \neq \emptyset}, \quad \sum_{S_i \cap b p_3 c^- \neq \emptyset}, \quad \text{and} \quad \sum_{S_i \cap c p_1 a^- \neq \emptyset}$$

carefully.

From Remark 2, Assertion 2, and the assumption that $S + T$ is a packing, we can easily deduce:

Assertion 3. *At most one translate S_i exists such that $S_i \cap [ap_2) \neq \emptyset$, and $Z(S_i, p_1 p_2 p_3) \geq \frac{1}{2}$.*

Assertion 4. *At most one translate S_i exists such that $S_i \cap ap_2 b^- \neq \emptyset$ and $Z(S_i, p_1 p_2 p_3) = 1$.*

Assertion 5. *If a translate S_j exists such that $S_j \cap (ap_2) \neq \emptyset$ and $Z(S_j, p_1 p_2 p_3) \neq 0$, then we have $\sum_{S_i \cap (p_2 b) \neq \emptyset} = 0$.*

Thus, by Assertions 3–5 and the assumption $I(p_1 p_2 p_3) = 0$, we get

$$\sum_{S_i \cap a p_2 b^- \neq \emptyset} = l + \frac{m}{2} + \frac{n}{3}, \quad (6)$$

where $l, m, n \in \{0, 1, 2\}$, $l \leq 1$, $l + m \leq 2$, and $n \leq 1$. Obviously, (6) holds for $\sum_{S_i \cap b p_3 c^- \neq \emptyset}$ and $\sum_{S_i \cap c p_1 a^- \neq \emptyset}$ also. Without loss of generality, we take

$$M = \sum_{S_i \cap a p_2 b^- \neq \emptyset} = \max \left\{ \sum_{S_i \cap a p_2 b^- \neq \emptyset}, \sum_{S_i \cap b p_3 c^- \neq \emptyset}, \sum_{S_i \cap c p_1 a^- \neq \emptyset} \right\}.$$

Then, from (6), we get

$$M \leq 1 + \frac{1}{2} + \frac{1}{3}.$$

Now we complete our proof by dealing with the following cases:

Case 1: $M = 1 + \frac{1}{2} + \frac{1}{3}$. Recalling Assertions 3–5, we see that

$$M = \sum_{S_i \cap a p_2 b^- \neq \emptyset} = 1 + \frac{1}{2} + \frac{1}{3}$$

implies the existence of two translates S_i and S_j such that

$$\begin{cases} S_i \cap S = a, \\ Z(S_i, p_1 p_2 p_3) \geq \frac{1}{2} \end{cases}$$

and

$$\begin{cases} S_j \cap S \in (p_2 b), \\ Z(S_j, p_1 p_2 p_3) \geq \frac{1}{2}. \end{cases}$$

Then, applying Assertions 3–5 to bp_3c^- and cp_1a^- , we get

$$\begin{cases} \sum_{S_i \cap bp_3c^- \neq \emptyset} \leq 1 + \frac{1}{3}, \\ \sum_{S_i \cap cp_1a^- \neq \emptyset} \leq 1 + \frac{1}{2}. \end{cases}$$

Keeping the assumption $I(p_1 p_2 p_3) = 0$ in mind we obtain

$$\begin{aligned} G(p_1 p_2 p_3) &= \sum_{S_i \cap ap_2b^- \neq \emptyset} + \sum_{S_i \cap bp_3c^- \neq \emptyset} + \sum_{S_i \cap cp_1a^- \neq \emptyset} \\ &\leq 4\frac{2}{3}. \end{aligned}$$

Case 2: $M < 1 + \frac{1}{2} + \frac{1}{3}$. In this case from (6) we get $M \leq 1 + \frac{1}{2}$. Then, from the definition of M and the assumption $I(p_1 p_2 p_3) = 0$, we get

$$\begin{aligned} G(p_1 p_2 p_3) &= \sum_{S_i \cap ap_2b^- \neq \emptyset} + \sum_{S_i \cap bp_3c^- \neq \emptyset} + \sum_{S_i \cap cp_1a^- \neq \emptyset} \\ &\leq 4\frac{1}{2}. \end{aligned}$$

