Free Arrangements and Relation Spaces*

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Abstract. Yuzvinsky [7] has shown that free arrangements are formal. In this note we define a more general class of arrangements which we call k -formal, and we show that free arrangements are k -formal. We close with an example which distinguishes k -formal arrangements from formal arrangements.

1. Introduction

Let $\mathbb{K}$ be a field and let V be an l -dimensional vector space over $\mathbb{K}$. A *hyperplane* in V is a codimension 1 subspace of V . An *arrangement* $\mathcal{A}$ in V is a finite set of hyperplanes.

Let $\{x_1, \dots, x_l\}$ be a basis for the dual V^* and let S be the symmetric algebra of V^* which is isomorphic to the polynomial algebra $\mathbb{K}[x_1, \dots, x_l]$. Then each hyperplane H in V has a defining form

$$\alpha_H = a_1x_1 + \cdots + a_lx_l$$

with $\ker(\alpha_H) = H$, unique up to a constant multiple. Thus an arrangement $\mathcal{A}$ can

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be described by the product of such forms. Given an arrangement $\mathcal{A}$, let

$$Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H.$$

Note that Q is unique up to constant multiple. The arrangement $\mathcal{A}$ is completely determined by Q .

Let $D(\mathcal{A})$ be the S -module which consists of all derivations $\theta: S \rightarrow S$ such that $\theta(Q)$ is a multiple of Q . We call $D(\mathcal{A})$ the module of $\mathcal{A}$ -derivations. When $D(\mathcal{A})$ is a free S -module, $\mathcal{A}$ is said to be *free*. There are some interesting connections between the lattice $L(\mathcal{A})$ of intersections of elements of $\mathcal{A}$ and the S -module $D(\mathcal{A})$. For a survey, see the book by Orlik and Terao [3]. In this work we study these connections further. In particular, we are interested in the study of formal arrangements, introduced by Falk and Randell [1]. An arrangement $\mathcal{A}$ is formal if all linear dependencies among the defining forms of the hyperplanes of $\mathcal{A}$ are generated by dependencies corresponding to the codimension 2 subspaces in $L(\mathcal{A})$. Using techniques developed in [6], Yuzvinsky has shown in [7] that free arrangements are formal. In this paper we introduce a more general class of arrangements called k -formal arrangements. By definition, $\mathcal{A}$ is k -formal if $\mathcal{A}$ is formal. The main result of this paper is that if $\mathcal{A}$ is free, then $\mathcal{A}$ is k -formal for $2 \leq k \leq r - 1$, where r is the codimension of the common intersection of all the hyperplanes of $\mathcal{A}$. Our proof does not use techniques from [6]. This theorem imposes strong conditions on the ways in which the hyperplanes of a free arrangement intersect each other. In particular, we have some combinatorial inequalities for free arrangements. We close the paper with an example of an arrangement which is 2-formal (formal) but not 3-formal. This distinguishes k -formal arrangements from formal arrangements.

2. Preliminaries

We begin with some definitions and a few basic facts which are used throughout this paper. For further background, see [3]. Fix an arrangement $\mathcal{A}$. Let $n = |\mathcal{A}|$. Let S and $Q(\mathcal{A})$ be as described in the introduction. We call $Q(\mathcal{A})$ the *defining polynomial* of $\mathcal{A}$.

Definition 2.1. Let $L = L(\mathcal{A})$ be the set of intersections of elements of $\mathcal{A}$. Define a *partial order* on L by

$$X \leq Y \iff Y \subseteq X.$$

Note that this is *reverse* inclusion. Thus V is the unique minimal element.

Definition 2.2. Define a rank function on L by $r(X) = \text{codim } X$. Thus $r(V) = 0$

and $r(H) = 1$ for $H \in \mathcal{A}$. Call H an *atom* of L . Let $X, Y \in L$. Then their *meet* $X \wedge Y$ is equal to $\bigcap \{Z \in L \mid X \cup Y \subseteq Z\}$. Their *join* $X \vee Y$ is equal to $X \cap Y$.

With this rank function, $L(\mathcal{A})$ is a geometric lattice. It is called the *intersection lattice* of $\mathcal{A}$.

Definition 2.3. Let $T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$ be the *unique maximal element* of $L(\mathcal{A})$. Define the *rank* of $\mathcal{A}$ by $r = r(\mathcal{A}) = r(T(\mathcal{A}))$. Call the l -arrangement $\mathcal{A}$ *essential* if $r(\mathcal{A}) = l$.

Definition 2.4. For each $X \in L(\mathcal{A})$, define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \leq X\}.$$

Definition 2.5. Let $\text{Der}_{\mathbb{K}}(S)$ be the graded S -module of $\mathbb{K}$ -linear maps $\theta: S \rightarrow S$ such that

$$\theta(fg) = f\theta(g) + g\theta(f), \quad f, g \in S.$$

An element of $\text{Der}_{\mathbb{K}}(S)$ is called a *derivation* of S over $\mathbb{K}$.

