The Cubical d -Polytopes with Fewer than 2^{d+1} Vertices

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Abstract. A cubical polytope is a convex polytope of which every facet is a combinatorial cube. We give here a complete enumeration of all the cubical d -polytopes with fewer than 2^{d+1} vertices, for $d \geq 4$.

1. Introduction

Let P be a convex d -polytope in $\mathbb{R}^d$, $d \geq 3$, and let $f_j(P)$ be the number of j -dimensional faces of P , $0 \leq j \leq d-1$. A basic problem is to determine the possible $f_j(P)$ and the possible f -vectors $(f_0(P), \dots, f_{d-1}(P))$ for all convex polytopes P or for interesting classes of convex polytopes P . Until now, comprehensive results have only been achieved for the class of simplicial (or, by duality, simple) polytopes. Their possible f -vectors have been completely characterized in [1] and [10]. Using this result, the possible vertex numbers $f_0(P)$ for simple polytopes P have been studied [9].

We deal here with the class of *cubical* polytopes, that is, convex polytopes of which every facet is a combinatorial cube (see, e.g., [6] or [7]). For cubical polytopes it is known that $f_j(P) \geq f_j(C)$, where C is a combinatorial cube [2]. In [3] it is shown that $f_0(P)$ is even for all even $d \geq 4$, and it is reasonable to expect that more gaps occur, especially for numbers slightly larger than $f_0(C) = 2^d$. We determine here all the gaps up to 2^{d+1} , that is, we determine all the possible $f_0(P)$ up to 2^{d+1} .

However, we do much more. For $d \geq 4$, we determine all the cubical d -polytopes with fewer than 2^{d+1} vertices, which turn out to be the so-called elementary ones. Roughly speaking, a k -elementary cubical d -polytope C_k^d , $0 \leq k \leq d$, is the union of $k+1$ combinatorial d -cubes with a common $(d-k)$ -face, as shown in Fig. 1 for $d = 3$.

Fig. 1. The elementary cubical d -polytopes for $d = 3$.

For a rigorous definition, we must start with a $(d + 1)$ -cube C^{d+1} in $\mathbb{R}^{d+1}$. A given $(d - k)$ -face K of C^{d+1} , $0 \leq k \leq d$, is contained in $k + 1$ facets $F_0, \dots, F_k$ of C^{d+1} , which span a subcomplex $\mathcal{F}_k^d$ of the boundary complex of C^{d+1} . Let $\mathcal{E}_k^d$ be the subcomplex of the outer faces of $\mathcal{F}_k^d$, that is, of those faces of C^{d+1} that are contained in both a facet of $\mathcal{F}_k^d$ and a facet not belonging to $\mathcal{F}_k^d$. Now a cubical d -polytope is called k -elementary if its boundary complex is combinatorially equivalent to $\mathcal{E}_k^d$, and it is then denoted by C_k^d .

Such polytopes do exist. Let $v \in \mathbb{R}^{d+1}$ be a point beyond every facet F_i , $0 \leq i \leq k$, and beneath every other facet of C^{d+1} , and let π_k be the projection from v onto a d -space of $\mathbb{R}^{d+1}$ intersecting every ray from v to a point of P . Then $\pi_k(C^{d+1})$ is a convex d -polytope, and the boundary complex of $\pi_k(C^{d+1}) = \pi_k(F_0) \cup \dots \cup \pi_k(F_k)$ is in fact combinatorially equivalent to $\mathcal{E}_k^d$. Using the fact that $\pi_k(F_i)$ is a combinatorial d -cube, we also see that a prototype of C_k^d can actually be constructed as the union of some $k + 1$ combinatorial d -cubes of $\mathbb{R}^d$ with a common $(d - k)$ -face.

Thus C_0^d is a combinatorial d -cube, and, for $k \geq 1$, C_k^d arises from C_{k-1}^d by pasting a combinatorial d -cube (after a suitable deformation of C_{k-1}^d) to it. For $k = 1, \dots, d - 1$, every vertex of C_{k-1}^d becomes a vertex of C_k^d , but C_{d-1}^d has one vertex that is not a vertex of C_d^d . Hence C_{d-1}^d and C_d^d have the same vertex number.

Therefore the vertex number of C_k^d is

$$|\text{vert } C_k^d| = \begin{cases} 2^d + 2^{d-1} + \dots + 2^{d-k} = 2^{d+1} - 2^{d-k} & \text{for } 0 \leq k \leq d-1, \\ 2^d + 2^{d-1} + \dots + 2 = 2^{d+1} - 2 & \text{for } k = d. \end{cases}$$

It is easy to see that C_{d-1}^d and C_d^d have not only the same vertex number, but also the same f -vector.

Before stating our results, we remark that cubical d -polytopes with 2^{d+1} vertices do exist. Take, e.g., the convex hull of two appropriate combinatorial d -cubes.

Our main result is:

Theorem. *For $d \geq 4$, let P be a cubical d -polytope with fewer than 2^{d+1} vertices. Then P is k -elementary for some k , $0 \leq k \leq d$.*

From this it follows immediately that:

Corollary 1. *For $d \geq 4$, the $d+1$ smallest possible vertex numbers for cubical d -polytopes are $2^d = 2^{d+1} - 2^d, \dots, 2^{d+1} - 2^{d-k}, \dots, 2^{d+1} - 2$ and 2^{d+1} .*

Since C_{d-1}^d and C_d^d have the same f -vectors, it also follows that:

Corollary 2. *For $d \geq 4$, let P be a cubical d -polytope with fewer than 2^{d+1} vertices. Then its f -vector is uniquely determined by $f_0(P)$.*

For $d = 3$, the situation is quite different. Cubical 3-polytopes with $2^d = 8, 10, 11, 12, \dots$ vertices exist. However, Corollary 2 remains true for $d = 3$, even without restriction on the number of vertices, since the affine hull of the f -vectors of cubical 3-polytopes has dimension 1 [5, p. 156].

Note that, for $d \geq 4$, the restriction on the number of vertices in Corollary 2 is essential. Cubical polytopes with 2^{d+1} vertices can be constructed as the convex hull of two appropriate combinatorial d -cubes. Thereby k facets of each cube may contain interior points of the convex hull, the k facets passing through a common $(d-k)$ -face, $1 \leq k \leq d-1$. The arising cubical polytopes for $k = 1$ and $k = 2$ have different facet numbers, so their f -vectors differ, and the bound in Corollary 2 is sharp.

The proof of our Theorem is carried out in two steps. First we study the $(d-3)$ -faces of P , and then we change parts of the boundary complex of P to get a more simple polytope.

2. Some Terminology and Notation

Let P be a convex d -polytope. Then ∂P is the boundary complex of P , and $\text{vert } P$ is the set of vertices of P . A *subfacet* (of a facet F) of P is a $(d-2)$ -face of P (contained in F).

Let $\mathcal{A}$ be a set of faces of P . Then the *complex spanned by $\mathcal{A}$* is the smallest subcomplex of ∂P containing $\mathcal{A}$; clearly, it consists of the elements of $\mathcal{A}$ and of all their faces.

We deal here only with subcomplexes S of ∂P such that set S is a topological n -ball for some n . Then a face of S is called an *outer face* of S provided it is contained in the boundary of set S , otherwise it is called an *inner face* of S . For example, let S be spanned by some set of subfacets of P , and let set S be a topological $(d - 2)$ -ball. Then a $(d - 3)$ -face of S is an outer or an inner one according to whether it lies in one or in two subfacets of S .

Throughout, a *cube* is a combinatorial cube.

We need the following properties of the cube and some related terminology. Let F be any facet of an n -cube C^n , then there is just one facet F' disjoint from F . F' is called the facet *opposite* to F in C^n . Let K be a j -face of C^n , $0 \leq j \leq n - 1$. Let $F_1, \dots, F_{n-j}$ be the facets containing K , and let $F'_1, \dots, F'_{n-j}$ be the corresponding opposite facets. Then $K' := F'_1 \cap \dots \cap F'_{n-j}$ is also a j -face and is called the j -face *opposite* to K in C^n . We extend this terminology also to $j = n$ saying that C^n is the n -face *opposite* to C^n in C^n .

Let F and F' be opposite facets of C^n . Every j -face not contained in F or F' , $1 \leq j \leq n - 1$, intersects F and F' in $(j - 1)$ -faces, which are called *related faces* of F and F' .

3. Regular and Singular $(d - 3)$ -Faces

Throughout the following, let P be a cubical d -polytope, $d \geq 4$, with fewer than 2^{d+1} vertices.

Calling a vertex of P k -valent if it is contained in just k edges, we have the key lemma:

Lemma 3.1. *The vertices of P are either d -valent or $(d + 1)$ -valent.*

Proof. This is the main theorem of [4]. □

This is of consequence to the $(d - 3)$ -faces of P .

Lemma 3.2. *Every $(d - 3)$ -face of P lies in at most four facets of P .*

Proof. Assume to the contrary that K is a $(d - 3)$ -face of P lying in at least five facets of P , that is, K lies in at least five subfacets of P . Let e be a vertex of K , then e lies in $d - 3$ edges of K , and it lies in one more edge for every subfacet containing K . Thus e is at least $(d + 2)$ -valent, which contradicts Lemma 3.1. □

Therefore, a $(d - 3)$ -face of P is contained in either three or four facets, accordingly it is called either *regular* or *singular*.

Lemma 3.3. *Every vertex of a singular $(d - 3)$ -face of P is $(d + 1)$ -valent.*

Proof. Analogously to the proof of Lemma 3.2. □

Information about how singular $(d - 3)$ -faces occur is given by:

Lemma 3.4.

- (a) *A $(d - 4)$ -face of P is contained either in no or in just three singular $(d - 3)$ -faces of P .*
- (b) *Let $\mathcal{F}$ be a facet of P . Then a $(d - 4)$ -face of $\mathcal{F}$ is contained either in no or in just two singular $(d - 3)$ -faces of $\mathcal{F}$.*

Proof. Let L be a $(d - 4)$ -face of P , and let e be a vertex of L . If e is d -valent, then all the $(d - 3)$ -faces of P through e are regular, hence L is contained in no singular $(d - 3)$ -face.

So let e be $(d + 1)$ -valent. Then the vertex figure $P|e$ of P at e is a $(d - 1)$ -polytope with $d + 1$ vertices, and it is simplicial, since the facets of P are simple. Such polytopes T are known to be the convex hull of an r -simplex and an s -simplex, $r \geq 1$, $s \geq 1$, $r + s = d - 1$, lying in complementary subspaces of $\mathbb{R}^{d-1}$, such that their intersection is a single relatively interior point of each simplex, see p. 139 of [8]. Distinguishing the vertices of the two simplices by different colors, say black and white, a subset of vert T spans a face of T if and only if the complementary subset of vert T contains both a black and a white vertex. Thus a $(d - j)$ -face of P through e corresponds to a set of $j + 1$ vertices of T where both colors occur; in particular, a facet corresponds to one black and one white vertex, and a set of four vertices corresponds to a singular $(d - 3)$ -face if and only if it consists of two black and two white vertices. We denote black and white vertices by B and W , respectively.

Now assume that L lies in a singular $(d - 3)$ -face K . Then K corresponds to a set $\{B_1, B_2, W_1, W_2\}$. Without loss of generality, $L \subset K$ corresponds to a set $\{B_1, B_2, W_1, W_2, W_3\}$, and some facet $\mathcal{F} \supset K$ corresponds to $\{B_1, W_1\}$. It follows that L is contained in just five $(d - 3)$ -faces, namely, $\{B_2, W_1, W_2, W_3\}$, $\{B_1, W_1, W_2, W_3\}$, $\{B_1, B_2, W_2, W_3\}$, $\{B_1, B_2, W_1, W_3\}$, $\{B_1, B_2, W_1, W_2\} = K$. Three of them are singular, which proves our assertion (a). Two singular ones lie in the facet $\{B_1, W_1\}$, which proves our assertion (b). $\square$

The following lemma is an immediate consequence of Lemma 3.1:

Lemma 3.5. *Every two facets of P with a common vertex have at least a common $(d - 3)$ -face.*

4. The Singular $(d - 3)$ -Faces of a Given Facet

Let $\mathcal{F}$ be a given facet of P , and let $\mathcal{S}$ be the set of singular $(d - 3)$ -faces of $\mathcal{F}$. We want to give a complete description of $\mathcal{S}$. To that end, in addition to Lemma 3.4(b) we need Lemmas 4.1 and 4.2 below.

We begin with some terminology. A sequence $\{B_1, \dots, B_n = B_0\}$ of k -faces of P is called a k -belt, if every B_i intersects B_{i-1} and B_{i+1} in opposite $(k - 1)$ -faces of B_i . The number n is called the *length* of the belt. If $B_i \in \mathcal{S}$ for all i , then $\{B_i\}_{i=1}^n$ is called an $\mathcal{S}$ -belt.

It is readily seen that in a belt of length 4, $B_i \cap B_{i+1}$ and $B_j \cap B_{j+1}$ are disjoint for $i \neq j$.

Lemma 4.1. *Let $\mathcal{S}$ be the set of the singular $(d-3)$ -faces of a given facet $\mathcal{F}$ of P . Then the length of every $\mathcal{S}$ -belt is at most 6.*

Proof. Let $B = \{B_1, \dots, B_n\}$ be an $\mathcal{S}$ -belt. We begin by specifying a 3-face D of the cube $\mathcal{F}$ reflecting some properties of B . Let e be a vertex of the $(d-4)$ -face $B_1 \cap B_2$, and let D be the unique 3-face of $\mathcal{F}$ such that $D \cap (B_1 \cap B_2) = \{e\}$. Then it follows successively that $b_i := B_i \cap D$ is an edge of D , $i = 1, \dots, n$, so that $b := \{b_1, \dots, b_n\}$ is a circuit on D . Now every k -face of D , $0 \leq k \leq 2$, is contained in a unique $(k+d-4)$ -face of $\mathcal{F}$ intersecting D in that given face. Using Lemma 3.4(b), it follows that b has no self-intersections, hence $n = 4, 6$, or 8 . Thus we prove Lemma 4.1 by assuming $n = 8$ and showing that P then has at least 2^{d+1} vertices, thus obtaining a contradiction to the assumption on P .