Thus, Lemma 2 is proved. □

Lemma 3. *If $I(p_1 p_2 p_3) = 1$, then*

$$G(p_1 p_2 p_3) \leq 5,$$

where the equality holds only if a translate S_i exists such that $S_i \cap p_1 p_2 p_3 \in \text{int}(abc)$.

Proof. To show our lemma, we prove the following assertions instead:

Assertion 6. *If a translate S_i exists such that*

$$\begin{cases} S_i \cap p_1 p_2 p_3 \in \text{int}(p_1 p_2 p_3) \setminus \text{int}(abc), \\ Z(S_i, p_1 p_2 p_3) = 1, \end{cases}$$

then we have

$$G(p_1 p_2 p_3) \leq 4\frac{2}{3}.$$

Without loss of generality, we can take

$$S_i \cap p_1 p_2 p_3 \in ap_2 b \setminus \{ap_2 \cup p_2 b\}.$$

Then, from Remark 2, Assertion 2, and the assumption that $S + T$ is a packing, we see that at most one other translate S_j exists such that $S_j \cap ap_2 b \neq \emptyset$ and $Z(S_j, p_1 p_2 p_3) > 0$ and, for that S_j ,

$$Z(S_j, p_1 p_2 p_3) \leq \frac{1}{2}$$

holds. So, we get

$$\sum_{S_i \cap ap_2 b^- \neq \emptyset} \leq 1 + \frac{1}{2}. \quad (7)$$

Simultaneously, applying Assertions 3–5 to $cp_1 a^-$ and $bp_3 c^-$ we get

$$\begin{cases} \sum_{S_i \cap cp_1 a^- \neq \emptyset} \leq 1 + \frac{1}{2} + \frac{1}{3}, \\ \sum_{S_i \cap bp_3 c^- \neq \emptyset} \leq 1 + \frac{1}{3}. \end{cases} \quad (8)$$

Thus, from (7) and (8) we obtain

$$\begin{aligned} G(p_1 p_2 p_3) &= \sum_{S_i \cap ap_2 b^- \neq \emptyset} + \sum_{S_i \cap bp_3 c^- \neq \emptyset} + \sum_{S_i \cap cp_1 a^- \neq \emptyset} \\ &\leq 4\frac{2}{3}. \end{aligned}$$

Assertion 7. *If a translate S_i exists such that*

$$\begin{cases} S_i \cap \text{int}(abc) \neq \emptyset, \\ Z(S_i, p_1 p_2 p_3) = 1, \end{cases}$$

then we have

$$G(p_1 p_2 p_3) \leq 5.$$

In this case there is no translate S_j such that

$$\begin{cases} S_j \cap p_1 p_2 p_3 \in \{a, b, c\}, \\ Z(S_j, p_1 p_2 p_3) > 0 \end{cases}$$

(otherwise, applying Remark 2 and Assertion 1 with some detailed argument we could show that $S + T$ is not a packing). So, applying Assertions 3–5 to $ap_2 b^-$, $bp_3 c^-$, and

cp_1a^- we obtain

$$\begin{cases} \sum_{S_i \cap ap_2b^- \neq \emptyset} \leq 1 + \frac{1}{3}, \\ \sum_{S_i \cap bp_3c^- \neq \emptyset} \leq 1 + \frac{1}{3}, \\ \sum_{S_i \cap cp_1a^- \neq \emptyset} \leq 1 + \frac{1}{3}. \end{cases} \quad (9)$$

