A New Necessary Condition for the Vertex Visibility Graphs of Simple Polygons*

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Abstract. A new necessary condition conjectured by Everett [2], which is essentially a stronger version of a necessary condition by Ghosh [3], for a graph to be the vertex visibility graph of a simple polygon is established.

1. Introduction

A *visibility graph* is a combinatorial structure which has proven useful for a variety of visibility questions in computational geometry. The visibility graph of a set of objects is a graph with a vertex for every object and an edge between a pair of vertices if the corresponding pair of objects are “visible” to each other under some notion of visibility. In this paper we study the problem of deciding whether a given graph is the visibility graph on the vertices of a simple polygon where a pair of vertices are *visible* to each other if the line segment between them does not intersect the exterior of the polygon.

Several necessary conditions for a given graph to be the visibility graph of a simple polygon have been proposed [3], [2], [1]. These, however, have not yet yielded a sufficient set of conditions for a graph to be the visibility graph of a simple polygon. The best-known bound on the complexity of the visibility graph recognition problem is that it is in PSPACE by Everett [2]. It is well known that

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a graph must necessarily be hamiltonian for it to be a visibility graph. Ghosh [3] first considered the simpler problem of trying to determine if the given graph *along with a hamiltonian cycle* on it is the visibility graph of a simple polygon whose boundary corresponds to the given hamiltonian cycle. If this is possible in polynomial time, then it would at least mean that the recognition problem is in NP. Ghosh [3] had proposed a set of necessary conditions for this problem and conjectured that this was sufficient as well. Everett [2, Section 2.2.3] later showed that this set of conditions is in fact not sufficient. She also conjectured a stronger version of one of these conditions. We settle this conjecture in the affirmative in this paper.

The rest of this paper is organized as follows. In Section 2 we introduce the definitions and the notation used in the paper besides quoting the earlier work which sets the stage for the work herein. In Section 3 we state and prove Everett's conjecture. Finally, in Section 4 we end with some concluding remarks.

2. Definitions, Notation, and Earlier Work

Consider a graph $G = (V, E)$ and a hamiltonian cycle $H = v_0, v_1, v_2, \dots, v_{n-1}, v_0$, $v_i \in V$ on it. For any pair of vertices (v_i, v_j) , $v_i \neq v_j$, $\text{chain}(v_i, v_j)$ is defined as the set $\{v_i, v_{i+1}, v_{i+2}, \dots, v_j\}$ of vertices where all index summations are modulo n . A pair of vertices (u, v) would be called an invisible pair if $(u, v) \notin E$ (we denote an edge between vertices u and v of G as (u, v)). A vertex $w = v_i \in \text{chain}(u, v) - \{u, v\}$ is called a *blocking vertex* of the pair (u, v) if, for any two vertices $u' \in \text{chain}(u, v_{i-1})$ and $v' \in \text{chain}(v_{i+1}, v)$, (u', v') is an invisible pair. An invisible pair is said to be *minimal with respect to w* if w is its only blocking vertex. Two invisible pairs (u, v) and (u', v') are said to be *separable with respect to $w = v_i$* if u' and v' are encountered before u and v (or vice versa) in $\text{chain}(v_i, v_{i-1})$.

Given an oriented, continuous, simple curve C , for any two points x and y on C , $C[x, y]$ ($C(x, y)$) denotes the part of C from x to y inclusive (exclusive) of both, if it exists, in the direction of orientation. We also use the curved (" $($ " or " $)$ ") and square (" $[$ " or " $]$ ") brackets appropriately at either end to denote part of a curve which includes one of its endpoints and not the other. If C is piecewise linear, then it is called a *polygonal chain* or simply a *chain* with the straight line segments of C called the *edges* and the "bends" called the *vertices*. If C is closed, i.e., is a Jordan curve, then $\text{int}(C)$ and $\text{cl}(C)$ denote respectively the interior and the closure (i.e., $C \cup \text{int}(C)$) of the region bounded by C . A *polygon* is a closed polygonal chain. As a convention we take all our polygons, unless otherwise mentioned, to be directed clockwise with the interior lying on the right of the polygon during a directed traversal. A vertex v of a polygon is said to be *reflex (convex)* if the interior angle between the edges adjacent to v is greater (less) than π . The vertices of a polygon are labeled in such a way that the ordering on the labels is consistent with the order in which the vertices occur during a directed traversal of the polygon. We assume throughout that our polygons are in *general position*, i.e., no three vertices are collinear.

A line through points x and y is denoted as $\overleftrightarrow{xy}$, the line $\overrightarrow{xy}$ directed from $x(y)$ to

$y(x)$ as $\vec{xy}$ ($\overleftarrow{xy}$), and the line segment joining the two as $\overline{xy}$. We use the notation (x, y) also to denote the open set $\overline{xy} - \{x, y\}$. The intended meaning of (x, y) —whether it denotes the open set or just the pair of points x and y —will be clear from the context. Henceforth throughout this paper we consider only directed lines unless otherwise mentioned. For a point x on a line L , $(x^+)_L$ and $(x^-)_L$ denote respectively the two components of $L - x$ directed *away from* and *into* x , while $[x^\pm]_L = (x^\pm)_L \cup \{x\}$. Also $[x^+]_L$ ($[x^-]_L$) would be called a *ray* whose *tail* (*head*) is x . The subscript L is omitted if the line being referred to is understood. For a pair of points x and y on L we say that x *precedes* or is *before* y if $y \in (x^+)$ and that it *follows* or is *after* y if $x \in (y^+)$. A wedge $\mathcal{W} = R_1 \cup R_2$ is a “broken” line formed by a pair of rays R_1 and R_2 such that the point $p = R_1 \cap R_2$, called the *apex* of $\mathcal{W}$, is the head of R_1 and the tail of R_2 . Clearly, a wedge also partitions the plane into two disjoint regions. It is now easy to extend the notions of the left and right half-planes defined by a line to that of the left and right regions of the partition induced by a wedge. Extending some of our other definitions for lines to wedges, we say that a point x precedes a point y on $\mathcal{W}$ if either both x and y lie on the same ray of $\mathcal{W}$ and x precedes y or $x \in R_1$ and $y \in R_2$. For convenience we also denote the portion of $\mathcal{W}$ between a pair of points x and y , where x precedes y , as $(\overline{xy})_{\mathcal{W}}$ (we call it a *segment with respect to* $\mathcal{W}$, or just a *segment* when the wedge being referred to is understood). Note that $(\overline{xy})_{\mathcal{W}} = \overline{xy}$ if both x and y lie on the same ray and $(\overline{xy})_{\mathcal{W}} = \overline{xp} \cup \overline{py}$ otherwise. We denote the open set $(\overline{xy})_{\mathcal{W}} - \{x, y\}$ as $(x, y)_{\mathcal{W}}$. For a point x preceding p , we denote $\overline{xp} \cup R_2$, i.e., the set of points following x on $\mathcal{W}$, as $[x^+]_{\mathcal{W}}$ and $[x^-]_{\mathcal{W}}$ is defined as $[x^-]_{R_1}$. Similarly, for a point x following p , $[x^+]_{\mathcal{W}}$ would mean $[x^+]_{R_2}$ and $[x^-]_{\mathcal{W}}$ would mean $R_1 \cup \overline{px}$. A similar notation with curved brackets “ $()$ ” is used to denote open sets obtained by removing the head or the tail of a ray. See Fig. 1 where the above definitions for a wedge $\mathcal{W}$ are illustrated. Finally, we mention that our definition of a wedge includes the case where R_1 and R_2 are collinear. So a line is also, in general, a wedge.