We denote k -faces of D with small letters, $0 \leq k \leq 2$, and the uniquely corresponding $(k+d-4)$ -faces of $\mathcal{F}$ by the corresponding capital letters. Let h_m , $1 \leq m \leq 6$, be the 2-faces of D , and let b_i and h_m be labeled as in Fig. 2. Every $(d-2)$ -face $H_m \subset \mathcal{F}$ is contained in a further facet $\mathcal{H}_m$ of P . Then every singular $(d-3)$ -face B_i is contained in addition to $\mathcal{F}$ in two facets of $\{\mathcal{H}_m\}$ and in one further facet $\mathcal{F}_i$. Concerning mutual intersections, we note: If $h_m \cap h_n$ is an edge not in $\{b_i\}$, then, by Lemma 3.4(b), $H_m \cap H_n$ is a regular $(d-3)$ -face of $\mathcal{F}$, so that $\mathcal{H}_m \cap \mathcal{H}_n$ is a $(d-2)$ -face. However, if $h_m \cap h_n = b_i$ for some i , then $\mathcal{H}_m \cap \mathcal{H}_n = B_i$, and both $\mathcal{H}_m \cap \mathcal{F}_i$ and $\mathcal{H}_n \cap \mathcal{F}_i$ are $(d-2)$ -faces. Furthermore, $\mathcal{F}_i \cap \mathcal{F} = B_i$.

By Lemma 3.3, every vertex of $B_i \cap B_{i+1}$ is $(d+1)$ -valent. In detail, a vertex of, e.g., $B_2 \cap B_3$ is contained in $d-4$ edges of the $(d-4)$ -face $B_2 \cap B_3$, in the two edges b_2 and b_3 , and in one further edge of $\mathcal{F}$. Thus the remaining two edges lie in both $\mathcal{F}_2$ and $\mathcal{F}_3$, and one of them must lie in $\mathcal{H}_4$, the other one in $\mathcal{H}_1 \cap \mathcal{H}_2$ (see Fig. 2). From these facts it follows that $\mathcal{F}_2 \cap \mathcal{H}_2 = \mathcal{F}_2 \cap \mathcal{H}_1 \cap \mathcal{H}_2$ and $\mathcal{F}_2 \cap \mathcal{H}_2 \subset \mathcal{F}_2 \cap$

Fig. 2. The 3-face D .

$\mathcal{F}_3$, which we use later. Moreover, $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ is a $(d-2)$ -face, $1 \leq i \leq 8$, and $\mathcal{F}_i \cap \mathcal{F}_{i-1}$ and $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ are opposite subfacets of $\mathcal{F}_i$, so that $\{\mathcal{F}_i\}_{i=1}^8$ is a $(d-1)$ -belt.

So we get the desired set of 2^{d+1} vertices of P by showing that the $(d-2)$ -faces $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and $\mathcal{F}_j \cap \mathcal{F}_{j+1}$ are disjoint for $i \neq j$, for then $\bigcup_{i=1}^8 \text{vert}(\mathcal{F}_i \cap \mathcal{F}_{i+1})$ contains just $8 \cdot 2^{d-2} = 2^{d+1}$ distinct vertices of P .

In order to show this property of $\{\mathcal{F}_i\}_{i=1}^8$, we notice that certain vertices of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ are also vertices of some $\mathcal{H}_m$, and so we prove in a first step that $\mathcal{H}_m \cap \mathcal{H}_n = \phi$ if $h_m \cap h_n = \phi$, that is, $\mathcal{H}_2 \cap \mathcal{H}_5 = \phi$, $\mathcal{H}_1 \cap \mathcal{H}_3 = \phi$, and $\mathcal{H}_4 \cap \mathcal{H}_6 = \phi$ (see Fig. 2). Now, $\mathcal{H}_2 \cap \mathcal{H}_5 = \phi$ follows from the fact that the vertices of $\mathcal{H}_2$ are the vertices of its opposite subfacets $\mathcal{H}_1 \cap \mathcal{H}_2$ and $\mathcal{H}_3 \cap \mathcal{H}_2$, that is, $\text{vert}(\mathcal{H}_2 \cap \mathcal{H}_5) \subset \text{vert}(\mathcal{H}_1 \cap \mathcal{H}_5) \cup \text{vert}(\mathcal{H}_3 \cap \mathcal{H}_5) = \text{vert } B_1 \cup \text{vert } B_5$, which is in fact disjoint from $\mathcal{H}_2$.

Let us assume now that $\mathcal{H}_1 \cap \mathcal{H}_3 \neq \phi$, and let e be a vertex of $\mathcal{H}_1 \cap \mathcal{H}_3$. We get a contradiction by showing that e is at least $(d+2)$ -valent. Clearly, $e \notin \mathcal{F}$. Moreover, $e \notin \mathcal{H}_2$, since otherwise the opposite subfacets $\mathcal{H}_1 \cap \mathcal{H}_2$ and $\mathcal{H}_3 \cap \mathcal{H}_2$ of $\mathcal{H}_2$ have a common vertex, which is impossible. Hence $e \in \mathcal{H}_1 \cap \mathcal{F}_1$ and $e \in \mathcal{H}_3 \cap \mathcal{F}_5$. From $e \in \mathcal{H}_1 \cap \mathcal{F}_1$ it follows that e lies in $d-1$ edges of $\mathcal{H}_1$ and in one more edge belonging to $\mathcal{F}_1$, say the edge $\overline{ee'}$ with vertices e and e' . Then, clearly, $e' \notin \mathcal{H}_1$, and because of $e' \in \mathcal{F}_1$ we have $e' \in \mathcal{F}_1 \cap \mathcal{F}_8$ or $e' \in \mathcal{F}_1 \cap \mathcal{F}_2$, say $e' \in \mathcal{F}_1 \cap \mathcal{F}_2$. Since, by the above, $\mathcal{F}_2 \cap \mathcal{H}_2 = \mathcal{F}_2 \cap \mathcal{H}_1 \cap \mathcal{H}_2$, it follows that $e' \notin \mathcal{H}_2$, and also $e' \notin \mathcal{F}$ because of $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F} = B_1 \cap B_2 \subset \mathcal{H}_1$. On the other hand, from $e \in \mathcal{H}_3 \cap \mathcal{F}_5$ it follows that there is an edge k_1 joining e with a vertex of $B_5 \subset \mathcal{F}$, so that $k_1 \not\subset \mathcal{H}_1$, and $k_1 \neq \overline{ee'}$. From $e \in \mathcal{H}_3 \cap \mathcal{F}_5$ it also follows that an edge k_2 joining e with a vertex of $\mathcal{H}_3 \cap \mathcal{H}_2$ exists. Clearly, $k_2 \neq k_1$, and because of $e' \notin \mathcal{H}_2$ we also have $k_2 \neq \overline{ee'}$. Moreover, $k_2 \not\subset \mathcal{H}_1$, since otherwise $\mathcal{H}_1$ and $\mathcal{H}_3 \cap \mathcal{H}_2$ have a common vertex, which means that the opposite subfacets $\mathcal{H}_1 \cap \mathcal{H}_2$ and $\mathcal{H}_3 \cap \mathcal{H}_2$ of $\mathcal{H}_2$ have a common vertex, which is impossible. So e is at least $(d+2)$ -valent in contradiction to Lemma 3.1. Hence, in fact, $\mathcal{H}_1 \cap \mathcal{H}_3 = \phi$, and likewise $\mathcal{H}_4 \cap \mathcal{H}_6 = \phi$.

In the next step we prove that $\mathcal{F}_i \cap \mathcal{F}_{i+1} \cap \mathcal{H}_m = \phi$ whenever $b_i \cap b_{i+1} \cap h_m = \phi$. Without loss of generality we may assume $i = 1$, so that $m = 2, 3$, and 6. Then, in fact, $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{H}_2 = \phi$, since $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{H}_2 = (\mathcal{F}_1 \cap \mathcal{F}_2) \cap (\mathcal{F}_2 \cap \mathcal{H}_2)$ and, by the above, $\mathcal{F}_2 \cap \mathcal{H}_2 \subset \mathcal{F}_2 \cap \mathcal{F}_3$, which is opposite to $\mathcal{F}_1 \cap \mathcal{F}_2$ in $\mathcal{F}_2$. Likewise $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{H}_6 = \phi$. We remark that passing to general i , we also have $\mathcal{F}_2 \cap \mathcal{F}_3 \cap \mathcal{H}_3 = \phi$.

Returning to $i = 1$ it remains to show that $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{H}_3 = \phi$. Assume to the contrary that e is a vertex of $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{H}_3$. We get a contradiction by showing that e is at least $(d+2)$ -valent. Clearly, e is contained in $d-1$ edges of $\mathcal{H}_3$. In order to find more edges through e we notice that $e \notin \mathcal{F}$. As we have seen in the first step, from $e \in \mathcal{H}_3$ it follows that $e \notin \mathcal{H}_1$. Moreover, from $e \notin \mathcal{F}$ and $e \in \mathcal{H}_3$ it follows that $e \notin \mathcal{H}_5$. So $e \in \mathcal{F}_1$ is joined by an edge k_1 with a vertex of $\mathcal{F}_1 \cap \mathcal{H}_1$ and by an edge k_2 with a vertex of $\mathcal{F}_1 \cap \mathcal{H}_5$. Clearly, $k_1 \neq k_2$, and $k_1, k_2 \not\subset \mathcal{H}_3$. Furthermore, from $e \in \mathcal{F}_1 \cap \mathcal{F}_2$ it follows that there is an edge $k \subset \mathcal{F}_2$ joining e with a vertex of $\mathcal{F}_2 \cap \mathcal{F}_3$. Since k_1 and k_2 are edges of $\mathcal{F}_1$, we have $k \neq k_1, k_2$. We also have $k \not\subset \mathcal{H}_3$, since otherwise $\mathcal{F}_2 \cap \mathcal{F}_3 \cap \mathcal{H}_3 \neq \phi$, which is already known to be impossible. So e is at least $(d+2)$ -valent in contradiction to Lemma 3.1. Hence, in fact, $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{H}_3 = \phi$.

We are now in a position to show that $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and $\mathcal{F}_j \cap \mathcal{F}_{j+1}$ are disjoint for $i \neq j$. For that purpose we call the vertices of $\mathcal{H}_1, \dots, \mathcal{H}_6$ briefly $\mathcal{H}$ -vertices. Let B'_i be

the $(d - 3)$ -face of $\mathcal{F}_i$ opposite to B_i . Then $B'_i \cap B'_{i+1}$ is the $(d - 4)$ -face of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ opposite to $B_i \cap B_{i+1}$, and the remaining vertices of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ are $\mathcal{H}$ -vertices. However, the result of the last step yields that the vertices of $B'_i \cap B'_{i+1}$ are not $\mathcal{H}$ -vertices. It also yields that $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and $\mathcal{F}_j \cap \mathcal{F}_{j+1}$ have no common $\mathcal{H}$ -vertex for $i \neq j$. Thus if $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and $\mathcal{F}_j \cap \mathcal{F}_{j+1}$ have a common vertex, then $B'_i \cap B'_{i+1}$ and $B'_j \cap B'_{j+1}$ must have a common vertex.

So assume that e is a common vertex of $B'_i \cap B'_{i+1}$ and $B'_j \cap B'_{j+1}$. Again we get a contradiction by showing that e is at least $(d + 2)$ -valent. From $e \in B'_i \cap B'_{i+1}$ it follows that e lies in $d - 2$ edges of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and in two further edges $k_1 \subset \mathcal{F}_i$, $k_2 \subset \mathcal{F}_{i+1}$ joining e with a vertex of $B'_i \cap B'_{i-1}$ and $B'_{i+1} \cap B'_{i+2}$, respectively. Thus all these d edges join e either with a non- $\mathcal{H}$ -vertex or with an $\mathcal{H}$ -vertex of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$. However, from $e \in B'_j \cap B'_{j+1}$ it follows that there are two edges of $\mathcal{F}_j \cap \mathcal{F}_{j+1}$ joining e with a vertex of $\mathcal{H}_m$ and a vertex of $\mathcal{H}_n \cap \mathcal{H}_l$, respectively, for some m, n, l ; so these two edges join e with an $\mathcal{H}$ -vertex of $\mathcal{F}_j \cap \mathcal{F}_{j+1}$. Thus all these $d + 2$ edges are different, in contradiction to Lemma 3.1. $\square$

Lemma 4.2. *Let $\mathcal{S}$ be the set of the singular $(d - 3)$ -faces of a given facet $\mathcal{F}$ of P , and let H and H' be opposite subfacets of $\mathcal{F}$. Then no two $\mathcal{S}$ -belts $\{B_1, \dots, B_4\}$ and $\{B'_1, \dots, B'_4\}$ exist such that $B_i \subset H$, $B'_i \subset H'$, and B_i, B'_i are related $(d - 3)$ -faces of H and H' .*

Proof. Assume to the contrary that $\{B_1, \dots, B_4 = B_0\}$ and $\{B'_1, \dots, B'_4 = B'_0\}$ are $\mathcal{S}$ -belts such that $B_i \subset H$, $B'_i \subset H'$, and B_i, B'_i are related in H and H' . We show that P then has at least 2^{d+1} vertices, thus obtaining a contradiction to the assumption on P .

