Special Representations for n -Bridge Links*

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Abstract. We describe a simple algorithm to obtain a catalogue of 3-bridge links by computer, depending upon 6-tuples of positive integers. This permits us to represent the genus two 3-manifolds by standardly constructed graphs with colored edges. Finally, we prove some results about the topological structure of these manifolds and extend the combinatorial representation to n -bridge links.

1. Introduction

Throughout this paper we work in the piecewise-linear category in the sense of [24]. Manifolds are always assumed to be closed and orientable. For basic definitions and results about 3-manifold topology and knot theory we refer to [2], [9], [13], and [23].

A subset L of $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ is a *link* if L is homeomorphic to a disjoint union of 1-spheres. If L has one component, then it is called a *knot*. Two links $L = L_1 \cup L_2 \cup \cdots \cup L_n$ and $L' = L'_1 \cup L'_2 \cup \cdots \cup L'_n$ are said to be *equivalent* if there is a homeomorphism $f: \mathbb{S}^3 \rightarrow \mathbb{S}^3$ such that $f(L_i) = L'_i$ for $i = 1, 2, \dots, n$. The equivalence class of a link is called the *link type* of it. A link L is said to be *represented by an n -bridge diagram* if L can be represented as a union of n disjoint arcs $a_1, a_2, \dots, a_n$ in the plane together with n connecting arcs which can be thought of as bridges with their feet on the plane. The $2n$ feet of the bridges join the $2n$ endpoints of the arcs in pairs. The number of components is at most n . Figures 1 and 3 give examples with $n = 2$ and Figs. 4–6 give examples with $n = 3$.

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A link L has *bridge index* n if L has an n -bridge diagram but no m -bridge diagram for any $m < n$.

In 1956 Schubert succeeded in solving a major problem in mathematics when he found a special way to represent 2-bridge knots and links (see [25] and [26]). His representation begins with a partial normal form in which the links are characterized by a pair of integers, say (p, q) . More precisely, he showed that any 2-bridge link diagram can be reduced, using the Reidemeister moves, either to a diagram with no crossing, or else to a diagram $K(p, q)$ of the type illustrated in Fig. 1, called the *Schubert normal form*. To construct $K(p, q)$ join each “end” $(0, i)$ to $(1, i + q)$, $i \in \mathbb{Z}_{2p}$, without introducing any new crossing. We assume, of course, that $1 \leq q < p$. Schubert found necessary and sufficient conditions for $K(p, q)$ to have the same link type as $K(p', q')$. Thus, he “solved the knot problem” in this special case. Notice that 1-bridge links are necessarily unknotted circles, so in characterizing the cases which give nontrivial knots and links he also showed when the bridge index is precisely 2. Finally, it is easily seen that $K(p, q)$ is a diagram of a knot if and only if $(p, q) = 1$ and p is odd. In this case Schubert proved that the twofold cover of $\mathbb{S}^3$ branched over $K(p, q)$ is the lens space $\mathbb{L}(p, q)$ (compare, for example, with [1], [10], and [16]). Moreover, Hodgson has shown that 2-bridge knots are the only knots whose twofold branched covers are lens spaces (see [12]).

In this paper we give partial generalizations of Schubert’s work in the case $n = 3$

Fig. 1. (a) The two bridges B_0 and B_1 of $K(p, q)$. (b) The trefoil knot $K(3, 1)$. (c) The figure-eight knot $K(5, 2)$.

and obtain some results about the case of general n . More precisely, we show that each 3-bridge link can be represented by a special type of diagram $L(p, q, k, h, t, s)$ which is characterized by six integers. Our representation is not unique as $L(13, 12, 4, 11, 1, 10)$ and $L(11, 1, 0, 11, 0, 1)$ are equivalent to the torus knot $T(11, 3)$ (see (3) of Corollary 4 and the proof of (3) of Proposition 6). At present we have not found any general criterion for solving the knot problem in our case. Furthermore, we do not know whether $L(p, q, k, h, t, s)$ represents a knot or a link. Our result and the graphical construction given in [6] yield a combinatorial representation of 3-manifolds of Heegaard genus 2. Indeed, they are twofold cyclic coverings of $\mathbb{S}^3$ branched along 3-bridge links (see [1]). This allows us to generate a catalogue of all 3-bridge links (and hence their twofold cyclic covers) by computer. Finally, we study the topological structure of the named manifolds and extend some results to the case of n -bridge links.

2. Link Moves

Here we describe a simple algorithm to construct all 3-bridge links, starting from the Schubert normal form. More precisely, we prove that any 3-bridge link can be obtained from a 2-bridge link $K(p, q)$ by means of a cut-and-paste move depending upon four integers (k, h, t, s) . Thus any 3-bridge link admits a representation of type $L(p, q, k, h, t, s)$. For this, let B_0, B_1 be the bridges of $K(p, q)$, $1 \leq q < p$, and let e_i be the arc of $K(p, q)$ with ends $(0, i), (1, i + q)$ for any $i \in \mathbb{Z}_{2p}$.

The cut-and-paste move of type I on $K(p, q)$ is defined by the following rules:

- (1) Cut h consecutive arcs $e_k, e_{k+1}, \dots, e_{k+h-1}$ of $K(p, q)$ by using a new bridge, B_2 say, which is disjoint and parallel to B_0 and B_1 . Here we assume that $2 \leq h \leq 2p$ and $0 \leq k \leq p - 1$. The bridge B_2 has exactly $h - 1$ undercrossings numbered as shown in Fig. 2(a).
- (2) Join the “hanging” arc with vertex $(0, k + i)$ to the vertex $(2, s + i)$, $i \in \mathbb{Z}_h$, $0 \leq s \leq h$, without introducing any new crossing.
- (3) Join the “hanging” arc with vertex $(1, k + h + q - i - 1)$ to the vertex $(2, s + h + i)$, $i \in \mathbb{Z}_h$, without introducing any new crossing.

