Covering the Plane with Two Kinds of Circles*

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Abstract. A sharp lower bound is given for the density of a covering of the plane with two kinds of circles

1. Introduction

Let us recall the classical results of Thue [10] and Kershner [8] according to which the density of a packing of equal circles in the plane is at most $\pi/\sqrt{12}$ and the density of a covering of the plane with equal circles is at least $2\pi/\sqrt{27}$. Both density bounds are sharp and are attained by the incircles and circumcircles of a regular hexagonal tiling, respectively. The book *Lagerungen in der Ebene, auf der Kugel und im Raum* [5] which laid the foundations for the theory of packing and covering contains a great variety of generalizations of these theorems. As a starting point of the investigations of the present paper, we mention the following result [3], [5, p. 72]:

The density of a packing in the plane consisting of n kinds of circles is at most

$$1 - \left(1 - \frac{\pi}{\sqrt{12}}\right)^n.$$

The geometric meaning of this density bound is the following: If we want to pack the plane possibly densely with n kinds of circles of arbitrary size, then we must pack the plane with unit circles as densely as possible, fill the gaps with very small circles, fill the gaps still remaining with equal circles which are very small compared with the second kind of circles, and so on.

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Fig. 1

The dual counterpart of this problem, to cover the plane possibly thinly with two, three, or more kinds of circles, is much more difficult. Here we solve this problem for two kinds of circles.

We use the following notation: $\text{int } S$, $\text{bd } S$, $\text{cl } S$, and $\text{conv } S$ denote the interior, boundary, closure, and convex hull of the set $S \in E^2$. If S is Lebesgue measurable, then its area is denoted by $a(S)$. The cardinality of a set X is denoted by $|X|$.

We consider an arrangement $\mathcal{A}$ of congruent circles concentric with the tiles of a regular hexagonal tiling (see Fig. 1). Let d be the density of $\mathcal{A}$. An easy calculation shows that the quotient of the area of the portion of one hexagon left uncovered by the circles and the area of the hexagon is

$$\sigma(d) = \begin{cases} 1 - d & \text{for } 0 < d \leq \frac{\pi}{\sqrt{12}}, \\ 1 - d + \frac{6d}{\pi} \arccos \frac{1}{\sqrt[4]{12}} \sqrt{\frac{\pi}{d}} & \text{for } \frac{\pi}{\sqrt{12}} < d \leq \frac{2\pi}{\sqrt{27}}, \\ -\sqrt{\frac{6\sqrt{3}d - 3\pi}{\pi}} & \text{for } \frac{2\pi}{\sqrt{27}} < d. \end{cases} \quad (1)$$

Next we consider a system $\tilde{\mathcal{A}}$ of congruent circles in a regular hexagonal arrangement covering the plane with density $2\pi/\sqrt{27}$. We obtain a covering with two kinds of circles by supplementing $\mathcal{A}$ by those circles from $\tilde{\mathcal{A}}$ which intersect the part of the plane left uncovered by the circles of $\mathcal{A}$. Choosing the circles of $\tilde{\mathcal{A}}$ sufficiently small compared with those of $\mathcal{A}$, the density of the covering so obtained is arbitrarily close to

$$d + \frac{2\pi}{\sqrt{27}} \sigma(d).$$

Taking the minimum for d , we see that there are coverings of the plane with two kinds of circles whose density is arbitrarily close to

$$d_2 = \min_{0 \leq d \leq 2\pi/\sqrt{27}} \left(d + \frac{2\pi}{\sqrt{27}} \sigma(d) \right) = 0.91438 \dots$$

$$+ \frac{2\pi}{\sqrt{27}} \sigma(0.91438 \dots) = 1.018955 \dots \quad (2)$$

On the other hand, we show that any covering of the plane with two kinds of circles has density at least d_2 . This follows immediately from the following:

Theorem. *Let R be a rectangle and suppose that some circles of two distinct radii with total area t cover R . Then*

$$t \geq d_2 a(R),$$

where d_2 is the constant defined by relations (1) and (2).

A lower bound for the density of a covering with two kinds of circles whose ratio of areas is $1 : q$, $q \geq 1$, was given by Florian [6]. The bound d_2 is larger than his for $q > 26.1$.