Definition 2.6. Define

$$D(\mathcal{A}) = \{\theta \in \text{Der}_{\mathbb{K}}(S) \mid \theta(Q) \in QS\}.$$

Note that $D(\mathcal{A})$ is a graded S -submodule of $\text{Der}_{\mathbb{K}}(S)$.

Proposition 2.7.

$$D(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D(\alpha_H).$$

Proof. See Proposition 4.8 of [3]. □

Corollary 2.8. If $\mathcal{B} \subseteq \mathcal{A}$, then $D(\mathcal{A}) \subseteq D(\mathcal{B})$.

Thus if $X, Y \in L(\mathcal{A})$, with $X \leq Y$, then $D(\mathcal{A}_Y) \subseteq D(\mathcal{A}_X)$.

Definition 2.9. An arrangement $\mathcal{A}$ is called a *free arrangement* if $D(\mathcal{A})$ is a free module over S .

Theorem 2.10. If $\mathcal{A}$ is free, then $\mathcal{A}_X$ is free for each $X \in L$.

Proof. See Theorem 4.37 of [3]. □

3. Formal Arrangements

Fix an arrangement $\mathcal{A}$. Choose a defining form α_H for each $H \in \mathcal{A}$. We wish to study the various relations among these forms. We consider the vector space $E(\mathcal{A})$ which has a basis consisting of the symbols $\{e_H | H \in \mathcal{A}\}$. We have a map

$$E(\mathcal{A}) = \bigoplus_{H \in \mathcal{A}} \mathbb{K}e_H \rightarrow V^*$$

given by $e_H \mapsto \alpha_H$. Now define $F(\mathcal{A})$ to be the kernel of this map. We call $F(\mathcal{A})$ the *relation space* and we refer to elements of $F(\mathcal{A})$ as *relations*. Then we have an exact sequence

$$0 \rightarrow F(\mathcal{A}) \rightarrow E(\mathcal{A}) \rightarrow T(\mathcal{A})^0 \rightarrow 0,$$

where $T(\mathcal{A})^0 \subseteq V^*$ denotes the annihilator of $T(\mathcal{A})$. Thus $\dim F(\mathcal{A}) = n - r(\mathcal{A})$.

Definition 3.1. An element $w = \sum c_H e_H$ of $E(\mathcal{A})$ has *length* p if precisely p of the $c_H \in \mathbb{K}$ are nonzero. We write $p = \text{length}(w)$.

Definition 3.2. Let $n = |\mathcal{A}|$. Let $2 \leq p \leq n - 1$. We define $F_p(\mathcal{A})$ to be the subspace of $F(\mathcal{A})$ spanned by all relations of length no greater than $p + 1$:

$$F_p(\mathcal{A}) = \langle \{w = \sum c_H e_H \in F(\mathcal{A}) | \text{length}(w) \leq p + 1\} \rangle.$$

Definition 3.3. We say $\mathcal{A}$ is *p-generated* if $F(\mathcal{A}) = F_p(\mathcal{A})$. Call $\mathcal{A}$ *formal* if $\mathcal{A}$ is 2-generated.

Remark 3.4. Let $r = r(\mathcal{A})$. For $r \leq p \leq n - 1$, $F_p(\mathcal{A}) = F(\mathcal{A})$. Thus all arrangements are r -generated.

Note that $\mathcal{A}$ formal is the strongest of the above conditions. Note that if $r \leq 2$, then it is always the case that $F(\mathcal{A}) = F_2(\mathcal{A})$. Thus, for the rest of the paper, we assume that $r \geq 3$.

Remark 3.5. For each $X \in L$, we have an inclusion $F_2(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$. If $\mathcal{A}$ is formal, then each relation in $F(\mathcal{A})$ can be written as a sum of relations in $F_2(\mathcal{A})$. Thus $\mathcal{A}$ is formal if and only if we have a surjection

$$\pi_2: \bigoplus_{\substack{X \in L \\ r(X) = 2}} F_2(\mathcal{A}_X) \rightarrow F(\mathcal{A}) \rightarrow 0.$$

Note that if $r(X) = 2$, then $F_2(\mathcal{A}_X) = F(\mathcal{A}_X)$.

Example 3.6. Consider the real 3-arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ defined by

$$Q_1 = x_1 x_2 x_3 (x_1 - x_2)(x_2 - x_3)(x_1 - 2x_3)$$

and

$$Q_2 = x_1 x_2 x_3 (x_1 + x_2 + x_3),$$

respectively. Then $\mathcal{A}_1$ is formal, whereas $\mathcal{A}_2$ is not.

Remark 3.7. Although each linear dependency among the defining forms of the hyperplanes of $\mathcal{A}$ is associated to a particular subspace in L , it is not true that the property of being formal depends only on L (see Example 2.2 of [7]). Note, however, that $\mathcal{A}$ formal does impose some conditions on L . By Remark 3.5, we have

$$\sum_{\substack{X \in L \\ r(X) = 2}} (|\mathcal{A}_X| - 2) \geq n - r(\mathcal{A}).$$

In [1] Falk and Randell asked if free arrangements are formal. This question was resolved by Yuzvinsky:

Theorem 3.8 [7, Corollary 2.5]. *If $\mathcal{A}$ is free, then $\mathcal{A}$ is formal.*

We reprove this by proving our main theorem in the next section.