For two curves (lines, polygonal chains, etc.) X and Y where X is oriented and open, $\text{first}(X \cap Y)$ and $\text{last}(X \cap Y)$ denote respectively the first and the last points of intersection between X and Y encountered during a directed traversal of X . We of course require that X and Y intersect at most finitely many times.

The visibility graph of a polygon P is the graph $G = (VP, E)$ such that $(u, v) \in E$

Fig. 1. Wedge $\mathcal{W} = R_1 \cup R_2$. x_i precedes y_i and the hatched portions denote $(\overline{x_i y_i})_{\mathcal{W}}$ for $1 \leq i \leq 3$.

if and only if $\overline{uv} \subset \text{cl}(P)$ where YP is the set of vertices of P . Also a pair of vertices (u, v) of P is an invisible pair if $\overline{uv} \not\subset \text{cl}(P)$. The correspondence between this and the graph-theoretic notion of an invisible pair we defined earlier, is obvious. We use the same names for both, the vertices of P and the corresponding ones of its visibility graph. Note that the ordered sequence of the vertices of P corresponds to a hamiltonian cycle of G . So trivially the existence of a hamiltonian cycle is a necessary condition for G to be the visibility graph of a polygon. Ghosh [3] considered the problem of determining if a graph along with a given hamiltonian cycle in it is the visibility graph of a polygon whose boundary corresponds to the given hamiltonian cycle. Two of the necessary conditions proposed by Ghosh [3] for this are:

Necessary Condition 1. *Every invisible pair in G has a blocking vertex.*

Necessary Condition 2. *If two invisible pairs are separable with respect to vertex b , then they cannot both be minimal with respect to b .*

3. Everett's Conjecture

Everett [2] conjectured the following stronger version of Necessary Condition 2 which we settle in the affirmative in this paper.

Theorem 1. *Given the graph $G = (V, E)$ with the hamiltonian cycle H of G , if G is the visibility graph of a simple polygon with H corresponding to the boundary of the polygon, then an assignment $B: (u, v) \mapsto w$ of blocking vertices to the invisible pairs of G exists such that, for any two distinct invisible pairs (u, v) and (u', v') , $B((u, v)) = B((u', v')) = w$ implies (u, v) and (u', v') are not separable with respect to w on H .*

An assignment of blocking vertices to invisible pairs satisfying the conditions of the above theorem is called a *consistent* assignment. Note that both Necessary Conditions 1 and 2 follow from Theorem 1. We now prove the theorem by first specifying, in Lemmas 3.1–3.3, an assignment for the visibility graph of any arbitrary simple polygon P and later proving its consistency. We use G throughout to denote the visibility graph of P .

We first need some more definitions here. A chain $P[x, y]$ of a polygon P is called a *forward chain* (abbreviated $\mathcal{F}$ -chain) with respect to a wedge $\mathcal{W}$ if $x, y \in \mathcal{W}$ with x preceding y on $\mathcal{W}$ and $P(x, y)$ lies wholly on one side of $\mathcal{W}$. $P[x, y]$ would be analogously called a *backward chain* ($\mathcal{B}$ -chain) if instead x follows y . We use $\mathcal{L}$ and $\mathcal{R}$ to denote the $\mathcal{B}$ - or $\mathcal{F}$ -chains lying wholly on the left and the right of $\mathcal{W}$, respectively. An $\mathcal{LF}$ -chain for instance is a forward chain on the left of $\mathcal{W}$. We say that an $\mathcal{R}$ - or $\mathcal{L}$ -chain $P[x, y]$ *encloses* a segment (point) $(\overline{ab})_{\mathcal{W}}$ (a) if $(\overline{ab})_{\mathcal{W}} \subset (\overline{xy})_{\mathcal{W}}$ ($a \in (x, y)_{\mathcal{W}}$). The *span* of such a chain, denoted as $\text{span}(P[x, y])$, is the interior of the region bounded by $P[x, y] \cup (\overline{xy})_{\mathcal{W}}$.

Let $P[x, y]$ and $P[x', y']$ be two $\mathcal{R}$ -chains of P with respect to a wedge $\mathcal{W}$.

Since P is simple, clearly, either $(\overline{xy})_{\mathcal{W}} \cap (\overline{x'y'})_{\mathcal{W}} = \emptyset$ or one of the two is contained in the span of the other. We can thus define a partial order “ $<$ ” such that, for any two of these $\mathcal{R}$ -chains $P[x, y]$ and $P[x', y']$, $P[x, y] < P[x', y']$ if and only if $P[x, y] \subset \text{span}(P[x', y'])$. A similar partial order can also be defined on the $\mathcal{L}$ -chains of P with respect to $\mathcal{W}$. A chain C' would be called an *immediate superior* of C if $C < C'$ and for no other chain C'' is $C < C'' < C'$. A chain C is called a *highest* (lowest) chain with respect to “ $<$ ” if for no other chain C' is $C < C'$ ($C' < C$).

