Sausages Are Good Packings

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Abstract. Let B^d be the d -dimensional unit ball and, for an integer n , let $C_n = \{x^1, \dots, x^n\}$ be a packing set for B^d , i.e., $|x^i - x^j| \geq 2$, $1 \leq i < j \leq n$. We show that for every $\rho < \sqrt{2}$ a dimension $d(\rho)$ exists such that, for $d \geq d(\rho)$,

$$V(\text{conv}(C_n) + \rho B^d) \geq V(\text{conv}(S_n) + \rho B^d),$$

where S_n is a “sausage” arrangement of n balls, holds. This gives considerable improvement to Fejes Tóth’s “sausage” conjecture in high dimensions. Further, we prove that, for every convex body K and $\rho < \frac{1}{32}d^{-2}$,

$$V(\text{conv}(C_n) + \rho K) \geq V(\text{conv}(S_n) + \rho K),$$

where C_n is a packing set with respect to K and S_n is a minimal “sausage” arrangement of K , holds.

1. Introduction

Throughout this paper E^d denotes the d -dimensional Euclidean space equipped with the Euclidean norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$. B^d denotes the d -dimensional unit ball with boundary S^{d-1} and $\text{conv}(P)$ denotes the convex hull of a set $P \subset E^d$. The volume of P with respect to the affine hull of P is denoted by $V(P)$ and let $\kappa_d = V(B^d)$. Finally, the spherical volume is denoted by $V_*(\cdot)$.

The well-known density concept of infinite packings (see, e.g., [GL], [CS], and [FK]) has the inherent disadvantage that it does not work for finite packings. Further, the usual finite packings are always considered with respect to a given

“large” convex body, e.g., a cube (a “bin” or “container”), into which the “small” convex bodies are packed (see, e.g., [Ro2], [GL], [FK], and [Gr2]). In contrast to these “bin packings” (for the name see, e.g., [KK] or [GW]), i.e., constrained finite packings, we are interested in “free packings”, i.e., finite packings independent of a given “large” convex body.

First results for these free finite packings in the plane have been obtained by Thue [Th], Rogers [Ro1], Groemer [Gr1], and Oler [Ol]. A way to study the density of free packings for balls in higher dimensions was observed by L. Fejes Tóth [Fe]. To this end let $\mathcal{P}_n(B^d)$ be the set of all possible packing arrangements of n translates of B^d in E^d , i.e.,

$$\mathcal{P}_n(B^d) = \{C_n \subset E^d: C_n = \{x^1, \dots, x^n\}, |x^i - x^j| \geq 2, i \neq j\}.$$

Fejes Tóth suggested defining the density of a (finite) packing $C_n \in \mathcal{P}_n(B^d)$ of n balls by

$$\delta(B^d, C_n) := \frac{n\kappa_d}{V(\text{conv}(C_n) + B^d)},$$

and he asked to determine, for every n and d ,

$$\delta(B^d, n) = \sup\{\delta(B^d, C_n): C_n \in \mathcal{P}_n(B^d)\}.$$

In other words: What is the best possible arrangement of n nonoverlapping d -dimensional unit balls with respect to the volume of the convex hull of the arrangement?

He observed that, at least for $d \geq 5$, the limit (with respect to n) of the densities of so-called “sausage” arrangements of balls are larger than the density of infinite packings. Here $S_n = \{x^1, \dots, x^n\} \in \mathcal{P}_n(B^d)$ is called a “sausage” arrangement if $x^i = 2i \cdot u$, $1 \leq i \leq n$, for some $u \in S^{d-1}$. More specifically he conjectured that, for $d \geq 5$, a “sausage” arrangement is a best possible arrangement:

“Sausage” Conjecture. For $d \geq 5$ and $n \in \mathbb{N}$,

$$\delta(B^d, n) = \delta(B^d, S_n)$$

holds.

For results related to this conjecture we refer to [GW]. Fejes Tóth’s concept was generalized by Betke *et al.* [BHW] to parametric densities by defining, for every $\rho \geq 0$ and $C_n \in \mathcal{P}_n(B^d)$,

$$\delta(B^d, C_n, \rho) = \frac{n\kappa_d}{V(\text{conv}(C_n) + \rho B^d)}$$

and

$$\delta(B^d, n, \rho) = \sup\{\delta(B^d, C_n, \rho): C_n \in \mathcal{P}_n(B^d)\}.$$

These parametric densities are not only of interest for finite packings but they also connect the finite and the infinite packing problem. In fact it was shown in [BHW] that a small ρ ($\rho < 2/\sqrt{3}$) favors “sausage” arrangements while a large ρ ($\rho \geq 2$) favors high-dimensional arrangements. Here we give an improvement of the bound $2/\sqrt{3}$. More precisely we prove:

Theorem 1.1. *For every $\rho < \sqrt{2}$ a dimension $d(\rho)$ exists such that for, $d \geq d(\rho)$ and $n \in \mathbb{N}$,*

$$\delta(B^d, n, \rho) = \delta(B^d, S_n, \rho)$$

holds.