Obviously, we operate in such a way that the resulting diagram is planar (see, for instance, Fig. 3).

The cut-and-paste move of type II on $K(p, q)$ is defined by the following rules:

- (1) Cut h consecutive arcs $e_k, e_{k+1}, \dots, e_{k+h-1}$ of $K(p, q)$ by using a new bridge, B_2 say, which is disjoint and parallel to B_0 and B_1 . Here we assume that $2 \leq h \leq 2p$, $0 \leq k \leq p - 1$, and that the arcs e_{p-1}, e_p belong to the set $\{e_{k+i} : i \in \mathbb{Z}_h\}$. Now we require that the bridge B_2 has exactly $h - t - 1$ ($t > 0$) undercrossings numbered as shown in Fig. 2(b).
- (2) Join the vertex $(0, p + i)$ to $(0, p - i - 1)$ by a new arc for any $i \in \mathbb{Z}_t$. Then it follows that $0 < t \leq t^* = \min\{p - k, k + h - p\} \leq [h/2]$, where $[x]$ denotes the integer part of x .
- (3) If $t < t^*$, then join the vertex $(0, k + i)$ to $(2, s + i)$, $i \in \mathbb{Z}_{p-t-k}$, the vertex

Fig. 2.

$(0, p + t + i)$ to $(2, s + p - t - k + i)$, $i \in \mathbb{Z}_{k+h-p-t}$, and the vertex $(1, k + h + q - i - 1)$ to $(2, s + h - 2t + i)$, $i \in \mathbb{Z}_h$, without introducing any new crossing (see, for instance, Fig. 4).

- (4) If $t = t^* < h/2$, then join the vertex $(1, k + h + q - i - 1)$ to $(2, s + h - 2t + i)$, $i \in \mathbb{Z}_h$, and the vertex $(0, \xi + i)$ to $(2, s + i)$, $i \in \mathbb{Z}_{h-2t}$, where $\xi = k$ (resp. $\xi = k + 2t$) if $t = t^* = k + h - p$ (resp. $t = t^* = p - k$). As usual, we require that the resulting diagram is planar (see, for example, Fig. 5(a) for $t = t^* = p - k$ and Fig. 5(b) for $t = t^* = k + h - p$).
- (5) If $t = t^* = h/2$, h even, then the h remaining edges with a free end are incident to vertices of B_1 . We connect these arcs to vertices of B_2 consecutively, so that the vertex $(1, k + h + q - i - 1)$ is joined to $(2, s + i)$, $i \in \mathbb{Z}_h$, and the resulting diagram is planar (see, for example, Fig. 6).

Theorem 1. Any 3-bridge link in $\mathbb{S}^3$ can be represented by a 3-bridge diagram $L(p, q, k, h, t, s)$ obtained from the Schubert normal form of a 2-bridge link $K(p, q)$ by a cut-and-paste move. Moreover, the 6-tuple of integers satisfies the following conditions:

$$1 \leq q \leq p - 1, \quad 0 \leq k \leq p - 1, \quad 2 \leq h \leq 2p, \\ 0 \leq t \leq \min\{p - k, k + h - p\}, \quad 1 \leq s \leq h - t.$$

Fig. 3. (a) $K(7, 3)$. (b) Link obtained from $K(7, 3)$ by a cut-and-paste move I, $k = 3$, $h = 5$, and $s = 2$.

Proof. Let L be a diagram of a 3-bridge link and let B_i denote the bridges of L drawn in the plane $z = 0$, $i \in \mathbb{Z}_3$. Let us define the following data:

- (1) α_{ij} is the number of arcs in L which connect B_i to B_j for any $i, j \in \mathbb{Z}_3$.
- (2) $\beta_i - 1$ is the number of undercrossings of the bridge B_i .
- (3) The pair (i, γ_i) denotes the first vertex of B_i which is joined to B_{i+1} (indices mod 3).

The link L is completely determined (up to isotopy) by the 6-tuple of integers $(\alpha_{01}, \beta_0, \beta_1, \gamma_0, \gamma_1, \gamma_2)$. Indeed, by planarity we obtain the following relations:

$$\begin{aligned} \alpha_{02} &= 2\beta_0 - \alpha_{01}, \\ \alpha_{12} &= 2\beta_1 - \alpha_{01}, \\ 2\beta_2 &= \alpha_{02} + \alpha_{12}. \end{aligned} \tag{I}$$

Fig. 4. Link obtained from $K(7, 3)$ by a cut-and-paste move II, $k = 4$, $h = 9$, $t = 2$, $s = 1$, and $t < t^* = 3$.

Fig. 5. Link obtained from $K(7, 3)$ by a cut-and-paste move II: (a) $k = 5$, $h = 6$, $t = 2$, $s = 0$, $t = t^* = 2 < h/2$, and $t^* = p - k$; (b) $k = 2$, $h = 9$, $t = 4$, and $s = 1$, hence $t = t^* = 4 < h/2$ and $t^* = k + h - p$.

Fig. 6. Link obtained from $K(7, 3)$ by a cut-and-paste move II, $k = 4$, $h = 6$, $t = 3$, and $s = 2$, hence $t = t^* = 3$.

