Knotted Lattice-Like Space Fillers*

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Abstract. The aim of this note is to construct a solid torus, knotted in an arbitrarily given knot type, which tiles $\mathbb{E}^3$ by a lattice. More generally, a lattice-like space filler is constructed that is a handlebody of arbitrary genus, whose handles are arbitrarily knotted and arbitrarily linked with each other, and, furthermore, through which mutually disjoint tunnels have been drilled, knotted, and linked with each other arbitrarily.

1. Introduction

Throughout this note, the universe is the three-dimensional Euclidean space $\mathbb{E}^3$, although some of the concepts considered here can be defined in a more general setting. A family of sets is a *tiling* (of the space) if the sets have mutually disjoint interiors, and the union of the sets is the whole space. Tilings whose members (tiles) are polyhedra are usually considered. A polyhedron is a *space filler* if a tiling consisting of congruent replicas of the polyhedron exists. If a tiling exists in which each tile is a translate of the given polyhedron, then the polyhedron is a *translative space filler*, and if things can be arranged so that the family of vectors of the translations sending a tile to a tile is a lattice, then the space filler is *lattice-like*. There is a considerable body of knowledge concerning tilings of the plane (see the impressive, encyclopedic account of this subject by Grünbaum and Shephard [GS]), but in three dimensions (let alone higher), our understanding of the topic is rather limited. The “extra room” provided by the additional dimension admits an astounding variety of possibilities which do not exist in the “flat world” of the plane.

Debrunner [De] shows that handlebodies of arbitrary genus can be space fillers. Schulte [Su] studies toroidal space fillers whose symmetry groups exhibit some

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interesting properties. In one of the questions, Schulte asks about the existence of a space-filling knotted torus. In relation to this question, Danzer [Da] (unpublished) constructs a tiling which is the union of four families of polyhedral knotted tori, mutually congruent within each family. Danzer starts by tiling space with linked, unknotted congruent tori, each of which he then dissects into four topologically equivalent (but not congruent) knotted tori. Subsequently, Schmitt [Sm] constructs a space-filling polyhedral torus knotted in the trefoil knot, as well other types of knots. McMullen [Mc] (unpublished) shows that the trefoil-knotted solid torus can be a space filler while preserving the natural symmetry of the knot.

The construction described in the present note shows that a toroidal lattice-like space filler can be knotted in an arbitrary way. More generally, in a similar manner a lattice-like space filler can be constructed that is a polyhedral handlebody of arbitrary genus, whose handles are arbitrarily knotted and arbitrarily linked with each other, and, furthermore, through which mutually disjoint tunnels have been drilled, knotted, and linked with each other arbitrarily. Concurrently and independently, Adams [Ad] has obtained a more general result, by a slightly different approach. According to Adams, any three-manifold with connected boundary, embedded in $\mathbb{E}^3$, can be made into a (lattice-like) space filler without altering its embedding type.

2. Knotted Toroidal Space Fillers

We begin with the following simple example of an unknotted torus whose translations by the vectors $[n, 0, 0]$ ($n \in \mathbb{Z}$) fill the square beam

$$B = \{(x, y, z) \in \mathbb{E}^3: 0 \leq y \leq 1; 0 \leq z \leq 1\}$$

(see Fig. 1).

The torus consists of an essential piece (the left-hand side of the solid), and an inessential one, which looks like a solid box attached to the right-hand side of the essential part. The inessential part is designed to fit precisely in the hole of the essential one. Notice the small bridge connecting the two parts, and a notch in the top left edge, cut there to accommodate the bridge. Obviously, this polyhedral torus is a lattice-like space filler, under the lattice of translations by vectors whose coordinates are all integers.

Fig. 1

Fig. 2

The procedure of the construction described below is gradually to modify this simple example making it more intricate each time, while preserving its lattice-like space-filling property. The first modification is shown in Fig. 2: one inessential piece is placed to the left of the essential part of the torus, and one to the right of it. Each of the inessential pieces is connected with the central part by a small bridge, and the essential part has two notches cut in it to accommodate the bridges. Again, the inessential pieces are designed to fit precisely in the hole of the central part.

Figure 3 shows that the essential part of the torus can be made almost flat and turned vertically, leaving “negative” imprints on the two protruding inessential parts. The essential part of this torus is adjacent to (and almost contained in) the plane P_1 : $y = 0$. Notice the two bridges connecting the essential part with the other two, and the corresponding notches, which are this time carved in the inessential pieces. The back view of the inessential pieces (invisible in Fig. 3) is not very interesting. The back surface of each of them is flat, with no carvings in it.

Figure 4 displays the possibility of placing the essential part of the torus in the neighborhood of the plane P_2 : $z = \frac{1}{2}$. The imprint in the inessential piece on the left is clearly visible, a similar imprint on the underside of the inessential piece on the right is hidden from our view. As the translates of the torus by the vectors $[n, 0, 0]$ fill the beam B , the essential piece of the torus is “sandwiched” between the two inessential pieces of the adjacent translates of the torus.

In Fig. 5 we can see a space filler which is a handlebody of genus 2, whose construction, combining the ideas of the previous two examples, utilizes the planes P_1 and P_2 at the same time. In order to show how the two planes could be utilized more efficiently (which will be needed later), one more vertical handle should be added. The imprint of the additional handle would show on the front of the

Fig. 3

Fig. 4

inessential piece on the right. This detail would make the example in Fig. 5 look more symmetric, but it is not included for it would obscure the picture.