Let d_i be the infimum of the densities of all coverings of the plane with i kinds of circles. Then, according to Kershner, we have $d_1 = 2\pi/\sqrt{27}$. Our theorem states that for $i = 2$ we have

$$d_i = \min_{0 \leq d \leq 2\pi/\sqrt{27}} (d + d_{i-1} \sigma(d)).$$

It can be conjectured that this is true for all i .

2. Proof of the Theorem

We start by collecting some known results used in the proof.

Lemma 1. *Let the function $f(q)$ be defined for $q \geq 1$ as*

$$f(q) = \min_{0 \leq m \leq 3, 0 < l} \frac{2\pi(mq + l)}{l(6 - m) \sin(2\pi/(6 - m)) + m(6 + l)q \sin(2\pi/(6 + l))}.$$

If a convex polygon H with at most six sides is covered by circles such that the ratio of areas of any two circles is at most q , then the total area of the circles is at least $a(H)f(q)$.

The proof of a little bit weaker result than Lemma 1 is given in [2] and reproduced on pp. 73–74 of [5]. On p. 79 of [5] the dual counterpart of Lemma 1 for packings is stated explicitly with the note that its proof is immediate. The proof of Lemma 1 is completely analogous. We observe that, in fact, the easy proof yields that it is sufficient to take the minimum for integers l and m .

The following result is due to Hadwiger [7] (see also p. 93 of [5] for a proof).

Lemma 2. *Any simply connected region with area a and perimeter p can be covered with*

$$\left[\frac{2}{\sqrt{27}}a + \frac{2}{\pi\sqrt{3}}p + 1 \right]$$

unit circles.

The following result of L. Fejes Tóth was first proved on pp. 80–84 of [5].

Lemma 3. *Let H be a convex polygon with at most six sides and let $C_1, \dots, C_n$ be congruent circles of total area t . Further, let U be the subset of H which is left uncovered by the circles. Then*

$$a(U) \geq a(H) \sigma \left(\frac{t}{a(H)} \right).$$

We need some preparation before the formulation of our last lemma. In what follows it is convenient to regard a half-plane as a circular disk of infinite radius. We consider three closed circular disks, two with unit radius and one with radius $\varrho \geq 1$. We vary the circles under the assumption that their intersection is nonempty and arrange them so that the area of the region covered by at least two of the circles is minimum. It is easily seen that the minimum is attained in the case when the intersection of the circles is a single point and the three chords common to each pair of the circles have equal lengths. The configuration is uniquely determined by this property, and an easy computation gives that, in this case, the area of the region covered by at least two of the circles equals

$$T(\varrho) = 4 \arcsin a + 2\varrho^2 \arcsin \frac{a}{\varrho} - 4a\sqrt{1-a^2} - 2a\sqrt{\varrho^2-a^2},$$

where

$$2a = \frac{-1 + \sqrt{1 + 8\varrho^2}}{2\varrho}$$

denotes the length of a chord common to a pair of circles.

We have

$$T(1) = \pi - \frac{\sqrt{27}}{2} = 0.543516\dots$$

and

$$T(\infty) = \pi - 2 = 1.141592\dots > 2T(1).$$

Further, it is clear from the geometric meaning of $T(\varrho)$ that it is an increasing continuous function. Therefore there is a unique number $\varrho_0 > 1$ such that

$$T(\varrho_0) = 2T(1).$$

Numerical calculation shows that

$$\varrho_0 = 16.7473\dots$$

Now we are ready to formulate our last lemma.

Lemma 4. *Let U be a bounded region which is the complement of the union of some circles of radius greater than or equal to ϱ_0 . Suppose that some unit circles of total area t cover U so that the area of each component of U which is covered by a single circle is at most $\sqrt{27}/2$. Then*

$$t \geq \frac{2\pi}{\sqrt{27}} a(U).$$

Lemma 4 generalizes a theorem of L. Fejes Tóth (see [4] and p. 67 of [5] which concerns the case when U is convex. The dual counterpart to our lemma for packings was given by Molnár [9, Theorem 12]. We prove our theorem anticipating the proof of Lemma 4 which we postpone to the next section.