Further, we can always assume that the inequalities, listed below, hold:

$$\begin{aligned}\beta_0 &\leq \min\{\beta_1, \beta_2\}, \\ 0 &\leq \gamma_i \leq \beta_i - 1, \quad i = 0, 1, \\ \gamma_0 &\leq \gamma_2 < \beta_2 + \gamma_0 - 1.\end{aligned}\tag{II}$$

Suppose B_1 is the bridge undercrossed by the arc with end $(0, \beta_0)$. Now let $K(p, q)$ be the Schubert normal form of a 2-bridge link. Then the 3-bridge link $L(p, q, k, h, t, s)$ is equivalent to L if the following relations are verified:

$$\begin{aligned}\alpha_{01} &= h - 2t, & \gamma_0 &= k, \\ \beta_0 &= p - t, & \gamma_1 &= s - 1, \\ \beta_1 &= h - t, & \gamma_2 &= k + q - 1.\end{aligned}\tag{III}$$

Substituting (I) in the inverse relations of (III) yields

$$\begin{aligned}p &= \beta_2, & h &= \alpha_{12}, \\ q &= \gamma_2 - \gamma_0 + 1, & t &= \beta_1 - \alpha_{01}, \\ k &= \gamma_0, & s &= \gamma_1 + 1.\end{aligned}\tag{IV}$$

Now we prove that the inequalities

$$\alpha_{01} \leq \beta_1 \leq \alpha_{12}\tag{V}$$

hold. Indeed, (I) and (II) imply that $\alpha_{12} = 2\beta_2 - \alpha_{02} \geq 2\beta_0 - \alpha_{02} = \alpha_{01}$, hence $\alpha_{12} \geq \alpha_{01}$. Thus, it follows that $2\alpha_{12} \geq 2\beta_1 = \alpha_{12} + \alpha_{01} \geq 2\alpha_{01}$ as claimed (use (I) too).

We have to show that the 6-tuple (p, q, k, h, t, s) , given by (IV), satisfies the inequalities of the statement. Obviously, we have that $p = \beta_2 > 1$ by definition (2). Since $\gamma_0 \leq \gamma_2 < \beta_2 + \gamma_0 - 1$ by (II) and $p = \beta_2$ by (IV), the integer $q = \gamma_2 - \gamma_0 + 1$ verifies the inequalities $1 \leq q \leq p - 1$ as requested. Further, (II) also gives $k = \gamma_0 \leq \beta_0 - 1 \leq \beta_2 - 1 = p - 1$, hence $0 \leq k \leq p - 1$. Since $\alpha_{12} \geq \beta_1$ by (V), we have that $h = \alpha_{12} \geq \beta_1 \geq 2$ by definition (2). Relations (I) and the inequality $\alpha_{01} \leq 2\beta_0$ imply that

$$h = \alpha_{12} = 2\beta_1 - \alpha_{01} \leq 2\beta_1 + 2\beta_0 - 2\alpha_{01} = \alpha_{02} + \alpha_{12} = 2\beta_2 = 2p,$$

hence it follows that $2 \leq h \leq 2p$.

Now we prove that

$$0 \leq t \leq \min\{p - k, k + h - p\}.$$

Obviously, the inequality $t \geq 0$ holds as $t = \beta_1 - \alpha_{01}$ by (IV) and $\beta_1 \geq \alpha_{01}$ by (V).

Thus, we have to show that $t \leq p - k$ and $t \leq k + h - p$. For the first one, (I) implies that $t = \beta_1 - \alpha_{01} \leq \beta_0 + \beta_1 - \alpha_{01} - \gamma_0 = (\alpha_{02} + \alpha_{12})/2 - \gamma_0 = \beta_2 - \gamma_0 = p - k$ as $\beta_0 \geq \gamma_0$ and $\beta_2 = p$. Hence, we have $t \leq p - k$ as requested.

The assumption that the arc with end $(0, \beta_0)$ undercrosses B_1 yields the inequality

$$\gamma_0 + \alpha_{01} \geq \beta_0.$$

This and (I) imply that

$$\gamma_0 + \alpha_{12} - \beta_0 \geq \alpha_{12} - \alpha_{01} = 2\beta_1 - 2\alpha_{01}.$$

Then we have

$$t = \beta_1 - \alpha_{01} \leq \gamma_0 + \alpha_{12} - \beta_0 - \beta_1 + \alpha_{01} = k + h - p$$

as $k = \gamma_0$, $h = \alpha_{12}$ by (IV), and $p = \beta_2 = (\alpha_{02} + \alpha_{12})/2 = \beta_0 + \beta_1 - \alpha_{01}$ by (I). Thus we have proved that

$$t \leq \min\{p - k, k + h - p\}$$

as claimed.

Now we have to show that $1 \leq s \leq h - t$. Obviously, $s = \gamma_1 + 1 \geq 1$ as $\gamma_1 \geq 0$. Further, (II) gives $\gamma_1 \leq \beta_1 - 1$ and hence by (I) and (IV) we obtain $s = \gamma_1 + 1 \leq \beta_1 = (2\beta_1 - \alpha_{01}) - (\beta_1 - \alpha_{01}) = h - t$ as $2\beta_1 - \alpha_{01} = \alpha_{12} = h$ and $t = \beta_1 - \alpha_{01}$. Thus the proof is completed. $\square$

It can be easily verified that the link $L(p, q, k, h, t, s)$ is again a 2-bridge link whenever one of the following cases holds:

$$k + h - 1 > p, \quad p - k = h - s, \quad (1)$$

$$k + h + q - p - 1 = t - s, \quad (2)$$

$$k + h + q - 1 > 2p, \quad k + h + q - 2p - 1 = t - s, \quad (3)$$

$$k + h + q - 1 > 2p, \quad 2p - q - k = s - 1, \quad (4)$$

$$k + h - 1 > 2p, \quad k + h + t - 2p = s. \quad (5)$$

However, this fact surely happens in more subtle ways but we do not know the general answer.

The next result describes the connected sum of 2-bridge knots in terms of our representation.