Using this theorem, we can give some easy lattice conditions which are necessary for an arrangement to be free. In addition to the inequality in Remark 3.7, we have the following proposition. The proof is straightforward and is left to the reader. Call $H \in \mathcal{A}$ a *separator* if $r(\mathcal{A} - \{H\}) < r(\mathcal{A})$.

Proposition 3.9. *Let $\mathcal{A}$ be an arrangement. Fix $H \in \mathcal{A}$ and let $\mathcal{A}' = \mathcal{A} - \{H\}$.*

- (i) *If H is a separator, then $\mathcal{A}$ is formal if and only if $\mathcal{A}'$ is formal.*
- (ii) *Suppose H is not a separator. If $\mathcal{A}$ is formal, then H contains a rank 2 element of $L(\mathcal{A}')$.*

Example 3.10. The real 4-arrangement $\mathcal{A}$ defined by

$$Q = x_1 x_2 x_3 x_4 (x_1 - x_2)(x_1 + x_3 - x_4)$$

is not free.

Proof. The plane $H = \ker(x_1 + x_3 - x_4)$ is not a separator, and does not contain any rank 2 subspace of $L(\mathcal{A} - \{H\})$. Thus by the previous proposition and Theorem 3.8, $\mathcal{A}$ is not free. Note, however, that H does contain the rank 3 subspace $\ker(x_1) \cap \ker(x_3) \cap \ker(x_4)$. $\square$

4. A Generalization of Formal Arrangements

We now define the class of k -formal arrangements and give a generalization of Theorem 3.8.

Suppose $\mathcal{A}$ is formal. Then we have an exact sequence

$$0 \rightarrow R_3(\mathcal{A}) \rightarrow \bigoplus_{\substack{X \in L \\ r(X)=2}} F_2(\mathcal{A}_X) \xrightarrow{\pi_2} F(\mathcal{A}) \rightarrow 0,$$

where $R_3(\mathcal{A})$ is the kernel of the map π_2 . Thus $R_3(\mathcal{A})$ is a space of relations *among* the relations corresponding to the rank 2 elements of L . Note that, for any $X \in L$, we have an inclusion $R_3(\mathcal{A}_X) \subset R_3(\mathcal{A})$. If the map

$$\bigoplus_{\substack{X \in L \\ r(X)=3}} R_3(\mathcal{A}_X) \xrightarrow{\pi_3} R_3(\mathcal{A})$$

is surjective, then we call $\mathcal{A}$ 3-formal. We can continue in this fashion to define k -formal for $k > 3$. We make this idea precise in the following definitions.

Definition 4.1. Let $R_0(\mathcal{A}) = T(\mathcal{A})^*$.

Remark 4.2. Note that, for each $X \in L$, there is a restriction map

$$R_0(\mathcal{A}_X) \rightarrow R_0(\mathcal{A}).$$

Definition 4.3. For $1 \leq k \leq r$, define $R_k(\mathcal{A})$ recursively to be the kernel of the map

$$\pi_{k-1} = \pi_{k-1}(\mathcal{A}): \bigoplus_{\substack{X \in L \\ r(X)=k-1}} R_{k-1}(\mathcal{A}_X) \rightarrow R_{k-1}(\mathcal{A}),$$

where the maps π_k are defined as follows. For each $k \geq 1$ and for each $Y \in L$ with $r(Y) \geq k$, we have a map $i_k(Y): R_k(\mathcal{A}_Y) \rightarrow R_k(\mathcal{A})$, which can be viewed as inclusion. The map π_k is a sum of these maps. Also, we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_{k+1}(\mathcal{A}_Y) & \longrightarrow & \bigoplus_{\substack{X \leq Y \\ r(X)=k}} R_k(\mathcal{A}_X) & \xrightarrow{\pi_k(\mathcal{A}_Y)} & R_k(\mathcal{A}_Y) \\ & & \downarrow i_{k+1}(Y) & & \downarrow j_k(Y) & & \downarrow i_k(Y) \\ 0 & \longrightarrow & R_{k+1}(\mathcal{A}) & \longrightarrow & \bigoplus_{\substack{X \in L \\ r(X)=k}} R_k(\mathcal{A}_X) & \xrightarrow{\pi_k(\mathcal{A})} & R_k(\mathcal{A}). \end{array}$$

Since $X \leq Y$, we have $(\mathcal{A}_Y)_X = \mathcal{A}_X$. The middle vertical map $j_k(Y)$ is a direct sum of identity maps and zero maps, so it can be viewed as inclusion.

Note that if $r(X) = 0$, then we have $X = V$. Thus $R_1(\mathcal{A})$ is the kernel of the restriction map $V^* \rightarrow T(\mathcal{A})^*$, so $R_1(\mathcal{A}) = T(\mathcal{A})^0$. Note also that we have identifications $R_1(\mathcal{A}_H) = T(\mathcal{A}_H)^0 \simeq H^0 \simeq \mathbb{K}e_H$. Hence $R_2(\mathcal{A})$ can be identified with the kernel of the map

$$\bigoplus_{H \in \mathcal{A}} \mathbb{K}e_H \rightarrow T(\mathcal{A})^0,$$

so we see that $R_2(\mathcal{A}) \simeq F(\mathcal{A})$.