For that purpose let H_i be the subfacet of $\mathcal{F}$ spanned by B_i and B'_i . Then H, H' , and H_i are contained in $\mathcal{F}$ and in one further facet $\mathcal{H}, \mathcal{H}'$, or $\mathcal{H}_i$, respectively. Every singular $(d - 3)$ -face B_i is contained in $\mathcal{F}, \mathcal{H}, \mathcal{H}_i$, and in one further facet $\mathcal{F}_i$; B'_i is contained in $\mathcal{F}, \mathcal{H}', \mathcal{H}_i$, and in one further facet $\mathcal{F}'_i$. Clearly, $\mathcal{F}_i \cap \mathcal{F} = \mathcal{H}_i \cap \mathcal{H} = B_i$, $\mathcal{F}'_i \cap \mathcal{F} = \mathcal{H}_i \cap \mathcal{H}' = B'_i$.

By Lemma 3.3, every vertex of $B_i \cap B_{i+1}$ is $(d + 1)$ -valent. It is contained in $d - 4$ edges of the $(d - 4)$ -face $B_i \cap B_{i+1}$, and in three more edges of $\mathcal{F}$, one of them contained in B_i and another one in B_{i+1} . So the remaining two edges are contained in both $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$, and one of them must lie in $\mathcal{H}$, the other one in $\mathcal{H}_i \cap \mathcal{H}_{i+1}$. From this it follows that $\mathcal{F}_i \cap \mathcal{H}_{i+1} \subset \mathcal{H}_i$ and likewise $\mathcal{F}_i \cap \mathcal{H}_{i-1} \subset \mathcal{H}_i$, which we use later. Moreover, $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ is a $(d - 2)$ -face, and $\mathcal{F}_i \cap \mathcal{F}_{i-1}, \mathcal{F}_i \cap \mathcal{F}_{i+1}$ are opposite subfacets of $\mathcal{F}_i$, so that $\{\mathcal{F}_i\}_{i=1}^4$ is a $(d - 1)$ -belt of length 4. From this it follows that $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and $\mathcal{F}_j \cap \mathcal{F}_{j+1}$ are disjoint for $i \neq j$. So $\{\mathcal{F}_i\}_{i=1}^4$ has 2^d different vertices, and the same is true for $\{\mathcal{F}'_i\}_{i=1}^4$. We show that $\{\mathcal{F}_i\}_{i=1}^4$ and $\{\mathcal{F}'_i\}_{i=1}^4$ have no common vertex, so that P has in fact 2^{d+1} vertices.

Now, for $d = 4$, $\{B_i\}_{i=1}^4$ is the family of all the $(d - 3)$ -faces of H , thus all the $(d - 3)$ -faces of H are singular. This need not be true in higher dimensions, so we must distinguish two cases.

Case 1: H has a regular $(d - 3)$ -face B , and $d \geq 5$. Then some more $(d - 3)$ -faces of H or H' turn out to be regular. Let $\bar{B}$ be opposite to B in H , and let B' and $\bar{B}'$ be

the $(d - 3)$ -faces of H' related to B and $\bar{B}$, respectively. Notice that $B \cap B_i$ is a $(d - 4)$ -face contained in B , B_i , and in just one more $(d - 3)$ -face A_i of $\mathcal{F}$, which must be singular by Lemma 3.4(b). Now, $B' \cap B'_i$ is also contained in A_i , so that Lemma 3.4(b) yields that B' is regular. Moreover, if $\bar{B}$ was singular, then, by Lemma 3.4(b), $\bar{B}$ is contained in an $\mathcal{S}$ -belt $\{\dots, B'_1, A_1, B_1, \bar{B}, B_3, A_3, B'_3, \dots\}$, which contradicts Lemma 4.1. Thus $\bar{B}$ is regular, and so is $\bar{B}'$.

Calling the vertices of $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}_i$, $1 \leq i \leq 4$, briefly $\mathcal{H}$ -vertices, we use this information to show first that $\{\mathcal{F}_i\}_{i=1}^4$ and $\{\mathcal{F}'_i\}_{i=1}^4$ have no common $\mathcal{H}$ -vertices. For that purpose we determine the $\mathcal{H}$ -vertices of $\mathcal{F}_i$, which turn out to be just the vertices of $\mathcal{F}_i \cap \mathcal{H}_i$ and of $\mathcal{F}_i \cap \mathcal{H}$.

We begin by determining the vertices of $\mathcal{F}_i$ lying in $\{\mathcal{H}_m\}_{m=1}^4$. Notice that $\{H_m\}_{m=1}^4$ is a $(d - 2)$ -belt, where $H_m \cap H_{m+1}$ is regular by Lemma 3.4(b), so that $\{\mathcal{H}_m\}$ is a $(d - 1)$ -belt. Hence every vertex of $\{\mathcal{H}_m\}$ is already a vertex of $\mathcal{H}_{i-1}$ or $\mathcal{H}_{i+1}$. Thus using the above inclusions $\mathcal{F}_i \cap \mathcal{H}_{i-1} \subset \mathcal{H}_i$ and $\mathcal{F}_i \cap \mathcal{H}_{i+1} \subset \mathcal{H}_i$, a vertex of $\mathcal{F}_i$ lying in $\{\mathcal{H}_m\}_{m=1}^4$ is in fact a vertex of $\mathcal{F}_i \cap \mathcal{H}_i$.

Our next aim is to determine the vertices of $\mathcal{F}_i$ lying in $\mathcal{H}$ or $\mathcal{H}'$. Let G_1 and G_2 be the subfacets of $\mathcal{F}$ spanned by B and B' or by $\bar{B}$ and $\bar{B}'$, respectively; then G_1 and G_2 are contained in $\mathcal{F}$ and in a further facet $\mathcal{G}_1$ or $\mathcal{G}_2$, respectively. Notice that $\{H, G_1, H', G_2\}$ is a $(d - 2)$ -belt, where the intersections of two consecutive elements are regular, so that $\{\mathcal{H}, \mathcal{G}_1, \mathcal{H}', \mathcal{G}_2\}$ is a $(d - 1)$ -belt. Hence every vertex of $\mathcal{H}$ or $\mathcal{H}'$ is also a vertex of $\mathcal{G}_1$ or $\mathcal{G}_2$. Now let e be a vertex of $B \cap B_i$. Then e is $(d + 1)$ -valent. In detail, e is contained in $d - 4$ edges of $B \cap B_i$ and in three more edges of $\mathcal{F}$, one of them, say b , contained in B and another one, say b_i , contained in B_i . Then $B \cap B_i$, b , and b_i span H , and $B \cap B_i$, b , and the third edge span G_1 . So the remaining two edges through e are both contained in $\mathcal{F}_i$, and one of them lies in $\mathcal{G}_1 \cap \mathcal{H}$. From this it follows that $\mathcal{F}_i \cap \mathcal{G}_1 \subset \mathcal{H}$. Likewise $\mathcal{F}_i \cap \mathcal{G}_2 \subset \mathcal{H}$. Thus a vertex of $\mathcal{F}_i$ lying in $\mathcal{H}$ or $\mathcal{H}'$ is in fact a vertex of $\mathcal{F}_i \cap \mathcal{H}$.

Likewise the $\mathcal{H}$ -vertices of $\mathcal{F}'_i$ are the vertices of $\mathcal{F}'_i \cap \mathcal{H}_i$ and of $\mathcal{F}'_i \cap \mathcal{H}'$.

So $\{\mathcal{F}_i\}$ and $\{\mathcal{F}'_i\}$ have no common $\mathcal{H}$ -vertices provided we can show that $(\mathcal{F}_i \cap \mathcal{H}) \cup (\mathcal{F}_i \cap \mathcal{H}_i)$ is disjoint from $(\mathcal{F}'_j \cap \mathcal{H}') \cup (\mathcal{F}'_j \cap \mathcal{H}_j)$, for $1 \leq i, j \leq 4$. However, this follows from the facts that $\mathcal{H}$ and $\mathcal{H}'$ are disjoint, being suitable members of a belt of length 4, that $(\mathcal{F}_i \cap \mathcal{H}) \cap (\mathcal{F}'_j \cap \mathcal{H}_j) = \emptyset$ because of $\mathcal{H} \cap \mathcal{H}_j = B_j$ and $\mathcal{F}'_j \cap B_j = \emptyset$, that likewise $(\mathcal{F}_i \cap \mathcal{H}_i) \cap (\mathcal{F}'_j \cap \mathcal{H}') = \emptyset$, and that $(\mathcal{F}_i \cap \mathcal{H}_i) \cap (\mathcal{F}'_j \cap \mathcal{H}_j) = \emptyset$, since $\{\mathcal{H}_m\}_{m=1}^4$ is a belt and $\mathcal{F}_i \cap \mathcal{H}_i$, $\mathcal{F}'_j \cap \mathcal{H}_i$ are opposite $(d - 2)$ -faces of $\mathcal{H}_i$. Thus $\{\mathcal{F}_i\}$ and $\{\mathcal{F}'_i\}$ have no common $\mathcal{H}$ -vertices.

We conclude the proof by assuming that $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and $\mathcal{F}'_j \cap \mathcal{F}'_{j+1}$, $1 \leq i, j \leq 4$, have a common vertex e . We get a contradiction by showing that e is at least $(d + 2)$ -valent. From $e \in \mathcal{F}_i \cap \mathcal{F}_{i+1}$ it follows that e lies in $d - 2$ edges of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ and in two further edges $k_1 \subset \mathcal{F}_i$, $k_2 \subset \mathcal{F}_{i+1}$. Since e is no $\mathcal{H}$ -vertex, all these d edges join e either with a non- $\mathcal{H}$ -vertex or with an $\mathcal{H}$ -vertex of $\mathcal{F}_i \cap \mathcal{F}_{i+1}$. However, from $e \in \mathcal{F}'_j \cap \mathcal{F}'_{j+1}$ it follows that there are two edges of $\mathcal{F}'_j \cap \mathcal{F}'_{j+1}$ joining e with $\mathcal{H}$ -vertices. Thus all these $d + 2$ edges are different, in contradiction to Lemma 3.1.

Case 2: All the $(d - 3)$ -faces of H and H' are singular. Then every $(d - 4)$ -face of H or H' is contained in two singular $(d - 3)$ -faces of H or H' . Since a $(d - 3)$ -face of

$\mathcal{F}$ is either contained in H or H' or intersects H in a $(d-4)$ -face, it follows from Lemma 3.4(b) that $\mathcal{F}$ has no other singular $(d-3)$ -faces.

In accordance with the notation B_i and B'_i , $1 \leq i \leq 4$, let us denote all pairs of related $(d-3)$ -faces of H and H' by B_i and B'_i , $1 \leq i \leq 2(d-2)$. Let H_i be the subfacet spanned by B_i and B'_i also for $i > 4$, and let $\mathcal{H}_i$ be the facet containing H_i with $\mathcal{H}_i \neq \mathcal{F}$.

Our first aim is to determine the set $\mathcal{S}(\mathcal{H}_i)$ of the singular $(d-3)$ -faces of $\mathcal{H}_i$, $1 \leq i \leq 2(d-2)$, by the aid of Lemma 3.4(b). So far we know that $\mathcal{H}_i$ has the singular $(d-3)$ -faces B_i and B'_i , and no other ones lying in the subfacet H_i . Let G_i be the subfacet of $\mathcal{H}_i$ meeting H_i in B_i , and let G'_i be the subfacet of $\mathcal{H}_i$ meeting H_i in B'_i . Then G_i and G'_i are opposite in $\mathcal{H}_i$, and B_i and B'_i are related $(d-3)$ -faces of G_i and G'_i . Now let L be a $(d-4)$ -face of B_i . Then L is contained in the $(d-3)$ -face B_i , in another $(d-3)$ -face of H_i , and in just one more $(d-3)$ -face of $\mathcal{H}_i$, which lies in G_i and must be singular by Lemma 3.4(b). This means that every $\mathcal{S}(\mathcal{H}_i)$ -belt containing B_i has three elements in G_i , and likewise every $\mathcal{S}(\mathcal{H}_i)$ -belt containing B'_i has three elements in G'_i . By Lemma 3.4(b), such three consecutive elements determine an $\mathcal{S}(\mathcal{H}_i)$ -belt uniquely. Hence if an $\mathcal{S}(\mathcal{H}_i)$ -belt containing B_i leaves G_i , it contains the related three $(d-3)$ -faces of G'_i and hence has length 8, which is impossible by Lemma 4.1. Thus an $\mathcal{S}(\mathcal{H}_i)$ -belt containing B_i does not leave G_i and so also contains the $(d-3)$ -face opposite to B_i in G_i . Thus all the $(d-3)$ -faces of G_i are singular, and so are all the $(d-3)$ -faces of G'_i , and Lemma 3.4(b) yields again that there are no more singular $(d-3)$ -faces of $\mathcal{H}_i$.

This result on $\mathcal{S}(\mathcal{H}_i)$ enables us to describe P more closely. Let $\bar{H}_i$ be the subfacet of $\mathcal{H}_i$ opposite to H_i , and let i, j be indices such that $H_i \cap H_j \neq \emptyset$. Then $H_i \cap H_j$ is a regular $(d-3)$ -face, so that $\mathcal{H}_i \cap \mathcal{H}_j$ is a $(d-2)$ -face, and $\bar{H}_i \cap \bar{H}_j$ is again a regular $(d-3)$ -face.

Therefore $\bar{H}_i \cap \bar{H}_j$ is contained in $\mathcal{H}_i, \mathcal{H}_j$ and in just one more facet $\bar{\mathcal{F}}_{ij}$. However, since $\bar{\mathcal{F}}_{ij}$ intersects $\mathcal{H}_i$ in $\bar{H}_i$ and $\mathcal{H}_j$ in $\bar{H}_j$, all these facets $\bar{\mathcal{F}}_{ij}$ coincide and thus may be denoted by $\bar{\mathcal{F}}$. Hence as well as $\mathcal{F}$ being a facet intersecting every $\mathcal{H}_i$ in H_i , $\bar{\mathcal{F}}$ is also a facet intersecting every $\mathcal{H}_i$ in $\bar{H}_i$. Thereby $\text{vert } \mathcal{F}$ and $\text{vert } \bar{\mathcal{F}}$ are disjoint, as is easily seen.

