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On Antipodal and Adjoint Pairs of Points for Two Convex Bodies

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Abstract. The numbers of antipodal and of adjoint pairs of points are estimated for a given pair of disjoint convex bodies in E^d.

1. Introduction

It is well known (see, for instance, [4]) that any two disjoint convex bodies K_1, K_2 in the Euclidean space E^d can be strictly separated by a hyperplane H, i.e., K_1, K_2 lie in distinct open half-spaces determined by H. This result easily implies the existence of two distinct parallel hyperplanes H_1, H_2 both separating K_1, K_2 such that H_1 supports K_1 and H_2 supports K_2. The last assertion has been improved by De Wilde [8], who showed that the above hyperplanes H_1, H_2 can be chosen so that the sets of contact $H_1 \cap K_1$, $H_2 \cap K_2$ are single points. Based on this result, we introduce the following definition. (As usual, $\exp K$ and $\operatorname{ext} K$ denote, respectively, the set of exposed points and the set of extreme points of K.)

Definition 1. Let K_1, K_2 be disjoint convex bodies in E^d. We say that points $x_1 \in \operatorname{ext} K_1$ and $x_2 \in \operatorname{ext} K_2$ are *adjoint* if there are distinct parallel hyperplanes H_1, H_2 through x_1, x_2, respectively, both separating K_1 and K_2. If, additionally, $H_1 \cap K_1 = \{x_1\}$ and $H_2 \cap K_2 = \{x_2\}$, the points x_1, x_2 are called *strictly adjoint*.

Dual to adjointness is the notion of antipodality, introduced for the case of two convex bodies as follows:

Definition 2. Let K_1, K_2 be disjoint convex bodies in E^d. We say that points $x_1 \in \operatorname{ext} K_1$ and $x_2 \in \operatorname{ext} K_2$ are *antipodal* provided there are parallel hyperplanes H_1, H_2 through x_1, x_2, respectively, such that both K_1, K_2 lie between H_1, H_2. If, additionally, $H_1 \cap K_1 = \{x_1\}$ and $H_2 \cap K_2 = \{x_2\}$, the points x_1, x_2 are called *strictly antipodal.*

Clearly, extreme points $x_1 \in K_1$, $x_2 \in K_2$ forming a strictly antipodal or strictly adjoint pair are exposed for K_1, K_2, respectively.

In our notation De Wilde's theorem states that any two disjoint convex bodies in E^d determine at least one strictly adjoint pair of points. Our purpose here is to sharpen De Wilde's result and to prove a few related assertions on the numbers of (strictly) adjoint and of (strictly) antipodal pairs determined by two disjoint translates of a given pair of convex bodies. For similar results on the numbers of antipodal pairs and strictly antipodal pairs of points of a single convex body in E^d see [6].

2. Main Results

Denote by $p(K_1, K_2)$ (by $\bar{p}(K_1, K_2)$) the number of antipodal (strictly antipodal) pairs of points $x_1 \in K_1$, $x_2 \in K_2$. Similarly, denote by $q(K_1, K_2)$ (by $\bar{q}(K_1, K_2)$) the number of adjoint (strictly adjoint) pairs of points $x_1 \in K_1$, $x_2 \in K_2$. Here and subsequently, we mean that two pairs $\{x_1, x_2\}$, $\{x_1', x_2'\}$ of points, where $x_1, x_1' \in K_1$ and $x_2, x_2' \in K_2$, are distinct if either $x_1 \neq x_1'$ or $x_2 \neq x_2'$. Define any of the values $p(K_1, K_2)$, $\bar{p}(K_1, K_2)$, $q(K_1, K_2)$, $\bar{q}(K_1, K_2)$ to be ∞ if the respective family of pairs is infinite.

Clearly, $p(K_1, K_2) \geq \bar{p}(K_1, K_2)$ and $q(K_1, K_2) \geq \bar{q}(K_1, K_2)$.

Theorem 1. *$\bar{p}(K_1, K_2) \geq 1$ and $\bar{q}(K_1, K_2) \geq d$ for any disjoint convex bodies K_1, K_2 in E^d.*

Examples 1 and 2 below demonstrate that the inequalities in Theorem 1 are sharp even for the values $p(K_1, K_2)$ and $q(K_1, K_2)$.