Proposition 2. *Let $K(p_i, q_i)$ be the Schubert normal form of a 2-bridge knot for $i = 1, 2$. Then the connected sum $K(p_1, q_1) \# K(p_2, q_2)$ is equivalent to the 3-bridge knot $L(p, q, k, h, t, s)$, where*

$$\begin{aligned} p &= p_2, & h &= 2p_2 - 1, \\ q &= q_2 + q_1 - p_1 \pmod{p_2}, & t &= p_2 - p_1, \\ k &= p_1 - q_1 + 1, & s &= p_2. \end{aligned}$$

Finally, we complete the section with some results about a certain class of 3-bridge knots, introduced by Morikawa in [18] and [19]. He defined a normal form $K(m, n; r)$ to study a class of 3-bridge knots, where m, n are coprime odd integers, $0 < |n| < m$, and $r \in \mathbb{Z}$. Now we give a description of this normal form in terms of our representation. This yield some consequences about the knot type of $L(p, q, k, h, t, s)$. First we sketch the construction of $K(m, n; r)$ as given in [18] and [19]. Let $K(m, n)$ be the Schubert diagram of a 2-bridge knot. Then $K(m, n)$ is the union of two (oriented) bridges AB and EF , whose interiors are contained in the upper half-space of $\mathbb{R}^3$, and of two (oriented) underpasses BE and FA in the plane $\mathbb{R}^2 \times 0$. Let A' and B' be the nearest points in $\rho(AB)$ to A and B , respectively. Here $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \times 0$ denotes the standard projection. Then the union of the (oriented) arcs AA' , $A'B$, BB' , and $B'A$ bounds a closed 2-cell of the plane $\mathbb{R}^2 \times 0$, called a *parallel area* (see [18]). Let C and D be points in the interior of the parallel area near A and B , respectively.

Now the knot $K(m, n; 0)$ is constructed from $K(m, n)$ by the following rules:

- (1) Add a new underpass CD in the interior of the parallel area.
- (2) Substitute the bridge AB of $K(m, n)$ with two disjoint parallel overpasses AD and CD .

To obtain $K(m, n; r)$, $r \neq 0$, from $K(m, n; 0)$ we perform $|r|$ twists around the arcs AF and CD . More precisely, if $r > 0$ (resp. $r < 0$), then we move the top vertex A (resp. C) so that it passes through the bridge CD (resp. AD) $|r|$ -times.

Now we can prove the following result:

Proposition 3. *Let $K(m, n)$ be the Schubert normal form of a 2-bridge knot, where m, n are odd coprime integers and $0 < |n| < m$. Let (α, β) be the unique pair of integers which satisfies the following conditions:*

$$m\beta - n\alpha = \pm 1, \quad 0 < |\beta| < \alpha < m, \quad \alpha \text{ even}.$$

Then the Morikawa knot $K(m, n; 0)$ is equivalent to the 3-bridge normal diagram $L(p, q, k, h, t, s)$, where $p = h = m + \alpha/2$, $q = n + (\beta + 1)/2 - 1$, and $k = s = t = 1$.

Further, the Morikawa knot $K(m, n; r)$, $r \neq 0$, has the same knot type as $L(p, q, k, h, t, s)$, where

$$p = m + \frac{\alpha}{2} + 2|r| - 1, \quad q = p - 1,$$

$$k = 2|r|, \quad h = m + \frac{\alpha}{2} + 1, \quad t = 1,$$

$$s = \begin{cases} m - n + \frac{\alpha}{2} - \frac{\beta + 1}{2} + 2 & \text{if } r < 0, \\ n + \frac{\beta + 1}{2} + 1 & \text{if } r > 0. \end{cases}$$

Applying Propositions 2–4 of [18] yields the following:

Corollary 4.

- (1) $L(4g + 2, 2g + 1, 1, 4g + 2, 1, 1)$ and $L(4g + 3, 4g + 2, 2, 4g + 3, 1, 2g + 2)$ are equivalent to the Schubert knots $K(8g + 1, 6g + 1)$ and $K(8g + 3, 2g + 1)$, respectively, for any $g \geq 1$.
- (2) $L(4g + 4, 2g + 1, 1, 4g + 4, 1, 1)$ and $L(4g + 5, 4g + 4, 2, 4g + 5, 1, 2g + 3)$ have the same knot types as the Schubert knots $K(8g + 5, 2g + 1)$ and $K(8g + 7, 6g + 5)$, respectively, for any $g \geq 0$. In particular, $L(4, 1, 1, 4, 1, 1) \cong K(5, 1)$ is the torus knot $T(5, 2)$ of type $(5, 2)$.
- (3) $L(3g + 2, 3g + 1, 2, 3g + 2, 1, 3g + 1)$ and $L(3g + 4, 3g + 3, 4, 3g + 2, 1, 3g + 1)$ are equivalent to the torus knots $T(3g + 1, 3)$ and $T(3g + 2, 3)$, respectively, for any $g \geq 1$.

3. Twofold Branched Covers

In this section we obtain a combinatorial representation of all closed orientable 3-manifolds of Heegaard genus 2 by means of special colored graphs, which are standardly constructed from the 6-tuples of integers (p, q, k, h, t, s) . First we recall some definitions and results about the representation of manifolds by means of edge-colored graphs, called *crystallizations* (see [7] and [22]). For basic graph theory we refer, for example, to [8]. We use the term *graph* instead of multigraph, i.e., loops are forbidden but multiple edges are allowed. An *edge-coloration* on a graph $G = (V(G), E(G))$ is a map $c: E(G) \rightarrow \Delta_n = \{0, 1, \dots, n\}$ such that $c(e) \neq c(f)$ for any two adjacent edges e, f of G . An $(n + 1)$ -colored graph is a pair (G, c) formed by a regular graph G of degree $n + 1$ together with an edge-coloration $c: E(G) \rightarrow \Delta_n$. An n -pseudocomplex (see [11]) $|G|$ can be associated to (G, c) by the

following rules:

- (1) Take an n -simplex $\sigma^n(v)$ for each vertex $v \in V(G)$ and label its vertices by Δ_n .
- (2) If $v, w \in V(G)$ are joined by an i -colored edge, $i \in \Delta_n$, then identify the $(n-1)$ -faces of $\sigma^n(v)$ and $\sigma^n(w)$ opposite to vertices labeled by i .