Since $\delta(B^d, n, \rho_1) = \delta(B^d, S_n, \rho_1)$ for a certain ρ_1 implies

$$\delta(B^d, n, \rho) = \delta(B^d, S_n, \rho) \quad \text{for all } \rho < \rho_1$$

(see [BHW]), this theorem shows that in high dimensions the “sausage” arrangements are much better packings than required by Fejes Tóth’s “sausage” conjecture.

Further, we may deduce from Theorem 1.1 the following upper bound for the density $\delta(B^d)$ of a densest infinite packing of balls (see Corollary 1.1 of [BHW]):

Corollary 1.1. *For every $\varepsilon > 0$ a $d(\varepsilon) \in \mathbb{N}$ exists such that, for $d \geq d(\varepsilon)$,*

$$\delta(B^d) < (\sqrt{2} - \varepsilon)^{-d}.$$

Observe that this bound is asymptotically of the same order as the classical bounds due to Rogers [Ro2] and Blichfeldt [Bl]. Though this bound is much weaker than the best-known one (see, e.g., [KL]) it shows that parametric densities also give a new approach to the study of infinite packings.

Now, the concept of parametric densities can easily be generalized to arbitrary convex bodies. To this end let $\mathcal{K}^d$ be the set of all convex bodies in E^d with nonempty interior and let $\mathcal{K}_0^d \subset \mathcal{K}^d$ be the set of all centrally symmetric convex bodies. The distance function of $K \in \mathcal{K}_0^d$ is denoted by f_K , i.e.,

$$f_K(x) = \min\{\lambda \geq 0: x \in \lambda K\}.$$

Further, for $K \in \mathcal{K}^d$ let $DK \in \mathcal{K}_0^d$ be the difference body, i.e., $DK = \frac{1}{2}(K - K)$. For $n \in \mathbb{N}$, $K \in \mathcal{K}^d$ let $\mathcal{P}_n(K)$ be the set of all possible packing arrangements of n translates of K , which can be defined by

$$\mathcal{P}_n(K) = \{C_n \subset E^d: C_n = \{x^1, \dots, x^n\}, x^i + \text{int}(K) \cap x^j + \text{int}(K) = \emptyset, i \neq j\},$$

where $\text{int}(\cdot)$ denotes the interior. Since

$$C_n \in \mathcal{P}_n(K) \quad \Leftrightarrow \quad C_n \in \mathcal{P}_n(DK), \quad (1.1)$$

we may write $\mathcal{P}_n(K)$ as

$$\mathcal{P}_n(K) = \{C_n \subset E^d: C_n = \{x^1, \dots, x^n\}, f_{DK}(x^i - x^j) \geq 2, i \neq j\}.$$

As in the case of the ball we consider the following parametric densities for $K \in \mathcal{K}^d$:

$$\delta(K, C_n, \rho) = \frac{n \cdot V(K)}{V(\text{conv}(C_n) + \rho K)},$$

$$\delta(K, n, \rho) = \sup\{\delta(K, C_n, \rho): C_n \in \mathcal{P}_n(K)\}.$$

In [BHW] it was shown that for $K \in \mathcal{K}_0^d$ a positive constant ρ_m exists depending only on the dimension such that, for $\rho < \rho_m$, $\delta(K, n, \rho)$ is attained for a certain “sausage” arrangement, where for $u \in S^{d-1}$ the set $S_n(u, K) = \{x^1, \dots, x^n\} \in \mathcal{P}_n(K)$ is called a “sausage” arrangement in direction u iff $x^i = 2i \cdot u / f_{DK}(u)$ for $1 \leq i \leq n$. Here we give an explicit bound for such a ρ_m for arbitrary convex bodies. To this end we denote by $r(K)$ and $R(K)$ the inradius and circumradius of a convex body.