Example 1. Let K_1 be the triangle with vertices $x_1 = (0; 0)$, $x_2 = (0; 5)$, and $x_3 = (5; 0)$, and let K_2 be the triangle with vertices $y_1 = (4; 4)$, $y_2 = (3; 4)$, and $y_3 = (4; 3)$ in the coordinate plane E^2. There is exactly one antipodal pair of points determined by K_1, K_2, namely, $\{x_1, y_1\}$, whence $p(K_1, K_2) = 1$.

Example 2. Let K_1 be the triangle with vertices $x_1 = (0; 0)$, $x_2 = (0; 5)$, and $x_3 = (5; 0)$, and let K_2 be the triangle with vertices $z_1 = (4; 4)$, $z_2 = (4; 9)$, and $z_3 = (9; 4)$ in the coordinate plane E^2. There are exactly two adjoint pairs determined by K_1, K_2, namely, $\{z_1, x_2\}$ and $\{z_1, x_3\}$, whence $q(K_1, K_2) = 2$.

Clearly, Examples 1 and 2 can be easily modified for the higher-dimensional case.

It is easily seen that the equalities $\bar{p}(K_1, K_2) = 1$ and $\bar{q}(K_1, K_2) = d$ are satisfied only for some special pairs $\{K_1, K_2\}$. The following theorem shows that any pair of convex bodies K_1, K_2 can be placed by suitable translations in order to obtain bigger values of $\bar{p}(K_1, K_2)$ and $\bar{q}(K_1, K_2)$.

Theorem 2. *For any convex bodies K_1, K_2 in E^d, $d \geq 2$, there are translates K_2', K_2'' of K_2 both disjoint to K_1 such that $\bar{p}(K_1, K_2') \geq d + 1$ and $\bar{q}(K_1, K_2'') \geq d + 1$.*

In fact, we can restrict our attention in Theorem 2 to the case when both K_1 and K_2 are polytopes.

Theorem 3. *For convex bodies K_1, K_2 in E^d the following conditions are equivalent*:

(1) $p(K_1, K_2')$ *is finite for every translate K_2' of K_2 disjoint to K_1.*
(2) $\bar{p}(K_1, K_2')$ *is finite for every translate K_2' of K_2 disjoint to K_1.*
(3) $q(K_1, K_2')$ *is finite for every translate K_2' of K_2 disjoint to K_1.*
(4) $\bar{q}(K_1, K_2')$ *is finite for every translate K_2' of K_2 disjoint to K_1.*
(5) *Both K_1, K_2 are polytopes.*

In connection with Theorem 2 the following question appears. For which pairs of convex bodies K_1, K_2 in E^d are the inequalities $\bar{p}(K_1, K_2') \geq d + 1$ and $\bar{q}(K_1, K_2'') \geq d + 1$ sharp? The answer to this question gives Theorem 4 below. Recall that K' is a positive (negative) homothetic copy of a convex body K provided $K' = a + \lambda K$ for a vector $a \in E^d$ and a real number $\lambda > 0$ ($\lambda < 0$).

Theorem 4. *For convex bodies K_1, K_2 in E^d, $d \geq 2$, the following conditions are equivalent*:

(1) $\bar{p}(K_1, K_2') \leq d + 1$ *for every positive homothetic copy K_2' of K_2 disjoint to K_1.*
(2) $\bar{p}(K_1, K_2') \leq d + 1$ *for every translate K_2' of K_2 disjoint to K_1.*
(3) $\bar{q}(K_1, K_2') \leq d + 1$ *for every positive homothetic copy K_2' of K_2 disjoint to K_1.*
(4) $\bar{q}(K_1, K_2') \leq d + 1$ *for every translate K_2' of K_2 disjoint to K_1.*
(5) (i) *K_1, K_2 are two simplices negatively homothetic to each other if $d \geq 3$.*
(ii) *K_1, K_2 are either triangles negatively homothetic to each other or parallelograms with parallel sides if $d = 2$.*

Conjecture 1. *For any convex bodies K_1, K_2 in E^d, $d \geq 2$, there are translates K_2', K_2'' of K_2 both disjoint to K_1 such that $p(K_1, K_2') \geq d^2$ and $q(K_1, K_2'') \geq d^2$.*

Conjecture 2. *For convex bodies $K_1, K_2 \subset E^d$, $d \geq 2$, the following conditions are equivalent*:

(1) $p(K_1, K_2') \leq d^2$ *for every positive homothetic copy K_2' of K_2 disjoint to K_1.*
(2) $p(K_1, K_2') \leq d^2$ *for every translate K_2' of K_2 disjoint to K_1.*
(3) $q(K_1, K_2') \leq d^2$ *for every positive homothetic copy K_2' of K_2 disjoint to K_1.*
(4) $q(K_1, K_2') \leq d^2$ *for every translate K_2' of K_2 disjoint to K_1.*
(5) *K_1, K_2 are two simplices positively homothetic to each other.*

Problem. Determine sharp lower and sharp upper bounds for the values $p(T_m, T_n')$, $\bar{p}(T_m, T_n')$, $q(T_m, T_n')$, and $\bar{q}(T_m, T_n')$ as functions of d, m, and n, where T_m and T_n are d-polytopes in E^d with m and n vertices, respectively, and T_n' is a translate of T_n disjoint to T_m.

A similar problem for the case of a single convex d-polytope is studied in [2] and [3].

3. Auxiliary Lemmas

Usual abbreviations conv, int, and bd are used for convex hull, interior, and boundary, respectively; $[x, y]$ and $[x, y\rangle$ denote the closed line segment with the endpoints x, y and the ray with apex x through y. Let v be a point exterior to K. A point $x \in K$ is called *exposed relative to* v if $\{x, v\}$ is a strictly adjoint pair for the sets K, $\{v\}$. We say that a closed half-space P of E^d *exposedly supports* a convex body K provided P contains K and the boundary hyperplane of P intersects K at one (exposed) point only. Following [7], a boundary point x of a closed convex set K in E^d is said to be *visible from an exterior point* w provided $[x, w] \cap K = \{x\}$.

The following lemmas are necessary in what follows.

Lemma 1 [4, Corollary 9.6.1]. *For a convex body K in E^d and a point $v \in E^d \setminus K$, the cone*

$$C_K(v) = \{(1 - \lambda)v + \lambda y: \lambda \geq 0, y \in K\}$$

is convex, closed, and contains no line.

Lemma 2 [4, Theorem 18.7]. *A closed convex cone in E^d containing no line is the closed convex hull of its exposed rays.*

A point $v \in E^d \setminus K$ is called *special* for K provided every ray starting at v and supporting K has exactly one common point with K.

Lemma 3 [1]. *For a given compact convex set K in E^d the set of special points for K is dense in $E^d \setminus K$.*

Lemma 4 [8]. *Two convex bodies K_1, K_2 in E^d are separated (strictly separated) by a hyperplane parallel to a given hyperplane H if and only if the difference $K_1 - K_2$ is separated (strictly separated) from 0 by a hyperplane parallel to H.*

Lemma 5 (see [5]). *For any convex bodies K_1, K_2 in E^d, one has* $\exp(K_1 + K_2) \subset \exp K_1 + \exp K_2$.

We need two more lemmas.

Lemma 6. *Let K_1, K_2 be disjoint convex bodies in E^d, let H be a hyperplane strictly separating K_1, K_2, and let l be the one-dimensional subspace in E^d orthogonal to H. For any $\varepsilon > 0$, there are distinct parallel hyperplanes H_1', H_2', and also distinct parallel hyperplanes H_1'', H_2'', such that:*

1. *H_1', H_1'' exposedly support K_1 and H_2', H_2'' exposedly support K_2.*
2. *Both H_1', H_2' separate K_1, K_2 and the strip between H_1'', H_2'' contains both K_1, K_2.*
3. *The one-dimensional subspace orthogonal to H_1', H_2' (resp. to H_1'', H_2'') both form with l an angle at most ε.*

Proof. Consider the convex body $M = K_1 - K_2$. We prove by induction on $d\ (\geq 2)$ the following assertion:

$(*)$ *For any* $\varepsilon > 0$ *and any orientation of* l *there is a half-space* P *exposedly supporting* M *and such that the outer normal of this half-space forms with* l*, taken in the given orientation, an angle at most* ε.