We identify $|G|$ with the underlying polyhedron and say that (G, c) represents $|G|$ and every PL homeomorphic space. The colored graph (G, c) is said to be *contracted* if $|G|$ has exactly $n+1$ vertices. A *crystallization* of a closed connected n -manifold M is a contracted $(n+1)$ -colored graph which represents M . It was proved in [22] that each closed connected PL n -manifold admits a crystallization. In particular, a simple algorithm to construct a crystallization (called *2-symmetric*) of the twofold covering of $\mathbb{S}^3$ branched over a bridge-presentation of a link L exists (for details, see [6]). Figures 7 and 8 show the construction for $K(5, 3)$ and $L(5, 2, 3, 4, 0, 3)$.

Now let $G(p, q, k, h, t, s)$ denote the 2-symmetric crystallization of the twofold branched covering over the 3-bridge diagram $L(p, q, k, h, t, s)$. As a direct consequence of [6] and Theorem 1 we obtain the following result:

Proposition 5. *Let M be a closed connected orientable 3-manifold of Heegaard genus ≤ 2 . Then a 6-tuple of integers exists such that $G(p, q, k, h, t, s)$ is a crystallization of M .*

Thus the set of 6-tuples of integers satisfying the conditions of Theorem 1 gives a complete catalogue of all 3-manifolds of Heegaard genus ≤ 2 . Let $M(p, q, k, h, t, s)$ be the twofold covering of $\mathbb{S}^3$ branched over the 3-bridge diagram $L(p, q, k, h, t, s)$.

Fig. 7. (a) $K(5, 3)$ and (b) the twofold branched covering over it: $\cdots$, 0; $++++$, 1; $---$, 2; $---$, 3.

Fig. 8. (a) $L(5, 2, 3, 4, 0, 3)$ and (b) the twofold branched covering of it.

The following proposition describes the topological structure of the manifolds $M(p, q, k, h, t, s)$ for several 6-tuples of integers.

Proposition 6.

(1) *Lens spaces:*

- (1.1) $M(2q - 1, q, 0, 2q - 1, 0, q) \simeq \mathbb{L}(2q - 1, q)$.
- (1.2) $M(p, q, 0, 2, 0, 1) \simeq \mathbb{L}(p + x, q)$, $x \equiv q^{\varphi(p)-1} \pmod{p}$, $0 \leq x < p$, where $\varphi(p)$ denotes the Gauss indicator of p .
- (1.3) $M(4g + 2, 2g + 1, 1, 4g + 2, 1, 1) \simeq \mathbb{L}(8g + 1, 6g + 1)$, $g \geq 1$.
- (1.4) $M(4g + 3, 4g + 2, 2, 4g + 3, 1, 2g + 2) \simeq \mathbb{L}(8g + 3, 2g + 1)$, $g \geq 1$.
- (1.5) $M(4g + 4, 2g + 1, 1, 4g + 4, 1, 1) \simeq \mathbb{L}(8g + 5, 2g + 1)$, $g \geq 0$.
- (1.6) $M(4g + 5, 4g + 4, 2, 4g + 5, 1, 2g + 3) \simeq \mathbb{L}(8g + 7, 6g + 5)$, $g \geq 0$.

(2) *Seifert manifolds:*

- (2.1) $M(3g + 2, 3g + 1, 2, 3g + 2, 1, 3g + 1)$ is homeomorphic to either the Seifert manifold $(o \ 0 \ o: -2(3, 1)(3, 1)(3g + 1)/2, 2g + 1))$ or $(o \ 0 \ o: -1(2, 1)(3, 1)(3g + 1, g/2))$ if g is odd or even, respectively.
- (2.2) $M(3g + 4, 3g + 3, 4, 3g + 2, 1, 3g + 1)$ is homeomorphic to either the Seifert manifold $(o \ 0 \ o: -1(2, 1)(3, 1)(3g + 2, (g + 1)/2))$ or $(o \ 0 \ o: -2(3, 1)(3, 1)(3g + 2)/2, 2g + 1))$ if g is odd or even, respectively.
- (2.3) $M(4g + 2r + 3, 4g + 2r + 2, 2r, 4g + 5, 1, 2g + 3)$ is homeomorphic to the Seifert manifold $(o \ 0 \ o: 0(2r - 1, 1)(2, 1)(4g + 3, -(2g + 1)))$, $r > 0$.
- (2.4) $M(4g + 2r + 1, 4g + 2r, 2r, 4g + 3, 1, 2g + 3)$ is homeomorphic to the Seifert manifold $(o \ 0 \ o: 0(2r + 1, 1)(2, 1)(4g + 1, -(2g + 1)))$, $r > 0$.

(3) *Brieskorn manifolds:*

$M(p, 1, 0, p, 0, 1)$ is homeomorphic to the Brieskorn manifold obtained by intersecting the complex algebraic surface $z_1^3 + z_2^3 + z_3^3 = 0$ of $\mathbb{C}^3$ with the unit sphere. In particular, if p is odd and $(p, 3) = 1$, then $M(p, 1, 0, p, 0, 1)$ is the Seifert manifold $(o \ 0 \ o: -1(2, 1)(3, 1)(p, (p \pm 1)/6))$. If p is even, $p \neq 2$ and $(p, 3) = 1$, then $M(p, 1, 0, p, 0, 1)$ is homeomorphic to $(o \ 0 \ o: -2(3, 1)(3, 1)(p/2, (2p \pm 1)/3))$.