This means that the subcomplex of ∂P spanned by $\mathcal{F}, \bar{\mathcal{F}}$, and $\{\mathcal{H}_i\}$ is combinatorially equivalent to the subcomplex of a d -cube, where two opposite facets have been removed. Therefore the remaining facets of P form two disjoint topological $(d-1)$ -balls. One of them contains $\mathcal{H}$ and so $\{\mathcal{F}_i\}_{i=1}^4$, the other contains $\mathcal{H}'$ and so $\{\mathcal{F}'_i\}_{i=1}^4$, so that these two sets have no common vertices, as claimed. $\square$

These two lemmas together with Lemma 3.4(b) are sufficient to describe completely the set of the singular $(d-3)$ -faces of any given facet $\mathcal{F}$ of P . For that purpose, let K be a k -face of $\mathcal{F}$, $0 \leq k \leq d-2$. Let $\mathcal{A}(K)$ be the subcomplex of $\partial \mathcal{F}$ spanned by the subfacets of $\mathcal{F}$ containing K , and let $\mathcal{S}_K$ be the set of the outer $(d-3)$ -faces of $\mathcal{A}(K)$. Then the union of the faces of $\mathcal{S}_K$ is the boundary of $\mathcal{F}(K)$, and is a topological $(d-3)$ -sphere. It is shown in [5] that a set $\mathcal{S}$ of set $(d-3)$ -faces of $\mathcal{F}$ with the properties stated in Lemmas 3.4(b), 4.1, and 4.2 is just a set $\mathcal{S}_K$ for some K .

Until now, we have tacitly assumed that $\mathcal{S} \neq \phi$. However, empty $\mathcal{S}$ are important, they occur, e.g., for every facet of a d -cube. So we would like to represent $\mathcal{S} = \phi$ also as some family $\mathcal{S}_K$. This can be done by putting $K = \mathcal{F}$, since then $\mathcal{A}(K) = \phi$ and hence $\mathcal{S}_K = \phi$.

K is called the *root* of $\mathcal{S}_K$. The root of a family $\mathcal{S}_K$ need not be unique, for let $\dim K = 0$, and let K' be the vertex of $\mathcal{F}$ opposite to K ; then, clearly, $\mathcal{S}_K = \mathcal{S}_{K'}$. Thus for $\dim K = 0$, $\mathcal{S}_K$ has just two roots, namely, the pair of opposite vertices K and K' of $\mathcal{F}$. However, it is easily seen that, for $1 \leq \dim K \leq d - 1$, the root of $\mathcal{S}_K$ is unique.

This means that the set of the singular $(d - 3)$ -faces of a given facet $\mathcal{F}$ of P can be represented by its—essentially unique—root.

5. The Complexes $\mathcal{A}(K)$ and $\mathcal{F}^\cap(K)$

Let $\mathcal{F}$ be a facet of P , and let K and K' be opposite k -faces of $\mathcal{F}$, $0 \leq k \leq d - 2$. In Section 4 we have defined the complex $\mathcal{A}(K)$ spanned by the subfacets of $\mathcal{F}$ containing K , and also the family $\mathcal{S}_K$ of the outer $(d - 3)$ -faces of $\mathcal{A}(K)$. We enlighten these two concepts by using them to describe the families of the subfacets and $(d - 3)$ -faces of $\mathcal{F}$. We also use another complex $\mathcal{F}^\cap(K)$ spanned by the subfacets of $\mathcal{F}$ having nonempty intersection with K . Clearly, for $k = 0$, we have $\mathcal{A}(K) = \mathcal{F}^\cap(K)$.

At first we note that $\mathcal{F}$ has precisely the following subfacets:

- The $(d - 1) - k$ subfacets containing K , that is, the subfacets of $\mathcal{A}(K)$.
- The $(d - 1) - k$ subfacets disjoint from K , that is, the subfacets of $\mathcal{A}(K')$. Every subfacet of $\mathcal{A}(K')$ is opposite to some subfacet of $\mathcal{A}(K)$.
- The remaining $2k$ subfacets, which intersect K but do not contain it. They are said to *intersect K strongly*. They intersect K in a $(k - 1)$ -face and there is, in fact, a bijection between the subfacets of $\mathcal{F}$ intersecting K strongly and the $(k - 1)$ -faces of K . Thus these subfacets form a set $\{F_L | L \dots (k - 1)\text{-face of } K\}$. Note that this characterization remains true if we replace K by K' .

Clearly, the subfacets of $\mathcal{F}$ split into the subfacets of $\mathcal{A}(K)$ and $\mathcal{F}^\cap(K')$.

Therefore the $(d - 3)$ -faces of $\mathcal{F}$ split into the inner $(d - 3)$ -faces of $\mathcal{A}(K)$, the inner $(d - 3)$ -faces of $\mathcal{F}^\cap(K')$, and the outer $(d - 3)$ -faces of both $\mathcal{A}(K)$ and $\mathcal{F}^\cap(K')$, which form the family $\mathcal{S}_K$.

It is easily seen that the inner $(d - 3)$ -faces of $\mathcal{A}(K)$ are precisely the $(d - 3)$ -faces of $\mathcal{F}$ containing K . The inner $(d - 3)$ -faces of $\mathcal{F}^\cap(K')$ are precisely the $(d - 3)$ -faces of $\mathcal{F}$ having nonempty intersection with K' .

We need the following lemma:

Lemma 5.1. *Let $\mathcal{F}$ be a facet of P , and let K be a k -face of $\mathcal{F}$, $1 \leq k \leq d - 2$. Let F_L be the unique subfacet of $\mathcal{F}$ intersecting K in a given $(k - 1)$ -face L . Let $\mathcal{S}_K$ be the set of the outer $(d - 3)$ -faces of $\mathcal{A}(K)$. Then the common $(d - 3)$ -faces of F_L and $\mathcal{S}_K$ are the $(d - 3)$ -faces of F_L containing L .*

Proof. Let K' be opposite to K in $\mathcal{F}$. Since F_L intersects K in L , it intersects K' in a face L' , which is opposite to L in F_L . According to the above, the $(d-3)$ -faces of F_L either contain L , or intersect both L and L' strongly, or contain L' . Precisely in the first case they are disjoint from K' . Now $F_L \in \mathcal{F}^\cap(K')$, so that precisely the $(d-3)$ -faces of F_L disjoint from K' are outer $(d-3)$ -faces of $\mathcal{F}^\cap(K')$, that is, belong to $\mathcal{S}_K$. $\square$

6. The Singular $(d-3)$ -Faces of a Given Subfacet

From now on, we abbreviate the term “set of the singular $(d-3)$ -faces” by *sf-set*.

We describe the sf-sets of the subfacets of P , and we study to what extent the sf-set of a given facet is determined by the sf-set of one of its subfacets. We need the following terminology. Let F be a subfacet of P , and let K be a k -face of F , $0 \leq k \leq d-2$. Then the set of all the $(d-3)$ -faces of F containing K is called a *K-set*; note that for $k = d-2$ the *K-set* is empty. The set of all the $(d-3)$ -faces of F not containing K is said to be *complementary to the K-set of F*; note that for $k = d-2$ it consists of all the $(d-3)$ -faces of F .

Recall the result of Section 4. For a given facet $\mathcal{F}$, the sf-set of $\mathcal{F}$ is a set $\mathcal{S}_K$, that is, the set of the outer $(d-3)$ -faces of the complex $\mathcal{F}(K)$ for some k -face K of $\mathcal{F}$, $0 \leq k \leq d-1$. K was called the root of $\mathcal{S}_K$; the root is unique for $k \geq 1$, and for $k = 0$ the vertex of $\mathcal{F}$ opposite to K is also a root.

Recall the convention stated in Section 2 that, for a cube C , its face C is opposite to itself.

The sf-set of a subfacet of P is given by:

Lemma 6.1. *Let $\mathcal{F}$ be a facet of P and let K be a root of its sf-set. Let $k := \dim K$, so that $0 \leq k \leq d-1$. Let F be a subfacet of $\mathcal{F}$.*

- (a) *If F contains K , then the sf-set of F is complementary to the K -set of F .*
- (b) *If $k > 0$ and F intersects K in a $(k-1)$ -face L , then the sf-set of F is the L -set of F .*
- (c) *If F is disjoint from K , then the sf-set of F is the K'' -set, where K'' is given as follows. Let K' be opposite to K in $\mathcal{F}$, and let K'' be opposite to K' in F .*

Proof. For $k = d-1$ this is an immediate consequence of the definition.

So let $k \leq d-2$. In case (a) F is a subfacet of the complex $\mathcal{F}(K)$, and the assertion follows from the fact that the inner $(d-3)$ -faces of $\mathcal{F}(K)$ are those ones containing K . In case (b) the assertion is Lemma 5.1. In case (c) note that F contains the face K' opposite to K in $\mathcal{F}$. Among the subfacets of $\mathcal{F}(K)$, one is opposite to F in $\mathcal{F}$, and the other ones intersect F in a $(d-3)$ -face, which belongs to $\mathcal{S}_K$. Thus the sf-set of F contains just $((d-1)-k)-1$ elements. Every such element belongs to a subfacet of $\mathcal{F}(K)$, hence it is disjoint from K' and so contains the face K'' opposite to K' in F . Therefore it is a member of the K'' -set of F having just $(d-2)-k = ((d-1)-k)-1$ elements. This proves assertion (c). $\square$

Conversely, the sf-set of a subfacet to some extent determines the sf-set of a facet containing it, that is, its root.

Lemma 6.2. *Let $\mathcal{F}$ be a facet of P and let F be a subfacet of $\mathcal{F}$. Let K be a k -face of F , $0 \leq k \leq d - 2$, and let H be the $(k + 1)$ -face of $\mathcal{F}$ containing K and not lying in F .*

- (a) *If the sf-set of F is the K -set of F , then a root of the sf-set of $\mathcal{F}$ is either given by H or by the face of H opposite to K .*
- (b) *If $k \geq 1$, and if the sf-set of F is complementary to the K -set of F , then the root of the sf-set of $\mathcal{F}$ is K .*

Proof. By Lemma 6.1, two types of sf-sets of subfacets occur, namely, K -sets and sets complementary to K -sets. For $\dim K = 0$, these notions coincide, for let K and K' be opposite vertices of a subfacet, then the K -set is complementary to the K' -set. However, for $\dim K \geq 1$, a K -set is never complementary to some K' -set.

Firstly, let $k \geq 1$. If the sf-set of F is the K -set of F , then it must be the result of cases (b) or (c) of Lemma 6.1, from which our assertion (a) for $k \geq 1$ follows immediately. However, if the sf-set of F is complementary to the K -set of F , then it is the result of case (a) of Lemma 6.1, from which our assertion (b) follows immediately.

Secondly, let $k = 0$. Let the sf-set of F be the K -set of F . Such an sf-set can be the result of any of cases (a)–(c) of Lemma 6.1, so that a root of the sf-set of $\mathcal{F}$ is the face of F opposite to K , or H , or the face of H opposite to K . Since the first and the last of these three possible roots are opposite vertices of $\mathcal{F}$, they are roots of the same sf-set and one can be omitted, which proves (a) also for $k = 0$. $\square$

7. Topological Cubical Complexes and STC-Complexes

So far we have shown that a cubical polytope P with less than 2^{d+1} vertices has only two types of $(d - 3)$ -faces, namely, regular and singular ones, and abbreviating the term “set of the singular $(d - 3)$ -faces” by sf-set, we have seen that the sf-set of a facet $\mathcal{F}$ of P is a set $\mathcal{S}_K$, which is to some extent determined by the sf-set of a subfacet of $\mathcal{F}$.

From now on we need a more general concept. By a *topological cube* we mean the topological image of a cube together with its faces. A finite family $\mathcal{P}$ of topological cubes in $\mathbb{R}^d$ is called a *topological cubical complex* provided:

- (1) Every face of a member of $\mathcal{P}$ is itself a member of $\mathcal{P}$.
- (2) The intersection of any two members of $\mathcal{P}$ is a face of each of them.

If it is clear from the context, we call a topological cubical complex briefly a *complex*. As usual, a member of $\mathcal{P}$ is called a *face* of $\mathcal{P}$, and a *vertex*, *edge*, *subfacet*, or *facet* of $\mathcal{P}$ is a face of dimension 0, 1, $d - 2$, or $d - 1$, respectively. Notions using only the inclusion of faces are defined as for usual complexes, e.g., the k -valence of a vertex or the *isomorphism* of two complexes.

For every facet $\mathcal{F}$ of $\mathcal{P}$, the complexes $\mathcal{F}(K)$, $\mathcal{F}^\cap(K)$, and the set $\mathcal{S}_K$ are defined as for combinatorial cubes, see Sections 4 and 5.

We deal with topological cubical complexes $\mathcal{P}$ in $\mathbb{R}^d$ having additional properties, namely:

- (3) Set $\mathcal{P}$ is a topological $(d - 1)$ -sphere.

Then every $(d - 3)$ -face of $\mathcal{P}$ is contained in at least three facets, and we require:

- (4) Every $(d - 3)$ -face of $\mathcal{P}$ is contained in at most four facets.

So again, $(d - 3)$ -faces are called *regular* or *singular* according to whether they are contained in three or four facets, and the term “set of the singular $(d - 3)$ -faces” is again abbreviated by sf-set.

- (5) The sf-set of a facet $\mathcal{F}$ of $\mathcal{P}$ is a set $\mathcal{S}_K$.

The corresponding face K of $\mathcal{F}$ is called a *root of the sf-set* of $\mathcal{F}$, or briefly a *root* of $\mathcal{F}$. Remember that $0 \leq \dim K \leq d - 1$, and $\dim K = d - 1$ if and only if $\mathcal{S}_K = \phi$.