Suppose that the rectangle R is covered by circles of two different kinds, which we refer to as small circles and big circles. We assume without loss of generality that the radius of the small circles is 1, while the big ones have radius $r > 1$. Let t_1 denote the total area of the big circles, t_2 the total area of the small circles, and write $t = t_1 + t_2$. If $r < \varrho_0$, then, by Lemma 1,

$$\frac{t}{a(R)} \geq f(\varrho_0^2) > 1.02 > d_2.$$

Therefore we restrict ourselves to the case when

$$r \geq \varrho_0.$$

Let U be the set of those points of R which are not covered by the big circles. Suppose that U has a connected component U_0 such that U_0 is covered by a single small circle and $a(U_0) > \sqrt{27}/2$. Then we remove this single small circle, together

with all big circles bounding U_0 . We show that the arising gap can be covered by small circles whose total area is less than that of the removed circles.

U_0 is bounded by sides of R and by big circles. Let l_1 and l_2 be the number of sides of R and the number of big circles bounding U_0 , respectively. We observe that $\sqrt{27}/2$ is nothing else than the area of a regular hexagon inscribed in a unit circle. Since U_0 is contained in a unit circle, the assumption $a(U_0) > \sqrt{27}/2$ readily implies that $l_1 + l_2 \geq 7$.

First we consider the case $l_1 = 0$. Then the total area of the removed circles is at least $(1 + 7r^2)\pi$. All the removed circles are contained in a circle of radius $1 + 2r$. In view of Lemma 2 this circle can be covered by unit circles of total area

$$\left[\frac{2\pi}{\sqrt{27}}(1 + 2r)^2 + \frac{4}{\sqrt{3}}(1 + 2r) + 1 \right] \pi.$$

A simple computation shows that, for $r \geq \varrho_0$, this expression is less than $(1 + 7r^2)\pi$.

Next let $l_1 = 1$. Now the total area of the removed circles is at least $(1 + 6r^2)\pi$. The removed circles are again contained in a circle of radius $1 + 2r$, however, since we are interested only in the part of R which gets uncovered, we can decrease this circle by the segment of it cut off by the side of R bounding U_0 . Using the fact that the distance of this side from the center of the circle is at most 2, a rough estimate shows that removing the segment from the circle decreases the area by at least 40% and the perimeter decreases by at least 10%. Thus, the part of R which gets uncovered is contained in a simply connected region whose area is at most $0.6(1 + 2r^2)\pi$ and whose perimeter is at most $1.8(1 + 2r)\pi$. Hence, by Lemma 2, this part of R can be covered by unit circles of total area

$$\left[\frac{1.2\pi}{\sqrt{27}}(1 + 2r)^2 + \frac{2.4}{\sqrt{3}}(1 + 2r) + 1 \right] \pi.$$

Again, it is easy to check that, for $r \geq \varrho_0$, this expression is less than $(1 + 6r^2)\pi$. The cases when $l_1 = 2, 3$, or 4 can be settled in a similar way. We omit the details.

By the above operation of removing certain circles and replacing them by unit circles, the total area of the circles covering R does not increase. Thus we can assume that the area of no connected component of U which is covered by a single small circle exceeds $\sqrt{27}/2$.

In view of Lemma 3 we have

$$a(U) \geq a(R) \sigma \left(\frac{t_1}{a(R)} \right).$$

Further, U and the small circles satisfy the conditions of Lemma 4. Therefore

$$t_2 \geq \frac{2\pi}{\sqrt{27}} a(U).$$

Combining the last two inequalities and recalling the definition of d_2 we get

$$t = t_1 + t_2 \geq t_1 + \frac{2\pi}{\sqrt{27}} a(R) \sigma \left(\frac{t_1}{a(R)} \right) \geq a(R) d_2.$$

This completes the proof of our theorem modulo the proof of Lemma 4.

3. Proof of Lemma 4

We call a set a ϱ -set if it is the complement of the union of some closed circles of radius greater than or equal to ϱ . We say that such a set is *finitely generated* if it is the complement of the union of finitely many circles with radius at least ϱ . We recall that here a half-plane is considered to be a circle of radius ∞ .

Let U be a bounded ϱ_0 -set and let $\mathcal{S}$ be a system of closed unit circles covering U so that the area of each connected component of U which intersects a single circle from $\mathcal{S}$ is at most $\sqrt{27}/2$. Without loss of generality we may assume the following:

- (i) U is finitely generated.
- (ii) No four centers of circles from $\mathcal{S}$ lie on a circle and no three centers lie on a circle whose center is a boundary point of U .

