An Example Concerning the Translative Kissing Number of a Convex Body*

Chuanming Zong

Institut für Analysis, Technische Mathematik und Versicherungsmathematik,
Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria
and

Institute of Mathematics, The Chinese Academy of Sciences,
Beijing 100080, People's Republic of China.

Abstract. This article presents a class of convex bodies in E^d ($d \geq 3$) where their maximum kissing numbers in translative packings are larger than their maximum kissing numbers in lattice packings.

1. Introduction

Let K be a convex body, let $N_T(K)$ be the maximum kissing number of K in translative packings, and let $N_L(K)$ be the maximum kissing number of K in lattice packings. In other words, $N_T(K)$ is the maximum number of the nonoverlapping translates of K which can be brought into contact with K and $N_L(K)$ is the maximum number of the translates of K in lattice packings which can be brought into contact with K .

Concerning $N_T(K)$ and $N_L(K)$, the following question can be asked:

Is there a convex body K such that $N_T(K) > N_L(K)$?

In 1961 Grünbaum [3] proved a conjecture of Hadwiger which implies

$$N_T(K) = N_L(K)$$

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for all convex domains K in E^2 . Unexpectedly, 10 years later Watson [4] found that the maximum kissing number of a nine-dimensional sphere in lattice packings is 272 while a translative packing exists in which some nine-dimensional spheres are touched by 306 others. In a summary of the known results about this problem, Conway and Sloane [1] in 1988 wrote: “As regards the maximum kissing number, 9 is the first dimension in which nonlattice packings are known to be superior to lattice packings.”

In this article we prove that when $d \geq 3$ there are always d -dimensional convex bodies K such that

$$N_T(K) > N_L(K).$$

By this we solve our problem completely with regard to the dimensions.

2. The Three-Dimensional Case

In this section we present an example in E^3 . Defining

$$U = \{(x^1, x^2, x^3): |x^i| \leq \frac{1}{2}, i = 1, 2, 3\}$$

and

$$V = \{(x^1, x^2, x^3): |x^i| \leq \frac{1}{2}, i = 1, 2, 3, |x^1 + x^2 \pm x^3| \leq \frac{4}{3}\},$$

we show that V is a required example. Namely, we prove

Theorem 1. $N_L(V) = 22 < 24 \leq N_T(V)$.

In order to carry out the proof of Theorem 1 efficiently, we first show a very special assertion which plays a key role in the proof of $N_L(K) = 22$.

Assertion Z. Denote by W the square $\{(x^1, x^2): |x^1| \leq \frac{1}{2}, |x^2| \leq \frac{1}{2}\}$. Let T be a point set in E^2 such that $0 \notin T$, $W + T$ is a translative packing, and $W \cap (W + t) \neq \emptyset$ holds for every point $t \in T$, then we have

$$\text{card}\{T\} \leq 8$$

and the equality holds only if

$$\{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \subset T.$$

Sketch of the Proof. Let $T = \{t_1, t_2, \dots, t_n\}$, $t_i = (t_i^1, t_i^2)$, $\eta(t_i) = \max\{|t_i^1|, |t_i^2|\}$, $t_i^* = t_i/\eta(t_i)$, $i = 1, 2, \dots, n$, and $T^* = \{t_1^*, t_2^*, \dots, t_n^*\}$. By considering the related positions of t_i and t_j it can be verified that $W + T^* \cup \{o\}$ is a translative packing, $W \cap (W + t^*) \neq \emptyset$ holds for every point $t^* \in T^*$, and $t_i^* \neq t_j^*$ if $i \neq j$. Then, recalling Groemer's well-known result on the kissing numbers (see [2]), we get

$$\text{card}\{T\} = \text{card}\{T^*\} \leq 8$$

and the equality holds only if

$$T^* = \{(1, 1), (1, 0), (1, -1), (0, -1), (-1, -1), (-1, 0), (-1, 1), (0, 1)\}. \quad (1)$$

Then, from the relationship between T and T^* , we can see that (1) implies

$$\{(1, 1), (1, -1), (-1, -1), (-1, 1)\} \subset T.$$

Thus, our assertion is proved. □

Lemma 1. $N_T(V) \geq 24$.

Proof. Let T be the set of points $(0, 0, 0)$, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, $(\pm 1, 0, \pm 1)$, $(0, \pm 1, \pm 1)$, $\pm(-1, 1, 1)$, $\pm(1, -1, 1)$, $\pm(-1, 1, 0)$, $(1, 1, \pm \frac{1}{2})$, and $(-1, -1, \pm \frac{1}{2})$. It is easy to check that $V + T$ is a translative packing and

$$(V + t) \cap V \neq \emptyset$$

holds for every point $t \in T$. This means the kissing number of V in the packing $V + T$ is 24. This proves the lemma. □

Lemma 2. $N_L(V) = 22$.