Without loss of generality, we may assume that the origin 0 is in int M. Let S be the unit Euclidean sphere in E^d. It is known (see, for instance, Corollary 25.1.3 of [4]) that, for a vector $u \in S$, a half-space with outer normal u supports M nonexposedly if and only if the boundary surface of the polar body M^* is not differentiable at any point z with an outer normal cone containing u. Since the set of singular boundary points of M^* has $(d-1)$-Lebesgue measure 0, the set of all outer normals u of half-spaces exposedly supporting M is dense in S. Now assertion $(*)$ easily follows.

We continue the proof of Lemma 6. Let P be a half-space described in $(*)$. Denote by z the (exposed) point at which P supports M. Since $z \in \exp M$, we have $z = z_1 - z_2$, where $z_1 \in \exp K_1$ and $z_2 \in \exp K_2$ (see Lemma 5). From the above and from Lemma 4 it follows that hyperplanes H_1, H_2 through z_1, z_2 and parallel to the boundary hyperplane of P exposedly support K_1, K_2, respectively. Clearly, either both K_1, K_2 lie between H_1, H_2 or both H_1, H_2 separate K_1, K_2, according to the orientation of l. □

Lemma 7. *Let* P *be a half-space of* E^d *exposedly supporting a convex body* K *at a point* $x \in \operatorname{bd} K$*, and let* u *be the outer normal of* P*. For any neighborhood* $U(x)$ *of* x *in* bd K *there is an* $\varepsilon > 0$ *such that every half-space* P'*, whose outer normal forms with* u *an angle at most* ε*, may support* K *in a subset of* $U(x)$ *only.*

Proof. Indeed, assume for a moment the existence of half-spaces $P_1, P_2, \ldots$ supporting K at points $z_1, z_2, \ldots \in \operatorname{bd} K$ such that $\|z_i - x\| \geq \delta$ for a suitable $\delta > 0$ and such that the angles formed by the outer normals to $P_1, P_2, \ldots$ with u are respectively at most $\varepsilon_1, \varepsilon_2, \ldots$, where $\varepsilon_i \to 0$. Due to the compactness arguments, we can suppose that $z_i \to z\ (\neq x)$ and $P_i \to P$. However, in this case P does not support K exposedly, contradicting the assumption of the lemma. □

4. Proofs of Main Results

Proof of Theorem 1. Let H be a hyperplane strictly separating K_1 and K_2. According to Lemma 4, there is a hyperplane parallel to H and strictly separating 0 from the difference $M = K_1 - K_2$. Let H' be the hyperplane parallel to H and supporting M such that both 0 and M are in the same half-space determined by H'. By Lemma 6, there is a hyperplane H'' sufficiently close to H' and exposedly supporting M at a point z, say. Since $z \in \exp M$, we have $z = x_1 - x_2$, where $x_1 \in \exp K_1$ and $x_2 \in \exp K_2$ (see Lemma 5). From the above and from Lemma 4 it follows that the hyperplanes H_1, H_2 parallel to H'' and passing through x_1, x_2, respectively, exposedly support K_1, K_2 and both K_1, K_2 lie between H_1, H_2. Thus $\bar{p}(K_1, K_2) \geq 1$.

In order to prove the inequality $\bar{q}(K_1, K_2) \geq d + 1$, again consider the set $M = K_1 - K_2$. We claim that M has at least d points exposed relative to 0. It is easily seen that every exposed point of M visible from 0 and contained in the interior of the cone $C_M = \{\lambda y: \lambda \geq 0, y \in M\}$ is exposed relative to 0.

Choose a point $v \in \operatorname{int} C_M \setminus M$ such that every ray starting at v and supporting M has exactly one point in common with M (see Lemma 3). In this case every exposed ray of the cone $C_M(v) = \{(1 - \lambda)v + \lambda y: \lambda \geq 0, y \in M\}$ intersects M at an exposed point of M. Since M is a convex body disjoint to v, the cone $C_M(v)$ is closed and contains no line (by Lemma 1). Then, due to Lemma 2, $C_M(v)$ has at least d exposed rays. Let $l_1, \ldots, l_d$ be some d of these rays. Denote by H_i a hyperplane in E^d such that

$$H_i \cap C_M(v) = l_i, \qquad i = 1, \ldots, d.$$

Clearly, H_i strictly separates 0 from M. By the above, $H_i \cap M$ is a point exposed for M relative to 0. Thus we have found at least d points exposed relative to 0.

