On-Line Covering the Unit Cube by Cubes*

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Abstract. We consider two on-line methods of covering the unit cube of Euclidean d -space by sequences of cubes. The on-line restriction means that we are given the next cube from the sequence only after the preceding cube has been put in place without the possibility of changing the placement. The first method enables on-line covering of the unit cube by an arbitrary sequence of cubes whose total volume is at least $3 \cdot 2^d - 4$. The second method is more complicated, but, asymptotically, as d tends to infinity, it yields an efficiency of the order of magnitude 2^d with factor 1. So, asymptotically, it is as good as the best possible non-on-line method of covering the unit cube by cubes.

1. Introduction

Denote by E^d the Euclidean d -space. We say that a sequence $C_1, C_2, \dots$ of sets of E^d permits a covering of a set $C \subset E^d$ if there are rigid motions σ_i such that C is contained in the union of sets $\sigma_i C_i$, where $i = 1, 2, \dots$. In [3] Groemer presents a survey of results on coverings of convex bodies by sequences of convex bodies.

The present paper considers covering methods which work under the restriction that we learn C_i , where $i > 1$, only after C_{i-1} has been placed, i.e., only after the motion σ_{i-1} has been irreversibly provided. At the beginning we are given C_1 . Such a covering is called an *on-line covering* by analogy to the on-line packing discussed in [6].

Kuperberg [5] proved that the unit cube I^d can be on-line covered by an arbitrary sequence of cubes of total volume at least 4^d . Our aim is to improve this

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estimate. We present two on-line methods of covering I^d by cubes. The first method makes possible an on-line covering of I^d by every sequence of cubes of total volume at least $3 \cdot 2^d - 4$. The second method is more sophisticated but it enables an on-line covering of I^d by every sequence of cubes whose total volume is a number of the order of magnitude of 2^d with factor 1, the same as for the bound $2^d - 1$ of the best non-on-line method of Groemer [2] and Bezdek and Bezdek [1].

2. The Method of the Current Bottom

Points of the unit cube

$$I^d = \{(x_1, \dots, x_d): 0 \leq x_1 \leq 1, \dots, 0 \leq x_d \leq 1\}$$

will be successively on-line covered by a sequence $Q_1, Q_2, \dots$ of d -dimensional cubes. In every Q_i , when we learn it, we find the greatest cube K_i whose side is of length of the form 2^{-r} , where r is a nonnegative integer. Then we find the greatest number $b_i \leq 1$ such that every point of I^d whose last coordinate is smaller than b_i has been covered by the preceding cubes, i.e., by cubes K_j such that $j < i$. The set of all points of the cube I^d having the last coordinate b_i is called *the i th bottom* or *the current bottom of the cube*. Of course, if $b_i > 0$, then the i th bottom is covered by cubes K_j , where $j < i$. A point of the i th bottom is called a *surface point* if no point with the last coordinate greater and with the other coordinates unchanged has been covered by cubes K_j , where $j < i$. We find a rigid motion σ_i such that $\sigma_i K_i$ contains a surface point of the i th bottom and that $\sigma_i K_i$ has the form

$$\{(x_i, \dots, x_d): a_k 2^{-r} \leq x_k \leq (a_k + 1) 2^{-r} \text{ for } k = 1, \dots, d-1 \text{ and } b_i \leq x_d \leq b_i + 2^{-r}\},$$

where 2^{-r} is the side of K_i and where $a_1, \dots, a_{d-1} \in \{0, \dots, 2^r - 1\}$. Observe that the prescription guarantees that at every stage there are surface points of the current bottom. If $b_n = 1$ for an index n , then we can stop the covering process; at this moment I^d has been covered by cubes $\sigma_1 Q_1, \dots, \sigma_{n-1} Q_{n-1}$. The method just described is called *the method of the current bottom*.

Theorem 1. *The method of the current bottom enables on-line covering of the unit cube by an arbitrary sequence of cubes of total volume at least $3 \cdot 2^d - 4$.*

Proof. Obviously, every sequence containing at least one cube with sides of length at least 1 permits a covering of I^d by the method of the current bottom. So we can assume that all cubes $Q_1, Q_2, \dots$ have sides smaller than 1.

For every $i \in \{1, 2, \dots\}$ the ratio of the volume $|Q_i|$ to the volume $|K_i|$ is smaller than 2^d . Consequently, it is sufficient to show that the unit cube can be on-line covered by every sequence of cubes $K_1, K_2, \dots$ of total volume at least $3 - 4 \cdot 2^{-d}$ whose sides are of the form 2^{-r} , where r is a positive integer.

