Two Combinatorial Problems in the Plane

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Abstract. The authors discuss two problems involving n points in the plane and the t lines they form.

The First Problem

Let n points be given in the plane, not all lying on a line. Let t_k denote the number of lines containing exactly k of the points, and let $t = t_2 + t_3 + t_4 + \cdots$. Melchior [4] proved that

$$t_2 \geq 3 + t_4 + 2t_5 + 3t_6 + 4t_7 + \cdots. \quad (1)$$

The following answers a question which the authors asked in [3].

Theorem 1. *Absolute positive constants C_1 and C_2 exist such that the following is true: Let $0 < \varepsilon \leq 1$. If $t_3 \geq \varepsilon t_2$, then*

$$t \geq C_1 \varepsilon n^2 \quad (2)$$

and

$$t_3 \geq C_2 \varepsilon^2 n^2. \quad (3)$$

We need the following:

Lemma 1. *Let $n = r + s$ points be given, of which r lie on a line L and s points do not. Then $t_2 \geq rs - s(s - 1)$.*

Proof. By induction on s . □

Proof of Theorem 1. Suppose that n points are given and that $t_3 \geq \varepsilon t_2$. Let L be the line containing the most points. If $|L|$ denotes the number of the n points that are on L , then let $k = n - |L|$. Then no line contains more than $n - k$ points. It follows from Lemma 1 that $t_2 \geq k(n - k) - k(k - 1)$. We show that $t_3 \leq \frac{1}{2}k(k - 1)$ and $k > \frac{1}{2}n\varepsilon$. Let t'_i denote the number of lines containing exactly i of the k points not on L . To obtain the upper bound in t_3 , note that there are only two types of lines containing exactly three of the n points. There are at most t'_2 lines containing one point of L and at most t'_3 lines containing no points of L . By counting pairs of the k points, we see that

$$t_3 \leq t'_2 + t'_3 \leq t'_2 + 3t'_3 \leq \frac{1}{2}k(k - 1).$$

Hence,

$$\frac{1}{2}k(k - 1) \geq t_3 \geq \varepsilon t_2 \geq \varepsilon k(n - k) - \varepsilon k(k - 1).$$

Dividing through by $\frac{1}{2}k$ gives $k - 1 \geq 2\varepsilon(n - 3k + 1)$. From this, we may derive the lower bound on k . For, since $0 < \varepsilon \leq 1$, we have

$$4k \geq (3\varepsilon + 1)k \geq 1 + (2n + 1)\varepsilon > 2n\varepsilon,$$

so that $k > \frac{1}{2}n\varepsilon$. However, at most $n - k$ points lie on any line, and so $t \geq Ckn$ by a theorem of Szemerédi and Trotter [6] and Beck [1], where $C > 0$. This gives us (2) with $C_1 = \frac{1}{2}C$.

To obtain (3), we observe that

$$(2 + \varepsilon)t_3 = 2t_3 + \varepsilon t_3 \geq 2\varepsilon t_2 + \varepsilon t_3 = \varepsilon(2t_2 + t_3).$$

By adding $t_2 + t_3$ to both sides of Melchior's inequality (1), we obtain a useful inequality due to Elliott [2]:

$$2t_2 + t_3 \geq 3 + t_2 + t_3 + t_4 + 2t_5 + 3t_6 + \cdots \geq 3 + t.$$

Hence

$$3t_3 \geq (2 + \varepsilon)t_3 \geq \varepsilon(3 + t) > \varepsilon t > C_1\varepsilon^2n^2,$$

or

$$t_3 > C_2\varepsilon^2n^2, \quad \text{where } C_2 = \frac{1}{3}C_1,$$

and Theorem 1 is proved. □

The Second Problem

Another problem which the authors introduced in [3] is the following: Given a set S of n points in the plane, not all collinear, let $f(S)$ denote the minimum of $|T|$ over all sets of points T , such that T is disjoint from S and every line containing two or more points of S contains at least one point of T . We then define $f(n)$ to be the

minimum of $f(S)$ taken over all noncollinear sets S such that $|S| = n$. How large is $f(n)$? That $f(n) > C_3 n$ for some $C_3 > 0$ follows from the so-called “weak Dirac conjecture,” proved in [1]—see also [5]—with a very small constant C_3 . As for an upper bound, an example with $n - 2$ points collinear shows easily that $f(n) \leq n - 1$, but Dean Hickerson has found a construction showing that $f(n) \leq n - 2$ for all $n \geq 6$.

We end the discussion with Hickerson’s construction: Let k be a positive integer. Let S consist of the points $(0, j)$ and $(1, j)$ for $j = 0, 1, \dots, k$, together with the points $A(\frac{1}{2}, \frac{1}{2}(k-1))$, $B(\frac{1}{2}, \frac{1}{2}k)$, and $C(\frac{1}{2}, \frac{1}{2}(k+1))$. If we label four of the points of S as $D(0, 0)$, $E(1, 0)$, $F(0, k)$, and $G(1, k)$, then every line determined by S has slope $0, \infty, \pm 1, \pm 2, \dots, \pm(k-1)$, except the six lines AF , AG , CD , CE , DBG , and EBF .

Let H be the intersection of AF and CD , and let I be the intersection of CE and DBG . Finally let J be the intersection of the lines AG and EBF . The set T consists of the points H , I , and J , together with the points at infinity determined by the aforementioned slopes $0, \infty, \pm 1, \pm 2, \dots, \pm(k-1)$. Then S and T satisfy the desired conditions and $|S| = 2k + 5$ and $|T| = 2k + 3$, showing that $f(n) \leq n - 2$ if n is odd.

When n is even, the construction needs to be modified. Delete C from S and replace the points H and I in T by the point K , the intersection of AF and DBG . The two constructions together show that $f(n) \leq n - 2$ for $n \geq 6$.

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