Proof. (1.1) The link $L(2q - 1, q, 0, 2q - 1, 0, q)$ has exactly q unlinked components, which are $K(2q - 1, q)$ and $q - 1$ unknots.

(1.2) Let $G = G(p, q, 0, 2, 0, 1)$ be the 2-symmetric crystallization of the twofold branched covering $M = M(p, q, 0, 2, 0, 1)$. Let C and C' denote the $\{0, 1\}$ - and $\{2, 3\}$ -colored cycles of G with vertex sets $\{(2, 0), (2, 1), (2, 2), (2, 3)\}$ and $\{(0, 1), (0, 5), (0, 9), (1, 2), (1, 8), (2, 2)\}$, respectively. The subgraph θ of G determined by C and C' is a $(3, 5)$ -dipole in the sense of [7]. Let G' be the colored graph obtained from G by canceling the dipole θ (for details see [7]). Then G' also represents the manifold M . Now it is very easy to see that M is the lens space of the statement since G' has genus 1.

(1.3)–(1.6) are direct consequences of Corollary 5.

(2.1) and (2.2) follow from Corollary 5 and Theorem 3.8 of [19].

(2.3) Let G be the 2-symmetric crystallization of

$$M = M(4g + 2r + 3, 4g + 2r + 2, 2r, 4g + 5, 1, 2g + 3).$$

As shown in [7], a finite presentation of the fundamental group $\Pi_1(M)$ of M can be directly obtained from G by the following rules:

- (/) The generators are the connected components, except one (arbitrarily chosen), of the subgraph $G_{\{0, 1\}} = (V(G), c^{-1}(\{0, 1\}))$, where $c: E(G) \rightarrow \Delta_3 = \{0, 1, 2, 3\}$ is the coloring of G .
- (//) The relators are obtained by walking along the $\{2, 3\}$ -colored cycles of G , giving the exponent $+1$ (resp. -1) to each generator that is met if 0 (resp. 1) is the color of the edge by which it is arrived at. Furthermore, one relator (arbitrarily chosen) may be dropped too.

Now let x, y, z be the $\{0, 1\}$ -colored cycles of G . By construction, they correspond to the bridges B_0, B_1, B_2 of L in this order. As written above, x, y, z generate the group $\Pi_1(M)$. To determine the relators of $\Pi_1(M)$ we consider the $\{2, 3\}$ -colored cycles, except one, of G . Walking along these cycles yields the finite presentation

$$\begin{aligned} \Pi_1(M) &= \langle x, y, z: (xy^{-1})^{2r-1}(zy^{-1})^2 = 1, xz^{-1}(xz^{-1}xy^{-1}zy^{-1})^{2g+1} = 1, y = 1 \rangle \\ &\cong \langle x, z: x^{2r-1} = z^{-2}, xz^{-1}(xz^{-1}xz)^{2g+1} = 1 \rangle. \end{aligned}$$

Moreover, this presentation is geometric, i.e., it arises from a Heegaard diagram of M . This also means that the presentation corresponds to a spine of M as shown in [4]. Hence M must be a Seifert fibered space over the 2-sphere with three exceptional fibers as $\Pi_1(M)$ admits a relation of type $x^n = z^m$ (see Theorem 3.1 of [21], p. 485). Now we simplify the above presentation. Since z^2 is central in the group $\Pi_1(M)$, the second relation is equivalent to

$$xz^{-1}(xz^{-1}xz^{-1}z^2)^{2g+1} = 1,$$

$$(xz^{-1})^{4g+3}z^{4g+2} = 1,$$

hence

$$\Pi_1(M) = \langle x, z: x^{2r-1} = z^{-2}, (xz^{-1})^{4g+3} z^{4g+2} = 1 \rangle.$$

Now we prove that M is homeomorphic to the Seifert manifold

$$\Sigma = (o \ 0 \ o: 0(2r-1, 1)(2, 1)(4g+3, -(2g+1))).$$

For this, it suffices to show that the fibered manifolds have isomorphic fundamental groups (use Theorem 6, p. 97, and Section 5.4, p. 99, of [20]). By Section 5.3, p. 90, of [20], it is well known that $\Pi_1(\Sigma)$ can be presented by

$$\begin{aligned} \langle q_1, q_2, q_3, h: q_1^{2r-1} h = 1, q_2^2 h = 1, q_3^{4g+3} h^{-(2g+1)} = 1, \\ q_1 q_2 q_3 = 1, [q_j, h] = 1 (j = 1, 2, 3) \rangle. \end{aligned}$$

Deleting the generators $q_3 = q_2^{-1} q_1^{-1}$ and $h = q_2^{-2}$ yields

$$\Pi_1(\Sigma) = \langle q_1, q_2: q_1^{2r-1} = q_2^2, q_2^{4g+2} = (q_1 q_2)^{4g+3} \rangle.$$

Setting $x = q_1$ and $z = q_2^{-1}$ implies that $\Pi_1(\Sigma)$ is isomorphic to $\Pi_1(M)$ as required.

(2.4) We proceed as in case (2.3). Then we obtain the finite presentation

$$\begin{aligned} \Pi_1(M) &= \langle x, y, z: (xy^{-1})^{2r} z x^{-1} z y^{-1} = 1, (xz^{-1} x y^{-1} z y^{-1})^{2g} z y^{-1} = 1, x = 1 \rangle \\ &\cong \langle y, z: z^2 = y^{2r+1}, (z^{-1} y^{-1} z y^{-1})^{2g} z y^{-1} = 1 \rangle. \end{aligned}$$

Now the result follows by using arguments similar to the previous case.