Let $G_i(r)$ be a set containing a surface point of the i th bottom and having the

form

$$\{(x_1, \dots, x_d): a_k 2^{-r} \leq x_k \leq (a_k + 1)2^{-r} \text{ for } k = 1, \dots, d-1 \text{ and } b_i \leq x_d\},$$

where r is a nonnegative integer and $a_1, \dots, a_{d-1} \in \{0, \dots, 2^r - 1\}$. Denote by $H_i(r)$ the part of $G_i(r)$ which is covered by cubes $\sigma_j K_j$, where $j < i$. The volume of $H_i(r)$ is smaller than the sum of the volumes of $2^{d-1} - 1$ cubes of each of the sides $2^{-r-1}, 2^{-r-2}, \dots$. Applying the formula for the sum of the geometric sequence we obtain that

$$|H_i(r)| < \frac{2^{d-1} - 1}{2^d - 1} 2^{-rd}. \quad (1)$$

Putting $r = 0$ in (1) we get

$$|H_i(0)| < \frac{2^{d-1} - 1}{2^d - 1} \quad \text{for every } i \in \{1, 2, \dots\}. \quad (2)$$

From (1) we conclude that the volume of the common part of the cube $\sigma_i K_i$ and of the union of cubes $\sigma_j K_j$, where $j < i$, is smaller than

$$\frac{2^{d-1} - 1}{2^d - 1} |K_i|.$$

Consequently, the region covered by $\sigma_i K_i$ for the first time (i.e., covered by $\sigma_i K_i$ but not covered by the union of $\sigma_j K_j$, where $j < i$) is of volume greater than $1 - (2^{d-1} - 1)/(2^d - 1) = 2^{d-1}/(2^d - 1)$ times the volume of K_i . If we suppose that $b_i < 1$ for every $i \in \{1, 2, \dots\}$, then the sum of the volumes of the regions covered by the cubes $\sigma_i K_i$ for the first time, where $i = 1, 2, \dots$, is over

$$(3 - 4 \cdot 2^{-d}) \frac{2^{d-1}}{2^d - 1} = 1 + \frac{2^{d-1}}{2^d - 1}.$$

This and the assumption that all b_i are smaller than 1 lead to a contradiction with (2). Thus we conclude the existence of a positive integer n such that $b_n = 1$. So a finite number of cubes of our sequence covers the whole I^d . $\square$

3. The Method of the Current Bottom and Top

The idea of the method of the current bottom and top is in selecting the cubes K_i in the cubes Q_i , $i = 1, 2, \dots$, from amongst a larger family of cubes than in the method of the current bottom. So besides cubes of sides of lengths $1, \frac{1}{2}, \frac{1}{4}, \dots$, as in the method of the current bottom, this time we are also looking for cubes of sides of lengths $\frac{1}{4}\sqrt{2}, \frac{1}{8}\sqrt{2}, \dots$. The first kind of cubes are used for placing them

starting from the bottom of I^d , and the second kind starting from the top of I^d . The advantage is that, for cubes Q_i of sides less than $\frac{1}{2}\sqrt{2}$, the crucial factor 2^d from the method of the current bottom is now lessened to the factor $(\sqrt{2})^d$. However, we cannot lessen the factor for cubes of side between $\frac{1}{2}\sqrt{2}$ and 1. For such cubes we apply another trick. Here are the details.

We on-line cover points of I^d by a sequence $Q_1, Q_2, \dots$ of cubes. A cube of our sequence is called *very big* if its side is of length at least 1, and it is called *big* if its side is of length at least $\frac{1}{2}\sqrt{2}$ but smaller than 1.

In every Q_i , when we learn it, we find the greatest cube N_i whose side is of a length of the form 2^{-r} or $2^{-r-3/2}$, where r is a nonnegative integer. If the length of the side of N_i is of the form 2^{-r} , where r is a nonnegative integer, then we put $B_i = N_i$ and $T_i = \emptyset$. If the side of N_i has the form $2^{-r-3/2}$, where r is a nonnegative integer, we put $B_i = \emptyset$ and $T_i = N_i$.

The sequence $B_1, B_2, \dots$ is used for a covering of points of I^d by the method of the current bottom described in the preceding section. The difference is that if $B_i = \emptyset$, we do not find the rigid motion σ_i and we do not use B_i for the covering.