We now prove a fact about the arrangement of the $\mathcal{R}$ - and $\mathcal{L}$ -chains of P with respect to an arbitrary wedge $\mathcal{W}$. We use this fact later in our proofs.

Lemma 3.1. *Let $\mathcal{W} = R_1 \cup R_2$ be a wedge with apex $p = R_1 \cap R_2$ such that p is the head and the tail of rays R_1 and R_2 , respectively. Suppose $\mathcal{W}$ intersects P and let $x \in \mathcal{W} - P$ be a point on $\mathcal{W}$. If $C_1 < C_2 < C_3 < \dots < C_t$ are all the $\mathcal{R}$ -chains of P with respect to $\mathcal{W}$ which enclose x , then, for $x \in \text{int}(P)$ ($x \notin \text{int}(P)$), C_i is an $\mathcal{RB}$ -chain ($\mathcal{RF}$ -chain) for all odd i and an $\mathcal{RF}$ -chain ($\mathcal{RB}$ -chain) for every even i . Also t is odd (> 0) if $x \in \text{int}(P)$ and even otherwise. A similar statement holds for the $\mathcal{L}$ -chains of P as well.*

Remark. This lemma is not as obvious as it seems at first sight. That the directions in which any oriented polygon crosses a given line, alternate along the line, is obvious enough. What we seek to establish here instead is that a similar property holds for the chains with respect to $\mathcal{W}$ enclosing x as well. This seems to warrant a formal proof for the following reason. Consider a line L' through x such that $L' \cap \mathcal{W} = \{x\}$. There does not appear to be any orderly relationship between the crossings of P across L' and the chains enclosing x . In fact there could be arbitrarily complicated interleavings between the crossings of two chains with L' —the only constraint being that no two chains can intersect. The following is thus a formal proof of the lemma.

Proof of Lemma 3.1. Let L' be a line through x , directed toward the left side of $\mathcal{W}$ such that $L' \cap \mathcal{W} = \{x\}$. See Fig. 2. Consider an $\mathcal{R}$ -chain C of P with respect to $\mathcal{W}$. If C encloses x , since the two endpoints are on either side of L' the number

Fig. 2. The $\mathcal{R}$ -chains enclosing a point x on a wedge $\mathcal{W} = R_1 \cup R_2$ which intersects P .

of intersections of C with $(x^-)_L$ (we denote this henceforth as $(x^-)_\mathcal{W}$ and $(x^\pm)_\mathcal{W}$ as simply $(x^\pm)$) is odd. Note that $C \cap (x^+)_\mathcal{W} = \emptyset$. Similarly, if C does not enclose x , then $|C \cap (x^-)_\mathcal{W}|$ is even.

Suppose $x \in \text{int}(P)$. If $t = 0$, i.e., if there are no $\mathcal{R}$ -chains of P enclosing x , then clearly $|P \cap (x^-)_\mathcal{W}|$ is even. Let $P \cap (x^-)_\mathcal{W} = \{x_1, x_2, \dots, x_m\}$ where x_i precedes x_{i-1} on L' for every $i > 1$. Because $x \in \text{int}(P)$ and $(x, x_1) \cap P = \emptyset$, we have $(x, x_1) \subset \text{int}(P)$ implying in turn that $(x_i, x_{i+1}) \subset \text{int}(P)$ whenever i is even (this follows from the easily verifiable fact that the segments formed by the intersections of a line with a polygon are alternately inside and outside the region enclosed by the polygon). Therefore, m being even, it must be that $(x_m^-)_\mathcal{W} \subset \text{int}(P)$. However, this cannot happen because $\text{int}(P)$ is bounded. Thus we assume henceforth that $t \geq 1$ (we have nothing to prove if $t = 0$ for $x \notin \text{int}(P)$).

Let x_{k_i} be the point of intersection of C_i with $(x^-)_\mathcal{W}$ closest to x and let $C_i \cap (x^-)_\mathcal{W} = p_i$ and $C_i \cap (x^+)_\mathcal{W} = q_i$ for some i , $1 \leq i \leq t$. See Fig. 2. We first count the number of intersections of the $\mathcal{R}$ -chains of P with (x, x_{k_i}) . Consider an $\mathcal{R}$ -chain $C \subset \text{span}(C_i)$ (note that because P is simple, only such chains can intersect (x, x_{k_i})) and let Q be the polygon $(\overline{xq_i})_\mathcal{W} \cup \overline{xx_{k_i}} \cup (\text{the part of } C_i \text{ between } x_{k_i} \text{ and } q_i)$. We now use the fact that for any Jordan curve J and any other open continuous curve S , both of whose endpoints lie outside (inside) the region bounded by J , $|S \cap J|$ is even (or zero). Also if one of the endpoints is inside and the other outside, then $|S \cap J|$ is odd.

If C encloses x , then let $a = C \cap (p_i, x)_\mathcal{W}$ and let b be a point on C very close to $C \cap (x, q_i)_\mathcal{W}$. Consider the portion C' of C between a and b . From our observation above, it follows that $|C' \cap Q|$ is odd. However $C' \cap (\overline{xq_i})_\mathcal{W} = C' \cap (\text{part of } C_i \text{ between } x_{k_i} \text{ and } q_i) = \emptyset$. Thus $|C \cap \overline{xx_{k_i}}| = |C' \cap \overline{xx_{k_i}}|$ is odd. However, if C does not enclose x (then either both its endpoints are in $(p_i, x)_\mathcal{W}$ or both are in $(x, q_i)_\mathcal{W}$), it should be clear, from a similar argument, that $|C \cap \overline{xx_{k_i}}|$ is even. Hence since there are $(i-1)$ $\mathcal{R}$ -chains enclosing x in the span of C_i , clearly, $|(x, x_{k_i}) \cap P| = k_i - 1$ is even if and only if i is odd. Now if $x \in \text{int}(P)$ ($x \notin \text{int}(P)$), we have $(x, x_1) \subset \text{int}(P)$ ($(x, x_1) \not\subset \text{int}(P)$), implying that $(x_{k_i-1}, x_{k_i}) \subset \text{int}(P)$ if and only if $k_i - 1$ is even (odd). Therefore, since P is directed clockwise, C_i crosses L' at x_{k_i} from the right of L' to its left if and only if i is odd (even).