Theorem 1.2. *Let $K \in \mathcal{K}^d$, $n \in \mathbb{N}$, and $\rho \leq \frac{1}{32}d^{-1}r(K)/R(K)$. Then*

$$\delta(K, n, \rho) = \max_{u \in S^{d-1}} \delta(K, S_n(u, K), \rho).$$

By John’s [Jo] theorem we can always achieve, by an affine transformation A , that the ratio $r(AK)/R(AK)$ is bounded from below by d^{-1} for $K \in \mathcal{K}^d$. For $K \in \mathcal{K}_0^d$ even $r(AK)/R(AK) \geq d^{-1/2}$ holds. Since $\delta(K, C_n, \rho)$ is invariant under affine mappings (see [BHW]) we can deduce, from Theorem 1.2:

Corollary 1.2. *Let $K \in \mathcal{K}^d$, $n \in \mathbb{N}$, and $\rho \leq \frac{1}{32}d^{-2}$. Then*

$$\delta(K, n, \rho) = \max_{u \in S^{d-1}} \delta(K, S_n(u, K), \rho).$$

Corollary 1.3. *Let $K \in \mathcal{K}_0^d$, $n \in \mathbb{N}$, and $\rho \leq \frac{1}{32}d^{-3/2}$. Then*

$$\delta(K, n, \rho) = \max_{u \in S^{d-1}} \delta(K, S_n(u, K), \rho).$$

Let us remark that for centrally symmetric convex bodies in the plane the exact bound is $\frac{3}{4}$. Moreover, for $K \in \mathcal{K}_0^d$ ($K \in \mathcal{K}^d$) and for $\rho \geq 2$ ($\rho \geq d+1$), $\limsup \delta(K, n, \rho)$ converges to the density of the densest infinite packing as n tends to infinity [BHW].

2. Proof of Theorem 1.1

As mentioned in the Introduction the following weaker version of Theorem 1.1 was proved in [BHW]:

Theorem. *For every $\rho < 2/\sqrt{3}$ a dimension $d(\rho)$ exists such that, for $d \geq d(\rho)$ and $n \in \mathbb{N}$,*

$$\delta(B^d, n, \rho) = \delta(B^d, S_n, \rho)$$

holds.

To describe the main tool used for the proof of the theorem we need some more notation. For an arrangement $C_n \in \mathcal{P}_n(B^d)$ and $x^j \in C_n$ let

$$H_j = \{x \in E^d: |x - x^j| \leq |x - x^i| \text{ for all } x^i \in C_n\}$$

denote the Dirichlet–Voronoi cell of x^j with respect to the arrangement. Further, for a $(d - 2)$ -dimensional linear subspace $L \subset E^d$ and for $\rho > 0$ let

$$M_j(\rho, L) = \{z \in S^{d-1} \cap L: x^j + \rho z \notin H_j\},$$

$$K_j(\rho, L) = \{z \in S^{d-1} \cap L: x^j + \rho z \in H_j\}.$$

In [BHW] the following remark was made at the end of the proof of Theorem 4.1:

Remark. Let $\rho \in \mathbb{R}^{>0}$ have the following property: a $d_0 \in \mathbb{N}$ and a $c \in \mathbb{R}$ exist such that, for all $d \geq d_0$, $n \in \mathbb{N}$, and $C_n \in \mathcal{P}_n(B^d)$,

$$V_*(K_j(\rho, L)) \geq c \cdot V_*(S^{d-2})$$

holds for all $(d - 2)$ -dimensional linear subspaces L and for every Dirichlet–Voronoi cell H_j of C_n . Then a $d(\rho) \in \mathbb{N}$ exists such that $\delta(B^d, n, \rho) = \delta(B^d, S_n, \rho)$ for all $d \geq d(\rho)$ and $n \in \mathbb{N}$.

In [BHW] this was proved for every $\rho < 2/\sqrt{3}$ which implies the above theorem. Here we use Rogers' [Ro2] technique for the estimation of the size of the Dirichlet–Voronoi cells of packings to find:

Theorem 2.1. *Let $0 < \rho < \sqrt{2}$. Then there is a constant $c > 0$ and a $d_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ and every packing $C_n \in \mathcal{P}_n(B^d)$ with $d \geq d_0$, it holds that, for every $x^j \in C_n$ and every $(d - 2)$ -dimensional linear subspace L ,*

$$\frac{V_*(M_j(\rho, L))}{V_*(K_j(\rho, L))} \leq c.$$

Since $V_*(M_j(\rho, L)) + V_*(K_j(\rho, L)) = V_*(S^{d-2})$ we obtain Theorem 1.1 as an immediate consequence of the above remark.