The sequence $T_1, T_2, \dots$ is used for covering I^d by a method similar to the method of the current bottom which is called *the method of the current top*. Here is the description. For every T_i we find the greatest nonnegative number t_i such that every point of I^d with the last coordinate greater than $1 - t_i$ has been covered by cubes T_j , where $j < i$. The set of all points of I^d having the last coordinate $1 - t_i$ is called the *ith top* or *the current top*. A point of the *ith top* is called a *surface point* if no point with the last coordinate smaller and with the other coordinates unchanged has been covered by cubes T_j , where $j < i$. If $T_i \neq \emptyset$, we find a rigid motion σ_i such that $\sigma_i T_i$ contains a surface point of the *ith top* and that it has the form

$$\{(x_1, \dots, x_d): a_k 2^{-r-3/2} \leq x_k \leq (a_k + 1) 2^{-r-3/2} \\ \text{for } k = 1, \dots, d-1 \text{ and } 1 - t_i - 2^{-r-3/2} \leq x_d \leq 1 - t_i\},$$

where $2^{-r-3/2}$ is the side of T_i and where $a_1, \dots, a_{d-1} \in \{0, 1, \dots, 3 \cdot 2^r - 1\}$. Observe that the method of the current top guarantees that there are always surface points of the current top.

The above prescription finds a rigid motion σ_i for every index i which moves one of the sets, B_i or T_i . Every time we move the corresponding cube Q_i . When $b_n + t_n \geq 1$, we can stop the covering process; at this moment I^d has been covered by cubes $\sigma_1 Q_1, \dots, \sigma_{n-1} Q_{n-1}$.

Lemma 1. *The set $\{(x_1, \dots, x_d): 0 \leq x_1 \leq 1, \dots, 0 \leq x_d \leq a\}$ can be covered by the method of the current bottom by every sequence of cubes whose sides are of the form 2^{-r} , where $r \in \{1, 2, \dots\}$, if the total volume of the cubes is at least*

$$b(a) = \frac{2^d - 1}{2^{d-1}} \left[a + \frac{2^{d-1} - 1}{2^d - 1} \right].$$

Lemma 2. *The set $\{(x_1, \dots, x_d): 0 \leq x_1 \leq 1, \dots, a \leq x_d \leq 1\}$ can be covered by the method of the current top by every sequence of cubes whose sides are of the form $2^{-r-3/2}$, where $r \in \{1, 2, \dots\}$, if the total volume of the cubes is at least*

$$t(a) = \frac{2^d - 1}{2^{d-1}} \left[(1 - a) \left(\frac{3\sqrt{2}}{4} \right)^{d-1} + \left(3^{d-1} - \frac{2^{d-1}}{2^d - 1} \right) \left(\frac{\sqrt{2}}{4} \right)^d \right].$$

We omit the proofs of Lemmas 1 and 2 because they are similar to the proof of Theorem 1.

Theorem 2. *Every sequence of d -dimensional cubes of total volume at least $2^d + (\frac{4}{3}\sqrt{2} + \frac{2}{3})(\frac{3}{2})^d$ permits an on-line covering of I^d by the method of the current bottom and top.*

Proof. Of course, every sequence of cubes containing a very big cube or at least 2^d big cubes covers I^d by the method of the current bottom and top. Further, we consider only sequences of cubes without very big cubes and with at most $2^d - 1$ big cubes.

The linearity of $b(x) + t(x)$, where $b(x)$ and $t(x)$ are defined in Lemmas 1 and 2, easily implies that the function $b(x) + t(x)$ is decreasing. Thus $b(a) + t(a) \leq b(0) + t(0)$ for every a between 0 and 1. Consequently, we can cover I^d by the sequence $Q_1, Q_2, \dots$ by the method of the current bottom and top if the volume of the sequence $N_1, N_2, \dots$ (defined in the description of the method) is at least $b(0) + t(0)$.

There are up to $2^d - 1$ big cubes in the sequence $Q_1, Q_2, \dots$. For every big cube Q_i we have $|Q_i| - (\sqrt{2})^d |N_i| < 1 - (\frac{1}{2}\sqrt{2})^d$. So I^d can be covered by the method of the current bottom and top by every sequence of cubes $Q_1, Q_2, \dots$ of total volume at least

$$(\sqrt{2})^d [b(0) + t(0)] + (2^d - 1) [1 - (\frac{1}{2}\sqrt{2})^d] < 2^d + (4\sqrt{2} + \frac{2}{3})(\frac{3}{2})^d. \quad \square$$

More specific estimates resulting from the method of the current bottom and top are presented in [4].

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