Definition 4.4. For $2 \leq k \leq r - 1$, define the class of k -formal arrangements as follows:

- (i) An arrangement $\mathcal{A}$ is *2-formal* if $\mathcal{A}$ is formal.
- (ii) For $k \geq 3$, $\mathcal{A}$ is *k -formal* if $\mathcal{A}$ is $(k - 1)$ -formal and the map

$$\pi_k: \bigoplus_{\substack{X \in L \\ r(X) = k}} R_k(\mathcal{A}_X) \rightarrow R_k(\mathcal{A})$$

is a surjection.

Our main theorem states that if $\mathcal{A}$ is free, then $\mathcal{A}$ is k -formal for $2 \leq k \leq r - 1$. To prove this theorem, we first relate the vector spaces $R_k(\mathcal{A})$ to the S -module of derivations $D(\mathcal{A})$. We define S -modules $D_k(\mathcal{A})$ for $0 \leq k \leq r$. Our construction is analogous to that of the vector spaces $R_k(\mathcal{A})$.

Definition 4.5. Let $D_0(\mathcal{A}) = D(\mathcal{A})$.

Recall that for each $X \in L$, we have an inclusion map $D_0(\mathcal{A}) \rightarrow D_0(\mathcal{A}_X)$. Denote this map by $\varphi_0(X)$.

Definition 4.6. For $1 \leq k \leq r$, define the S -modules $K_k(\mathcal{A})$ and $D_k(\mathcal{A})$ recursively to be the kernel and cokernel respectively of the map

$$\tau_{k-1} = \tau_{k-1}(\mathcal{A}): D_{k-1}(\mathcal{A}) \rightarrow \bigoplus_{\substack{X \in L \\ r(X) = k-1}} D_{k-1}(\mathcal{A}_X),$$

where the map τ_k is a sum of maps $\varphi_k(Y): D_k(\mathcal{A}) \rightarrow D_k(\mathcal{A}_Y)$. For $Y \in L$ with

$r(Y) \geq k - 1$, these maps $\varphi_k(Y)$ are defined by the following diagram:

$$\begin{array}{ccccccc}
 D_{k-1}(\mathcal{A}) & \xrightarrow{\tau_{k-1}(\mathcal{A})} & \bigoplus_{\substack{X \in L \\ r(X) = k-1}} D_{k-1}(\mathcal{A}_X) & \longrightarrow & D_k(\mathcal{A}) & \longrightarrow & 0 \\
 \downarrow \varphi_{k-1}(Y) & & \downarrow p_{k-1}(Y) & & \downarrow \varphi_k(Y) & & \\
 D_{k-1}(\mathcal{A}_Y) & \xrightarrow{\tau_{k-1}(\mathcal{A}_Y)} & \bigoplus_{\substack{X \leq Y \\ r(X) = k-1}} D_{k-1}((\mathcal{A}_Y)_X) & \longrightarrow & D_k(\mathcal{A}_Y) & \longrightarrow & 0.
 \end{array}$$

The middle map $p_{k-1}(Y)$ is simply a projection (compare with Definition 4.3). It is not difficult to see that the left part of this diagram commutes, thus we may define $\varphi_k(Y): D_k(\mathcal{A}) \rightarrow D_k(\mathcal{A}_Y)$ so that the right part of the diagram commutes.

For $k \geq r + 1$ define $D_k(\mathcal{A}) = 0$.

Remark 4.7. Note that $K_1(\mathcal{A}) = 0$ for any arrangement, since it is the kernel of the inclusion map $\tau_0: D_0(\mathcal{A}) \rightarrow \text{Der}_{\mathbb{K}}(S)$. Also $K_2(\mathcal{A}) = 0$, for if $[\theta]$ is in the kernel of the map

$$\tau_1: D_1(\mathcal{A}) \rightarrow \bigoplus_{H \in \mathcal{A}} D_1(\mathcal{A}_H),$$

then $\theta \in D_0(\mathcal{A}_H)$ for each $H \in \mathcal{A}$.

Remark 4.8. If $\mathcal{A}$ is the empty arrangement, then $D_1(\mathcal{A}) = 0$, and hence $D_k(\mathcal{A}) = 0$ for $1 \leq k \leq r$.

Remark 4.9. Note that $D_1(\mathcal{A}) = \text{Der}_{\mathbb{K}}(S)/D(\mathcal{A})$. We have an isomorphism $g: T(\mathcal{A}) \rightarrow D_0(\mathcal{A})_0$ given by

$$g(v) = \sum_{i=1}^l x_i(v) D_i$$

for each $v \in T(\mathcal{A})$. Thus $\text{Der}_{\mathbb{K}}(S)_0 \simeq V$ and $D_1(\mathcal{A})_0 \simeq V/T(\mathcal{A})$.