To complete the proof, we only have to show that if C_i crosses L' at x_{k_i} , from the right (left) to its left (right), then C_i is an $\mathcal{RB}$ -chain ($\mathcal{RF}$ -chain) of P with respect to $\mathcal{W}$.

Suppose, on the contrary, C_i crosses L' at x_{k_i} from the left of L' to its right and is an $\mathcal{RB}$ -chain. Since x_{k_i} is the point in $C_i \cap (x^-)_\mathcal{W}$ which is closest to x , we have $P(q_i, x_{k_i}) \cap (x, x_{k_i}) = \emptyset$. Thus $Q = \overline{xx_{k_i}} \cup (\overline{xq_i})_\mathcal{W} \cup P[q_i, x_{k_i}]$ is a simple polygon with C_i directed into $\text{int}(Q)$ at x_{k_i} . Also $p \notin \text{int}(Q)$. Hence $P(x_{k_i}, p_i) \cap (x, x_{k_i}) \neq \emptyset$ (since $P(x_{k_i}, p_i) \cap Q \neq \emptyset$ and $P(x_{k_i}, p_i) \cap P(q_i, x_{k_i}) = P(x_{k_i}, p_i) \cap (x, q_i)_\mathcal{W} = \emptyset$). However, then x_{k_i} cannot be the point of $C_i \cap (x^-)_\mathcal{W}$ closest to x giving rise to a contradiction. The case where C_i crosses L' at x_{k_i} from the right to the left is analogous (we take the polygon $P[p_i, x_{k_i}] \cup \overline{xx_{k_i}} \cup (\overline{p_i x})_\mathcal{W}$ and then arrive at a contradiction just as above). Hence the lemma. $\square$

Now consider the set $\mathcal{S}$ of $\mathcal{R}$ -chains of $P[u, v]$ with respect to $\vec{uv}$, for some pair

Fig. 3. $P[a, b]$, $P[c, d]$, $P[e, f]$, and $P[g, v]$ are the $\mathcal{R}$ -chains of $P[u, v]$, of which $P[a, b]$ and $P[g, v]$ are also blocking chains.

of vertices (u, v) of P , along with the partial order “ $<$ ” as defined earlier, on them. A *blocking chain* of (u, v) in $P[u, v]$ is an $\mathcal{RF}$ -chain in $\mathcal{S}$ such that, with respect to the ordering “ $<$ ”, it is either a highest chain in $P[u, v]$ or its immediate superior in $P[u, v]$ is not an $\mathcal{RB}$ -chain. Similarly, (u, v) could also have blocking chains in $P[v, u]$ which are $\mathcal{RF}$ -chains with respect to $\vec{vu}$. Henceforth whenever we refer to an $\mathcal{R}$ - or an $\mathcal{L}$ -chain of $P[u, v]$, we mean a subchain of $P[u, v]$ which is an $\mathcal{R}$ - or $\mathcal{L}$ -chain with respect to $\vec{uv}$. Analogous definitions hold for $\mathcal{B}$ - and $\mathcal{F}$ -chains and also for the subchains of $P[v, u]$. Figure 3 illustrates the $\mathcal{R}$ -chains and the blocking chains of $P[u, v]$ for a pair of vertices (u, v) of P . Our definition of a blocking chain is motivated by the fact that every invisible pair of G has at least one blocking vertex which corresponds to a reflex vertex of P lying on a blocking chain of the pair. We eventually define B to assign only such blocking vertices to invisible pairs. However, first we confirm that every invisible pair does indeed have a blocking chain and later show the existence of such blocking vertices through the following two lemmas.

Lemma 3.2. *A blocking chain exists, either on $P[u, v]$ or $P[v, u]$, for any pair of invisible vertices (u, v) of P .*

Proof. Let x be a point in the open set (u, v) such that $x \notin \text{cl}(P)$. Such an x must clearly exist because $\vec{uv} \not\subset \text{cl}(P)$. Let $\mathcal{T}_c$ denote the number of subchains of chain c of type $\mathcal{T}$ which enclose x , where $c \in \{uv, vu\}$ and $\mathcal{T} \in \{\mathcal{RF}, \mathcal{RB}, \mathcal{LF}, \mathcal{LB}\}$. Here we use “ uv ” and “ vu ” for convenience to denote $P[u, v]$ and $P[v, u]$, respectively. Suppose $\mathcal{RF}_{uv} > \mathcal{RB}_{uv}$. Elementary combinatorics will now tell us that in any sequence (induced by $<$) of the $\mathcal{R}$ -chains of $P[u, v]$ enclosing x , there is at least one occurrence of an $\mathcal{RF}$ -chain which is not followed immediately by an $\mathcal{RB}$ -chain. Such a chain is clearly a blocking chain of (u, v) .

So let $\mathcal{RF}_{uv} \leq \mathcal{RB}_{uv}$. However, since u and v lie on either side of x on $\vec{uv}$, clearly, the number of $\mathcal{F}$ -chains of $P[u, v]$ enclosing x is exactly one more than the number

of $\mathcal{B}$ -chains. Thus $\mathcal{RF}_{uv} + \mathcal{LF}_{uv} = \mathcal{RB}_{uv} + \mathcal{LB}_{uv} + 1$. This implies that $\mathcal{LF}_{uv} > \mathcal{LB}_{uv}$. Now applying Lemma 3.1 to the left side of $\vec{uv}$ we find $\mathcal{LF}_{uv} + \mathcal{RB}_{vu} = \mathcal{LB}_{uv} + \mathcal{RF}_{vu}$. Hence $\mathcal{RF}_{vu} > \mathcal{RB}_{vu}$. From our earlier observation for the case $\mathcal{RF}_{uv} > \mathcal{RB}_{uv}$, we see that $P[v, u]$ must now have a blocking chain of (u, v) . $\square$

Henceforth to make our exposition less clumsy, we use the term “blocking vertex” of a pair of vertices (u, v) of P to denote the vertex of P corresponding to a blocking vertex of the pair of vertices of G corresponding to (u, v) . Here we stress the fact that the notion of a blocking vertex is strictly graph theoretic.