- (6) Every two facets of $\mathcal{P}$ with a common vertex have at least a common $(d - 3)$ -face.

- (7) Every circuit of edges of $\mathcal{P}$ has an even number of edges.

Such a topological cubical complex is called a *special topological cubical $(d - 1)$ -complex* or *$(d - 1)$ -STC-complex*. Clearly, the boundary complex of a cubical d -polytope with fewer than 2^{d+1} vertices is a $(d - 1)$ -STC-complex ((6) is Lemma 3.5, and (7) is satisfied for every cubical polytope, see [3]).

Since Lemmas 6.1 and 6.2 refer only to a given facet, they remain true for STC-complexes.

In the remainder of this paper let $\mathcal{P}$ be a $(d - 1)$ -STC-complex.

Clearly, a vertex of $\mathcal{P}$ is contained in at least d facets and in at least d edges. It is contained in just d facets if and only if it is contained in just d edges, and then it is called *simple* as usual.

It is readily seen that a simple vertex of $\mathcal{P}$ is only contained in regular $(d - 3)$ -faces. The following lemma shows that the converse is also true; we prove a slightly more general result.

Lemma 7.1. *Let e be a vertex of $\mathcal{P}$, and let $\mathcal{F}$ be a facet of $\mathcal{P}$ containing e such that every $(d - 3)$ -face of $\mathcal{F}$ containing e is regular. Then e is simple.*

Proof. Let $\mathcal{F}$ be a facet containing e , and let $F_1, \dots, F_{d-1}$ be the subfacets of $\mathcal{F}$ containing e . Then, by (3), each F_i is also contained in a facet $\mathcal{F}_i$ of $\mathcal{P}$ and, by assumption, every $(d - 3)$ -face of $\mathcal{F}$ containing e lies in precisely three facets, namely, in $\mathcal{F}$ and in two facets of $\{\mathcal{F}_i\}_{i=1}^{d-1}$.

If e were not simple, then, by (3), at least one more facet $\mathcal{F}_d$ containing e and meeting, e.g., $\mathcal{F}_1$ in a subfacet F_d exists. Then $F_1 \cap F_d$ is a $(d - 3)$ -face of $\mathcal{F}$ and so is contained, by the above, only in facets of $\{\mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_{d-1}\}$, but not in $\mathcal{F}_d$. This is a contradiction. $\square$

8. Cubal Subcomplexes of STC-Complexes

A topological cubical complex is said to be *cubal* if it is isomorphic to a subcomplex of the boundary complex of a cube. The aim of this section is to find cubal subcomplexes of a $(d - 1)$ -STC-complex $\mathcal{P}$.

The following subcomplexes of $\mathcal{P}$ are particularly important. Let K be a face of $\mathcal{P}$, so that $0 \leq \dim K \leq d - 1$. Then the subcomplex $\mathcal{P}(K)$ is spanned by the facets of $\mathcal{P}$ containing K , and the subcomplex $\mathcal{P}^\cap(K)$ is spanned by the facets of $\mathcal{P}$ having nonempty intersection with K . Note that, for $\dim K = 0$, $\mathcal{P}(K) = \mathcal{P}^\cap(K)$. The complexes $\mathcal{F}(K)$ and $\mathcal{F}^\cap(K)$ described in Section 5 are the analogous complexes relative to a facet $\mathcal{F}$; thereby $K = \mathcal{F}$ was admitted. Clearly, $\mathcal{F}(K)$ and $\mathcal{F}^\cap(K)$ are cubal, but in general $\mathcal{P}(K)$ and $\mathcal{P}^\cap(K)$ are not.

The following two lemmas enable us to find cubal subcomplexes $\mathcal{P}^\cap(K)$ of $\mathcal{P}$.

Lemma 8.1. *Let $\mathcal{F}$ be a facet of $\mathcal{P}$, let K be a root of $\mathcal{F}$, and let K' be opposite to K in $\mathcal{F}$. Then every vertex of K' is simple.*

Proof. Since K is a root of $\mathcal{F}$, the sf-set of $\mathcal{F}$ is $\mathcal{S}_K$, the set of the outer $(d - 3)$ -faces of $\mathcal{F}(K)$. However, $\mathcal{S}_K$ is also the set of the outer $(d - 3)$ -faces of $\mathcal{F}^\cap(K')$, where K' is opposite to K in $\mathcal{F}$ (this argument is also true for $\dim K = d - 1$). Thus a $(d - 3)$ -face of $\mathcal{F}$ intersecting K' is an inner $(d - 3)$ -face of $\mathcal{F}^\cap(K')$ and so is regular. Hence by Lemma 7.1, every vertex of K' is simple. $\square$

Lemma 8.2. *Let K be a face of $\mathcal{P}$ so that all the vertices of K are simple. Then $\mathcal{P}^\cap(K)$ is cubal.*

Proof. For $\dim K = 0$, K is a simple vertex and so is contained in precisely d facets. Hence $\mathcal{P}^\cap(K)$ is cubal.

For $\dim K = 1$, K is an edge with two simple vertices e_1 and e_2 . Thus $\mathcal{P}(K)$ is spanned by $d - 1$ facets, and $\mathcal{P}^\cap(K)$ is spanned by two more facets, say $\mathcal{F}_1$ through e_1 and $\mathcal{F}_2$ through e_2 . Then, clearly, $\mathcal{P}^\cap(K)$ is cubal if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint. To show this, let e'_i be the vertex of $\mathcal{F}_i$ opposite to e_i , $i = 1, 2$. Since all the other vertices of $\mathcal{F}_i$ are also vertices of $\mathcal{P}(K)$, it remains to show that $e'_1 \neq e'_2$. However, this follows from property (7) of an STC-complex.

For $\dim K \geq 2$, since the vertices of K are simple, a facet of $\mathcal{P}^\cap(K)$ either contains K or intersects K in a $(k - 1)$ -face. Let $e_1 e_2$ be an edge of K with the vertices e_1 and e_2 , and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be defined as above, so that $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint. $\mathcal{F}_1$ and $\mathcal{F}_2$ intersect K in $(k - 1)$ -faces L_1 and L_2 , respectively, and L_1 and L_2 are opposite $(k - 1)$ -faces of K . Now let $\mathcal{F}$ be a facet of $\mathcal{P}^\cap(K)$, $\mathcal{F} \neq \mathcal{F}_1, \mathcal{F}_2$. Then $\mathcal{F}$ intersects both L_1 and L_2 and so intersects both $\mathcal{F}_1$ and $\mathcal{F}_2$ in a subfacet. Thus the vertices of $\mathcal{F}$ are vertices of $\mathcal{F}_1$ and $\mathcal{F}_2$. From this it follows easily that $\mathcal{P}^\cap(K)$ is cubal also in this case, which completes the proof. $\square$

From Lemmas 8.1 and 8.2 it follows that $\mathcal{P}^\cap(K)$ is cubal for every face K that is opposite in some facet $\mathcal{F}$ to a root of $\mathcal{F}$. This is the base for finding an even more meaningful cubal subcomplex of $\mathcal{P}$.

We need some terminology. Let $\mathcal{F}$ be a facet of $\mathcal{P}$, and let k be the dimension of a root of $\mathcal{F}$, $0 \leq k \leq d - 1$. Then $\mathcal{F}$ is said to be of *type* k . Let m be the minimal type of all the facets of $\mathcal{P}$. It turns out that m is a significant parameter of $\mathcal{P}$.

So the following lemma states that $m = d - 1$ is equivalent to the fact that the whole complex $\mathcal{P}$ is cubal.

Lemma 8.3. *The following is equivalent.*

- The minimal type m of all the facets of $\mathcal{P}$ is $d - 1$.
- $\mathcal{P}$ is isomorphic to the boundary complex of the d -cube.

Proof. Clearly, it is enough to show that, for $m = d - 1$, $\mathcal{P}$ is isomorphic to the boundary complex of the d -cube. For that purpose, let $\mathcal{F}$ be a facet of $\mathcal{P}$. Then it follows from $m = d - 1$ together with Lemmas 8.1 and 8.2 that $\mathcal{P}^\cap(\mathcal{F})$ is cubal. Moreover, from $m = d - 1$ together with Lemma 8.1 it follows that all the vertices of $\mathcal{P}^\cap(\mathcal{F})$ are simple. Thus every outer vertex of $\mathcal{P}^\cap(\mathcal{F})$ is contained in just one more facet, which is one facet $\mathcal{F}$, so that $\mathcal{P}$ is combinatorially equivalent to the boundary complex of the d -cube. $\square$

For $0 \leq m \leq d - 2$ we get the desired cubal subcomplex of $\mathcal{P}$ as follows:

Lemma 8.4. *Let m be the minimal type of all the facets of $\mathcal{P}$, and let $0 \leq m \leq d - 2$.*

- (a) *Let $m = 0$ and assume that all the facets of $\mathcal{P}$ are of type 0. Then a 0-face $\mathcal{D}'$ of $\mathcal{P}$ exists so that $\mathcal{P}^\cap(\mathcal{D}')$ is cubal.*
- (b) *Let $m = 0$ and assume that not all the facets of $\mathcal{P}$ are of type 0. Then a 1-face $\mathcal{D}'$ of $\mathcal{P}$ exists so that $\mathcal{P}^\cap(\mathcal{D}')$ is cubal.*
- (c) *Let $1 \leq m \leq d - 2$. Then an $(m + 1)$ -face $\mathcal{D}'$ of $\mathcal{P}$ exists so that $\mathcal{P}^\cap(\mathcal{D}')$ is cubal.*

Moreover, for every facet $\mathcal{F}$ of $\mathcal{P}^\cap(\mathcal{D}')$, a root of $\mathcal{F}$ is given by the face opposite to $\mathcal{F} \cap \mathcal{D}'$ in $\mathcal{F}$, in each of cases (a)–(c).

Proof. (a) Let $\mathcal{F}_0$ be a facet of $\mathcal{P}$, so that $\mathcal{F}_0$ is of type 0. Let $\mathcal{D}$ and $\mathcal{D}'$ be the roots of $\mathcal{F}_0$. Then, by Lemmas 8.1 and 8.2, $\mathcal{P}^\cap(\mathcal{D}')$ is cubal, as required.

As to the roots of the facets of $\mathcal{P}^\cap(\mathcal{D}')$, the assertion is clearly true for $\mathcal{F}_0$. So let $\mathcal{F} \neq \mathcal{F}_0$ be a facet of $\mathcal{P}^\cap(\mathcal{D}')$. Then $\mathcal{F}$ has a common subfacet F with $\mathcal{F}_0$ containing $\mathcal{D}'$, which is a root of $\mathcal{F}_0$. By Lemmas 6.1(a) and 6.2(a), using the fact that $\mathcal{F}$ is of type 0, we see that a root of $\mathcal{F}$ is given by the vertex opposite to $\mathcal{D}'$ in $\mathcal{F}$, as claimed.

(b) The assumption of this case yields that facets $\mathcal{F}_0$ and $\mathcal{F}_1$ with a common subfacet F exist such that $\mathcal{F}_0$ is of type 0 and $\mathcal{F}_1$ is not. Then $\mathcal{F}_0$ has two roots, namely, two opposite vertices. Therefore F contains a root of $\mathcal{F}_0$, say the vertex E_0 . Let $\mathcal{D}'$ be the 1-face of $\mathcal{F}_1$ through E_0 and not lying in F . Then, by Lemmas 6.1(a) and 6.2(a), using the fact that $\mathcal{F}_1$ is not of type 0, we see that $\mathcal{D}'$ is opposite to the root of $\mathcal{F}_1$. Hence, by Lemmas 8.1 and 8.2, $\mathcal{P}^\cap(\mathcal{D}')$ is cubal, as required.

To determine the roots of the facets of $\mathcal{P}^\cap(\mathcal{D}')$, we denote the facets of $\mathcal{P}^\cap(\mathcal{D}')$ containing $\mathcal{D}'$ by $\mathcal{F}_j$, $1 \leq j \leq d - 1$; the remaining two facets are $\mathcal{F}_0$ intersecting $\mathcal{D}'$ in the vertex E_0 , and a facet $\mathcal{F}_d$ intersecting $\mathcal{D}'$ in a vertex E_d . Clearly, the assertion is true for $\mathcal{F}_0$ and $\mathcal{F}_1$.

We determine a root of $\mathcal{F}_j$, $2 \leq j \leq d-1$, by means of its subfacet $F_j := \mathcal{F}_j \cap \mathcal{F}_0$. F_j contains the root E_0 of $\mathcal{F}_0$, and so, by Lemmas 6.1(a) and 6.2(a), a root of $\mathcal{F}_j$ is given by its face opposite to $\mathcal{D}'$, or by its vertex opposite to E_0 . In the latter case all the vertices of $\mathcal{F}_j$, except E_0 and its opposite vertex, are contained in a singular $(d-3)$ -face. This contradicts the fact that the vertex E_d of $\mathcal{P}^\cap(\mathcal{D}')$ is simple. So the root of $\mathcal{F}_j$ is its face opposite to $\mathcal{D}'$, as claimed.

Finally, the facet $\mathcal{F}_d$ has a common subfacet F_{jd} with each $\mathcal{F}_j$, $1 \leq j \leq d-1$. Thereby F_{jd} intersects the root of $\mathcal{F}_j$ in a vertex E_{jd} opposite to E_d in F_{jd} , and, by Lemma 6.1(b), the sf-set of each F_{jd} is the E_{jd} -set. Thus, by Lemma 6.2(a), the vertex opposite to E_d in $\mathcal{F}_d$ is a root of $\mathcal{F}_d$, as claimed.