Proposition 4.10. For $0 \leq k \leq r$, we have $D_k(\mathcal{A})_0^* \simeq R_k(\mathcal{A})$.

Proof. Recall that $R_0(\mathcal{A}) = T(\mathcal{A})^*$. By the remark above, we have an isomorphism $f_0(\mathcal{A}) = g^*: D_0(\mathcal{A})_0^* \rightarrow R_0(\mathcal{A})$ for any arrangement $\mathcal{A}$. For $k \geq 0$, consider the following diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D_{k+1}(\mathcal{A})_0^* & \longrightarrow & \bigoplus_{\substack{X \in L \\ r(X) = k}} D_k(\mathcal{A}_X)_0^* & \xrightarrow{(\tau_k)_0^*} & D_k(\mathcal{A})_0^* \\
 & & & & \downarrow \oplus f_k(\mathcal{A}_X) & & \downarrow f_k(\mathcal{A}) \\
 0 & \longrightarrow & R_{k+1}(\mathcal{A}) & \longrightarrow & \bigoplus_{\substack{X \in L \\ r(X) = k}} R_k(\mathcal{A}_X) & \xrightarrow{\pi_k} & R_k(\mathcal{A}).
 \end{array}$$

First note that the rows are exact. It is straightforward to show that this diagram commutes for $k = 0$, and hence we can construct an isomorphism $f_1(\mathcal{A}): D_1(\mathcal{A})_0^* \rightarrow R_1(\mathcal{A})$, for any arrangement $\mathcal{A}$. We continue via induction, noting that it is sufficient to show that the following diagram commutes for each $X \in L$ with $r(X) = k$:

$$\begin{array}{ccc} D_k(\mathcal{A}_X)_0^* & \xrightarrow{\varphi_k(X)_0^*} & D_k(\mathcal{A})_0^* \\ \downarrow f_k(\mathcal{A}_X) & & \downarrow f_k(\mathcal{A}) \\ R_k(\mathcal{A}_X) & \xrightarrow{i_k(X)} & R_k(\mathcal{A}). \end{array}$$

By chasing this diagram, we can show that, for each $k \leq r$, the diagram commutes and hence we have an isomorphism $f_k(\mathcal{A}): D_k(\mathcal{A})_0^* \rightarrow R_k(\mathcal{A})$. The idea is to combine the above diagram with those from Definitions 4.3 and 4.6. $\square$

We may view the correspondence $X \mapsto D(X) = D(\mathcal{A}_X)$ for $X \in L$ as a contravariant functor on L , because there is the inclusion map $D(X) \rightarrow D(Y)$ whenever $X \geq Y$. Note that $D(T(\mathcal{A})) = D(\mathcal{A})$ and $D(V) = \text{Der}_{\mathbb{K}}(S)$. It turns out that this functor is local (in the sense of Definition 6.4 of [4]):

Proposition 4.11. *Let $X \in L$ and $\wp \in \text{Spec}(S)$, and define*

$$X(\wp) = \bigcap_{\substack{H \in \mathcal{A}_X \\ \alpha_H \in \wp}} H.$$

Then the localization at $\wp$ of the inclusion map $D(X) \hookrightarrow D(X(\wp))$ is an isomorphism.

Proof. This is contained in Example 4.123 of [3]. $\square$

Lemma 4.12. *For $0 \leq k \leq r$, the functors $D_k(X) = D_k(\mathcal{A}_X)$ for $X \in L$ are local.*

Proof. This follows using induction and the fact that localization preserves exact sequences. $\square$

The following result is essentially Lemma (5.15) of [5], but since our setup is a bit different, we give a detailed proof here.

Proposition 4.13. *If $\mathcal{A}$ is free, then, for $1 \leq k \leq r$, we have*

- (i) $\text{pd}_S D_{k-1}(\mathcal{A}) = k - 1$,
- (ii) $\text{ht } \wp = k - 1$ for each $\wp \in \text{Ass}_S D_{k-1}(\mathcal{A})$,
- (iii) $K_k(\mathcal{A}) = 0$.

Before the proof, let us recall a fundamental fact from commutative algebra. For details (in the local ring case) see Proposition 3.9 in Chapter VI of [2].

Theorem 4.14. *For any finitely generated graded S -module M ,*

$$\begin{aligned} \text{depth}_S M &\leq \min_{\wp \in \text{Ass}_S M} \{l - \text{ht } \wp\} \leq \max_{\wp \in \text{Ass}_S M} \{l - \text{ht } \wp\} = \max_{\wp \in \text{Supp}_S M} \{l - \text{ht } \wp\} \\ &= \dim_S M. \end{aligned}$$

Here, $\text{depth}_S M$ is defined with respect to the maximal homogeneous ideal of S .

Proof of 4.13. We use induction. When $k = 1$ we know $\text{pd}_S D_0(\mathcal{A}) = 0$ since $\mathcal{A}$ is free, so (i) is clear. Also, since $D_0(\mathcal{A})$ is free, the annihilator of each $\theta \in D_0(\mathcal{A})$ is the zero ideal. Thus $\text{Ass}_S D_0(\mathcal{A})$ consists of the zero ideal, and now (ii) follows. In Remark 4.7 we saw that $K_1(\mathcal{A}) = 0$ so (iii) holds.