Lemma 3.3. *If $P[u, v]$ contains blocking chains of an invisible pair of vertices (u, v) , then a blocking vertex of (u, v) exists on the convex hull of the set of blocking chains of (u, v) in $P[u, v]$.*

Proof. Let w be the vertex on the convex hull CH of the blocking chains which admits a tangent to the hull parallel to $\vec{uv}$. We show that w is a blocking vertex of (u, v) . Let the tangent through w parallel to $\vec{uv}$ be L and let the $\mathcal{RF}$ -chain containing w be C . See Fig. 4. We first show that a vertex $w' \in P(v, u)$ on the right of or on L exists such that $\overline{ww'} \subset \text{cl}(P)$. Let x and y be the points in $L \cap P$, closest to w on either side of it. Clearly, $\overline{xy} \subset \text{cl}(P)$ because P is oriented clockwise and w lies on an $\mathcal{RF}$ -chain. We now show that $x \in P(v, u)$ by contradiction. Suppose $x \in P(u, v)$ and let C_x be the chain of $P[u, v]$ containing x . Since $x \notin \text{cl}(CH)$, clearly, $C_x \not\subset \text{span}(C)$. If, however, $C < C_x$, then, because $\overline{wx} \subset \text{cl}(P)$, C_x must be the immediate superior of C . See Fig. 5(a). C being a blocking chain it now follows that C_x must be an $\mathcal{RF}$ -chain. However, this cannot be because now, for $\overline{wx}$ to be in $\text{cl}(P)$, $\text{int}(P)$ must lie on the left of C_x , contrary to our assumption that P is oriented clockwise. The only other possibility is where $\text{span}(C_x) \cap \text{span}(C) = \emptyset$. Suppose C_x is an $\mathcal{RB}$ -chain as in Fig. 5(b). Here too from the clockwise orientation

Fig. 4. The vertex w of CH farthest from $\vec{uv}$ is a blocking vertex of (u, v) .

Fig. 5. The points x and y have to be in $P[v, u]$.

of P it follows that this is not possible. Finally, let C_x be an $\mathcal{RF}$ -chain. See Fig. 5(c). Since $x \notin \text{cl}(CH)$, C_x is clearly not a blocking chain implying that it does have an immediate $\mathcal{RB}$ -superior C'_x . Now if $\overline{wx} \subset \text{cl}(P)$, it is easy to see that no $\mathcal{R}$ -chain of $P[u, v]$ can be such that it encloses just one of the two, C_x and C , and not the other. Thus C'_x is an immediate superior of C as well contradicting the fact that C is a blocking chain. Thus $x \in P(v, u)$. Similarly, we can show that $y \in P(v, u)$.

If either x or y is a vertex of P , we take w' as that vertex. Otherwise let v_x and v_y be respectively the endpoints of the edges of P containing x and y such that both v_x and v_y lie on the right of L (see Fig. 4). We now observe that since $\overline{wx} \subset \text{cl}(P)$, P must be directed from x to v_x at x and from v_y to y at y . This, along with the fact that both x and y are in $P[v, u]$, implies that the chain $P[x, y]$ which contains v_x and v_y is a subset of $P[v, u]$. Therefore $Q = P[x, y] \cup \overline{xy}$ is a simple polygon with $\text{cl}(Q) \subset \text{cl}(P)$. We now choose a vertex of $P(x, y)$ as our w' .

If $\overline{wv_x} \subset \text{cl}(Q)$, then $w' = v_x$. Otherwise vertices of Q must exist in the interior of the triangle $\triangle wv_x x$. In that case we take w' as the vertex z of Q in $\triangle wv_x x$ for which the angle between “positive” L and $\vec{wz}$ is the least. That such a w' is just what we want, can be verified easily.

Now to show that w is a blocking vertex we first observe that since P is a simple polygon with $\overline{ww'}$ its chord, for any pair of vertices (u', v') with $u' \in P(w, w')$ and $v' \in P(w', w)$, if $\overline{u'v'} \subset \text{cl}(P)$, then it must be that $(u', v') \cap (w, w') \neq \emptyset$. We now show that, for any pair of vertices (u', v') with $u', v' \in P[u, v] - w$ and $\overline{u'v'} \subset \text{cl}(P)$, $(u', v') \cap (w, w') = \emptyset$, thus showing that w is indeed a blocking vertex of (u, v) .

Firstly for a pair of vertices (u', v') , both of which are either to the left of $\vec{uv}$ or in $\text{cl}(CH)$, obviously $(u', v') \cap (w, w') = \emptyset$. So without loss of generality let u' be on the right of $\vec{uv}$ and outside $\text{cl}(CH)$. Let C' be the $\mathcal{R}$ -chain of $P[u, v]$ containing u' and since $u' \notin \text{cl}(CH)$, clearly, $C' \not\subset \text{span}(C)$. Here again we have four cases similar to those in Fig. 5.

1. $C < C'$.

(a) C' is an $\mathcal{RF}$ -chain (Fig. 6(a)). $\overline{ww'} \subset \text{span}(C')$ and $\overline{u'v'} \cap \text{span}(C') = \emptyset$ because $\overline{ww'}, \overline{u'v'} \subset \text{cl}(P)$ and P is oriented clockwise.

(b) C' is an $\mathcal{RB}$ -chain (Fig. 6(b)). C being a blocking chain, there is an immediate $\mathcal{RF}$ -superior C_1 of C in $P[u, v]$ such that $C < C_1 < C'$. Again because $\overline{ww'}, \overline{u'v'} \subset \text{cl}(P)$ we find $\overline{ww'} \subset \text{span}(C_1)$ and $\overline{u'v'} \cap \text{span}(C_1) = \emptyset$.

Fig. 6. For every pair (u', v') , $u'v' \in P[u, v] - w$, if $\overrightarrow{u'v'} \subset \text{cl}(P)$, then $\overrightarrow{ww'} \cap \overrightarrow{u'v'} = \emptyset$.

2. $\text{span}(C) \cap \text{span}(C') = \emptyset$.

- (a) C' is an $\mathcal{RF}$ -chain (Fig. 6(c)). We know that C is a blocking chain and that C' is not. From these it is easy to infer that there must be a chain C_1 enclosing just one of these two and not the other. Therefore assume $C < C_1$ without loss of generality. Again because $\overrightarrow{ww'}$, $\overrightarrow{u'v'} \subset \text{cl}(P)$ we find $\overrightarrow{ww'} \subset \text{span}(C_1)$ and $\overrightarrow{u'v'} \cap \text{span}(C_1) = \emptyset$.
- (b) C' is an $\mathcal{BB}$ -chain (Fig. 6(d)). Since P is clockwise oriented, clearly either v' is on the left of $\overrightarrow{uv}$ or $(u', v'] \subset \text{span}(C')$. Moreover, $\overrightarrow{ww'} \cap (\text{span}(C') \cup C') = \emptyset$.