Suppose that for some $C \in \mathcal{S}$ the intersection of $C \cap U$ is not connected. Then there are two circles, say $\tilde{C}_1$ and $\tilde{C}_2$, from the generating system of U such that $C \cap \tilde{C}_1 \cap \tilde{C}_2 \neq \emptyset$ and $C \cap (E^2 \setminus (\tilde{C}_1 \cup \tilde{C}_2))$ is not connected. Let a and b be the two points of intersection of $\text{bd } \tilde{C}_1$ and $\text{bd } \tilde{C}_2$. Let m be the midpoint of the segment ab and let L be the line through m perpendicular to ab . It is easily seen that C contains both a and b , so it contains m as well. Let L_1 and L_2 be the two lines parallel to L at distance 2 from m such that a is between L and L_1 and b is between L and L_2 . The set $L_i \cap (E^2 \setminus (\tilde{C}_1 \cup \tilde{C}_2))$ ($i = 1, 2$) consists of three components: of two rays and a segment. We denote the latter by S_i ($i = 1, 2$). Clearly, the set $C \cap U$ is contained in the quadrilateral $Q = \text{conv}(S_1 \cup S_2)$. It is easy to see that the length of S_i is at most $2(\varrho_0 - \sqrt{\varrho_0^2 - 4}) = 0.2397\dots$, thus $a(Q) \leq 8(\varrho_0 - \sqrt{\varrho_0^2 - 4}) = 0.9588\dots < 2.598\dots = \sqrt{27}/2$. We modify $\mathcal{S}$ and U by omitting the circle C from $\mathcal{S}$ and by cutting out the set $U \cap Q$ from U . The new system $\mathcal{S}$ still covers the new set U . Since $a(C)/a(U \cap Q) > \sqrt{27}/2$, it suffices to prove the inequality $t \geq (\sqrt{27}/2)a(U)$ for the new $\mathcal{S}$ and U .

Our new set U need not be a ϱ_0 -set any longer. However, it remains such a set locally. More precisely, the following condition holds:

- (iii) For each circle $C \in \mathcal{S}$ there is a finitely generated ϱ_0 -set $U(C)$ such that $C \cap U = C \cap U(C)$.

The generating set of circles for $U(C)$ is either the same as that for U or is obtained from it by adding one of the half-planes bounded by L_1 and L_2 and containing L . Since the distance of L_1 and L_2 is 4, it easily follows that:

- (iv) For two circles $C_1, C_2 \in \mathcal{S}$ for which $C_1 \cap C_2 \neq \emptyset$ we have $U(C_1) = U(C_2)$.

Now we can repeat the above operation so that properties (iii) and (iv) are preserved until the intersection of U with each circle from $\mathcal{S}$ is connected. Thus it suffices to prove the lemma in the case when:

(v) For each circle $C \in \mathcal{S}$ the intersection $C \cap U = C \cap U(C)$ is connected.

In view of (v) we can partition $\mathcal{S}$ into subclasses so that the circles from one subclass intersect the same connected component of U . Thus it suffices to prove the lemma for connected sets U . Henceforth, we assume that:

(vi) U is connected.

Let $C_1, \dots, C_n$ be the circles from $\mathcal{S}$. Without loss of generality we assume that:

(vii) No proper subset of $\mathcal{S}$ covers U .

The validity of the lemma is supposed for $n = 1$, so we have to consider only the case when

$$n \geq 2.$$

Let $D_1, \dots, D_n$ be the Dirichlet cells of the circles $C_1, \dots, C_n$, i.e., D_i ($i = 1, \dots, n$) consists of those points of the plane which are closer to the center of C_i than to the center of any circle C_j , $j = 1, \dots, n$, $j \neq i$. The sets D_i , $i = 1, \dots, n$, are mutually disjoint and their closures cover the plane. The *Dirichlet cell of C_i with respect to U* is the set $D_i \cap U$. Since the circles $C_1, \dots, C_n$ cover U , we have

$$D_i \cap U \subset C_i$$

for all $i = 1, \dots, n$. Further, in view of (vii), $D_i \cap U \neq \emptyset$ for $i = 1, \dots, n$. The set $D_i \cap U$ need not be connected. Next we describe a procedure which associates to each set $D_i \cap U$ which is not connected, a connected set containing it.