Suppose the result holds for some k ($1 \leq k \leq r - 1$). Then, since $K_k(\mathcal{A}) = 0$, we have an exact sequence

$$0 \rightarrow D_{k-1}(\mathcal{A}) \rightarrow \bigoplus_{\substack{X \in L \\ r(X) = k-1}} D_{k-1}(\mathcal{A}_X) \rightarrow D_k(\mathcal{A}) \rightarrow 0.$$

Thus $\text{pd}_S D_k(\mathcal{A}) \leq k$. Now let $\wp \in \text{Spec}(S)$ with $\text{ht } \wp \leq k - 1$. We write $\mathcal{A}_\wp = \mathcal{A}_{T(\mathcal{A})_{(\wp)}}$. Then we have the exact sequence

$$0 \rightarrow D_{k-1}(\mathcal{A}_\wp) \rightarrow \bigoplus_{\substack{X \in L \\ r(X) = k-1}} D_{k-1}(\mathcal{A}_{X(\wp)})_\wp \rightarrow D_k(\mathcal{A})_\wp \rightarrow 0.$$

Note that there is at most one $X \in L$ of rank $k - 1$ (in the case when $\text{ht } \wp = k - 1$) with $D_{k-1}(\mathcal{A}_{X(\wp)}) \neq 0$. Thus either

$$D_{k-1}(\mathcal{A}_\wp) \rightarrow \bigoplus_{\substack{X \in L \\ r(X) = k-1}} D_{k-1}(\mathcal{A}_{X(\wp)})_\wp$$

is an isomorphism or $D_{k-1}(\mathcal{A}_{X(\wp)}) = 0$ for all $X \in L$ with $r(X) = k - 1$. In either case, $D_k(\mathcal{A})_\wp = 0$. This shows that, for any $\wp \in \text{Supp}_S D_k(\mathcal{A})$, we have $\text{ht } \wp \geq k$, and hence, by Theorem 4.14, we know $\text{depth}_S D_k(\mathcal{A}) \leq \dim_S D_k(\mathcal{A}) \leq l - k$.

Now by the Auslander–Buchsbaum formula (for graded modules), we have

$$l = \text{pd}_S D_k(\mathcal{A}) + \text{depth}_S D_k(\mathcal{A}) \leq \text{pd}_S D_k(\mathcal{A}) + l - k.$$

We have seen that $\text{pd}_S D_k(\mathcal{A}) \leq k$, so it follows that $\text{pd}_S D_k(\mathcal{A}) = k$. This shows (i). Also,

$$l - k = \text{depth}_S D_k(\mathcal{A}) \leq \dim_S D_k(\mathcal{A}) \leq l - k.$$

Thus $D_k(\mathcal{A})$ is Cohen–Macaulay, and, by Theorem 4.14, each $\wp \in \text{Ass}_S D_k(\mathcal{A})$ has height k . This shows (ii).

To show (iii), consider the sequence

$$0 \rightarrow K_{k+1}(\mathcal{A}) \rightarrow D_k(\mathcal{A}) \rightarrow \bigoplus_{\substack{X \in L \\ r(X)=k}} D_k(\mathcal{A}_X).$$

Suppose there is some $\wp \in \text{Ass}_S K_{k+1}(\mathcal{A})$. Then, since $\text{Ass}_S K_{k+1}(\mathcal{A}) \subseteq \text{Ass}_S D_k(\mathcal{A})$, we know that $\text{ht } \wp = k$ by (ii). We localize to obtain the exact sequence

$$0 \rightarrow K_{k+1}(\mathcal{A})_{\wp} \rightarrow D_k(\mathcal{A})_{\wp} \rightarrow \bigoplus_{\substack{X \in L \\ r(X)=k}} D_k(\mathcal{A}_{X(\wp)})_{\wp}.$$

Since $\text{ht } \wp = k$, there is at most one $X \in L$ with $r(X) = k$ such that $D_k(\mathcal{A}_{X(\wp)}) \neq 0$, so either the map

$$D_k(\mathcal{A})_{\wp} \rightarrow \bigoplus_{\substack{X \in L \\ r(X)=k}} D_k(\mathcal{A}_{X(\wp)})_{\wp}$$

is an isomorphism or $D_k(\mathcal{A}_{\wp}) = 0$. This shows that $K_{k+1}(\mathcal{A})_{\wp} = 0$. This is impossible, however, since $\text{Ass}_S K_{k+1}(\mathcal{A}) \subseteq \text{Supp}_S K_{k+1}(\mathcal{A})$ [2, Proposition 2.9 in Chapter VI]. Thus it must be the case that $\text{Ass}_S K_{k+1}(\mathcal{A}) = \emptyset$. This implies that $K_{k+1}(\mathcal{A}) = 0$ [2, Proposition 2.4 in Chapter VI]. Hence (iii) is established and the proof is complete. $\square$