Then, from (9) and the assumption that there is a translate S_i such that $S_i \cap \text{int}(abc) \neq \emptyset$, we get

$$\begin{aligned} G(p_1 p_2 p_3) &= 1 + \sum_{S_i \cap ap_2b^- \neq \emptyset} + \sum_{S_i \cap bp_3c^- \neq \emptyset} + \sum_{S_i \cap cp_1a^- \neq \emptyset} \\ &\leq 5. \end{aligned}$$

Lemma 3 follows immediately from Assertions 6 and 7. $\square$

Lemma 4. *If p_2 and p_3 are interior points of $p_{1,2}p_{3,2}p_{4,2}$ and $p_{1,3}p_{2,3}p_{4,3}$, respectively, and*

$$\begin{cases} d(p_2, \overline{p_{1,2}p_{3,2}}) \geq \frac{1}{4}d(p_{4,2}, \overline{p_{1,2}p_{3,2}}), \\ d(p_3, \overline{p_{1,3}p_{2,3}}) \geq \frac{1}{4}d(p_{4,3}, \overline{p_{1,3}p_{2,3}}), \end{cases} \quad (10)$$

then we have

$$G(p_1 p_2 p_3) < 5.$$

Proof. If, on the contrary, $G(p_1 p_2 p_3) = 5$, then by Lemma 3 and its proof a translate S_j exists such that

$$p_{4,j} = (x^1, x^2, 0) = S_j \cap S \in \text{int}(abc)$$

and

$$\begin{cases} \sum_{S_i \cap \{(ap_2) \cup (p_2b)\} \neq \emptyset} = 1 + \frac{1}{3}, \\ [12pt] \sum_{S_i \cap \{(bp_3) \cup (p_3c)\} \neq \emptyset} = 1 + \frac{1}{3}. \end{cases} \quad (11)$$

From (11) and Assertion 5 we obtain two points y, z and two translates S_k, S_l such that

$$\begin{cases} y = (y^1, y^2, 0) = S_k \cap \{(ap_2) \cup (p_2b)\}, \\ Z(S_k, p_1 p_2 p_3) = 1 \end{cases}$$

and

$$\begin{cases} z = (z^1, z^2, 0) = S_l \cap \{(bp_3) \cup (p_3c)\}, \\ Z(S_l, p_1 p_2 p_3) = 1. \end{cases}$$

Thus, from Remark 2 we get that

$$\begin{cases} y + (0, 0, \frac{1}{2} + \eta) + (\frac{1}{2} + \eta)S \subset S_k, \\ z + (0, 0, \frac{1}{2} + \eta) + (\frac{1}{2} + \eta)S \subset S_l \end{cases} \quad (12)$$

holds with some positive number η . Then, since (10) implies

$$\begin{cases} p_2 + (0, 0, \frac{1}{4}) + \frac{1}{4}S \subset S_2, \\ p_3 + (0, 0, \frac{1}{4}) + \frac{1}{4}S \subset S_3, \end{cases}$$

keeping Assertion 2 and the assumption that $S + T$ is a packing in mind, we get

$$\begin{cases} y = (y^1, y^2, 0) \in (ap_2) \cup (p_2b), \\ \frac{1}{2} < y^1 \leq \frac{3}{4} \end{cases} \quad (13)$$

and

$$\begin{cases} z = (z^1, z^2, 0) \in (bp_3) \cup (p_3c), \\ \frac{1}{2} < z^2 \leq \frac{3}{4}. \end{cases} \quad (14)$$

Now we prove our lemma by studying the following cases:

Case 1: $y \in ap_2, z \in p_3c$. Since $p_{4,j} = (x^1, x^2, 0) = S_j \cap S \in \text{int}(abc)$, at least one of the two coordinates x^1, x^2 is greater than $\frac{1}{4}$. If, without loss of generality, $x^1 > \frac{1}{4}$, keeping (12) and (13) in mind we get that both $(y^1, x^2, \frac{1}{2})$ and $(y^1, x^2, \frac{1}{2} + \eta)$ belong to $S_j \cap S_k$. Hence, from Assertion 1 we get

$$\text{int}(S_j) \cap \text{int}(S_k) \neq \emptyset,$$

which is contrary to the assumption that $S + T$ is a packing.

