On the Connected Components of the Space of Line Transversals to a Family of Convex Sets*

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Abstract. Let $\mathcal{L}$ be the space of line transversals to a finite family of pairwise disjoint compact convex sets in $\mathbb{R}^3$. We prove that each connected component of $\mathcal{L}$ can itself be represented as the space of transversals to some finite family of pairwise disjoint compact convex sets.

Introduction

Let $\mathcal{A}$ be a family of convex sets in $\mathbb{R}^d$. A k -transversal for $\mathcal{A}$ is a k -flat, an affine subspace of dimension k , that intersects every member of $\mathcal{A}$. (For basic facts about transversal theory, see [1] or [3].) Let $\mathcal{A}^*$ be the set of all k -transversals of $\mathcal{A}$, considered as a subspace of the affine Grassmannian $G'_{k,d}$ of all k -flats in $\mathbb{R}^d$. We are interested in studying the connected components of this space $\mathcal{A}^*$ and, in particular, in the question of whether each of these connected components can be represented as the space of k -transversals of some other family $\mathcal{B}$ of convex sets.

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This question was posed in [2], where it was shown that a convexity structure can be defined on $G'_{k,d}$ in which the convex sets are precisely the sets of k -transversals to families of convex point sets in $\mathbb{R}^d$. This convexity structure coincides with the usual one on $\mathbb{R}^d$ if $k = 0$, and retains many of its properties for $k > 0$. In particular, the *convex hull* of a family $\mathcal{F}$ of k -flats can be defined by means of a duality operator, $*$, between sets of k -flats and families of convex point sets, as follows. If $\mathcal{F}$ is a family of k -flats and $\mathcal{A}$ is a family of convex point sets in $\mathbb{R}^d$, let $\mathcal{F}^*$ be the family of convex point sets meeting every member of $\mathcal{F}$ and let $\mathcal{A}^*$ be the family of k -flats meeting every member of $\mathcal{A}$. Then $\text{conv}(\mathcal{F})$, the convex hull of $\mathcal{F}$, is simply $\mathcal{F}^{**}$. A family $\mathcal{F}$ of k -flats is *convex* if $\text{conv}(\mathcal{F}) = \mathcal{F}$. It is easily seen that $\mathcal{A}^*$ is convex for any family $\mathcal{A}$ of convex point sets.

One can equally well obtain the convex hull as follows. A family $\mathcal{F}$ of k -flats is said to *surround* a k -flat f if there is some j -flat g containing f such that every $(j - 1)$ -flat h containing f and lying in g strictly separates two k -flats of $\mathcal{F}$ lying in g ; i.e., h cannot “escape to infinity” within g without containing a member of $\mathcal{F}$ after moving some positive distance. (If $j = k$ we interpret this condition as meaning that $f \in \mathcal{F}$.) A family $\mathcal{F}$ of lines in $\mathbb{R}^3$, for example, surrounds a line f if either

- (i) $f \in \mathcal{F}$,
- (ii) f lies strictly between two lines of $\mathcal{F}$ with all three contained in a plane g , or
- (iii) each plane h through f lies strictly between two lines of $\mathcal{F}$ parallel to h .

The family of all k -flats surrounded by $\mathcal{F}$ turns out to be exactly the convex hull of $\mathcal{F}$ [2]. Notice that this characterization of the convex hull applies equally well to 0-flats (or points) in $\mathbb{R}^d$, where it agrees with the standard one for point sets. (For a complete discussion of convexity on the affine Grassmannian, see [2].)

If $\mathcal{F}$ is a convex set of k -flats in $\mathbb{R}^d$, then $\mathcal{F} = \mathcal{A}^*$ for some family $\mathcal{A}$ of convex point sets in $\mathbb{R}^d$; $\mathcal{F}$ is then said to be *presented* by $\mathcal{A}$. A convex set $\mathcal{F}$ may be presented by many different families of convex point sets.