Proof. Let Λ be a lattice such that $V + \Lambda$ is a packing, and let N be the kissing number of V in the lattice packing $V + \Lambda$. Now we prove our lemma by dealing with the following cases:

Case 1: All $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ belong to Λ . In this case, Λ is uniquely determined by these three points and it can be easily calculated that

$$N = 22. \quad (2)$$

Case 2: One of these three points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ does not belong to Λ . Without loss of generality, if $(0, 0, 1) \notin \Lambda$, we define

$$J_1 = \{x = (x^1, x^2, x^3): x \in \Lambda, x^3 > \frac{1}{3}, (V + x) \cap V \neq \emptyset\},$$

$$J_2 = \{x = (x^1, x^2, x^3): x \in \Lambda \setminus \{o\}, |x^3| \leq \frac{1}{3}, (V + x) \cap V \neq \emptyset\},$$

and

$$J_3 = \{x = (x^1, x^2, x^3): x \in \Lambda, x^3 < -\frac{1}{3}, (V + x) \cap V \neq \emptyset\}.$$

Obviously,

$$N = \text{card}\{J_1\} + \text{card}\{J_2\} + \text{card}\{J_3\}. \quad (3)$$

Now, we estimate $\text{card}\{J_1\}$, $\text{card}\{J_2\}$, and $\text{card}\{J_3\}$, respectively, as follows (when $(1, 0, 0) \notin \Lambda$ or $(0, 1, 0) \notin \Lambda$, the three sets J_1 , J_2 , and J_3 can be defined according to x^1 or x^2 , respectively, and can be estimated in a similar way):

(i) $\text{card}\{J_1\}$. Let H be the hyperplane $x^3 = \frac{2}{3}$ and

$$H(J_1) = \{(x^1, x^2, \frac{2}{3}): (x^1, x^2, x^3) \in J_1\}.$$

Keeping the definitions of V and J_1 and the assumption that $(0, 0, 1) \notin J_1$ in mind, we see that

$$(0, 0, \frac{2}{3}) \notin H(J_1), \quad \text{card}\{H(J_1)\} = \text{card}\{J_1\},$$

$\bigcup_{x \in J_1} ((V + x) \cap H) = W + H(J_1)$ is a translative packing of W in the hyperplane H (different from the definition in Assertion Z, here W indicates the square $\{(x^1, x^2, 0): |x^1| \leq \frac{1}{2}, |x^2| \leq \frac{1}{2}\}$), and $(W + (0, 0, \frac{2}{3})) \cap (W + y) \neq \emptyset$ holds for every point $y \in H(J_1)$. So, by Assertion Z we get

$$\text{card}\{J_1\} = \text{card}\{H(J_1)\} \leq 8 \quad (4)$$

and the equality holds only if four points

$$(1, 1, \alpha_1), (-1, 1, \alpha_2), (1, -1, \alpha_3), (-1, -1, \alpha_4) \in J_1 \quad (5)$$

exist.

(ii) $\text{card}\{J_2\}$ and $\text{card}\{J_3\}$. Apparently, the previous assumption $(0, 0, 1) \notin \Lambda$ implies $(0, 0, -1) \notin J_3$. Thus by applying a similar argument to sets J_2 and J_3 we get

$$\text{card}\{J_2\} \leq 8, \quad (6)$$

$$\text{card}\{J_3\} \leq 8 \quad (7)$$

and the equalities in (6) and (7) hold only if there are points

$$(1, 1, \beta_1), (-1, 1, \beta_2), (1, -1, \beta_3), (-1, -1, \beta_4) \in J_2 \quad (8)$$

and

$$(1, 1, \gamma_1), (-1, 1, \gamma_2), (1, -1, \gamma_3), (-1, -1, \gamma_4) \in J_3. \quad (9)$$

However, (5), (8), and (9) cannot hold simultaneously since the three nonoverlapping translates $V + (1, 1, \alpha_1)$, $V + (1, 1, \beta_1)$, and $V + (1, 1, \gamma_1)$ cannot be brought into contact with V at the same time. So, the equalities in (4), (6), and (7) cannot hold simultaneously. Thus, combining with (3) we get

$$N = \text{card}\{J_1\} + \text{card}\{J_2\} + \text{card}\{J_3\} < 24. \quad (10)$$

Considering the fact that the kissing number of V in any lattice packing is an even number, (10) can be improved to

$$N \leq 22. \quad (11)$$

Thus, by these two cases we get

$$N_L(V) = 22.$$

Lemma 2 is proved. $\square$

Obviously, Theorem 1 is an immediate consequence of Lemmas 1 and 2.

3. Two Generalizations and One Problem

By a similar argument we can prove

Theorem 2. *Let K be a convex body such that $V \subset K \subset U$ and $\pm(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}) \notin K$, then*

$$N_L(K) = 22 < 24 \leq N_T(K).$$

In high dimensions we have the following analogy:

Theorem 3. *If $d \geq 3$,*

$$K_1 = \left\{ x = (x^1, x^2, \dots, x^d): |x^i| \leq \frac{1}{2}, \left| \pm x^1 + \sum_{j=2}^d x^j \right| \leq \frac{d-1}{2} + \frac{1}{3} \right\},$$

$$K_2 = \{x = (x^1, x^2, \dots, x^d): |x^i| \leq \frac{1}{2}\}, \quad K_1 \subset K \subset K_2,$$

and $\pm(\pm\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \notin K$, then

$$N_L(K) = 3^d - 5 < 3^d - 3 \leq N_T(K).$$

To end this article we raise the following problem:

Problem. Determine

$$\max_{K \in \mathcal{K}} (N_T(K) - N_L(K)),$$

where $\mathcal{K}$ denotes the set of all d -dimensional convex bodies.

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