It is trivially seen that in each of the above cases $\overrightarrow{u'v'} \cap \overrightarrow{ww'} = \emptyset$. The lemma is thus proven. $\square$

Now our assignment of blocking vertices to invisible pairs of P is just that implied by Lemma 3.3, i.e., for an invisible pair (u, v) we pick any one of the two chains $P[u, v]$ and $P[v, u]$ which contains blocking chains of (u, v) , compute the convex hull of all the blocking chains on it, and then assign a vertex on the hull which admits a tangent to the hull parallel to $\overrightarrow{uv}$ as the blocking vertex of (u, v) . All the vertices on the convex hull of the blocking chains are clearly reflex vertices of P . Thus our assignment assigns only reflex vertices of P as blocking vertices of invisible pairs. In fact we show in the following lemma that even the converse of this is true. Henceforth we assume that B is our assignment of blocking vertices to invisible pairs as described above.

Lemma 3.4. *For any three consecutive vertices u, v , and w of P such that $w \in P(u, v)$, if w is a reflex vertex of P , then $B((u, v)) = w$. Note that if w is a reflex vertex, then (u, v) is necessarily an invisible pair.*

Fig. 7. Every concave vertex of P is assigned as the blocking vertex of some invisible pair.

Proof. Consider any $\mathcal{RF}$ -chain in $P[v, u]$. We know from Lemma 3.1 that any such chain must certainly have an immediate superior chain in either $P[u, v]$ or $P[v, u]$ which is an $\mathcal{RB}$ -chain with respect to $\vec{uv}$. See Fig. 7. Obviously this cannot be in $P[u, v]$ as $P[u, v]$ has exactly one vertex w which is on the left of $\vec{uv}$. Thus every $\mathcal{RF}$ -chain in $P[v, u]$ has an immediate $\mathcal{RB}$ -superior in $P[v, u]$. Hence $P[v, u]$ has no blocking chain of (u, v) . It is now clear that $B((u, v)) = w$. $\square$

We can further show that the assignment B is in fact consistent.

Lemma 3.5. *The assignment B of blocking vertices to invisible pairs is consistent.*

Proof. Let w be the blocking vertex assigned to an invisible pair (u, v) and let (u', v') be another invisible pair separable from (u, v) with respect to w . Let us assume without loss of generality that $w \in P(u, v)$ and that $P[v', u'] \subset P[u, v]$. Note that then either $P[v', u'] \subset P[u, w]$ or $P[v', u'] \subset P[w, v]$. Also $w \in P(u', v')$. See Fig. 8 where the relative positions of w, u, v, u' , and v' on P are shown schematically. We now show that $B((u', v')) \neq w$, hence proving the consistency.

Suppose on the contrary that $B((u', v')) = w$. Then w lies on blocking chains of both $P[u, v]$ and $P[u', v']$. Let these two $\mathcal{RF}$ -chains with respect to $\vec{uv}$ and $\vec{u'v'}$

Fig. 8. Schematic diagram showing the relative positions of w, u, v, u' , and v' on P : (a) $P[v', u'] \subset P[u, w]$ and (b) $P[v', u'] \subset P[w, v]$.

Fig. 9. If $x_1 \in C_2$, then $\vec{u'v'}$ is directed from the left to the right of $\vec{uv}$.

be respectively $C_1 = P[x_1, y_1]$ and $C_2 = P[x_2, y_2]$. Clearly, $C_1 \cap C_2 \neq \emptyset$ because w lies on both. Also $C_1 \cap C_2$ is a contiguous chain of P . Let $p = \vec{uv} \cap \vec{u'v'}$. Henceforth we denote $(p^\pm)_{\vec{uv}}$ as $(p^\pm)$ and $(p^\pm)_{\vec{u'v'}}$ as $(p^\pm)'$.

We prove the lemma through a case analysis. We first assume that $C_1 \not\subset C_2$. If this is not true we could carry on with the same proof simply by swapping the roles of $P[u, v]$ and C_1 with those of $P[u', v']$ and C_2 , respectively. So we essentially have two possibilities here:

- C_2 contains neither x_1 nor y_1 .
- C_2 contains exactly one of x_1 and y_1 .

We first show that if $x_1 \in C_2$, then $\vec{u'v'}$ is necessarily directed from the left of $\vec{uv}$ to its right. Suppose on the contrary that $\vec{u'v'}$ is directed from the right of $\vec{uv}$ to its left. Then notice that x_1 must be on the right of $\vec{u'v'}$ and that $y_2 = \text{first}(P[x_1, y_1] \cap \vec{u'v'})$. Moreover, since C_2 is supposed to be an $\mathcal{RF}$ -chain with respect to $\vec{u'v'}$ it is clear that $x_2 \in (y_2^-)$. It is easy to see from Fig. 9 that no point in C_2 can lie on the convex hull of C_1 while admitting a tangent to the hull parallel to $\vec{uv}$. This contradicts our assumption that $w \in C_2$ is such a point. Thus $\vec{u'v'}$ must be directed from the left to the right of $\vec{uv}$. An identical argument will also tell us that if $y_1 \in C_2$, then $\vec{u'v'}$ is directed from the right to the left of $\vec{uv}$. We thus prove the lemma separately for the two cases, one for each direction in which $\vec{u'v'}$ can cross $\vec{uv}$. We make no assumptions on whether C_2 is contained in C_1 or not except that if $\vec{u'v'}$ is directed from the left (right) of $\vec{uv}$ to its right (left) and $C_2 \not\subset C_1$, then $x_1 \in C_2$ and $y_1 \notin C_2$ ($y_1 \in C_2$ and $x_1 \notin C_2$). It can be verified that the proofs work both when $C_2 \subset C_1$ and otherwise, though most of our illustrations are typically for the case $C_2 \subset C_1$.