Case 2: $y \in ap_2, z \in bp_3$. When $x^1 > \frac{1}{4}$, the argument of Case 1 can be repeated. When $x^2 > \frac{1}{4}$, it can be verified that both $z + (0, 0, \frac{1}{2})$ and $z + (0, 0, \frac{1}{2} + \eta)$ belong to $S_j \cap S_l$. Then Assertion 1 implies

$$\text{int}(S_j) \cap \text{int}(S_l) \neq \emptyset,$$

which is contrary to the assumption that $S + T$ is a packing.

Case 3: $y \in p_2b, z \in p_3c$. Apparently, this case is equivalent to Case 2.

Case 4: $y \in p_2b, z \in bp_3$. Similar to Case 1. If, without loss of generality, $x^1 > \frac{1}{4}$, keeping (12) and (13) in mind we can verify that both $y + (0, 0, \frac{1}{2})$ and $y + (0, 0, \frac{1}{2} + \eta)$ belong to $S_j \cap S_k$. Then we get that

$$\text{int}(S_j) \cap \text{int}(S_k) \neq \emptyset,$$

which is contrary to the assumption that $S + T$ is a packing.

By summarizing Cases 1–4, Lemma 4 is proved. $\square$

Proof of Theorem 1. From Lemmas 1–3 we have proved $M(S) \leq 20$. Now we prove $M(S) \neq 20$.

If, on the contrary, $M(S) = 20$ holds for some T , then from Lemmas 2 and 3 we see that

$$G(p_1 p_2 p_3) = G(p_1 p_2 p_4) = G(p_1 p_3 p_4) = G(p_2 p_3 p_4) = 5.$$

Hence indices $i(1), i(2), i(3), i(4), j(1), j(2), j(3)$, and $j(4)$ exist such that

$$\begin{cases} S_{i(k)} \cap S = p_k \in \text{int}(p_{l,i(k)} p_{m,i(k)} p_{n,i(k)}), \\ S_{j(k)} \cap S = p_{k,j(k)} \in \text{int}(p_l p_m p_n), \end{cases}$$

where k, l, m, n is an arbitrary permutation of 1, 2, 3, 4.

Define

$$\begin{aligned} \psi(p_k, p_k p_m p_n) &= \psi(p_k, p_m p_k p_n) = \psi(p_k, p_m p_n p_k) \\ &= \begin{cases} 0 & \text{if } d(p_k, \overline{p_{m,i(k)} p_{n,i(k)}}) < \frac{1}{4} d(p_{l,i(k)}, \overline{p_{m,i(k)} p_{n,i(k)}}), \\ 1 & \text{if } d(p_k, \overline{p_{m,i(k)} p_{n,i(k)}}) \geq \frac{1}{4} d(p_{l,i(k)}, \overline{p_{m,i(k)} p_{n,i(k)}}) \end{cases} \end{aligned}$$

and

$$\Psi(p_k p_m p_n) = \psi(p_k, p_k p_m p_n) + \psi(p_m, p_k p_m p_n) + \psi(p_n, p_k p_m p_n).$$

Suppose

$$p_k \in H_{\varphi(k)}(p_{l,i(k)} p_{m,i(k)} p_{n,i(k)})$$

($H_i(p_1 p_2 p_3)$ were defined under Remark 2), then from Remark 3 we get

$$\Psi(p_1 p_2 p_3) + \Psi(p_1 p_2 p_4) + \Psi(p_1 p_3 p_4) + \Psi(p_2 p_3 p_4) = \sum_{k=1}^4 \varphi(k).$$