In the course of our investigation we need to integrate over a part of a simplex. This can be done most conveniently by using the following “simplicial coordinates”:

Lemma 2.1. *Let $S \subset E^d$ be a d -simplex, let F_k be a k -face of S , $k \leq d-1$, and let $\bar{F}_k$ be the $(d-k-1)$ -face of S with $F_k \cap \bar{F}_k = \emptyset$. For a measurable subset $G \subset S$ and a continuous function f on S we have*

$$\int_G f dx = \frac{d!}{k!(d-1-k)!} \frac{V(S)}{V(F_k)V(\bar{F}_k)} \cdot \int_{F_k} \int_{\bar{F}_k} \int_{\mu\bar{x} + (1-\mu)x \in G} f(\mu\bar{x} + (1-\mu)x) \mu^{d-1-k} (1-\mu)^k d\mu d\bar{x} dx.$$

Remark. Here the notation $\int dx$ means integration in a space of appropriate dimension.

Proof. To compute the integral let $S = \text{conv}\{a^0, \dots, a^d\}$, $F_k = \text{conv}\{a^0, \dots, a^k\}$, and $\bar{F}_k = \text{conv}\{a^{k+1}, \dots, a^d\}$. Further, let A , A_k , and $\bar{A}_k$ denote the matrices

$$A = (a^1 - a^0, \dots, a^d - a^0), \quad A_k = (a^1 - a^0, \dots, a^k - a^0), \quad \text{and} \\ \bar{A}_k = (a^{k+2} - a^{k+1}, \dots, a^d - a^{k+1}).$$

Finally, we denote by T^d the standard simplex

$$T^d = \left\{ (x_1, \dots, x_d)^T \in E^d : x_i \geq 0, \sum_{i=1}^d x_i \leq 1 \right\}. \quad \square$$

Now we apply a number of transformations. First let $\varphi_1: E^d \rightarrow E^d$ be given by $\varphi_1(x) = Ax + a^0$. Then $T^d = \varphi_1^{-1}(S)$ and thus

$$\int_G f(x) dx = d! V(S) \int_{\varphi_1^{-1}(G)} f(\varphi_1(x)) dx. \quad (2.1)$$

Next we consider $\varphi_2: E^k \times E \times E^{d-1-k} \rightarrow E^d$ with

$$\varphi_2(\mu_1, \dots, \mu_k, \mu, \mu_{k+2}, \dots, \mu_d) \\ = (1-\mu) \cdot (\mu_1, \dots, \mu_k, 0, \dots, 0)^T \\ + \mu \cdot \left(0, \dots, 0, 1 - \sum_{i=k+2}^d \mu_i, \mu_{k+2}, \dots, \mu_d \right)^T.$$

φ_2 is a bijection from the interior of $T^k \times T^1 \times T^{d-1-k}$ onto the interior of T^d . By a straightforward calculation we find, for the determinant of the Jacobian $D\varphi_2$ of

$\varphi_2, \det(D\varphi_2) = \mu^{d-1-k}(1-\mu)^k$. Thus with $\mu' = (\mu_1, \dots, \mu_k)$, $\bar{\mu} = (\mu_{k+2}, \dots, \mu_d)$, and (2.1),

$$\begin{aligned} \int_G f(x) dx &= d! V(S) \int_{T^k} \int_{T^{d-1-k}} \int_{\varphi_2(\mu', \mu, \bar{\mu}) \in \varphi_1^{-1}(G)} f(\varphi_1(\varphi_2(\mu', \mu, \bar{\mu}))) \\ &\quad \times \mu^{d-1-k}(1-\mu)^k d\mu d\bar{\mu} d\mu'. \end{aligned}$$

As

$$\varphi_1(\varphi_2(\mu', \mu, \bar{\mu})) = (1-\mu)(A_k \mu' + a^0) + \mu(\bar{A}_k \bar{\mu} + a^{k+1}),$$

$$F_k = \{A_k \mu' + a^0 : \mu' \in T^k\},$$

and

$$\bar{F}_k = \{\bar{A}_k \bar{\mu} + a^{k+1} : \bar{\mu} \in T^{d-1-k}\}$$

we obtain the required formula. $\square$

Further, we need the quotient of two certain integrals to be bounded:

Lemma 2.2. *Let $k, \bar{k} \in \mathbb{N}$ with $\bar{k} \geq k+1$ and let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > \beta > 0$, $\alpha > 0$. Then for $a, b, c \in \mathbb{R}$, $d \in \mathbb{N}$ with $b, c \geq 0$, $b < c$, $a \geq \alpha$, $a^2 + c^2 \geq \gamma$, $a^2 + b^2 \leq \beta$, $d \geq \bar{k}$, the quotient*

$$\frac{\int_0^{\mu_0} \left(\sqrt{a^2 + (\mu c + (1-\mu)b)^2} \right)^{-(d+1)} \mu^{d-1-k}(1-\mu)^k d\mu}{\int_{\mu_0}^1 \left(\sqrt{a^2 + (\mu c + (1-\mu)b)^2} \right)^{-(d+1)} \mu^{d-1-k}(1-\mu)^k d\mu}, \quad (2.2)$$

where $\mu_0 \in [0, 1]$ is determined by $a^2 + (\mu_0 c + (1-\mu_0)b)^2 = \beta$, is maximal for $a = \alpha$, $b = 0$, $a^2 + c^2 = \gamma$, and $d = \bar{k}$.