A convex set of k -flats need not be connected for $k > 0$. For instance, a finite family of mutually skew lines in $\mathbb{R}^3$ is convex, by the “surrounding” criterion. Problem 7.2 of [2] asks whether the connected components of a convex set of k -flats are necessarily convex. In general the answer turns out to be negative, as we show by exhibiting an example of a convex family of lines in $\mathbb{R}^3$ that has a nonconvex connected component. If, however, a convex set of k -flats in $\mathbb{R}^d$ is presented by some finite family of suitably separated compact convex point sets, then we conjecture that the connected components of $\mathcal{F}$ will themselves be convex. We prove this conjecture for the case of a convex set of lines in $\mathbb{R}^3$ presented by a finite family of pairwise disjoint compact convex point sets; this is the main result of this paper.

A Nonconvex Connected Component

In this section we exhibit a connected set $\mathcal{L}$ of lines in $\mathbb{R}^3$ whose convex hull has two arcwise connected components. One of these components must contain $\mathcal{L}$, so it cannot be convex.

Start with the set of all the tangents to a unit circle centered at O in the (x, y) -plane. Let α be an angle irrational with respect to π and for each $n = 1, 2, \dots$ translate the pair of lines tangent at $(\cos n\alpha, \sin n\alpha)$ and $(-\cos n\alpha, -\sin n\alpha)$ toward O until they are each at distance $1/n$ from it. For each direction θ , $0 \leq \theta < \pi$, we now have a pair of parallel lines around O in direction θ ; fill in all the lines between these two parallels in each direction θ , getting a strip S_θ . It follows from the irrationality of α that arbitrarily close to each direction there are arbitrarily narrow strips.

Finally, for each θ , $0 \leq \theta < \pi$, translate this strip S_θ upward to a strip T_θ at height $z = \theta$. This gives a “spiral staircase” $\mathcal{L}$ having the z -axis as an axis of symmetry, closed at the height $z = 0$ and open at the height $z = \pi$.

Each line in a strip T_θ is connected to the member of T_θ passing through the z -axis. Since the latter form a connected family for $\theta \in [0, \pi)$, $\mathcal{L}$ is connected. We claim that the convex hull of $\mathcal{L}$ is $\mathcal{L} \cup \{z\text{-axis}\}$, so that $\text{conv}(\mathcal{L})$ has two components, $\mathcal{L}$ and the z -axis. We show this by using the “surrounding” criterion above.

The fact that the z -axis $\in \text{conv}(\mathcal{L})$ is seen by considering any plane through the z -axis; since each T_θ has positive width this plane will be trapped by lines in $\mathcal{L}$ strictly parallel to it, so that the z -axis is surrounded by $\mathcal{L}$.

It remains only to show that no other line $l \notin \mathcal{L}$ lies in $\text{conv}(\mathcal{L})$. If l is parallel to the (x, y) -plane, any plane Π through l not itself parallel to the (x, y) -plane has lines of only a single strip T_θ parallel to itself, with T_θ depending only on the line l , so that by rotating Π around l we can find a position in which Π lies entirely on one side of T_θ , hence escapes in the opposite direction. Suppose, then, that l is not horizontal. Let Π be any plane through l . Π meets the (x, y) -plane in a line having some direction θ . Look at the plane $z = \theta$. By varying Π slightly, if necessary, we may assume that the intersection l_1 of Π with that plane misses the z -axis, say by some distance $\varepsilon > 0$. Since l_1 varies continuously with the choice of Π through l , we can choose Π' sufficiently close to Π so that the corresponding line l' (at height θ' equal to its horizontal direction) lies at distance $> \varepsilon/2$ from the z -axis, but so that the strip in the direction parallel to l' lies entirely within $\varepsilon/2$ of the z -axis (i.e., has width $< \varepsilon$). Since the only lines that can “block” Π' in either direction are those parallel to Π' , hence in the strip in question, it follows that Π' can escape, i.e., that l is not in the convex hull of $\mathcal{L}$.