(c) Let $\mathcal{F}_0$ be a facet of $\mathcal{P}$ of type m , $1 \leq m \leq d-2$, and let K_0 be the m -dimensional root of $\mathcal{F}_0$. Let L_1 and L_2 be opposite $(m-1)$ -faces of K_0 . Then, for each i , $i = 1, 2$, just one subfacet F_i of $\mathcal{F}_0$ intersecting K_0 strongly in L_i exists. F_i is contained in a further facet $\mathcal{F}_i \neq \mathcal{F}_0$. We determine a root of $\mathcal{F}_i$ by means of its subfacet F_i . By Lemma 6.1(b), the sf-set of F_i is the L_i -set of F_i . Thus letting K_i be the m -face of $\mathcal{F}_i$ containing L_i and not lying in F_i , it follows from Lemma 6.2(a) that a root of $\mathcal{F}_i$ is either given by K_i or by the face of K_i opposite to L_i . However, since m is the minimal type of the facets of $\mathcal{P}$, the root of $\mathcal{F}_i$ must be K_i , $i = 1, 2$.

Now let K'_0 be opposite to K_0 in $\mathcal{F}_0$, and let K'_i be opposite to K_i in $\mathcal{F}_i$. Then the intersection of K'_i and K'_0 is again an $(m-1)$ -face, which is contained in F_i . Moreover, by Lemma 8.1, the vertices of K'_0 , K'_1 , and K'_2 are simple. Hence they span an $(m+1)$ -face $\mathcal{D}'$, of which K'_1 and K'_2 are opposite faces. Thus all the vertices of $\mathcal{D}'$ are simple, and Lemma 8.2 yields that $\mathcal{P}^\cap(\mathcal{D}')$ is cubal, as required.

We now turn to the problem of determining the roots of the facets of $\mathcal{P}^\cap(\mathcal{D}')$. We already know that the root of $\mathcal{F}_i$ is K_i , the face opposite to $K'_i = \mathcal{F}_i \cap \mathcal{D}'$, as claimed. So, firstly, let $\mathcal{F} \neq \mathcal{F}_i$ be a facet of $\mathcal{P}^\cap(\mathcal{D}')$ intersecting $\mathcal{D}'$ strongly. Then $\mathcal{F} \cap \mathcal{D}'$ is an m -face K' different from K'_i , so K' and, e.g., K'_1 have a common $(m-1)$ -face, and $\mathcal{F}$ and $\mathcal{F}_1$ have a common subfacet F . We determine the root of $\mathcal{F}$ by means of F . Let K be opposite to K' in $\mathcal{F}$, then K and the root K_1 of $\mathcal{F}_1$ again have a common $(m-1)$ -face contained in F . Thus, by Lemmas 6.1(b) and 6.2(a) together with the assumption that m is the minimal type of the facets of $\mathcal{P}$, it follows that K is the root of $\mathcal{F}$, as claimed.

Secondly, let $\bar{\mathcal{F}}$ be a facet of $\mathcal{P}^\cap(\mathcal{D}')$ containing $\mathcal{D}'$. Then $\bar{\mathcal{F}}$ intersects $\mathcal{F}_i$ in a subfacet $\bar{F}_i$ containing K'_i . We determine the root of $\bar{\mathcal{F}}$ by means of the sf-sets of $\bar{F}_i$, $i = 1, 2$. Since the root K_i of $\mathcal{F}_i$ is opposite to K'_i in $\mathcal{F}_i$, $\bar{F}_i \supset K'_i$ is disjoint from K_i . Hence, by Lemma 6.1(c), the sf-set of $\bar{F}_i$ is the $\bar{K}_i$ -set of $\bar{F}_i$, where $\bar{K}_i$ is opposite to K'_i in $\bar{F}_i$. Now let $\bar{\mathcal{D}}$ be opposite to $\mathcal{D}'$ in $\bar{\mathcal{F}}$, so that $\bar{K}_1$ and $\bar{K}_2$ are opposite faces of $\bar{\mathcal{D}}$. Then by Lemma 6.2(a) applied to both $\bar{F}_1$ and $\bar{F}_2$, the root of $\bar{\mathcal{F}}$ is either $\bar{\mathcal{D}}$ or each $\bar{K}_i$, $i = 1, 2$. Since the latter case is impossible, the root of $\bar{\mathcal{F}}$ must be $\bar{\mathcal{D}}$, as claimed. $\square$

9. Decomposing $\mathcal{P}$ into Two Complexes

As in Lemma 8.4, let m be the minimal type of the facets of $\mathcal{P}$, and let $0 \leq m \leq d-2$. We decompose $\mathcal{P}$ into two subcomplexes, namely, into the subcomplex

$\mathcal{P}^\cap(\mathcal{D}')$ of Lemma 8.4, and the subcomplex $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ spanned by the facets of $\mathcal{P}$ not belonging to $\mathcal{P}^\cap(\mathcal{D}')$. Thus every face of $\mathcal{P}$ belongs to $\mathcal{P}^\cap(\mathcal{D}')$ or $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$, and their common faces are just their outer faces.

Clearly, $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ consists of all the faces of $\mathcal{P}$ disjoint from $\mathcal{D}'$; in the context of convex polytopes it is the antistar of $\mathcal{D}'$ in $\mathcal{P}$.

In the following we deal with the outer subfacets and outer $(d-3)$ -faces of $\mathcal{P}^\cap(\mathcal{D}')$ and $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$. Recall that the outer subfacets of $\mathcal{P}^\cap(\mathcal{D}')$ are those ones lying in just one facet of $\mathcal{P}^\cap(\mathcal{D}')$, and, since $\mathcal{P}^\cap(\mathcal{D}')$ is cubal, the outer $(d-3)$ -faces of $\mathcal{P}^\cap(\mathcal{D}')$ are those ones lying in just one or two facets of $\mathcal{P}^\cap(\mathcal{D}')$.

There is a criterion whether an outer $(d-3)$ -face of $\mathcal{P}^\cap(\mathcal{D}')$ is regular or singular.

Lemma 9.1. *An outer $(d-3)$ -face of $\mathcal{P}^\cap(\mathcal{D}')$ is regular if it lies in just one facet of $\mathcal{P}^\cap(\mathcal{D}')$, and it is singular if it lies in just two facets of $\mathcal{P}^\cap(\mathcal{D}')$.*

Proof. Recall that the singular $(d-3)$ -faces of a facet $\mathcal{F}$ with root K are the outer $(d-3)$ -faces of $\mathcal{F}(K)$.

Now let S be an outer $(d-3)$ -face of $\mathcal{P}^\cap(\mathcal{D}')$. Then S is contained in two outer subfacets F_1 and F_2 of $\mathcal{P}^\cap(\mathcal{D}')$. As outer subfacets, F_1 and F_2 are disjoint from $\mathcal{D}'$.

Firstly, let S be contained in one facet $\mathcal{F}$ of $\mathcal{P}^\cap(\mathcal{D}')$, that is, $F_1, F_2 \subset \mathcal{F}$. Since F_1 and F_2 are disjoint from $\mathcal{D}'$, Lemma 8.4 yields that they contain a root K of $\mathcal{F}$ and so belong to $\mathcal{F}(K)$ for a root K of $\mathcal{F}$. Then $S = F_1 \cap F_2$ is an inner $(d-3)$ -face of $\mathcal{F}(K)$ and so is regular.

Secondly, let S be contained in just two facets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{P}^\cap(\mathcal{D}')$, that is, $F_1 \subset \mathcal{F}_1$, $F_2 \subset \mathcal{F}_2$ without loss of generality. Since F_1 is disjoint from $\mathcal{D}'$, Lemma 8.4 yields that F_1 contains a root K of $\mathcal{F}_1$ and so belongs to $\mathcal{F}_1(K)$ for a root K of $\mathcal{F}_1$. Thereby K is opposite to $\mathcal{F}_1 \cap \mathcal{D}'$ in $\mathcal{F}_1$. On the other hand, S is contained in the subfacet $F := \mathcal{F}_1 \cap \mathcal{F}_2$ intersecting $\mathcal{D}'$ in $\mathcal{F}_1 \cap \mathcal{D}'$ or a face of it. So $F \subset \mathcal{F}_1$ does not belong to $\mathcal{F}_1(K)$. Hence $S = F_1 \cap F$ is an outer $(d-3)$ -face of $\mathcal{F}_1(K)$ and so is singular. $\square$

An immediate consequence of Lemma 9.1 is:

Lemma 9.2. *Every outer $(d-3)$ -face of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ lies in precisely two facets of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$.*

There is another important property of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$.

Lemma 9.3. *Every facet $\mathcal{F}$ of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ has at most one outer subfacet.*

Proof. Assume to the contrary that $\mathcal{F}$ has two subfacets F_1 and F_2 that are outer subfacets of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$. Then Lemma 9.2 yields that F_1 and F_2 are opposite in $\mathcal{F}$.

F_1 and F_2 are also outer subfacets of $\mathcal{P}^\cap(\mathcal{D}')$, and we must distinguish three cases according to the dimension of $\mathcal{D}'$.

Firstly, let $\dim \mathcal{D}' = 0$. Then all the facets of $\mathcal{P}^\cap(\mathcal{D}')$ contain the vertex $\mathcal{D}'$. There are d edges of $\mathcal{P}^\cap(\mathcal{D}')$ passing through $\mathcal{D}'$; let $e_1, \dots, e_d$ be their other

vertices. Every outer subfacet of $\mathcal{P}^\cap(\mathcal{D}')$ contains one such vertex, and so do F_1 and F_2 . Since F_1 and F_2 are disjoint, we may assume $e_1 \in F_1$ and $e_2 \in F_2$. Then $e_1, e_2 \in \mathcal{F}$, so that $\mathcal{F}$ contains the 2-face spanned by e_1 and e_2 . Thus $\mathcal{F}$ contains $\mathcal{D}'$, which is impossible.

Secondly, let $\dim \mathcal{D}' = 1$. An outer subfacet of $\mathcal{P}^\cap(\mathcal{D}')$ is contained in one facet of $\mathcal{P}^\cap(\mathcal{D}')$ and is disjoint from $\mathcal{D}'$, so, by Lemmas 8.4 and 6.1(a), its sf-set is complementary to a K -set with $\dim K = 0$ or 1. This yields two types of outer subfacets of $\mathcal{P}^\cap(\mathcal{D}')$ according to whether $\dim K = 0$ or 1, and we remark that every subfacet of the one type has a common $(d - 3)$ -face with some subfacet of the other type.

In particular, the sf-set of F_i , $i = 1, 2$, is complementary to a K_i -set with $\dim K_i = 0$ or 1. If $\dim K_i = 1$, then, by Lemma 6.2(b), the root of $\mathcal{F}$ is K_i ; if $\dim K_i = 0$, then, by Lemma 6.2(a), the root of $\mathcal{F}$ is either one-dimensional and intersecting F_i , or there are two zero-dimensional roots. Since F_1 and F_2 are opposite in $\mathcal{F}$, it follows that $\dim K_1 = \dim K_2 = 0$.

Now let F_3 be an outer subfacet of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ such that F_3 has a common $(d - 3)$ -face with F_1 . Then Lemma 9.2 yields that F_3 is contained in a facet of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ also having two outer subfacets. Thus the sf-set of F_3 is also complementary to a K_3 -set with $\dim K_3 = 0$. However, this contradicts the above remark.

Finally, let $\dim \mathcal{D}' \geq 2$. An outer subfacet of $\mathcal{P}^\cap(\mathcal{D}')$ is contained in one facet of $\mathcal{P}^\cap(\mathcal{D}')$ and is disjoint from $\mathcal{D}'$, so, by Lemmas 8.4 and 6.1(a), its sf-set is complementary to a K -set with $\dim K \geq 1$. By Lemma 6.2(b), it follows that both F_1 and F_2 contain the at least one-dimensional root of $\mathcal{F}$, which is impossible. $\square$

10. The Replacing Operation

As in Lemma 8.4, let m be the minimal type of the facets of $\mathcal{P}$, and let $0 \leq m \leq d - 2$. In this section we replace the cubal subcomplex $\mathcal{P}^\cap(\mathcal{D}')$ in $\mathcal{P}$ by some “smaller” complex.

For that purpose we use $\mathcal{E}^d$, the boundary complex of the d -cube, which is a $(d - 1)$ -STC-complex as well as $\mathcal{P}$. Since $\mathcal{P}^\cap(\mathcal{D}')$ is cubal, it is isomorphic to some subcomplex of $\mathcal{E}^d$. More closely, choose a $(\dim \mathcal{D}')$ -dimensional face of $\mathcal{E}^d$ and denote it also by $\mathcal{D}'$; let $\mathcal{D}$ be the face of $\mathcal{E}^d$ opposite to $\mathcal{D}'$. In accordance with the previous notation, let $\mathcal{E}^\cap(\mathcal{D}')$ and $\mathcal{E}(\mathcal{D})$ be the subcomplexes of $\mathcal{E}^d$ spanned by the facets of $\mathcal{E}^d$ intersecting $\mathcal{D}'$ or containing $\mathcal{D}$, respectively. Then $\mathcal{P}^\cap(\mathcal{D}')$ is isomorphic to $\mathcal{E}^\cap(\mathcal{D}')$.