Let $x_1, \ldots, x_d$ be d points in M exposed relative to 0. Denote by H_i a hyperplane strictly separating 0 from M such that $H_i \cap M = \{x_i\}$. Since $M = K_1 - K_2$, every point x_i is of the form $x_i = z_i' - z_i''$, where z_i', z_i'' are exposed points for K_1, K_2, respectively, and the hyperplanes H_i', H_i'' through z_i', z_i'' parallel to H_i satisfy

$$H_i' \cap K_1 = \{z_i'\}, \qquad H_i'' \cap K_2 = \{z_i''\}.$$

Trivially, both H_i', H_i'' separate K_1 and K_2. Hence $\{z_i', z_i''\}$, $i = 1, \ldots, d$, are pairwise distinct strictly adjoint pairs for K_1, K_2. Therefore $\bar{q}(K_1, K_2) \geq d$. □

Proof of Theorem 3. (1) ⇒ (2) and (3) ⇒ (4) are trivial.

(2) ⇒ (5) and (4) ⇒ (5). Assume, in order to obtain a contradiction, that K_1 is not a polytope. Then the number of exposed points of K_1 is infinite and, by the compactness of $\operatorname{bd} K_1$, there is a point $x \in \operatorname{bd} K_1$ any of whose neighborhoods contains infinitely many exposed points of K_1. Choose a point $y \in \operatorname{int} K_1$, and let $U(x)$ be a neighborhood of x in $\operatorname{bd} K_1$ such that every hyperplane supporting K_1 at a point in $U(x)$ intersects the ray $[y, x\rangle$ inside a given line segment $[x, z]$, $z \in [y, x\rangle \setminus K$. Now we can translate K_2 in a position K_2' such that:

(1) Any hyperplane H supporting K_1 at a point in $U(x)$ strictly separates K_2' from K_1.
(2) There is a hyperplane H' parallel to H, supporting K_2' and strictly separating K_1, K_2'.

Similarly, there is a translate K_2'' of K_2 disjoint to K_1 and satisfying the property: for every hyperplane H supporting K_1 at a point in $U(x)$ there is a hyperplane H'' parallel to H and supporting K_2'' such that both K_1, K_2'' lie between H, H'' and $H \cap K_2'' = \varnothing$, $H'' \cap K_1 = \varnothing$.

For a given integer m, let $x_1, \ldots, x_m$ be distinct exposed points of K_1 lying in $U(x)$. Denote by $V(x_i)$, $i = 1, \ldots, m$, some pairwise disjoint neighborhoods of $x_1, \ldots, x_m$ contained in $U(x)$. By Lemmas 6 and 7, there are m pairs of parallel hyperplanes $H_1, H_1', \ldots, H_m, H_m'$ such that, for every $i = 1, \ldots, m$:

(1) Both H_i, H_i' separate K_1 and K_2'.
(2) H_i exposedly supports K_1 at an exposed point $z_i \in V(x_i)$ and H_i' exposedly supports K_2'.

Hence $\bar{q}(K_1, K_2') \geq m$. Similarly, $\bar{p}(K_1, K_2'') \geq m$. Since m is chosen arbitrarily, $\bar{q}(K_1, K_2') = \bar{p}(K_1, K_2'') = \infty$.

(5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3). Since any antipodal or adjoint pair of points for polytopes K_1, K_2 consists of their vertices, any of $p(K_1, K_2)$, $q(K_1, K_2)$ is at most $m_1 m_2$, where m_1, m_2 are the numbers of vertices of K_1, K_2, respectively. □

Proof of Theorem 2. Due to Theorem 3, it is sufficient to consider the case when both K_1, K_2 are polytopes. Fix any vertex v of K_1 and let C be the cone generated by K_1 at v: $C = \{(1 - \lambda)v + \lambda y \colon \lambda \geq 0,\ y \in K_1\}$. Clearly, C is a convex polyhedral cone with apex x, so is the cone C' symmetric to C relative to v. Denote by $F_1, \ldots, F_m$ all facets of K_1 containing v. Now translate K_2 in a position $K_2' \subset \operatorname{int} C$ disjoint to K_1 such that for every vertex $w \in F_i \setminus \{v\}$ there are hyperplanes H, H' parallel to each other, both K_1, K_2 contained in the strip between them, with H exposedly supporting K_1 at w and H' supporting K_2'. We can slightly move H and H' simultaneously such that both H, H' will support K_1, K_2' exposedly. Hence every vertex $w \in F_1 \cup \cdots \cup F_m$ determines a strictly antipodal pair of vertices for K_1, K_2'.