(3) It suffices to note that $L(p, 1, 0, p, 0, 1)$ is equivalent to the torus link of type $(p, 3)$. Then the result is a direct consequence of Theorem 3.8 of [17] and Lemma 1.1 of [14]. $\square$

Remark. Proposition 6(2.3) and (2.4) gives an alternative proof of Theorem 4 of [19] by using the theory of 3-manifold spines. This also allows us to obtain simple geometric presentations of the fundamental group of the considered manifolds.

4. n -Bridge Links

In this section we define a cut-and-paste move on an n -bridge link depending upon a 5-tuple of integers (k, h, t, s, r) . For $r = 0$, we essentially obtain again the move of type I/II described in Section 2. The case $r \neq 0$ yields a new cut-and-paste move of type III. Then we prove that any n -bridge link L in $\mathbb{S}^3$, $n \geq 4$, can be represented by a special type of diagram obtained from a 3-bridge diagram $L(p, q, k, h, t, s)$ by a sequence of $(k_i, h_i, t_i, s_i, r_i)$ -moves, for $i = 1, 2, \dots, n-3$. Thus any n -bridge link can be standardly constructed from a set of $5n-9$ integers

Fig. 9.

which satisfy certain inequalities ($n \geq 3$). Now applying the construction of [3] to our n -bridge diagrams yields a combinatorial representation of closed 3-manifolds by special edge-colored graphs (also compare [1], [10], [15], and [16]).

Let L be a diagram of an n -bridge link. Let B_i denote the bridges of L drawn in the plane $z = 0$ and let $\beta_i - 1$ be the number of undercrossings of B_i . We denote by $B_{i_1}, B_{i_2}, \dots, B_{i_m}$ the bridges of L which are connected to B_i by some arcs of L and assume the numbering of vertices of B_{i_j} as shown in Fig. 9(a).

Let $e_{j,l}$ be the arc of L with ends (i, j) and $(i_1, j + q_l)$ for $i = 0, 1, \dots, 2\beta_i - 1$ and $l = 1, 2, \dots, m$. By definition, the arc $e_{j,l}$ joins the bridge B_i to B_{i_l} . Let us consider h consecutive arcs $d_1, d_2, \dots, d_h$, $2 \leq h \leq 2\beta_0$, of L which connect B_i to B_{i_l} , starting from $d_1 = e_{k,l}$, $0 \leq k \leq \beta_0 - 1$.

Now we define a (k, h, t, s, r) -move on L . If $r = 0$, then operate on $d_1, d_2, \dots, d_h$ by a cut-and-paste move of type I/II as shown in Section 2. Here we insert a new bridge, B_n say, which is disjoint and parallel to B_i and B_{i_l} . If $r \neq 0$, then the move of type III is defined by the following rules:

- (1) Cut the arcs $d_1, d_2, \dots, d_h$ by a new bridge B_n as before. Let x_j, y_j denote the ends of d_j incident with B_i, B_{i_l} , respectively. Suppose that an index α , $1 < \alpha \leq h$, exists such that $x_\alpha = (i, 0)$. Then we require that B_n has exactly $h - t - |r| - 1$, $t > 0$, undercrossings numbered as shown in Fig. 9(b).
- (2) Add t undercrossings $d'_1, d'_2, \dots, d'_t$ (resp. $d''_1, d''_2, \dots, d''_t$) between the vertices (i, β_i) and $(i, \beta_i - 1)$ (resp. (i, β_i) and $(i, \beta_i + 1)$). Then change the numbering

on the bridge obtained from B_i as follows. Call a_ξ the arc with end x_ξ for any $\xi = 1, 2, \dots, \alpha - 1$, and set $a_{\eta+\alpha} = d'_\eta$, $a_{\eta+\alpha+t+1} = d''_\eta$ for any $\eta = 1, 2, \dots, t$. Finally, let $a_{\alpha+t+1}$ be the arc with end x_α and $a_{\gamma+\alpha+2t+1}$ the one with end x_γ for any $\gamma = 0, 1, \dots, h - \alpha - 1$.

- (3) ($r > 0$) Join the hanging arc a_v with the vertex y_v for any $v = 1, 2, \dots, r$, $1 \leq r < h$. Then connect the $2(h - t - r)$ remaining arcs with a free end to vertices of B_n consecutively so that a_{r+v+1} (resp. y_μ) is joined to $(n, s + v)$ (resp. $(n, s - \mu - 1)$) for $0 \leq v \leq h - 2t + r - 1$ and $r + 1 \leq \mu \leq h$.
- (3') ($r < 0$) Join the hanging arc $a_{h+2t+1-v}$ with the vertex y_{h-v-1} for any $v = 1, 2, \dots, -r$, $-h < r < 0$. Then connect the $2(h - t + r)$ remaining arcs with a free end to vertices of B_n , consecutively, so that a_{v+1} (resp. $y_{\mu+1}$) is joined to $(n, s + v)$ (resp. $(n, s - \mu - 1)$) for $0 \leq v \leq h - 2t + r - 1$ and $0 \leq \mu \leq h + r - 1$. Observe that there is a unique way (up to isotopy) to do these constructions in order to preserve the planarity and without introducing any new crossing (see, for example, Fig. 10 for the case $r = 0$ and Fig. 11 for $r \neq 0$).

