On Graver's Conjecture Concerning the Rigidity Problem of Graphs

Hiroshi Maehara

College of Education, Ryukyu University,
Nishihara, Okinawa, Japan

Abstract. We show that the four-dimensional case of Graver's conjecture is not true.

1. Introduction

A *framework* in Euclidean d -space R^d is a graph whose vertices are points in R^d . A framework F in R^d is said to be *d-flexible* if we can continuously move the vertices of F in R^d so that

- (1) the distance between adjacent vertices are kept unchanged, and
- (2) at least a pair of vertices of F change their mutual distance.

If a framework is not d -flexible, then it is said to be *d-rigid*.

Let G be an abstract graph with n vertices $1, 2, \dots, n$. Take n points $p_1, \dots, p_n$ in R^d , and define p_i, p_j to be adjacent whenever i, j are adjacent in G . Then we have a framework in R^d . This framework is denoted by $G(p)$, where $p = (p_1, \dots, p_n) \in R^{nd}$. If all the coordinates of p are algebraically independent over the rational field, then we call $G(p)$ a *generic realization* of G . The d -rigidity or d -flexibility of a generic realization $G(p)$ of G is determined only by the abstract graph structure of G and independent of the choice of p (see, e.g., [1] and [5]). If a generic realization $G(p)$ of G is d -rigid, then the graph G is said to be *generically d-rigid*, otherwise it is said to be *generically d-flexible*.

Some techniques are proposed to build up a larger, generically d -rigid graph from a smaller one. The following two operations are valid:

- (1) Addition of a new d -valent vertex.
- (2) Exchange an edge xy for a new $(d + 1)$ -valent vertex z attached to x, y and any other $d - 1$ vertices.

These are known as *Henneberg methods*. Laman's theorem [4] states that every "edge-minimal" generically 2-rigid graph is obtained from $K_2 = \circ - \circ$ by applying

Fig. 1. Operation X_4 .

operations (1) and (2). However, these are not sufficient for higher-dimensional cases. So other operations are sought.

Let G be a graph with more than $d + 1$ vertices. Consider the following operation X_d on G : add G a new vertex v and $d + 2$ new edges connecting v to some $d + 2$ vertices of G , and remove any two nonadjacent edges among those $d + 2$ vertices. The resulting graph is denoted by $X_d G$. Figure 1 shows the operation X_4 .

Conjecture 1 [6]. *If $G = (V, E)$ is a generically d -rigid graph with $|V| \geq d + 2$ and $|E| = d|V| - d(d + 1)/2$, then $X_d G$ is also generically d -rigid.*

This would be a simple consequence of the following Graver's conjecture (see Conjectures 4.7, 4.8, 5.5, and 5.6 of [6]):

Conjecture 2 [3]. *If v is an $(n + 2)$ -valent vertex in G adjacent to the vertex set S , $G - v$ is generically n -independent, $d(S, G - v) \geq 2$ and $d(S', G - v) \geq 1$ for all subsets of $n + 1$ elements in S , then G is generically n -independent.*

For the definitions of "generic n -independence" and "the generic n -degree of freedom" $d(U, G)$ of a vertex set U in a graph G , see, e.g., [6]. (We do not use these definitions in this paper.)

We prove

Theorem 1. *There exists a generically 4-rigid graph $G = (V, E)$ with $|V| \geq 6$, $|E| = 4|V| - 10$, such that $X_4 G$ is not generically 4-rigid.*

Thus Conjecture 1 is not true, and hence the four-dimensional case of Graver's conjecture is also not true.

Preceding the proof, we state the following theorem, which is an easy corollary to Theorem 1 of [7].

Theorem 2. *A complete bipartite graph $K_{m,n}$ ($\neq K_{1,1}$) is generically d -rigid if and only if*

$$m, n \geq d + 1 \quad \text{and} \quad m + n \geq (d + 1)(d + 2)/2.$$

This theorem is essentially proved in [2], but they never stated it in this form.

Fig. 2

Example. $K_{6,7}$, $K_{6,8}$, and $K_{7,7}$ are generically 4-flexible.

2. Proof of Theorem 1

Note that

$$G_0 := K_6 - (\text{an edge})$$

is generically 4-rigid, since it is the “suspension” of K_4 . The graph G_0 has 14 edges, and they can be partitioned into seven nonadjacent edge-pairs, see Fig. 2.

Consider the sequence of graphs

$$G_0, G_1, \dots, G_7,$$

where G_i is obtained from G_{i-1} by applying the operation X_4 : connecting the new vertex to the six vertices of G_{i-1} that are inherited from G_0 , and eliminating the i th pair of edges in Fig. 2. Then G_7 is isomorphic to the complete bipartite graph $K_{6,7}$. However, $K_{6,7}$ is generically 4-flexible by the above example. It is easy to see that in each G_i

$$(\text{number of edges}) = 4 (\text{number of vertices}) - 10.$$

Hence one of G_i ($0 \leq i \leq 6$) is a desired graph. □

Remark. Professor W. Whiteley pointed out the following: The fifth graph G_5 ($K(6, 5)$ plus four bars) is generically rigid, but the sixth graph G_6 ($K(6, 6)$ plus two bars) is generically flexible. The proofs rely on theorems in [7]. $K(6, 6)$ plus two bars is infinitesimally flexible if all joints plus the two lines of the added bars lie on a quadric in 4-space—i.e., if 12 points plus the midpoints of the two added bars lie on a quadric. However, every 14 points in 4-space lie on a quadric. On the other hand, $K(6, 5)$ plus four bars requires $11 + 4$ midpoints to lie on a quadric—and five points can be placed to avoid a quadric or the other degeneracies.

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