Proof. First we show that the quotient is monotonely decreasing in a, b, c, d . To do this we set

$$f(a, b, c, d, \mu) = \left(\sqrt{a^2 + (\mu c + (1-\mu)b)^2} \right)^{-(d+1)} \mu^{d-1-k}(1-\mu)^k.$$

For d we have, by the mean value theorem of integral calculus,

$$\begin{aligned} \frac{\int_0^{\mu_0} f(a, b, c, d+1, \mu) d\mu}{\int_{\mu_0}^1 f(a, b, c, d+1, \mu) d\mu} &= \frac{\mu_1 / \sqrt{a^2 + (\mu_1 c + (1-\mu_1)b)^2}}{\mu_2 / \sqrt{a^2 + (\mu_2 c + (1-\mu_2)b)^2}} \\ &\quad \cdot \frac{\int_0^{\mu_0} f(a, b, c, d, \mu) d\mu}{\int_{\mu_0}^1 f(a, b, c, d, \mu) d\mu} \end{aligned}$$

for suitable

$$0 \leq \mu_1 \leq \mu_0 \leq \mu_2 \leq 1.$$

Since the function $\mu / \sqrt{a^2 + (\mu c + (1 - \mu)b)^2}$ is monotonely increasing in μ we get that (2.2) is monotonely decreasing in d .

Now let $c' > c$ and let $\mu'_0 \in [0, 1]$ be given by $a^2 + (\mu'_0 c' + (1 - \mu'_0)b)^2 = \beta$. We apply the linear transformation $\mu = (c - b)/(c' - b)\mu'$ and obtain

$$\begin{aligned} & \frac{\int_0^{\mu'_0} f(a, b, c', d, \mu) d\mu}{\int_{\mu'_0}^1 f(a, b, c', d, \mu) d\mu} \\ &= \frac{\left(\frac{c-b}{c'-b}\right)^{d-k} \int_0^{\mu'_0} f(a, b, c, d, \mu') \left(\frac{1 - (c-b)/(c'-b)\mu'}{1 - \mu'}\right)^k d\mu'}{\left(\frac{c-b}{c'-b}\right)^{d-k} \int_{\mu'_0}^1 f(a, b, c, d, \mu') \left(\frac{1 - (c-b)/(c'-b)\mu'}{1 - \mu'}\right)^k d\mu'}. \end{aligned}$$

As $(c' - b)/(c - b) > 1$ and $(1 - (c - b)/(c' - b)\mu')/(1 - \mu')$ is monotonely increasing for $0 \leq \mu' \leq 1$ we find, again from the mean value theorem of integral calculus,

$$\frac{\int_0^{\mu'_0} f(a, b, c', d, \mu) d\mu}{\int_{\mu'_0}^1 f(a, b, c', d, \mu) d\mu} \leq \frac{\int_0^{\mu'_0} f(a, b, c, d, \mu') d\mu'}{\int_{\mu'_0}^1 f(a, b, c, d, \mu') d\mu'}.$$

Quite analogously we find that (2.2) is monotonely decreasing in b .

Finally, let $a' > a$. For the corresponding value μ'_0 we have $\mu'_0 < \mu_0$ and further we may write $a'^2 + (\mu c + (1 - \mu)b)^2 = g(\mu)(a^2 + (\mu c + (1 - \mu)b)^2)$ for an appropriate function g . We easily check that g is monotonely decreasing in μ for $\mu \geq 0$. Thus we find, as before,

$$\frac{\int_0^{\mu'_0} f(a', b, c, d, \mu) d\mu}{\int_{\mu'_0}^1 f(a', b, c, d, \mu) d\mu} < \frac{\int_0^{\mu_0} f(a', b, c, d, \mu) d\mu}{\int_{\mu_0}^1 f(a', b, c, d, \mu) d\mu} \leq \frac{\int_0^{\mu_0} f(a, b, c, d, \mu) d\mu}{\int_{\mu_0}^1 f(a, b, c, d, \mu) d\mu}.$$