Let U be a ϱ -set and let C be a circle whose radius is less than ϱ . Suppose that $U \cap C$ is connected, and observe that then $U \cap C$ is simply connected. Moreover, if $C \cap U \neq U$, then each component of $C \setminus U$ is simply connected. In this case, if p and q are points of $\text{bd } U$ belonging to the same component of $\text{cl}(C \setminus U)$, then there is a well-defined *arc of $\text{bd } U$ connecting p to q inside C* .

Now, let D be a polygon such that $D \cap U$ is contained in C . Suppose that $D \cap U$ is not connected. Then we associate to $D \cap U$ a connected set $\hat{D}$ as follows:

We observe that $\text{bd}(D \cap C)$ falls into arcs lying inside and outside U alternately following each other. We consider the arcs lying outside U . If such an arc connects the point p to q , then we replace it by the arc of $\text{bd } U$ connecting p to q inside C . In this way we obtain a closed continuous curve which we denote by B . Some points of B are visited twice, however, B does not cross itself. Now the set $\hat{D} = (D \cap U) \cup B$ is connected and its boundary is exactly the curve B .

If an arc of $\text{bd } U$ is visited twice by the curve B , then we think of the two arcs of B as being separated by an infinitesimal corridor. Thus we think of $\hat{D}$ as of topological disks (see Fig. 2).

Let $\hat{D}_i$ be the set obtained by applying the above procedure to the sets D_i , C_i , and $U(C_i)$ in the place of D , C , and U . The new sets $\hat{D}_i$, $i = 1, \dots, n$, will still form a

Fig. 2

packing. Let $\hat{D}_{n+1}, \dots, \hat{D}_N$ ($N > n$) be the connected components of the complement of U . Since U is connected and bounded, with the exception of one unbounded component, all components of its complement are simply connected. We consider the cell complex $\mathcal{E}$ whose faces are $\hat{D}_1, \dots, \hat{D}_N$. An edge e of $\mathcal{E}$ is a maximal arc common to two faces, say $\hat{D}_i$ and $\hat{D}_j$, of $\mathcal{E}$. A vertex of $\mathcal{E}$ is the common endpoint of some edges of $\mathcal{E}$. The set of edges and vertices of $\mathcal{E}$ are denoted by E and V , respectively.

Assumption (ii) guarantees that in all vertices of $\mathcal{E}$ exactly three edges meet. Hence it follows immediately by Euler's theorem that $|V| = 2(N - 2)$. Since $N > n$, we have

$$|V| \geq 2n - 2. \quad (3)$$

An edge which is common to the cells $\hat{D}_i$ and $\hat{D}_j$ with $i, j \leq n$ is called *internal*, otherwise it is called *external*. A vertex is called *internal* if all edges meeting in it are internal and it is called *external* otherwise. It is easily seen that in each external vertex exactly one internal and two external edges meet. We denote by V_i and V_e the set of internal and external vertices, respectively.

With each edge $e \in E$ we associate a set $L(e)$ as follows: Let e be an edge common to $\hat{D}_i$ and $\hat{D}_j$. If $i, j \leq n$, then let $L(e) = C_i \cap C_j$. If, on the other hand, $i \leq n$ and $j > n$, then let $L(e)$ be the connected component of $C_i \cap \hat{D}_j$, the boundary of which contains e . For a vertex $v \in V$ in which the edges e_1 , e_2 , and e_3 meet, we define $I(v)$ as the connected component of $L(e_1) \cap L(e_2) \cap L(e_3)$ which contains v . We write

$$t(v) = a(L(e_1)) + a(L(e_2)) + a(L(e_3)) - 2a(I(v)).$$

We note that in the case when v is an internal vertex common to the cells $\hat{D}_i$, $\hat{D}_j$, and $\hat{D}_k$, with $i, j, k \leq n$, then $t(v)$ is nothing else than the area of the part of the plane covered at least twice by the circles C_i , C_j , and C_k . Thus we have

$$t(v) > T(1) = \pi - \frac{\sqrt{27}}{2} \quad \text{if } v \in V_i. \quad (4)$$