So, if

$$\sum_{k=1}^4 \varphi(k) > 4,$$

we can get a triangle, for example $p_1 p_2 p_3$, such that

$$\Psi(p_1 p_2 p_3) \geq 2,$$

which implies an inequality like (10). Then a contradiction with $G(p_1 p_2 p_3) = 5$ immediately occurs. Hence, we have

$$\sum_{k=1}^4 \varphi(k) = 4,$$

$$\varphi(k) = 1, \quad k = 1, 2, 3, 4,$$

and therefore

$$p_k \in H_1(p_{l,i(k)} p_{m,i(k)} p_{n,i(k)}). \quad (15)$$

It is easy to see that $H_1(p_{1,i(2)} p_{3,i(2)} p_{4,i(2)})$ consists of three separated parallelograms each containing a vertex of $p_{1,i(2)} p_{3,i(2)} p_{4,i(2)}$. Considering the special case $k = 2$ of (15), if p_2 is a point of the isolated parallelogram of $H_1(p_{1,i(2)} p_{3,i(2)} p_{4,i(2)})$ which contains $p_{4,i(2)}$, we get

$$d(p_2, \overline{p_{1,i(2)} p_{3,i(2)}}) > \frac{1}{2} d(p_{4,i(2)}, \overline{p_{1,i(2)} p_{3,i(2)}}),$$

which implies

$$(1, 0, \frac{1}{2} + \eta) + (\frac{1}{2} + \eta)S \subset S_{i(2)}$$

for some positive η . Then, from Assertion 2 with some routine argument, we can get

$$\sum_{S_i \cap ap_2 b^- \neq \emptyset} \leq \frac{1}{3}$$

and hence

$$G(p_1 p_2 p_3) < 5.$$

By this contradiction we get

$$N(S) \leq 19.$$

In addition, since $N^*(S)$ cannot be an odd number and $N^*(S) \leq N(S)$, we obtain

$$N^*(S) \leq 18.$$

On the other hand, letting S^* be the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ and letting Λ^* be the lattice with a basis $\frac{1}{2}(-1, 1, 1)$, $\frac{1}{2}(1, -1, 1)$, and $\frac{1}{2}(1, 1, -1)$, it is routine to verify that $S^* + \Lambda^*$ is a lattice packing and every tetrahedron in this packing is in contact with 18 others. Thus, we have

$$N^*(S) = 18.$$

The proof of Theorem 1 is complete. □

Obviously, the proof of Theorem 1 implies

Corollary 1. $18 \leq N(S) \leq 19$.

3. A Strange Phenomenon and Two Open Problems

Let S' be the regular tetrahedron with vertices $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, and $(-1, -1, -1)$ and let Λ' be the lattice with a basis $(2, -\frac{1}{3}, -\frac{1}{3})$, $(-\frac{1}{3}, 2, -\frac{1}{3})$, and

$(-\frac{1}{3}, -\frac{1}{3}, 2)$. In 1970 Hoylman [8] proved that *the greatest possible density for a lattice packing of S' is $\frac{18}{49}$; furthermore, the lattice Λ' which gives this packing density is unique up to reflections in the coordinate planes, rotations of 90° about any of the coordinate axes, and combinations of these*. It is easy to see that every tetrahedron in these packings is in contact with 14 others. However, although every one is in contact with 18 others in $S^* + \Lambda^*$, the density of this packing is only one-third. This verifies:

Phenomenon 1. *Every tetrahedron is in contact with 14 others in its densest lattice packings. However, in a lattice packing with a much smaller density every one is in contact with 18 others.*

To end this article we formulate two conjectures:

Conjecture 1. *The translative kissing number of a tetrahedron is 18.*

Conjecture 2. *For a given tetrahedron S the lattice which provides the lattice kissing number of S is unique.*

Acknowledgments

I am obliged to Professor Peter M. Gruber and Professor Edmund Hlawka for discussion and encouragement. I am obliged to the referee for helpful comments.

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Received August 29, 1994.