Similarly, K_2 can be translated in a position $K_2'' \subset \operatorname{int} C'$ such that for every vertex $w \in F_i \setminus \{v\}$ there are hyperplanes G, G' parallel to each other, both separating K_1, K_2'', with G exposedly supporting K_1 at w and G' supporting K_2''. As above, we can slightly move G and G' simultaneously such that both G, G' will support K_1, K_2'' exposedly. This implies that every vertex $w \in F_1 \cup \cdots \cup F_m$ determines a strictly adjoint pair of vertices for K_1, K_2''.

Since the number of vertices of K_1 lying in $F_1 \cup \cdots \cup F_m$ is at least $d + 1$, we have $\bar{p}(K_1, K_2') \geq d + 1$ and $\bar{q}(K_1, K_2'') \geq d + 1$. □

Proof of Theorem 4. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (5). Due to Theorem 3, it can be assumed that both K_1 and K_2 are polytopes.

First consider the case $d \geq 3$. From the proof of Theorem 2 it follows that under condition (2) of the theorem, every vertex v of K_1 belongs to exactly d facets $F_1, \ldots, F_d$ and every facet $F_1, \ldots, F_d$ is a $(d - 1)$-simplex. Clearly, in this situation K_1 is a d-simplex if $d \geq 3$. Similarly, K_2 is a d-simplex. Moreover, if K_2 is translated in a position $K_2' \subset \operatorname{int} C$, where $C = \{(1 - \lambda)v + \lambda y \colon \lambda \geq 0,\ y \in K_1\}$, then each vertex of K_1 lying in $F_1 \cup \cdots \cup F_d$ belongs to exactly one strictly antipodal pair for K_1, K_2'. In particular, v determines exactly one strictly antipodal pair with a vertex z, say, of K_2'. It means that, for every hyperplane H supporting K_1 exposedly at v, the hyperplane H', parallel to H and supporting K_2' such that both K_1, K_2' are between H, H', has with K_2' exactly one point in common, namely,

z. This implies that the cones C and $C_2 = \{(1-\lambda)z + y\colon \lambda \geq 0,\ y \in K_2'\}$ are symmetric to each other. Since this conclusion holds for each vertex of K_1, K_2 is a negative homothetic copy of K_1.

Now let $d = 2$. Assume for a moment that one of K_1, K_2, say K_1, is not a triangle or a parallelogram. Then there are four consecutive vertices of K_1, say a, b, c, e, such that the half-lines $[a, b\rangle$ and $[e, c\rangle$ have a common point, x, exterior to K_1. Denote by D the cone with apex x bounded by $[b, a\rangle$ and $[c, e\rangle$. If we translate K_2 in a position $K_2' \subset \operatorname{int} D$ sufficiently far from K_1, then each of a, b, c, e determines a strict antipodal pair, i.e., $\bar{p}(K_1, K_2') \geq 4$, contradicting the hypothesis. Hence each of K_1, K_2 is either a triangle or a parallelogram. As in the case $d \geq 3$, for any vertex v_1 of K_1 there is a vertex v_2 of K_2 such that the sides of K_1 congruent to v_1 are parallel to the respective sides of K_2 congruent to v_2, and the outer normals to these sides of K_1 are opposite to the respective sides of K_2. Now it easily follows that either K_1 and K_2 are two parallelograms with parallel sides, or K_1 and K_2 are triangles negatively homothetic to each other.

Similar arguments are true under condition (4) of the hypothesis of Theorem 4.

(5) $\Rightarrow$ (1). If K_1, K_2 are convex d-polytopes in E^d, $d \geq 2$, and if $x_1 \in K_1$, $x_2 \in K_2$ form a strictly antipodal pair, the open outer normal cones of K_1 at x_1 and of $-K_2$ at $-x_2$ intersect. Since for a pair of negatively homothetic simplices K_1, K_2 there are at most $d+1$ pairs of intersecting open outer normal cones, we have $\bar{p}(K_1, K_2) \leq d + 1$. Similarly, any pair of disjoint parallelograms in E^2 with parallel sides, determines at most three pairs of strictly antipodal vertices.

The proof of (5) $\Rightarrow$ (3) is similar. □

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