Let v be an external vertex of $\mathcal{E}$ in which the cells $\hat{D}_i$, $\hat{D}_j$, and $\hat{D}_k$ meet so that $i, j \leq n$ and $k > n$. Let e_1 , e_2 , and e_3 be the edges of $\mathcal{E}$ emanating from v so that e_1 is common to $\hat{D}_i$ and $\hat{D}_j$, e_2 is common to $\hat{D}_i$ and $\hat{D}_k$, and, finally, e_3 is common to $\hat{D}_j$ and $\hat{D}_k$. Since $C_i \cap C_j \neq \emptyset$, we have, by (v), $U(C_i) = U(C_j)$. Let $\tilde{C}$ be a circle from the generating system of $U(C_i) = U(C_j)$ such that $v \in \text{cl}\tilde{C}$. Then we have $L(e_2) \subset C_i \cap \tilde{C}$ and $L(e_3) \subset C_j \cap \tilde{C}$. Hence it easily follows that $t(v)$ cannot be less than the area of the part of the plane covered at least twice by the circles C_i , C_j , and $\tilde{C}$. Thus we have

$$t(v) \geq T(\varrho_0) = 2T(1) \quad \text{if } v \in V_e. \quad (5)$$

The main tool to the proof of Lemma 4 is the following:

Proposition 1. *We have*

$$a(U) = n\pi - \frac{1}{2} \sum_{v \in V} t(v).$$

For the case when U is convex this formula was proved in [4]. A similar formula for the area of a convex disk covered by n translates of a centrally symmetric convex disk has been proved in [1].

Before proceeding with the proof of Proposition 1, we show that it readily implies Lemma 4. In view of the assumption $n \geq 2$ we have $|V_e| \geq 2$. Thus, using Proposition 1 and inequalities (3)–(5) we get

$$a(U) = n\pi - \frac{1}{2} \sum_{v \in V} t(v) \leq n\pi - \frac{1}{2}(|V_i|T(1) + |V_e|2T(1)) \leq n\pi - nT(1) = \pi \frac{\sqrt{27}}{2}.$$

We consider the circle C_i ($i \leq n$) and the cell $\hat{D}_i$ associated with it. The edges of $\mathcal{E}$ bounding $\hat{D}_i$ are called the *sides* of $\hat{D}_i$. We assign a positive orientation so that the interior of $\hat{D}_i$ is on the left while travelling on $\text{bd } \hat{D}_i$ in the positive direction. Let $v_1, \dots, v_r$ be the vertices of $\mathcal{E}$ lying on the boundary of $\hat{D}_i$ enumerated in the positive cyclic order and write $v_{r+1} = v_1$. Let s_m denote the side of $\hat{D}_i$ connecting v_m to v_{m+1} .

With each side s_m , $m = 1, \dots, r$, we associate an arc s_m^* which is obtained by elongating s_m in both directions to the boundary of C_i . We call this arc the *chord of C_i determined by s_m* . The way of elongation depends on the type of s_m and its vertices.

If s_m is an external edge of $\mathcal{E}$, then let s_m^* be the connected component of $C_i \cap \text{bd } U$ containing s_m .

Next consider the case when s_m is an internal edge of $\mathcal{E}$ which is common to $\hat{D}_i$ and $\hat{D}_j$ ($i, j \leq n$). Let p_m and q_m be the points of intersection of $\text{bd } C_i$ and $\text{bd } C_j$ so that $\hat{D}_i$ is on the left-hand side of the oriented line $p_m q_m$. We describe how s_m is elongated beyond v_m ; the process for v_{m+1} is similar. If v_m is an external vertex of $\mathcal{E}$, then it lies on the segment $p_m q_m$. However, if it is an internal vertex, then it may occur that it does not lie on $p_m q_m$ but on an arc of $\text{bd } U$ (Fig. 3). In this latter case the segment $p_m q_m$ intersects the component of $C_i \cap \text{bd } U$ containing v_m . Then, in the first step of the construction, we elongate s_m beyond v_m through the correspond-

Fig. 3

ing arc of $C_i \cap \text{bd } U$ to the first point of intersection with the segment $p_m q_m$. Let us denote this point by x_m . Further, write $x_m = v_m$ if $v_m \in p_m q_m$.

Now we define the arc of s_m^* between p_m and x_m . This is done as follows: We travel on $p_m q_m$ from p_m in the direction of q_m . If we arrive at a point $y \in \text{bd } U$ such that the connected component of $C_i \cap \text{bd } U$ intersects the segment of $x_i y$ in a point different from y , then we travel along this arc to the last point of intersection with $p_m q_m$ before x_m . From here on we travel further on $p_m q_m$ and repeat the procedure above, if necessary. In finitely many steps we arrive at x_m .