Hence we may assume $b = 0$ and $c^2 = \gamma - a^2$. So $\mu_0 = \sqrt{(\beta - a^2)/(\gamma - a^2)}$ and we see that μ_0 , regarded as a function in a , is monotonely decreasing in a . Now let $a' > a$, $c' = \gamma - (a')^2$, and μ'_0 be the value of μ with respect to a' and c' . The function $g(\mu)$ defined by $a'^2 + (\mu c')^2 = g(\mu)(a^2 + (\mu c)^2)$ is monotonely decreasing in μ and as above we find

$$\frac{\int_0^{\mu'_0} f(a', 0, c', d, \mu) d\mu}{\int_{\mu'_0}^1 f(a', 0, c', d, \mu) d\mu} < \frac{\int_0^{\mu_0} f(a', 0, c', d, \mu) d\mu}{\int_{\mu_0}^1 f(a', 0, c', d, \mu) d\mu} \leq \frac{\int_{T_0}^{\mu_0} f(a, 0, c, d, \mu) d\mu}{\int_{\mu_0}^1 f(a, 0, c, d, \mu) d\mu}.$$

Hence the integral in (2.2) is increased if a is decreased and $a^2 + c^2$ is kept constant. Thus the lemma is proved. $\square$

Now we come to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\rho < \sqrt{2}$, $d_0 \in \mathbb{N}$ with $d_0 > \max\{4, (4 - \rho^2)/(2 - \rho^2)\}$ and let $n \in \mathbb{N}$, $d \geq d_0$, and $C_n \in \mathcal{P}_n(B^d)$. Further, let H be an arbitrary Dirichlet–Voronoi cell of the arrangement with respect to x^1 , say, and let L be a $(d - 2)$ -dimensional linear subspace. Obviously we may assume $x^1 = 0$. In the following we show that a constant $c \in \mathbb{R}$ exists such that

$$\frac{V_*(M_1(\rho, L))}{V_*(K_1(\rho, L))} \leq c. \quad (2.3)$$

To this end we consider the bounded polyhedron $H \cap W^d$, where W^d denotes the d -cube with midpoint 0 and inradius $\sqrt{2}$. Observe that

$$M_1(\rho, L) = \{z \in S^{d-1} \cap L : \rho z \notin H \cap W^d\},$$

$$K_1(\rho, L) = \{z \in S^{d-1} \cap L : \rho z \in H \cap W^d\}.$$

Now we use a technique due to Rogers [Ro2] which gives, by an inductive construction, a dissection of the intersection $(H \cap W^d) \cap L$ into simplices S of the form $S = \text{conv}\{c^0, \dots, c^{d-2}\}$, such that each c^i is contained in a $(d - 2 - i)$ -face F of $(H \cap W^d) \cap L$ which contains $\text{conv}\{c^i, \dots, c^{d-2}\}$ and is the nearest point of F to 0.

Let $c^0 = 0$ and assume we have chosen a sequence of points $\{c^0, \dots, c^i\}$ such that c^j lies in a $(d - 2 - j)$ -face F_{d-2-j} of $(H \cap W^d) \cap L$ and is the nearest point of the face to the origin, $0 \leq j \leq i$. Let $F_{d-3-i}^1, \dots, F_{d-3-i}^k$ be the facets of F_{d-2-i} which do not contain c^i . Then for $1 \leq j \leq k$ we define the point $c^{i+1,j}$ to be the point in F_{d-3-i}^j nearest to 0. So we obtain sequences $\{c^0, \dots, c^i, c^{i+1,j}\}$, $0 \leq j \leq k$, and at the end of the construction ($i = d - 2$) we obtain a dissection of the desired form.

If c^i belongs to a face of W^d , then we clearly have $|c^i| \geq \sqrt{2}$. Otherwise, if c^i lies in a $(d - 2 - i)$ -face of H we have, by a result of Rogers [Ro2], $|c^i| \geq \sqrt{2} \sqrt{i/(i+1)}$. Thus we get, for $0 \leq i \leq d - 2$,

$$|c^i| \geq \sqrt{2} \sqrt{\frac{i}{i+1}}. \quad (2.4)$$

Let $S = \text{conv}\{c^0, \dots, c^{d-2}\}$ be an arbitrary but fixed simplex of this dissection, let C be the cone generated by $c^1, \dots, c^{d-2}$, and let

$$M = \{z \in (S^{d-1} \cap L) \cap C : \rho z \notin S\}, \quad K = \{z \in (S^{d-1} \cap L) \cap C : \rho z \in S\}.$$