Convex Connected Components

In this section it is more convenient to work in the affine oriented Grassmannian, $\bar{\mathbf{G}}'_{1,d}$, consisting of all oriented lines in $\mathbb{R}^d$. If $\mathcal{A}$ is a family of convex sets in $\mathbb{R}^d$, let $\mathcal{A}^\circledast$ represent the family of oriented k -flats meeting all the members of $\mathcal{A}$. Clearly, each unoriented k -transversal of $\mathcal{A}$ gives rise to a pair of oriented k -transversals of $\mathcal{A}$.

Let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^d$ and let $\mathcal{A}^\circledast$ be its set of oriented line transversals. Each line in $\mathcal{A}^\circledast$ induces a linear ordering on the sets in $\mathcal{A}$ corresponding to the order in which the line meets the members of $\mathcal{A}$. Each transversal within a given connected component of $\mathcal{A}^\circledast$ induces

the same linear ordering on $\mathcal{A}$. (This follows easily from the pairwise disjointness; for more details, and a generalization to k -transversals for $k > 1$, see [3].) Thus each connected component of $\mathcal{A}^\circ$ is associated with some linear ordering of $\mathcal{A}$. Similarly, each connected component of the space $\mathcal{A}^*$ of unoriented line transversals is associated with a pair of linear orderings of $\mathcal{A}$ (usually referred to as a *geometric permutation* of $\mathcal{A}$ [4].)

Two or more connected components of $\mathcal{A}^\circ$ may be associated with the same linear ordering of $\mathcal{A}$. (For an example, see p. 174 of [3]). The exception is in $\mathbb{R}^2$, where components of $\mathcal{A}^\circ$ are associated with distinct linear orderings, and (likewise) components of $\mathcal{A}^*$ with distinct pairs of linear orderings. More generally, if $\mathcal{A}$ is suitably separated, distinct components of the space of hyperplane transversals are associated with distinct *order types* (see [3]). A proof for unoriented hyperplane transversals in $\mathbb{R}^d$, which can easily be adapted to the oriented case, can be found in [5]. We include a proof here for the case of oriented lines in $\mathbb{R}^2$ for the convenience of the reader.

Lemma 1. *Let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^2$. Each connected component of $\mathcal{A}^\circ$ determines a distinct linear ordering of $\mathcal{A}$.*

Proof. Suppose directed lines $l, l' \in \mathcal{A}^\circ$ meet the members of $\mathcal{A}$ in the same order $a_1, \dots, a_n$. We show that l can be moved continuously to l' through $\mathcal{A}^\circ$.

If $\mathcal{A}$ consists of a single set, then $\mathcal{A}^\circ$ is connected and the theorem is trivially true.

Assume $\mathcal{A}$ contains two or more sets. If l is parallel to l' , then it must have the same orientation as l' . Then all the parallel lines between l and l' with the same orientation as l lie in $\mathcal{A}^\circ$ and connect l to l' .

Suppose l intersects l' at a point p . Lines l and l' divide the plane into four quadrants labeled 1, 2, 3, 4 in clockwise order as in Fig. 1. For each set $a_i \in \mathcal{A}$, choose points $\alpha_i \in l \cap a_i$ and $\alpha'_i \in l' \cap a_i$. Each line segment $\overline{\alpha_i \alpha'_i}$ lies in a single quadrant. (If a segment lies on one of the quadrant boundaries, assign it arbitrarily to one of the adjacent quadrants.)

Two or more line segments cannot lie in quadrants 2 and 4, since l and l' would meet these segments in different orders. If a single segment, $\overline{\alpha_k \alpha'_k}$, is contained in these two quadrants, no endpoint α_i or α'_i lies between α_k and p or α'_k and p . Rotate l around α_k until it passes through α'_k and then rotate it around α'_k to l' . If no segments are contained in quadrants 2 and 4, rotate l around p to l' . $\square$

Fig. 1. Directed lines inducing the same linear ordering.

Lemma 1 has a simple corollary for line transversals in any dimension.