It follows that a homeomorphism φ from $\mathcal{P}^\cap(\mathcal{D}')$ onto $\mathcal{E}^\cap(\mathcal{D}')$ exists. Moreover, since both set $\mathcal{E}^\cap(\mathcal{D}')$ and set $\mathcal{E}(\mathcal{D})$ are topological $(d - 1)$ -balls, and since their boundaries coincide, the identity map on their common boundary is extendible to a homeomorphism ψ from set $\mathcal{E}^\cap(\mathcal{D}')$ onto set $\mathcal{E}(\mathcal{D})$. Then $(\psi \circ \varphi)^{-1}$ is a homeomorphism from set $\mathcal{E}(\mathcal{D})$ onto set $\mathcal{P}^\cap(\mathcal{D}')$, and it is also a homeomorphism between the subcomplexes of the outer faces of $\mathcal{E}(\mathcal{D})$ and $\mathcal{P}^\cap(\mathcal{D}')$. Identifying $\mathcal{E}(\mathcal{D})$ with its image under $(\psi \circ \varphi)^{-1}$, we can replace the subcomplex $\mathcal{P}^\cap(\mathcal{D}')$ in $\mathcal{P}$ by $\mathcal{E}(\mathcal{D})$, thus again obtaining a finite family $\tilde{\mathcal{P}}$ of topological cubes, with

set $\bar{\mathcal{P}} = \text{set } \mathcal{P}$. The next lemma states that $\bar{\mathcal{P}}$ is also a $(d - 1)$ -STC-complex with further nice properties.

Lemma 10.1. *$\bar{\mathcal{P}}$ as defined above is a $(d - 1)$ -STC-complex. Furthermore, letting m be, as before, the minimal type of the facets of $\mathcal{P}$, $0 \leq m \leq d - 2$, letting $\bar{m}$ be the same for $\bar{\mathcal{P}}$, and recalling the connection between m and $\dim \mathcal{D}'$ given in Lemma 8.4, the following is true:*

- (a) *Let $\dim \mathcal{D}' = 0$, that is, $m = 0$ and all the facets of $\mathcal{P}$ are of type 0. Then $\bar{m} = 0$, and not all the facets of $\bar{\mathcal{P}}$ are of type 0. Moreover, the vertex number of $\bar{\mathcal{P}}$ equals the vertex number of $\mathcal{P}$.*
- (b) *Let $\dim \mathcal{D}' = 1$, that is, $m = 0$ and not all the facets of $\mathcal{P}$ are of type 0. Then $\bar{m} = 0$, and not all the facets of $\bar{\mathcal{P}}$ are of type 0, or $\bar{m} = 1$. Moreover, the vertex number of $\bar{\mathcal{P}}$ is smaller than the vertex number of $\mathcal{P}$ by 2.*
- (c) *Let $\dim \mathcal{D}' \geq 2$, that is, $m \geq 1$ and $\dim \mathcal{D}' = m + 1$. Then $m \leq \bar{m} \leq m + 1$. Moreover, the vertex number of $\bar{\mathcal{P}}$ is smaller than the vertex number of $\mathcal{P}$ by 2^{m+1} .*

Proof. Observe that $\bar{\mathcal{P}}$ consists of the two complexes $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ and $\mathcal{E}(\mathcal{D})$, pasted together along their outer faces. We begin by showing that $\bar{\mathcal{P}}$ satisfies the conditions of a $(d - 1)$ -STC-complex.

Conditions (1) and (3) are satisfied by definition.

To prove (2) it is enough to show that, for every two faces $S_1 \in \mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ and $S_2 \in \mathcal{E}(\mathcal{D})$, $S_1 \cap S_2$ is a face of both of them. Clearly, $S_1 \cap S_2$ is either empty and there is nothing to prove or it is the union of outer faces of both $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ and $\mathcal{E}(\mathcal{D})$. So let E_1 be an outer vertex of S_1 . Then S_1 is contained in a facet $\mathcal{F}_1 \in \mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ through E_1 . Using property (6) of $\mathcal{P}$ we see that $\mathcal{F}_1$ has an outer $(d - 3)$ -face through E_1 , and from Lemma 9.2 it follows that $\mathcal{F}_1$ has an outer subfacet F_1 through E_1 . The same argument together with Lemma 9.3 yields that every outer vertex of $\mathcal{F}_1$ is contained in F_1 . Thus $S_1 \cap S_2 \subset F_1$, and hence $S_1 \cap S_2 = (S_1 \cap F_1) \cap (S_2 \cap F_1)$. Since F_1 is a face of both $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ and $\mathcal{E}(\mathcal{D})$, it follows that $S_1 \cap S_2$ is a face of S_1 and S_2 , so that $\bar{\mathcal{P}}$ satisfies (2).

To prove (6), choose facets $\mathcal{F}_1 \in \mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ and $\mathcal{F}_2 \in \mathcal{E}(\mathcal{D})$ with a common vertex E_1 . As we have already seen, $\mathcal{F}_1$ then has an outer subfacet F_1 through E_1 . It remains to show that $F_1 \cap \mathcal{F}_2$ is at least $(d - 3)$ -dimensional. However, this is clear, since both F_1 and $\mathcal{F}_2$ are faces of $\mathcal{E}(\mathcal{D}) \subset \mathcal{E}^d$ with a common vertex.

Property (7) follows immediately from the corresponding property of $\mathcal{P}$ and $\mathcal{E}^d$.

In order to prove (4) it is enough to show that every outer $(d - 3)$ -face S of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ is contained in only three or four facets of $\bar{\mathcal{P}}$. However, this is clear, since S is contained in just two facets of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$ by Lemma 9.2, and in at most two facets of $\mathcal{E}(\mathcal{D})$. We can say more. Since every $(d - 3)$ -face of $\mathcal{E}^d$ is contained in just three facets, a common $(d - 3)$ -face of $\mathcal{E}^\cap(\mathcal{D}')$ and $\mathcal{E}(\mathcal{D})$ is contained either in one facet of $\mathcal{E}^\cap(\mathcal{D}')$ and in two facets of $\mathcal{E}(\mathcal{D})$, or vice versa. Observing that $\mathcal{P}^\cap(\mathcal{D}')$ is isomorphic to $\mathcal{E}^\cap(\mathcal{D}')$, from this together with Lemma 9.2 it follows that S is regular in $\mathcal{P}$ if and only if it is singular in $\bar{\mathcal{P}}$. Thus transition from

$\mathcal{P}$ to $\bar{\mathcal{P}}$ interchanges regularity and singularity of the outer $(d-3)$ -faces of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$.

In proving (5), we simultaneously determine the types of the facets of $\bar{\mathcal{P}}$. At first we consider the facets of $\mathcal{E}(\mathcal{D})$. The inner $(d-3)$ -faces of $\mathcal{E}(\mathcal{D})$ are contained in just three facets of $\bar{\mathcal{P}}$ and so are regular; they are just the $(d-3)$ -faces of $\mathcal{E}(\mathcal{D})$ containing $\mathcal{D}$. By Lemma 9.2, an outer $(d-3)$ -face of $\mathcal{E}(\mathcal{D})$ is singular if and only if it lies in two facets of $\mathcal{E}(\mathcal{D})$, that is, if and only if it lies in a subfacet of $\mathcal{E}(\mathcal{D})$ passing through $\mathcal{D}$. Thus a given $(d-3)$ -face of $\mathcal{E}(\mathcal{D})$ is singular if and only if it is contained in a subfacet passing through $\mathcal{D}$, but does not contain $\mathcal{D}$. Since every facet $\mathcal{F} \in \mathcal{E}(\mathcal{D})$ contains $\mathcal{D}$, from this it follows that the sf-set of $\mathcal{F}$ is the set of the outer $(d-3)$ -faces of the complex $\mathcal{F}(\mathcal{D})$. Thus $\mathcal{F} \in \mathcal{E}(\mathcal{D})$ satisfies (5). The root of $\mathcal{F}$ is $\mathcal{D}$, so that the type of $\mathcal{F} \in \mathcal{E}(\mathcal{D})$ is $\dim \mathcal{D}'$.

Now we consider the facets of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$. Clearly, it is enough to consider those having an outer $(d-3)$ -face, and as we have already seen, all the outer $(d-3)$ -faces of such a facet are contained in one subfacet. So let $\mathcal{F}$ be a facet of $\mathcal{P} \setminus \mathcal{P}^\cap(\mathcal{D}')$, with the outer subfacet F . The sf-set of $\mathcal{F}$ differs according to whether $\mathcal{F} \in \mathcal{P}$ or $\mathcal{F} \in \bar{\mathcal{P}}$; by the above, each sf-set arises from the other one by interchanging regular and singular $(d-3)$ -faces of F . Since $\mathcal{P}$ satisfies (5), the sf-set of $\mathcal{F} \in \mathcal{P}$ is the set of the outer $(d-3)$ -faces of some complex $\mathcal{F}(K)$, where K is a root of $\mathcal{F} \in \mathcal{P}$.

Two possibilities for F and K turn out to be important. Firstly, let $K \subset F$, that is, $F \in \mathcal{F}(K)$. Then it is easy to see that interchanging regular and singular $(d-3)$ -faces in F yields an sf-set of $\mathcal{F} \in \bar{\mathcal{P}}$ that is the set of the outer $(d-3)$ -faces of the complex $\mathcal{F}(K')$, where $\mathcal{F}(K')$ is spanned by the subfacets of $\mathcal{F}(K)$ except F . Thus in this case the type of $\mathcal{F}$ increases by one. Secondly, let K be one-dimensional and intersecting F . Then the sf-set of $\mathcal{F} \in \bar{\mathcal{P}}$ is the set of the outer $(d-3)$ -faces of the complex $\mathcal{F}(K')$, where $\mathcal{F}(K')$ is spanned by the subfacets of $\mathcal{F}(K)$ plus F .

We must now find a concrete root of $\mathcal{F} \in \mathcal{P}$. For this purpose we distinguish three cases, according to the statement of the lemma.

(a) Let $\dim \mathcal{D}' = 0$. Then all the facets of $\mathcal{P}$ are of type 0, and so is $\mathcal{F} \in \mathcal{P}$. Thus F contains a zero-dimensional root K of $\mathcal{F}$. Hence, by the above, the sf-set of $\mathcal{F} \in \bar{\mathcal{P}}$ has the desired structure, and the type of $\mathcal{F}$ is 1.

Thus for $\dim \mathcal{D}' = 0$, not all the facets of $\bar{\mathcal{P}}$ are of type 0, and the facets of $\mathcal{E}(\mathcal{D})$ yield $\bar{m} = 0$, as claimed.

(b) Let $\dim \mathcal{D}' = 1$. We find a root of $\mathcal{F} \in \mathcal{P}$ by using the fact that F is also contained in a facet $\mathcal{F}_0 \in \mathcal{P}^\cap(\mathcal{D}')$. Lemma 8.4 yields that F contains a root K of $\mathcal{F}_0$, which may be zero- or one-dimensional. If K is one-dimensional, then Lemmas 6.1(a) and 6.2(b) yield that $K \subset F$ is also a root of $\mathcal{F} \in \mathcal{P}$. Thus, by the above, the sf-set of $\mathcal{F} \in \bar{\mathcal{P}}$ has the desired structure. However, if K is zero-dimensional, then, by Lemmas 6.1(a) and 6.2(a), a root of $\mathcal{F} \in \mathcal{P}$ is either K , or is one-dimensional and intersecting F . Thus, by the above, in both cases the sf-set of $\mathcal{F} \in \bar{\mathcal{P}}$ has the desired structure.

Here, in case $\dim \mathcal{D}' = 1$, the facets of $\mathcal{E}(\mathcal{D})$ yield that $\bar{m} \leq 1$, and that not all the facets of $\bar{\mathcal{P}}$ are of type 0, as claimed.

(c) Let $\dim \mathcal{D}' \geq 2$. Again we find a root of $\mathcal{F} \in \mathcal{P}$ by using the fact that F is also contained in a facet $\mathcal{F}_0 \in \mathcal{P}^\cap(\mathcal{D}')$. Lemma 8.4 yields that F contains a root K

of $\mathcal{F}_0$, which is at least one-dimensional. By Lemmas 6.1(a) and 6.2(b), $K \subset F$ is also a root of $\mathcal{F} \in \mathcal{P}$. Thus, by the above, the sf-set of $\mathcal{F} \in \mathcal{P}$ has the desired structure, and the type of $\mathcal{F}$ increases by one.

From this it follows that $\bar{m} \geq m$, and the facets of $\mathcal{E}(\mathcal{D})$ yield $\bar{m} \leq m + 1$, as claimed.

Thus $\bar{\mathcal{P}}$ also satisfies (5), so that it is a $(d - 1)$ -STC-complex as claimed, and the stated assertions about the types of the facets of $\bar{\mathcal{P}}$ are true.

To complete the proof it remains to determine the vertex number of $\bar{\mathcal{P}}$. However, this follows immediately from comparing the vertex numbers of $\mathcal{E} \cap (\mathcal{D}')$ and $\mathcal{E}(\mathcal{D})$, which are equal for $\dim \mathcal{D}' = 0$, and otherwise differ by the vertex number of $\mathcal{D}'$. $\square$

11. Iterating the Replacing Operation

As before, let $m = m(\mathcal{P})$ be the minimal type of the facets of a $(d - 1)$ -STC-complex $\mathcal{P}$. As we have seen in Section 10, if $m(\mathcal{P}) \leq d - 2$, then $\mathcal{P}$ admits a replacing operation. Lemma 10.1 states that the resulting object $\bar{\mathcal{P}}$ is again a $(d - 1)$ -STC-complex, so the replacing operation can be iterated, provided $m(\mathcal{P}) \leq d - 2$, too. This yields a sequence $\mathcal{P} =: \mathcal{P}_0, \bar{\mathcal{P}} =: \mathcal{P}_1, \bar{\bar{\mathcal{P}}} =: \mathcal{P}_2, \dots$ that stops if and only if we have got a $\mathcal{P}_k$ with $m(\mathcal{P}_k) = d - 1$, that is, $\mathcal{P}_k$ is isomorphic to $\mathcal{E}^d$, the boundary complex of the d -cube, by Lemma 8.3.