Now we have our main theorem:

Theorem 4.15. *If $\mathcal{A}$ is free, then $\mathcal{A}$ is k -formal for $2 \leq k \leq r - 1$.*

Proof. Suppose $2 \leq k \leq r - 1$. Then, by part (iii) of the previous proposition, we have an exact sequence of vector spaces

$$0 \rightarrow D_k(\mathcal{A})_0 \rightarrow \bigoplus_{\substack{X \in L \\ r(X)=k}} D_k(\mathcal{A}_X)_0 \rightarrow D_{k+1}(\mathcal{A})_0 \rightarrow 0.$$

Now, by Proposition 4.10, we know that $D_k(\mathcal{A})_0^* \simeq R_k(\mathcal{A})$ for $0 \leq k \leq r$. Once again we have a commutative diagram with the same maps as in the proof of Proposition 4.10. Recall that the maps in the second and third columns are isomorphisms.

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_{k+1}(\mathcal{A})_0^* & \longrightarrow & \bigoplus_{\substack{X \in L \\ r(X)=k}} D_k(\mathcal{A}_X)_0^* & \longrightarrow & D_k(\mathcal{A})_0^* \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R_{k+1}(\mathcal{A}) & \longrightarrow & \bigoplus_{\substack{X \in L \\ r(X)=k}} R_k(\mathcal{A}_X) & \xrightarrow{\pi_k} & R_k(\mathcal{A}). \end{array}$$

Now we see that the map π_k is onto. Hence $\mathcal{A}$ is k -formal. $\square$

This theorem gives additional conditions on L for free arrangements. Suppose $\mathcal{A}$ is free. Then, by Theorem 2.10, $\mathcal{A}_Y$ is free and hence k -formal for each $Y \in L$, $0 \leq k \leq r(Y) - 1$. Consider $Y \in L$ with $r(Y) \geq 2$. By Remark 3.7, we have

$$\sum_{\substack{X \leq Y \\ r(X)=2}} (|\mathcal{A}_X| - 2) \geq |\mathcal{A}_Y| - r(Y).$$

From the exact sequence

$$0 \rightarrow R_3(\mathcal{A}_Y) \rightarrow \bigoplus_{\substack{X \leq Y \\ r(X)=2}} F_2(\mathcal{A}_X) \xrightarrow{\pi_2} F(\mathcal{A}_Y) \rightarrow 0,$$

we see that

$$\dim R_3(\mathcal{A}_Y) = r(Y) - |\mathcal{A}_Y| + \sum_{\substack{X \leq Y \\ r(X)=2}} (|\mathcal{A}_X| - 2).$$

Thus, for example, since $\mathcal{A}$ is 3-formal, the surjectivity of

$$\pi_3: \bigoplus_{\substack{Y \in L \\ r(Y)=3}} R_3(\mathcal{A}_Y) \rightarrow R_3(\mathcal{A})$$

yields the following:

Corollary 4.16. *If $\mathcal{A}$ is free and $r(\mathcal{A}) \geq 4$, then*

$$\sum_{\substack{Y \in L \\ r(Y)=3}} \left(3 - |\mathcal{A}_Y| + \sum_{\substack{X \leq Y \\ r(X)=2}} (|\mathcal{A}_X| - 2) \right) \geq r(\mathcal{A}) - n + \sum_{\substack{X \in L \\ r(X)=2}} (|\mathcal{A}_X| - 2).$$

There are similar inequalities corresponding to k -formal for $k \geq 4$.

5. An Example

We now distinguish k -formal arrangements from formal arrangements.

Example 5.1. The real 4-arrangement $\mathcal{A}$ described below is formal but not 3-formal.

Let $Q(\mathcal{A}) = \prod_{j=1}^{10} \alpha_j$, where

$$\begin{aligned} \alpha_1 &= x_3, & \alpha_6 &= x_1 + x_3 - 2x_4, \\ \alpha_2 &= x_3 - x_4, & \alpha_7 &= x_2 + 2x_3 - 2x_4, \\ \alpha_3 &= x_2, & \alpha_8 &= x_1 + 2x_3 - 2x_4, \\ \alpha_4 &= x_2 + x_3 - 2x_4, & \alpha_9 &= x_1 + x_2 + x_3 - 2x_4, \\ \alpha_5 &= x_1, & \alpha_{10} &= x_4. \end{aligned}$$

We computed $L = L(\mathcal{A})$ by hand and later verified our calculations using a computer program written by John Keaty. Note that relations among the defining forms come from subspaces $X \in L$ such that $r(X) < |\mathcal{A}_X|$. Thus, in describing L , we do not include the so-called “generic” subspaces $X \in L$ where $r(X) = |\mathcal{A}_X|$. We use the following notation to describe the remaining elements of L . The intersection $H_i \cap H_j$ is denoted by the corresponding indices ij . We do the same for intersections of three or more hyperplanes. Each lattice element is denoted by its maximal index set. For example $14 = 17 = 147$, so 147 is listed below.