Corollary 1. *Let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^d$. If $l, l' \in \mathcal{A}^\circ$ meet the members of $\mathcal{A}$ in the same order and are coplanar, then l and l' lie in the same connected component of $\mathcal{A}^\circ$.*

Proof. Assume that $l, l' \in \mathcal{A}^\circ$ generate the same ordering on $\mathcal{A}$ and that l and l' both lie in a plane Π . Let $\mathcal{A}' = \{\Pi \cap a \mid a \in \mathcal{A}\}$, considered as a family of subsets of Π . By Lemma 1, l and l' are connected through the space of transversals of $\mathcal{A}'$. Since every transversal of $\mathcal{A}'$ is a transversal of $\mathcal{A}$, it follows that l and l' lie in the same connected component of $\mathcal{A}^\circ$. $\square$

We refer to a vector v as a *positive combination* of vectors $v_1, v_2, \dots, v_n$ if $v = \sigma_1 v_1 + \sigma_2 v_2 + \dots + \sigma_n v_n$ for real numbers $\sigma_i \geq 0$. The set of all positive combinations of finitely many vectors belonging to a set V is the *positive hull* of V ; it is a convex cone centered at the origin O .

As before, let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^d$. For each directed line l , let v_l be the unit vector in the direction of l . For each connected component $\mathcal{L} \subseteq \mathcal{A}^\circ$, define $C(\mathcal{L})$ to be the positive hull of the vectors $v_l, l \in \mathcal{L}$.

Lemma 2. *Let $\mathcal{A}$ be a finite family of at least two pairwise disjoint compact convex point sets in $\mathbb{R}^3$ and let $\mathcal{L}$ be a connected component of $\mathcal{A}^\circ$. A directed line $l \in \mathcal{A}^\circ$ belongs to $\mathcal{L}$ if and only if v_l belongs to $C(\mathcal{L})$.*

Proof. If l is in $\mathcal{L}$, then by definition $C(\mathcal{L})$ contains v_l . Assume that l lies in $\mathcal{A}^\circ$ but not in $\mathcal{L}$. We wish to show that v_l does not lie in $C(\mathcal{L})$.

Let $\tilde{l}$ be some line in $\mathcal{L}$, and suppose first that l and $\tilde{l}$ meet $\mathcal{A}$ in different orders. There is some pair of sets $a, b \in \mathcal{A}$ such that l meets a before b and $\tilde{l}$ meets b before a . Let h be a plane separating a from b . Every line in $\mathcal{L}$ must point toward the half-space bounded by h containing a , while line l points toward the half-space bounded by h containing b . Thus v_l cannot be a positive combination of the vectors $v_{l'}, l' \in \mathcal{L}$.

Now suppose l meets the members of $\mathcal{A}$ in the same order that $\tilde{l}$ does. Let Π be a plane perpendicular to l , and denote by $\Pi(a)$, $\Pi(l')$, and $\Pi(v)$ the orthogonal projection of any set $a \in \mathcal{A}$, any line $l' \in \mathbb{R}^3$, and any vector $v \in \mathbb{R}^3$ onto Π . If v_l is a positive combination of vectors $v_{l'}, l' \in \mathcal{L}$, then the origin, O , is in the convex hull of $\Pi(v_{l'}), l' \in \mathcal{L}$.

Let $\kappa = \bigcap_{a \in \mathcal{A}} \Pi(a)$. (See Fig. 2.) Notice that if a line l' is parallel to l , then l' is in $\mathcal{A}^\circ$ if and only if $l' \cap \Pi$ lies in κ . Each directed line of $\mathcal{A}^\circ$ parallel to l and oriented in the same direction is clearly in the same connected component of $\mathcal{A}^\circ$ as l . By Corollary 1, no such line intersects a line of $\mathcal{L}$. Thus $\Pi(l')$, for each line $l' \in \mathcal{L}$, misses κ .

Without loss of generality, assume that κ lies to the left of $\Pi(\tilde{l})$. Since $\Pi(l')$ is a directed line that misses κ for every $l' \in \mathcal{L}$ and $\mathcal{L}$ is connected, κ must be to the left of $\Pi(l')$ for every $l' \in \mathcal{L}$. Let t be the translate of $\Pi(\tilde{l})$ supporting κ on its left.