Case 1: $\vec{u'v'}$ is directed from the right to the left of $\vec{uv}$. Consider the wedge $\mathcal{W}$, formed by the rays $(p^-)'$ and $(p^+)'$ with p as the apex. Let $q = \text{first}(P[x_2, y_2] \cap \mathcal{W})$. Note that if $P(x_2, y_1) \cap \mathcal{W} = \emptyset$, then $q \equiv y_1$ and $q \equiv y_2$ otherwise. Let $C_3 = P[x_2, q]$. Observe that $w \in C_3 \subseteq C_2$ and that C_3 is an $\mathcal{RF}$ -chain with respect to $\mathcal{W}$. We first observe that no subchain of C_1 can be a superior of C_3 with respect to $\mathcal{W}$. We show this by contradiction. Suppose there is some such subchain $C \subset C_1$ superior to C_3 . Let the endpoints of C be x and y where x follows y on $\mathcal{W}$. Clearly, $y \in (x_2^-)$ and $x \in (q^+)_{\mathcal{W}}$. It is now easy to see that w lies in the interior of the polygon

Fig. 10. No subchain C of C_1 can be a superior of C_2 with respect to $\mathcal{W} = (p^-) \cup (p^+)$.

$C \cup (\overline{y_1x})_{\mathcal{W}}$. See Fig. 10 (in this and the subsequent figures up to Fig. 13, $q \equiv y_2$ and $C_3 \equiv C_2$). Therefore w cannot be on the convex hull of C_1 , contradicting our choice of w .

Now Lemma 3.1 assures us that an immediate superior $C = P[x, y]$ of C_3 in P with respect to $\mathcal{W}$ exists and that it is necessarily a $\mathcal{B}$ -chain. We already know that $C \not\subset C_1$. Thus $C \cap C_1 \neq \emptyset$ implies that either $x_1 \in C$ or $y_1 \in C$. Now since $P(x, y) \cap \overrightarrow{uv} = \emptyset$, $x_1 \in C$ means that $x \equiv x_1$ and in turn that $C \subset C_1$, which we already know is not possible. However, if $y_1 \in C$, then $x \equiv y_1$ making y_1 a vertex of P . The general position assumption about P then implies that $y_1 \equiv v$. All these follow from the orientations of C and C_1 across $\overrightarrow{uv}$ at x and x_1 , respectively. We have thus shown that if $C \cap C_1 \neq \emptyset$, then $C \cap C_1 = \{v\}$ and $C \subset P[v, u]$.

Now, since $P(x, y) \cap \overrightarrow{uv} = \emptyset$, either $C \subset P[u, v]$ or $C \subset P[v, u]$. We treat these two cases separately below.

Case 1.1: $C \subset P[u, v]$. We know from the observation above that then $C \cap C_1 = \emptyset$. Let $a = \text{last}(P[u, x] \cap \overrightarrow{uv})$. Note that if $x \in (p^+)$, then $a \equiv x$. Clearly, $a \notin \overline{x_1y_1}$, because x is outside the polygon $\overline{x_1y_1} \cup P[x_1, y_1]$ and if $a \in \overline{x_1y_1}$, then $P[u, x]$ would enter the interior of the polygon at a . So either $a \in (x_1^-)$ or $a \in (y_1^+)$. We treat these two cases separately below.

Case 1.1.1: $a \in (x_1^-)$. Note that then $q \equiv y_2 \in (p^-)$, $C_3 \equiv C_2$, and $x \in \overline{y_2p}$. It is now easy to see that $\text{last}(P[a, x] \cap \overrightarrow{uv}) \in \overline{y_2p}$. Let $c = \text{first}(P[a, x] \cap \overline{y_2p})$ and $b = \text{last}(P[a, c] \cap \overrightarrow{uv})$. Clearly, $b \in (y^-)$ and $c \in \overline{xp}$. Note that the only other possibility is $b \in (y, x_2)$ and $c \in (y_2, x)$. In that case C cannot be the immediate superior of C_3 with respect to $\mathcal{W}$ because $P[b, c]$ would be “between” C_3 and C . See Fig. 11. Now if $z = \text{first}(P[y, v] \cap \overrightarrow{uv})$, then $z \in (a, x_1)$. This follows from the observations that $z \notin (x_1, y_1)$ and that if $z \in (y_1^+)$ or $z \in (a^-)$, then C cannot be the immediate superior of C_3 with respect to $\mathcal{W}$. Note that if $z \in (y_1^+)$ or $z \in (a^-)$ there must necessarily be a subchain of $P[y, z]$ which lies “between” C_3 and C . Similarly, it is easy to see that there will be a subchain “between” C_3 and C in any $\mathcal{B}$ -chain with respect to $\overrightarrow{uv}$, one of whose endpoints is in (a, x_1) and the other in either (a^-) or (y_1^+) . Thus if an $\mathcal{B}$ -chain encloses either $\overline{az}$ or $\overline{x_1y_1}$, then in fact it encloses both. Now $z \in (a, x_1)$ means that $P[a, z]$ is an $\mathcal{AF}$ -chain of $P[u, v]$. If $P[a, z]$ is not a blocking chain, then its immediate superior in $P[u, v]$, say C' , is an $\mathcal{AB}$ -chain.

Fig. 11. The case where $C \subset P[u, v]$ and $\text{last}(P[u, x] \cap \vec{ub}) \in (x_1^-)$.

However, then C' must also be the immediate superior of C_1 . This clearly contradicts our assumption that C_1 is a blocking chain. Thus $P[a, z]$ is a blocking chain as well. Note here that $C \subset P[u, v]$ also implies $P[a, z] \subset P[u, v]$. However, then w , being in the interior of the polygon $C \cup \overline{xy}$, cannot be on the convex hull of $C_1 \cup P[a, z]$ which it must have been if $B((u, v)) = w$, as we had assumed at the beginning. This is again a contradiction.

Case 1.1.2: $a \in (y_1^+)$. Let $z = \text{first}(P[y, v] \cap \vec{ub})$. Arguments similar to the ones used for settling Case 1.1.1 would tell us easily that $z \notin [x_1, a]$. Also if $z \in (x_1^-)$ (shown as z_1 in Fig. 12), then $P[a, z]$ would be an immediate superior of C_1 in $P[u, v]$ and if $z \in (a^+)$ (shown as z_2 in Fig. 12), then $P[a, z]$ is a blocking chain of (u, v) in $P[u, v]$. All these follow from the crucial fact that C is the immediate superior of C_3 with respect to $\mathcal{W}$. Now if $z \in (x_1^-)$, then C_1 could not have been a blocking chain and if $z \in (a^+)$, then w would not be on the convex hull of $C_1 \cup P[a, z]$. In either case w could not have been assigned as the blocking vertex of (u, v) , contradicting our earlier assumptions.