Proposition 7. *With the above notation, any n -bridge link in $\mathbb{S}^3$, $n \geq 4$, can be obtained from an $(n - 1)$ -bridge link by a (k, h, t, s, r) -move, where*

$$\begin{aligned} 0 \leq k \leq \beta_0 - 1, \quad 2 \leq h \leq 2\beta_0, \\ 0 \leq t \leq \min\{\beta_0 - k, k + h - \beta_0\}, \quad 1 \leq s \leq h - t - |r|, \\ 0 \leq |r| < h. \end{aligned}$$

Proof. Assume that $\beta_0 \leq \beta_1$ and that the vertex $(0, \beta_0)$ is joined to the bridge B_1 . Let $(0, \beta)$ and $(1, \gamma)$ denote the ends of the first arc, with respect to the numbering, which connects B_0 and B_1 . The inequalities $1 \leq \beta \leq \beta_0$ hold by a possible rotation of B_0 through an angle π . Now we construct an $(n - 1)$ -bridge link L' from which L is obtained by a suitable cut-and-paste move. For this, take out the bridge B_1

Fig. 10. (a) $L(7, 1, 4, 7, 1, 2)$ and (b) link obtained from it by a $(3, 6, 1, 1, 0)$ -move.

Fig. 11. Link obtained from $L(7, 1, 4, 7, 1, 2)$ by (a) a $(3, 7, 1, 1, 2)$ -move and (b) a $(4, 5, 1, 1, -1)$ -move.

from L . Then there are $2\beta_1$ hanging arcs, named $x_1, x_2, \dots, x_{h_1}, y_1, y_2, \dots, y_{h_2}$, where x_j (resp. y_i) is (resp. is not) joined to B_0 . Here we have $h_1 \geq 1$ and $h_2 = 2\beta_1 - h_1 \geq 1$. By hypothesis, the vertex $(0, \beta_0)$ belongs to an arc x_α for some α , $1 \leq \alpha \leq h_1$. We have to consider three cases.

Case 1: $h_1 \leq h_2$. We construct L' by the following rules:

- (1) Add $t = (h_2 - h_1)/2 = \beta_1 - h_1$ undercrossings between the vertices $(0, \beta_0)$ and $(0, \beta_0 - 1)$ (resp. $(0, \beta_0)$ and $(0, \beta_0 + 1)$), named $d'_1, d'_2, \dots, d'_t$ (resp. $d''_1, d''_2, \dots, d''_t$). Now there are exactly $h_1 + 2t$ hanging arcs coming out the bridge B'_0 derived from B_0 . We renumber the vertices of B'_0 as follows. Any vertex $(0, j)$ of B_0 preserves the same coordinates for $j \in \mathbb{Z}_{\beta_0}$. Let $(0, \beta_0 + i)$ and $(0, \beta_0 + t + i)$ be the vertices incident to d'_i and d''_i , respectively, for each $i = 1, 2, \dots, t$. Finally, the vertex $(0, \beta_0)$ of B_0 becomes $(0, \beta_0 + t)$ in B'_0 and $(0, j)$ becomes $(0, \beta_0 + 2t + j)$ for $\beta_0 + 1 \leq j \leq 2\beta_0 - 1$.
- (2) Join the free end of x_j (resp. x_α) to y_j (resp. y_α) for each $j = 1, 2, \dots, \alpha - 1$.
- (3) Join the free end of d'_j (resp. d''_j) to $y_{j+\alpha-1}$ (resp. $y_{j+\alpha+t}$) for each $j = 1, 2, \dots, t$.

As usual, we perform the construction preserving the planarity (see, for example, Fig. 10). Thus L is obtained from L' by a (k', h', t', s', r') -move, where $k' = \beta$, $h' = h_2 = h_1 + 2t$, $t' = t = (h_2 - h_1)/2$, $s' = \gamma$, and $r' = 0$. It can be easily verified that the 5-tuple of integers satisfies the inequalities of the statement.

Case 2: $h_1 > h_2$, $t = (h_1 - h_2)/2 \leq t^*$, where $t^* = \min\{\beta_0 - \beta, \beta + h_1 - \beta_0\}$. The above proof can be easily adjusted in order to show that L is obtained from L' by a (k', h', t', r') -move, where $k' = \beta$, $h' = h_1 - 2t$, $t' = t = (h_1 - h_2)/2$, $s' = \gamma$, and $r' = 0$.

Case 3: $h_1 > h_2$, $t = (h_1 - h_2)/2 > t^*$. For convenience, we examine the case $t^* = \beta_0 - \beta$. For $t^* = \beta + h_1 - \beta_0$, we can proceed in a similar way. If $t^* = \beta_0 - \beta$, then we cut $r = t - t^*$ consecutive arcs $d'_1, d'_2, \dots, d'_r$ of L with ends $(0, 1)$,

$(0, 2), \dots, (0, r)$, respectively. Let a_j (resp. b_j) denote the end of d'_j which is (resp. is not) incident to the bridge B_0 for each $j = 1, 2, \dots, r$. Further, let a_{r+j} (resp. b_{r+i}) be the arc with end x_j (resp. y_i) for each $j = 1, 2, \dots, h_1$ (resp. $i = 1, 2, \dots, h_2$). Now there are exactly $h_1 + h_2 + 2r$ hanging arcs, of which $h_1 + r$ come out from B_0 . To construct L' , we connect the free ends of the arcs a_{i+a+r} and a_{r+i-1} for each $i \in \mathbb{Z}$, and the ones of $a_{r+i+j-1}$ and b_j for each $j = 1, 2, \dots, \beta_1 - \beta_0 + \beta$, preserving the planarity. Then L is obtained from L' by a (k', h', t', s', r') -move, where $k' = \beta - r$ (resp. β) if $t^* = \beta_0 - \beta$ (resp. $t^* = \beta + h_1 - \beta_0$), $h' = h_1 - 2t + r$, $t' = t$, $s' = \gamma$, and $r' = r$ (resp. $-r$) if $t^* = \beta_0 - \beta$ (resp. $\beta + h_1 - \beta_0$), as required (see, for example, Fig. 11). $\square$

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