To show (2.3) it is clearly sufficient to show that $V_*(M)/V_*(K)$ is bounded from above. In order to evaluate these spherical volumes we apply several transformations. First we replace the sets M, K by the associated parts M_B, K_B of the d -ball, i.e.,

$$M_B = \left\{ z \in (B^d \cap L) \cap C : \frac{\rho z}{|z|} \notin S \right\}, \quad K_B = \left\{ z \in (B^d \cap L) \cap C : \frac{\rho z}{|z|} \in S \right\}.$$

Clearly, we have $V_*(M)/V_*(K) = V(M_B)/V(K_B)$. In the next step we express $V(M_B)$, $V(K_B)$ as integrals over the simplex S . To this end let $v \in S^{d-1} \cap L$ be the outward unit normal vector of the facet $F = \text{conv}\{c^1, \dots, c^{d-2}\}$ with respect to S and let $\eta \in \mathbb{R}$ be the distance of the affine hull of F to the origin. We consider the map $\varphi: E^d \cap L \rightarrow E^d \cap L$ given by $\varphi(x) = (\langle v, x \rangle / (\eta|x|))x$. For the determinant of the Jacobian we find $\det(D\varphi) = (\langle v, x \rangle / (\eta|x|))^{d-2}$ and with

$$M_S = \varphi^{-1}(M_B) = \left\{ z \in S: \frac{\rho z}{|z|} \notin S \right\}, \quad K_S = \varphi^{-1}(K_B) = \left\{ z \in S: \frac{\rho z}{|z|} \in S \right\}$$

we obtain

$$V(M_B) = \int_{M_S} \left(\frac{\langle v, x \rangle}{\eta|x|} \right)^{d-2} dx, \quad V(K_B) = \int_{K_S} \left(\frac{\langle v, x \rangle}{\eta|x|} \right)^{d-2} dx.$$

Now we use Lemma 2.1 with $k = 0$, $F_0 = 0$, and $\bar{F}_0 = F$. As $\langle v, x \rangle / (\eta|x|)$ is constant on rays emanating from 0 we get

$$V(M_B) = \frac{\eta}{d-2} \int_{M_F} \left(\frac{1}{|x|} \right)^{d-2} dx, \quad V(K_B) = \frac{\eta}{d-2} \int_{K_F} \left(\frac{1}{|x|} \right)^{d-2} dx, \quad (2.5)$$

with $M_F = \{z \in F: |z| \leq \rho\}$ and $K_F = \{z \in F: |z| \geq \rho\}$.

Finally, by the definition of d_0 we may choose a $k \in \mathbb{N}$, $d-4 \geq k$, such that $\sqrt{2} \sqrt{(k+2)/(k+3)} > \rho$ (see (2.4)). We apply Lemma 2.1 to the $(d-3)$ -simplex F and the faces $F_k = \text{conv}\{c^1, \dots, c^{k+1}\}$, $\bar{F}_k = \text{conv}\{c^{k+2}, \dots, c^{d-2}\}$. Defining

$$\xi(F, F_{k-1}) = \frac{\eta}{d-2} \frac{(d-3)!}{k! (d-4-k)!} \frac{V(F)}{V(F_k)V(\bar{F}_k)}$$

we may write, by (2.5),

$$V(M_B) = \xi(F, F_k) \int_{F_k} \int_{\bar{F}_k} \int_{|\mu\bar{x} + (1-\mu)x| \leq \rho} \frac{\mu^{d-4-k}(1-\mu)^k}{|\mu\bar{x} + (1-\mu)x|^{d-2}} d\mu d\bar{x} dx,$$

$$V(K_B) = \xi(F, F_k) \int_{F_k} \int_{\bar{F}_k} \int_{|\mu\bar{x} + (1-\mu)x| \geq \rho} \frac{\mu^{d-4-k}(1-\mu)^k}{|\mu\bar{x} + (1-\mu)x|^{d-2}} d\mu d\bar{x} dx.$$

Thus to prove that $V_*(M)/V_*(K)$ is bounded from above it is sufficient to show

$$\frac{\int_{|\mu\bar{x} + (1-\mu)x| \leq \rho} |\mu\bar{x} + (1-\mu)x|^{-(d-2)} \mu^{d-4-k}(1-\mu)^k d\mu}{\int_{|\mu\bar{x} + (1-\mu)x| \geq \rho} |\mu\bar{x} + (1-\mu)x|^{-(d-2)} \mu^{d-4-k}(1-\mu)^k d\mu} \leq \tilde{c} \quad (2.6)$$

for all $x \in F_k$, $\bar{x} \in \bar{F}_k$, and a suitable constant $\tilde{c}$.