Rank 3:

1 2 9 10, 3 6 9 10, 4 5 9 10
1 3 6 8 9, 1 4 5 7 9, 1 4 6 7 8, 2 3 5 7 8, 2 3 6 7 9, 2 4 5 8 9, 3 4 5 6 9
1 2 3 4 7 10, 1 2 5 6 8 10

Rank 2:

1 2 10, 1 4 7, 1 6 8, 2 3 7, 2 5 8, 3 6 9, 4 5 9

We first show that $\mathcal{A}$ is formal. We count seven nongeneric rank 2 elements, each an intersection of three hyperplanes. Thus we have seven relations of length 3. They are:

$$\begin{aligned} \alpha_1 - \alpha_2 - \alpha_{10} &= 0, & 2\alpha_2 + \alpha_5 - \alpha_8 &= 0, \\ \alpha_1 + \alpha_4 - \alpha_7 &= 0, & \alpha_3 + \alpha_6 - \alpha_9 &= 0, \\ \alpha_1 + \alpha_6 - \alpha_8 &= 0, & \alpha_4 + \alpha_5 - \alpha_9 &= 0, \\ 2\alpha_2 + \alpha_3 - \alpha_7 &= 0, \end{aligned}$$

This leads to the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

This matrix has rank 6, and thus $\dim F_2(\mathcal{A}) = 6$. Recall that we have an exact sequence

$$0 \rightarrow F(\mathcal{A}) \rightarrow \bigoplus_{j=1}^{10} \mathbb{K}e_j \rightarrow T(\mathcal{A})^0 \rightarrow 0.$$

Since $\mathcal{A}$ is essential, $\dim T(\mathcal{A})^0 = 4$, and thus $\dim F(\mathcal{A}) = 6$. Hence $F_2(\mathcal{A}) = F(\mathcal{A})$ and $\mathcal{A}$ is formal.

Now show that $\mathcal{A}$ is not 3-formal. Since $\mathcal{A}$ is formal, we have an exact sequence

$$0 \rightarrow R_3(\mathcal{A}) \rightarrow \bigoplus_{\substack{X \in L \\ r(X)=2}} F_2(\mathcal{A}_X) \rightarrow F(\mathcal{A}) \rightarrow 0.$$

Note that each (nongeneric) rank 2 element $X \in L$ is an intersection of three hyperplanes, so each $F_2(\mathcal{A}_X)$ will have dimension 1. Thus the middle term in the above sequence has dimension 7, and $\dim R_3(\mathcal{A}) = 1$. In particular, $R_3(\mathcal{A}) \neq 0$. To show that the map

$$\bigoplus_{\substack{Y \in L \\ r(Y)=3}} R_3(\mathcal{A}_Y) \rightarrow R_3(\mathcal{A})$$

is not surjective, we show that, for each $Y \in L$ with $r(Y) = 3$, we have $R_3(\mathcal{A}_Y) = 0$. This amounts to showing that, for each such Y , the map

$$\bigoplus_{\substack{X \in L(\mathcal{A}_Y) \\ r(X)=2}} F_2((\mathcal{A}_Y)_X) \rightarrow F(\mathcal{A}_Y)$$

is an injection.

Each $T(\mathcal{A}_Y)^0$ has dimension 3, so $\dim F(\mathcal{A}_Y) = |\mathcal{A}_Y| - 3$. The chart below lists the nongeneric rank 3 elements $Y \in L$, along with the nongeneric rank 2 elements of $L(\mathcal{A}_Y)$:

Rank 3 element	Contained by rank 2 element	Difference
4 5 9 10	4 5 9	3
3 6 9 10	3 6 9	3
1 2 9 10	1 2 10	3
1 3 6 8 9	1 6 8, 3 6 9	3
1 4 6 7 8	1 4 7, 1 6 8	3
1 4 5 7 9	1 4 7, 4 5 9	3
2 3 5 7 8	2 3 7, 2 5 8	3
2 3 6 7 9	2 3 7, 3 6 9	3
2 4 5 8 9	2 5 8, 4 5 9	3
3 4 5 6 9	3 6 9, 4 5 9	3
1 2 3 4 7 10	1 4 7, 2 3 7, 1 2 10	3
1 2 5 6 8 10	1 6 8, 2 5 8, 1 2 10	3

For each Y in the chart, the relations corresponding to the rank 2 elements of $L(\mathcal{A}_Y)$ are linearly independent. Thus $R_3(\mathcal{A}_Y) = 0$ for each Y of rank 3 so $\mathcal{A}$ is not 3-formal. In particular, $\mathcal{A}$ is not free.

From this chart, we also see that the number of nongeneric rank 2 elements of $L(\mathcal{A}_Y)$ is always equal to $|\mathcal{A}_Y| - 3$. Thus each $\mathcal{A}_Y$ is formal, so that $\mathcal{A}$ is actually locally formal [7, Definition 2.3].

Note that we can show $\mathcal{A}$ is not free directly, by using Corollary 4.16 and the information in the chart above. Alternatively, $\mathcal{A}$ is not free by Yuzvinsky's inequality [6, Corollary 3.2] as well as by Terao's factorization theorem [3, Theorem 4.137].

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