Case 1.2: $C \subset P[v, u]$. We first note that in this case $C \subset P[u', v']$. Now if $x \in (p^-)$, then C is obviously also the immediate superior of C_3 in $P[u', v']$. This contradicts our assumption that C_2 is a blocking chain of (u', v') in $P[u', v']$. Recall again that $x \in (p^-)$ implies $C_3 \equiv C_2$. Thus $x \in (y_1^+)$. Let $a = \text{last}(P[y_2, x] \cap \vec{u'v'})$. Clearly, $P[a, y]$ is an $\mathcal{A}$ -chain of $P[u', v']$ because $C \subset P[a, y]$.

Now again two possibilities arise. Either $a \in (y^-)$ or $a \in (y_2^+)$. Note that if $a \in (y, x_2)$, then $P[a, x]$ would prevent C from being the immediate superior of C_3

Fig. 14. The case where $\vec{u'v'}$ is directed from the left to the right of $\vec{uv}$ and (a) $a \in (x_1^-)$, (b) $a \in (y_1^+)$. Here $q = x_2$ and $C_3 \equiv C_2$.

Case 2: $\vec{u'v'}$ is directed from the left to the right of $\vec{uv}$. This case is completely analogous to Case 1. This time we consider the wedge $\mathcal{W}$ formed by the pair of rays (p^-) and (p^+) . Also consider $q = \text{last}(P[x_1, y_2] \cap \mathcal{W})$ with C_3 defined as $P[q, y_2]$. Again let $C = P[x, y]$ be the immediate superior of C_3 with respect to this wedge. Just as we did for the earlier case we can show that if $C \subset P[u, v]$, then $C \cap C_1 = \emptyset$. We also know that C is an $\mathcal{RB}$ -chain with respect to $\mathcal{W}$. Thus $x \in (y_2^+)$ and $y \in (q^-)$. Suppose $C \subset P[u, v]$. We now look at $a = \text{first}(P[u, x] \cap \vec{uv})$. If $a \in (x_1^-)$ (see Fig. 14(a)), then $z = \text{first}(P[y, v] \cap \vec{uv})$ lies in $\vec{ax}_1$. Then w would not be on the convex hull of the blocking chains $(P[a, z])$ would also be a blocking chain if C_1 is one). If, however, $a \in (y_1^+)$ (see Fig. 14(b)), then either $z \in (x_1^-)$ (shown as z_1) or $z \in (a^+)$ (shown as z_2). In the former case C_1 would not be a blocking chain and in the latter w would not be on the convex hull of $C_1 \cup P[a, z]$. All these lead to contradictions. Finally, if $C \subset P[v, u]$, then we only have to consider $z = \text{first}(P[y, v] \cap \vec{u'v'})$ to see that if $z \in (p^+)$, then $P[x, z]$ will be the immediate $\mathcal{RB}$ -superior of C_2 in $P[u', v']$ and if $z \in (x^+)$, then w would not be on the convex hull of $P[x, z] \cup C_2$. These again are contradictions.

Hence the lemma. $\square$

We have thus proved Theorem 1 by constructing a consistent invisible pair to blocking vertex assignment for the visibility graph of any simple polygon P . Similar to Everett's restatement of Ghosh's Necessary Condition 1 [2, Section 2.2.4], we can now state a version of Theorem 1 which firmly establishes the relationship between the strictly graph-theoretic notion of a blocking vertex and the intuitive geometric notion of a reflex vertex "blocking" the visibility between a pair of vertices of a simple polygon. Our restatement follows easily from Lemma 3.4.

Theorem 2. *Let G be the visibility graph of a simple polygon and let P be any polygon in general position with G as its visibility graph. Then a consistent assignment*

B of blocking vertices to invisible pairs of G exists such that $B((u, v)) = w$ for some invisible pair (u, v) if and only if w corresponds to a reflex vertex of P .

4. Concluding Remarks

In this paper we have proved Everett's conjecture and that gives us a new necessary condition for Ghosh's version of the visibility graph recognition problem. We conclude with a look at an example, due to Everett [2], of a graph (shown in Fig. 15) which fails to be a visibility graph with $v_1, \dots, v_{10}$ as the chosen hamiltonian cycle in spite of satisfying all of Ghosh's necessary conditions. We show that this graph does not admit a consistent blocking vertex to invisible pair assignment. It would then follow from Theorem 1 that this graph is indeed not a visibility graph. We thus have an alternative proof of the fact that Ghosh's necessary conditions are insufficient, in place of the rather *ad hoc* proof by Everett.

Theorem 3. *The graph in Fig. 15 does not admit a consistent blocking vertex to invisible pair assignment with respect to the hamiltonian cycle $v_1, v_2, \dots, v_{10}$.*

Proof. Clearly, v_4, v_8 , and v_{10} are the only three vertices which can be blocking vertices. Note that for any other vertex v_i , v_{i-1} , and v_{i+1} are adjacent, where the index arithmetic is modulo n . The pair (v_1, v_4) and (v_4, v_7) are minimal with respect to v_{10} and v_8 , respectively. Now consider (v_4, v_9) . Only v_{10} and v_8 are blocking vertices of this pair. If v_{10} is assigned to it, then clearly (v_1, v_4) and (v_4, v_9) are separable with respect to v_{10} and the assignment is inconsistent. If, however, v_8 is assigned to (v_4, v_9) , then there is an inconsistency with (v_4, v_7) which is also assigned v_8 . Thus clearly a consistent assignment does not exist. $\square$

Sufficiency or otherwise of the result established here, together with the already

Fig. 15. The graph G with the hamiltonian cycle $v_1, v_2, \dots, v_{10}, v_1$ does not admit a consistent blocking vertex to invisible pair assignment.

known set of conditions for a graph with a given hamiltonian cycle on it, to be the visibility graph of a simple polygon, is open. Finally, it also remains to be seen if the consistency condition can be checked for a given graph and a hamiltonian cycle in polynomial time.

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