The sequence $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots$ corresponds to a sequence of replacing operations r_i transforming $\mathcal{P}_i$ into $\mathcal{P}_{i+1}$. Letting $\mathcal{D}'(\mathcal{P}_i)$ be the face of $\mathcal{P}_i$ specified in Lemma 8.4, a replacing operation r_i is characterized by the value of $\dim \mathcal{D}'(\mathcal{P}_i)$, $0 \leq \dim \mathcal{D}'(\mathcal{P}_i) \leq d - 1$, which is called the *type* of the replacing operation. By Lemma 10.1, every replacing operation reduces the vertex number, except a replacing operation of type 0, which can occur only once. Thus the sequence $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots$ must be finite, namely, $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k$, with $\mathcal{P}_k$ isomorphic to $\mathcal{E}^d$. The so far excluded case of a $\mathcal{P}$ with $m(\mathcal{P}) = d - 1$ can now be included by putting $k = 0$.

If $\mathcal{P}$ has less than 2^{d+1} vertices, then the types of all the occurring replacing operations can be determined explicitly.

Lemma 11.1. *Let $\mathcal{P}$ be a $(d - 1)$ -STC-complex with fewer than 2^{d+1} vertices. Let $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k$ be the sequence resulting from $\mathcal{P}$ by successive replacing operations r_i transforming $\mathcal{P}_i$ into $\mathcal{P}_{i+1}$, and let $n(r_i)$ be the type of r_i . Then $k \leq d$, and $n(r_i) = i + (d - k)$, $0 \leq i \leq k - 1$.*

Proof. Lemma 10.1 asserts that $n(r_i)$ is an increasing sequence running through every natural number between $n(r_0)$ and $n(r_{k-1})$. Moreover, if $n(r_0) = 0$, then $n(r_1) = 1$, and since $m(\mathcal{P}_k) = d - 1$, we must have $n(r_{k-1}) = d - 1$.

For the moment, let us put $2^0 := 0$. Then, by Lemma 10.1, the vertex number of $\mathcal{P}$ is $\text{vert } \mathcal{P} = 2^{n(r_0)} + 2^{n(r_1)} + \dots + 2^{n(r_{k-1})} + 2^d$. Now, if we had $n(r_i) = n(r_{i+1})$ for some i , then, by the above, $\mathcal{P}$ has at least 2^{d+1} vertices, which contradicts the assumption on $\mathcal{P}$. Thus $n(r_i) \neq n(r_{i+1})$, so that $n(r_{i+1}) = n(r_i) + 1$, therefore $n(r_i) = n(r_0) + i$. Together with $n(r_{k-1}) = d - 1$ this yields $n(r_i) = i + (d - k)$. Thus $k \leq d$ follows from $n(r_0) \geq 0$. $\square$

As a consequence of Lemma 11.1 we can relabel the sequence $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k$ in order to standardize it, getting a sequence $\mathcal{P} = \mathcal{P}_{d-k}, \mathcal{P}_{d-k+1}, \dots, \mathcal{P}_d$, with $d - k \geq 0$, $\mathcal{P}_d$ is isomorphic to $\mathcal{E}^d$, and the type $n(r_i)$ of a replacing operation r_i transforming $\mathcal{P}_i$ into $\mathcal{P}_{i+1}$ is just $n(r_i) = i$, $d - k \leq i \leq d - 1$.

This enables us to describe completely $\mathcal{P} = \mathcal{P}_{d-k}$ up to isomorphisms.

Lemma 11.2. *Let $\mathcal{P}$ be a $(d - 1)$ -STC-complex with fewer than 2^{d+1} vertices. Let $\mathcal{P} = \mathcal{P}_{d-k}, \dots, \mathcal{P}_d$ be the sequence associated with $\mathcal{P}$ as above, $k \leq d$. Then $\mathcal{P}$ is isomorphic to the boundary complex of the k -elementary cubical d -polytope C_k^d .*

Proof. We denote the boundary complex of C_k^d by $\mathcal{E}_k^d$, so that $\mathcal{E}_0^d$ is the same as $\mathcal{E}^d$, the boundary complex of the d -cube. We do not distinguish between $\mathcal{E}_k^d$ and isomorphic $(d - 1)$ -STC-complexes.

Recall the notation $\mathcal{P}(K)$, $\mathcal{P}^\cap(K)$, $\mathcal{F}(K)$, $\mathcal{F}^\cap(K)$ for certain subcomplexes of a $(d - 1)$ -STC-complex $\mathcal{P}$ and of a facet $\mathcal{F}$ of $\mathcal{P}$, see the beginning of Section 8. Thereby $0 \leq \dim K \leq d - 1$ is admitted.

If, in particular, $\mathcal{P} = \mathcal{E}^d$, we again write $\mathcal{E}(K)$ and $\mathcal{E}^\cap(K)$. We need the following remark. The outer $(d - 3)$ -faces of $\mathcal{E}(K)$ that are contained in two facets of $\mathcal{E}(K)$ are just the outer $(d - 3)$ -faces of $\mathcal{F}(K)$ for all facets $\mathcal{F} \in \mathcal{E}(K)$; this follows from the fact that a $(d - 3)$ -face of $\mathcal{E}(K)$ is contained in two facets of $\mathcal{E}(K)$ if and only if it is contained in a subfacet of $\mathcal{E}(K)$ containing K , and a $(d - 3)$ -face of $\mathcal{E}(K)$ as well as a $(d - 3)$ -face of $\mathcal{F}(K)$ is an inner one if and only if it contains K . Likewise, the outer $(d - 3)$ -faces of $\mathcal{E}^\cap(K)$ that are contained in two facets of $\mathcal{E}^\cap(K)$ are just the outer $(d - 3)$ -faces of $\mathcal{F}^\cap(K \cap \mathcal{F})$ for all facets $\mathcal{F} \in \mathcal{E}^\cap(K)$.

We begin our proof by studying $\mathcal{E}_k^d$ more closely, in particular, we determine the roots of the facets of $\mathcal{E}_k^d$. The polytope C_k^d can be imagined as the union of certain $k + 1$ d -cubes with a common $(d - k)$ -face K . These $k + 1$ d -cubes span a complex in $\mathbb{R}^d$, which we also denote by C_k^d , as well as every combinatorial isomorphic complex. Clearly, removing one of the d -cubes of C_k^d yields C_{k-1}^d .

The $(d - 1)$ -STC-complex $\mathcal{E}_k^d$ is, up to isomorphism, the subcomplex spanned by the outer facets of C_k^d . In the following, let $k \geq 1$. Let C be a d -cube of C_k^d , so that $C \supset K$, and let K' be the face opposite to K in C . Let $\mathcal{E}$ be the boundary complex of C . Then transforming C_k^d into C_{k-1}^d by removing C clearly induces a transformation of $\mathcal{E}_k^d$ into $\mathcal{E}_{k-1}^d$ by replacing $\mathcal{E}^\cap(K')$ by $\mathcal{E}(K)$.

This provides a way to determine those singular $(d - 3)$ -faces of $\mathcal{E}_k^d$ that lie in $\mathcal{E}^\cap(K')$. Clearly, $(d - 3)$ -faces of $\mathcal{E}^\cap(K')$ can only be singular if they are outer ones. Note that $\mathcal{E}^\cap(K')$ and $\mathcal{E}(K)$ have the same outer $(d - 3)$ -faces, and such an outer $(d - 3)$ -face lies in one facet of $\mathcal{E}^\cap(K')$ if and only if it lies in two facets of $\mathcal{E}(K)$, and vice versa. Since every $(d - 3)$ -face of both $\mathcal{E}_k^d$ and $\mathcal{E}_{k-1}^d$ is contained in only three or four facets, it follows that an outer $(d - 3)$ -face of $\mathcal{E}^\cap(K')$ is singular if and only if it lies in two facets of $\mathcal{E}^\cap(K')$. So, by the above remark, the singular $(d - 3)$ -faces of $\mathcal{E}^\cap(K')$ are just the outer $(d - 3)$ -faces of $\mathcal{F}^\cap(K' \cap \mathcal{F})$ for all facets $\mathcal{F} \in \mathcal{E}^\cap(K')$. Thus the face opposite to $K' \cap \mathcal{F}$ in $\mathcal{F}$ is a root of $\mathcal{F}$. So if $\mathcal{F} \in \mathcal{E}^\cap(K')$ is disjoint from K , then $\mathcal{F}$ contains K' , and a root of $\mathcal{F}$ has dimension $\dim K' = \dim K = d - k$. However, if $\mathcal{F} \in \mathcal{E}^\cap(K')$ intersects K , then $K \cap \mathcal{F}$ is

opposite to $K' \cap \mathcal{F}$ in $\mathcal{F}$, so that $K \cap \mathcal{F}$ is a root of $\mathcal{F}$, which has dimension $d - k - 1 \geq 0$.

Since C is any d -cube of C_k^d , this result is true in general. Thus, for any facet $\mathcal{F} \in \mathcal{E}_k^d$, if $\mathcal{F} \cap K = \emptyset$, then the root of $\mathcal{F}$ is $(d - k)$ -dimensional, and if $\mathcal{F} \cap K \neq \emptyset$, then $\mathcal{F} \cap K$ is a $(d - k - 1)$ -dimensional root of $\mathcal{F}$, with $d - k - 1 \geq 0$.

Therefore, for $1 \leq k \leq d - 1$, the $(d - k - 1)$ -faces of K are roots of all the facets of $\mathcal{E}_k^d$ containing it, and it is easily seen that they are the only such $(d - k - 1)$ -faces of $\mathcal{E}_k^d$. Clearly, this result is true even for $0 \leq k \leq d - 1$.

We are now in a position to prove our lemma. Clearly, it is true for $k = 0$. We prove it generally by induction on k and assume, therefore, that it is true for some k , $0 \leq k < d$. Now let $\mathcal{P}$ be a $(d - 1)$ -STC-complex such that some replacing operation of type $d - (k + 1)$ transforms $\mathcal{P}$ into $\mathcal{E}_k^d$. We must show that $\mathcal{P}$ is isomorphic to $\mathcal{E}_{k+1}^d$.

The above assumption on $\mathcal{P}$ means that $\mathcal{P}$ has a $(d - (k + 1))$ -face $\mathcal{D}'$ such that $\mathcal{P} \cap (\mathcal{D}')$ is cubal, and replacing $\mathcal{P} \cap (\mathcal{D}')$ by some $\mathcal{E}(\mathcal{D})$ yields $\mathcal{E}_k^d$; thereby $\mathcal{P} \cap (\mathcal{D}') \subset \mathcal{P}$ and $\mathcal{E}(\mathcal{D}) \subset \mathcal{E}_k^d$, pasted together along their common outer faces, form some $\mathcal{E}^d$ with opposite faces $\mathcal{D}$ and $\mathcal{D}'$, and $\mathcal{P} \cap (\mathcal{D}') = \mathcal{E} \cap (\mathcal{D}')$.

The crucial point of the proof is to specify the $(d - (k + 1))$ -face $\mathcal{D}$ of $\mathcal{E}_k^d$. We do this by means of the singular $(d - 3)$ -faces of $\mathcal{E}(\mathcal{D}) \subset \mathcal{E}_k^d$. Clearly, $(d - 3)$ -faces of $\mathcal{E}(\mathcal{D})$ can only be singular if they are outer ones. Note that $\mathcal{E}(\mathcal{D}) \subset \mathcal{E}_k^d$ and $\mathcal{E} \cap (\mathcal{D}') \subset \mathcal{P}$ have the same outer $(d - 3)$ -faces, and such an outer $(d - 3)$ -face lies in one facet of $\mathcal{E}(\mathcal{D})$ if and only if it lies in two facets of $\mathcal{E} \cap (\mathcal{D}')$, and vice versa. Since every $(d - 3)$ -face of both $\mathcal{E}_k^d$ and $\mathcal{P}$ lies in only three or four facets, it follows that an outer $(d - 3)$ -face of $\mathcal{E}(\mathcal{D})$ is singular if and only if it lies in two facets of $\mathcal{E}(\mathcal{D})$. So, by the above remark, the singular $(d - 3)$ -faces of $\mathcal{E}(\mathcal{D})$ are just the outer $(d - 3)$ -faces of $\mathcal{F}(\mathcal{D})$ for all facets $\mathcal{F} \in \mathcal{E}(\mathcal{D})$. Thus $\mathcal{D}$ is a root of all the facets of $\mathcal{E}(\mathcal{D})$, that is, $\mathcal{D}$ is a root of all the facets containing it. Moreover, $\mathcal{D}$ is $(d - k - 1)$ -dimensional. Thus $\mathcal{D}$ is a $(d - k - 1)$ -face of the $(d - k)$ -face $K \in C_k^d$.

We started with $\mathcal{P}$ and got $\mathcal{E}_k^d$ by replacing $\mathcal{P} \cap (\mathcal{D}')$ by $\mathcal{E}(\mathcal{D})$. Conversely, we may start with $\mathcal{E}_k^d$ and replace $\mathcal{E}(\mathcal{D}) = \mathcal{E}_k^d \cap (\mathcal{D})$ by $\mathcal{E} \cap (\mathcal{D}')$ thus getting $\mathcal{P}$. Thereby, as we have seen, $\mathcal{D}$ is a $(d - k - 1)$ -face of $K \in C_k^d$. However, all the $(d - 1)$ -STC-complexes resulting from $\mathcal{E}_k^d$ by replacing $\mathcal{E}(\mathcal{D})$ by $\mathcal{E} \cap (\mathcal{D}')$, for any such $\mathcal{D}$, are isomorphic. Clearly, $\mathcal{E}_{k+1}^d$ results from $\mathcal{E}_k^d$ this way, so that $\mathcal{P}$ is in fact isomorphic to $\mathcal{E}_{k+1}^d$. $\square$

Lemma 11.2 states that every $(d - 1)$ -STC-complex with less than 2^{d+1} vertices is isomorphic to the boundary complex of some k -elementary cubical polytope C_k^d , $0 \leq k \leq d$. So, in particular, the same is true for every cubical d -polytope with less than 2^{d+1} vertices, and, by definition, the isomorphism is just a combinatorial isomorphism. Thus our theorem is proved.

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