The chord s_m^* divides C_i into two connected components; the one which is disjoint from $\text{int } \hat{D}_i$ is denoted by S_m and is called the *segment of C_i cut off by s_m^** .

We note that if s_m is an external edge of $\mathcal{E}$, then we have $S_m = L(s_m)$. Further, if e is an edge of $\mathcal{E}$ common to $\hat{D}_i$ and $\hat{D}_j$, then the chords of C_i and C_j determined by e coincide. Furthermore, the union of the segments of C_i and C_j cut off by e^* is just the set $L(e)$.

Let a_1 be an arc contained in C_i and let a_2 be a subarc of s_m^* ($1 \leq m \leq r$). We say that a_1 *crosses* the chord a_2 if $a_1 \cap a_2 \neq \emptyset$ and a_1 contains points both in $\text{int } S_m$ and in $\text{int}(C_i \setminus S_m)$. We say that a_1 crosses a_2 more than once if a_1 contains two disjoint subarcs both crossing a_2 .

Let s_m^+ and s_m^- be the subarcs of s_m^* from v_m through v_{m+1} to q_m and from p_m through v_m to v_{m+1} , respectively. The construction of the chords s_m^* readily implies the following properties:

- (viii) For every pair j, k ($1 \leq j \leq k \leq r$) the arcs s_j^- and s_k^- do not cross and neither do the arcs s_j^+ and s_k^+ . Furthermore, the arcs s_j^- and s_k^+ , as well as the arcs s_j^+ and s_k^- , cross at most once.

We have

$$a(D_i \cap U) = a(\hat{D}_i) = a(C_i) - a\left(\bigcup_{m=1}^r S_m\right) \quad (i = 1, \dots, n). \quad (6)$$

Writing $N(r) = \{1, \dots, r\}$ we have

$$a\left(\bigcup_{m=1}^r S_m\right) = \sum_{m=1}^r (-1)^{m-1} \sum_{\substack{M \subset N(r) \\ |M|=m}} a\left(\bigcap_{j \in M} S_j\right). \quad (7)$$

Let us suppose that for some j and k we have $S_j \cap S_k \neq \emptyset$. Let S be a connected component of $S_j \cap S_k$. Then property (viii) implies that S is bounded either by a subarc of s_j^+ , by a subarc of s_k^- , and an arc of $\text{bd } C_i$, or it is bounded by a subarc of s_j^- , by a subarc of s_k^+ , and an arc of $\text{bd } C_i$. Without loss of generality we assume the first case. Let s_l be a side of $\hat{D}_i$ which lies between s_j and s_k when we go around $\text{bd } \hat{D}_i$ in the positive direction. It is easily seen that the points $p_j, p_l, p_k, q_j, q_l, q_k$ lie on $\text{bd } C_i$ in the cyclic order as enumerated here. Since by (viii) the chord s_l^* does not cross the arcs s_j^+ and s_k^- , it follows that

$$S \subset S_l.$$

Suppose that there are t sides of D_i between s_j and s_k . Then there are $\binom{t}{m-2}$ possibilities of obtaining S as the intersection of m different segments of C_i . This gives a contribution to the sum on the right-hand side of (7) equal to

$$a(S) \sum_{m=2}^{t+2} (-1)^{m-1} \binom{t}{m-2}.$$

This term is equal to $-a(S)$ if $t = 0$, that is, if s_j and s_k are adjacent sides of D_i , and it is 0 otherwise. Thus, writing $S_0 = S_r$ and denoting by J_m the connected component of $S_{m-1} \cap S_m$ containing the vertex v_m , we have

$$a\left(\bigcup_{m=1}^r S_m\right) = \sum_{m=1}^r (a(S_m) - a(J_m)).$$

Hence we obtain, by (6),

$$a(D_i \cap U) = a(\hat{D}_i) = \pi - \sum_{m=1}^r (a(S_m) - a(J_m)) \quad (i = 1, \dots, n).$$

Summing up this for $i = 1, \dots, n$ we get

$$a(U) = \sum_{i=1}^n a(D_i \cap U) = n\pi - \sum_{e \in E} a(L(e)) + \sum_{v \in V} a(I(v)).$$

From here Proposition 1 follows by observing that

$$\sum_{e \in E} a(L(e)) - \sum_{v \in V} a(I(v)) = \frac{1}{2} \sum_{v \in V} t(v).$$

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