Observe that the set $(P_1 \cup P_2) \cap B$ is the Cartesian product $I \times T$ of the interval

$$I = \{(x, 0, 0) : 0 \leq x \leq 1\}$$

and the simple triod

$$T = \{(0, y, z) : y = 0, 0 \leq z \leq 1 \text{ or } z = \frac{1}{2}, 0 \leq y \leq 1\}.$$

The next modification, resulting in a space-filling knotted torus, concentrates on the set $I \times T$ as the place at which the knotting will occur. The following lemma sets ground for this modification.

Lemma 1. *Every knot in $\mathbb{E}^3$ can be embedded in $I \times T$.*

Proof. Any given knot can be placed in $\mathbb{E}^3$ in general position with respect to the xz -plane and so that its projection into that plane is contained in the relative interior of the square $0 \leq x \leq 1; 0 \leq z \leq 1$. (For a definition of the basic notion of general position of a knot with respect to a plane, a proof of existence of general positions, giving rise to the familiar diagram of a knot drawn as a planar polygonal loop with

Fig. 5

Fig. 6

cross-overs, see, for instance, [CF]). Moreover, the knot can be placed so that:

- (i) All crossing points of the projection lie on the line $y = 0, z = \frac{1}{2}$.
- (ii) Each cross-over is horizontal.
- (iii) Each cross-over lies on the side $y > 0$ of the xz -plane (this condition actually describes the defining characteristic of a cross-over, as if we were looking at the projection from the positive half of the y -axis).

A projection satisfying the above conditions can be easily turned into an embedding of the knot into $I \times T$. It suffices to push the cross-overs into the square $\{(x, y, \frac{1}{2}): 0 \leq x \leq 1, 0 \leq y \leq 1\}$ contained in $I \times T$. Figure 6 shows such an embedding of the trefoil knot. $\square$

With the above lemma as a tool, it is now easy to construct a knotted torus (of a given knot type) which tiles $\mathbb{E}^3$ by a lattice. Begin with the embedding of the given knot in $I \times T$, then fatten up the knot by a narrow tubular neighborhood so it becomes a torus (the segments of the knot which lie in the xz -plane are fattened up only to the side $y \geq 0$, to keep the torus within the beam B). Then attach to it two inessential pieces, much in the same way as shown in Fig. 5, with corresponding valleys carved in each of the pieces, the precise impressions of the knot.

A complete drawing to illustrate this construction, even in the case of the trefoil knot, would be too large to be included here or too crowded to be clear. Figure 7 shows three important details of the construction: a single cross-over in the central (essential) part and its imprints in the left and right (inessential) parts, taken out from the complete solid and magnified. Combining Figs. 6 and 7, the reader should have a clear image of the space-filling trefoil knot.

The construction described above results in the following theorem:

Theorem 1. *A lattice-like space-filling toroidal polyhedron, knotted in an arbitrary knot, exists.*

Fig. 7

Remark 1. No two tori of the tiling constructed above are linked (as intended), but if one wishes some linking to occur, a minor modification of the construction can produce linkings of various kinds. For instance, each torus can be linked with its two neighbors in the same beam, forming an infinite chain. Additionally, the tori from adjacent chains can be linked, too.

3. Knotted Handlebodies with Tunnels

A handlebody embedded in $\mathbb{E}^3$ is equivalent to a regular neighborhood of an embedded graph. Thus, for the generalization of the construction of the previous section to arbitrarily embedded handlebodies (without tunnels), all we need is the following generalization of Lemma 1.

Lemma 2. *Every embedding of a (finite) graph in $\mathbb{E}^3$ is topologically equivalent to an embedding of the graph in $I \times T$.*

The proof of this lemma is almost identical to the proof of Lemma 1, the only difference is that we adjust the position of a graph instead of just a simple closed curve. The fact that any piecewise-linear embedding of a graph in $\mathbb{E}^3$ can be approximated by an embedding in general position is proven in the same way as it is done for knots.

Proceeding as in the previous section, we construct a handlebody embedded in $\mathbb{E}^3$ in prescribed types of knots and links, which tiles by a lattice, and looks very much like the solid in Fig. 5. The main part of the space filler (the very handlebody) is in the middle, and it has two inessential (in the same sense as before) pieces attached to it, one in the front and one in the back, each having valleys carved in it to fit the central part.

If we now wish to drill tunnels (resembling wormholes) through the handlebody, we can do so without much trouble, one tunnel at a time. A tunnel begins at an entry disk and terminates at an exit disk, disjoint from the entry. We may assume that both disks, the entry and the exit, lie on the boundary of the central part. Moreover, things can be arranged so that the entry's imprint lies on the left inessential part, and the exit's imprint lies on the right one. Now, the portion of the handlebody to be carved out from it to create the tunnel (which looks like a long and

Fig. 8

narrow curved cylinder) can be divided into two topological balls by a disk partitioning the tunnel somewhere in the tunnel's interior, and the two parts shifted to their corresponding left and right inessential parts, where they get attached along the imprinted entry disk and exit disk, respectively (see Fig. 8).

Additional tunnels can be created in the same manner. The entire construction can be made piecewise-linear, so as to result in a polyhedron. This detail concludes the construction which demonstrates the following theorem:

Theorem 2. *A lattice-like space-filling polyhedral handlebody with tunnels, whose handles and tunnels are arbitrarily knotted and linked, exists.*

Remark 2. (In the spirit of Remark 1.) No two solids of the tiling constructed above are linked, but if one wishes some linking to occur, a minor modification of the construction can produce linkings of various kinds. For instance, each solid can be linked with its two neighbors in the same beam, forming an infinite chain. Furthermore, the solids from adjacent chains can be linked as well.

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