Fig. 2. Projection of sets onto plane Π .

Line t does not support any $\Pi(a)$, $a \in \mathcal{A}$, or else $\tilde{l}$ would not intersect a . Thus $t \cap \kappa$ consists of only a single point p , since otherwise some line strictly between $\tilde{l}$ and t would already support κ . Let t^+ be the open half-plane in Π bounded by t not containing κ . By Helly's theorem, there are two distinct point sets $\Pi(a)$, $\Pi(b)$, with $a, b \in \mathcal{A}$, such that $\Pi(a) \cap \Pi(b) \cap t^+ = \emptyset$.

Let t_a and t_b be supporting lines to $\Pi(a)$ and $\Pi(b)$ at p . Orient t_a and t_b in directions v_a and v_b , respectively, so that $\Pi(v_{\tilde{l}})$ lies in the open cone between v_a and v_b . For each $l' \in \mathcal{L}$, the vector $\Pi(v_{l'})$ is not a positive scalar multiple of v_a or v_b or else l' would not intersect a or b . Since $\mathcal{L}$ is connected, all the vectors $\Pi(v_{l'})$, $l' \in \mathcal{L}$, also lie in the open cone between v_a and v_b . Thus 0 is not a convex combination of $\Pi(v_{l'})$, $l' \in \mathcal{L}$, and so v_l is not in the positive hull of $v_{l'}$, $l' \in \mathcal{L}$. $\square$

The following simple lemma holds for k -transversals in $\mathbb{R}^d$ for any k and d .

Lemma 3. *Let $\mathcal{A}$ be a nonempty family of convex point sets in $\mathbb{R}^d$ and let $\mathcal{A}^*$ and $\mathcal{A}^\circ$ represent the sets of unoriented and oriented k -transversals of $\mathcal{A}$, respectively. If every point set in $\mathcal{A}$ is compact, then so are $\mathcal{A}^*$ and $\mathcal{A}^\circ$.*

Proof. For each compact convex set $a \in \mathcal{A}$, the set of (oriented) k -transversals of a is compact. $\mathcal{A}^*$ ($\mathcal{A}^\circ$) is the intersection of these compact sets of (oriented) k -transversals and so is also compact. $\square$

We return to lines in $\mathbb{R}^3$ where $\mathcal{A}^\circ$ is the set of oriented line transversals of $\mathcal{A}$. Given a set $\mathcal{L}$ of directed lines, let us denote the set of lines with reverse orientation by $\mathcal{L}^{\text{rev}}$. If $\mathcal{L}$ is a connected component of $\mathcal{A}^\circ$, then clearly $\mathcal{L}^{\text{rev}}$ is also.

Theorem 1. *Let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^3$. If $\mathcal{L}$ is a connected component of $\mathcal{A}^\circ$, then $\mathcal{L} \cup \mathcal{L}^{\text{rev}}$ is convex. Moreover,*

$\mathcal{L} \cup \mathcal{L}^{\text{rev}}$ is itself the space of oriented line transversals of some finite family of pairwise disjoint compact convex point sets.

Proof. If $\mathcal{A}$ consists of a single convex set, then $\mathcal{A}^{\circledast}$ has just one connected component and the theorem is trivially true. We therefore assume that $\mathcal{A}$ consists of more than one convex set.

Let $\mathcal{L}$ be a connected component of $\mathcal{A}^{\circledast}$. Let h be some plane strictly separating a pair of sets in $\mathcal{A}$ translated to pass through O . The vectors v_l , $l \in \mathcal{L}$, all lie in one of the open half-spaces, say h^+ , bounded by h . Thus $C(\mathcal{L})$ lies in h^+ as well.

By Lemma 3, since $\mathcal{A}$ is compact, so is $\mathcal{A}^{\circledast}$. Hence so are the sets $\mathcal{L}$ and $\mathcal{A}^{\circledast} \setminus \mathcal{L}$, and the convex cone $C(\mathcal{L})$ is closed. By Lemma 2, if $l \in \mathcal{A}^{\circledast} \setminus \mathcal{L}$, then v_l does not lie in $C(\mathcal{L})$. Hence, by compactness, there is an angle $\alpha > 0$ such that the angle between v_l , for any $l \in \mathcal{A}^{\circledast} \setminus \mathcal{L}$, and any vector $v \in C(\mathcal{L})$ is always greater than α . We can therefore enlarge $C(\mathcal{L})$ slightly to form a new closed convex cone C' with vertex O such that $C(\mathcal{L})$ lies in the interior of C' , $C' \subset h^+$, and v_l does not lie in C' for any $l \in \mathcal{A}^{\circledast} \setminus \mathcal{L}$.

Consider a directed line $l \in \mathcal{A}^{\circledast}$. Line l belongs to $\mathcal{L} \cup \mathcal{L}^{\text{rev}}$ if and only if v_l or $-v_l$ lies in the interior of C' . Hence $l \in \mathcal{L} \cup \mathcal{L}^{\text{rev}}$ if and only if $l \cap C'$ is unbounded. By compactness, there is a ball B centered at O such that $C \cap l$ is contained in B for every l in $\mathcal{A}^{\circledast} \setminus (\mathcal{L} \cup \mathcal{L}^{\text{rev}})$. Similarly by compactness, there is a ball B' centered at O such that $C \cap l$ meets B' for every $l \in \mathcal{L}$. Finally, let B'' be a ball centered at O containing all the sets $a \in \mathcal{A}$. Call the largest of these three balls B''' . Translate the plane h to a plane h' that intersects the cone C' but not the ball B''' . Let $c = C' \cap h'$, and notice that since $C' \subset h^+$ was closed, c is compact. Then every line in $\mathcal{L} \cup \mathcal{L}^{\text{rev}}$ intersects c , while no line in $\mathcal{A}^{\circledast} \setminus (\mathcal{L} \cup \mathcal{L}^{\text{rev}})$ does. Thus $\mathcal{A} \cup \{c\}$ consists of pairwise disjoint compact convex sets, and we have $\mathcal{L} \cup \mathcal{L}^{\text{rev}} = (\mathcal{A} \cup \{c\})^{\circledast}$. $\square$

Our desired result then follows as a corollary.

Corollary 2. *Let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^3$. If $\mathcal{L}$ is a connected component of $\mathcal{A}^*$, then $\mathcal{L}$ is convex. Moreover, $\mathcal{L}$ is itself the space of oriented line transversals of some finite family of pairwise disjoint compact convex point sets.*

Proof. Again, we may assume without loss of generality that $\mathcal{A}$ consists of at least two convex sets. Since the sets in $\mathcal{A}$ are disjoint, $\mathcal{L}$ gives rise to two connected components of $\mathcal{A}^{\circledast}$ coming from the two orientations of lines in $\mathcal{L}$. By Theorem 1, there is a compact convex set c disjoint from each of the sets in $\mathcal{A}$ such that the two components together constitute the space of directed line transversals of $\mathcal{A} \cup \{c\}$. Thus $\mathcal{L}$ is the space of undirected line transversals of $\mathcal{A} \cup \{c\}$. $\square$

References

1. Eckhoff, J. Helly, Radon, and Carathéodory type theorems. In *Handbook of Convex Geometry*, P. M. Gruber and J. M. Wills, eds. North-Holland, Amsterdam, 1993, pp. 389–448.

2. Goodman, J. E., and Pollack, R. Foundations of a theory of convexity on affine Grassmann manifolds. *Mathematika* (to appear).
3. Goodman, J. E., Pollack, R., and Wenger, R. Geometric transversal theory, In *New Trends in Discrete and Computational Geometry*, J. Pach, ed. Springer-Verlag, Heidelberg, 1993, pp. 163–198.
4. Katchalski, M., Lewis, T., and Zaks, J. Geometric permutations for convex sets. *Discrete Math.* **54** (1985), 271–284.
5. Wenger, R. Geometric permutations and connected components. Tech. Report TR-90-50, DIMACS, 1990.

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