By construction of S we have that $|\mu\bar{x} + (1 - \mu)x|$ is strictly increasing in μ . Further, we have $|x| \geq 1$ and, by the choice of k , $|\bar{x}| > \rho$ (see (2.4)). Thus for $|x| \geq \rho$ there is nothing to prove. Otherwise we find that the inequality (2.6) is of the form

$$\frac{\int_0^{\mu_0} \left(\sqrt{a^2 + (\mu c + (1 - \mu)b)^2} \right)^{-(d-2)} \mu^{d-4-k} (1 - \mu)^k d\mu}{\int_{\mu_0}^1 \left(\sqrt{a^2 + (\mu c + (1 - \mu)b)^2} \right)^{-(d-2)} \mu^{d-4-k} (1 - \mu)^k d\mu},$$

where $a \geq \alpha = 1$ denotes the distance of the line through $\bar{x}, x$ from 0, b is given by $a^2 + b^2 = |x|^2$, c by $a^2 + c^2 = |\bar{x}|^2$, and μ_0 is determined by

$$a^2 + (\mu_0 c + (1 - \mu_0)b)^2 = \rho^2.$$

However, now (2.6) follows from Lemma 2.2 with $\beta = \rho^2$ and

$$\gamma = \sqrt{2} \sqrt{(k+2)/(k+3)}$$

and thus Theorem 2.1 is proved. $\square$

3. Proof of Theorem 1.2

From the definition it follows that the calculation of $\delta(K, C_n, \rho)$ for $K \in \mathcal{K}^d$ and $C_n \in \mathcal{P}_n(K)$ requires information on $V(\text{conv}(C_n) + \rho K)$, which can be written as a polynomial in ρ with the mixed volumes $V_i(\text{conv}(C_n), K)$ (see [Sc]) as coefficients:

$$V(\text{conv}(C_n) + \rho K) = \sum_{i=0}^d \binom{d}{i} \rho^{d-i} V_i(\text{conv}(C_n), K). \quad (3.1)$$

In particular $V_d(\text{conv}(C_n) + \rho K) = V(\text{conv}(C_n))$, $V_0(\text{conv}(C_n), K) = V(K)$, and $V_i(\text{conv}(C_n), K) = 0$ if $\dim(\text{conv}(C_n)) < i$. Formula (3.1) is an essential tool in the proof.

For a sausage configuration $S_n(u, K) = \{x^1, \dots, x^n\}$ we have

$$\dim(\text{conv}(S_n(u, K))) = 1$$

and thus

$$V_i(\text{conv}(S_n(u, K)), K) = 0, \quad i = 2, \dots, d.$$

Moreover, since

$$f_{DK}(x^i - x^{i+1}) = 2, \quad 1 \leq i \leq n-1,$$

we get

$$V_1(\text{conv}(S_n(u, K)), K) = 2(n-1)(f_{DK}(u))^{-1}V(K|u^\perp),$$

where $K|u^\perp$ denotes the orthogonal projection of K onto the hyperplane with normal vector u (see [Sc]). Hence

$$V(\operatorname{conv}(S_n(u, K)) + \rho K) = 2(n-1) \frac{V(K|u^\perp)}{f_{DK}(u)} \rho^{d-1} + V(K) \rho^d. \quad (3.2)$$

Proof of Theorem 1.2. Let $n \in \mathbb{N}$, $n > 1$, $K \in \mathcal{K}^d$, $C_n = \{x^1, \dots, x^n\} \in \mathcal{P}_n(K)$, $\rho \leq \frac{1}{32} d^{-1}(r(K)/R(K))$, and let $m = \dim(\operatorname{conv}(C_n))$. To simplify the notation we write R instead of $R(K)$. Let $z_R \in E^d$ with $K \subset z_R + RB^d$ and let $\bar{r}$ be the inradius of DK . Then

$$\bar{r} \geq r(K). \quad (3.3)$$

The proof now proceeds in four steps:

(1) We construct iteratively from C_n distances h_j and associated subspaces L_j as follows: Without loss of generality let $x^1 = 0$ and let the diameter of $\operatorname{conv}(C_n)$ be given by the distance of the points x^1, x^2 . We set $h_1 = |x^1 - x^2|/2$ and $L_1 = \operatorname{lin}\{x^1, x^2\}$, where $\operatorname{lin}(S)$ denotes the linear hull of a set S . Since $C_n \in \mathcal{P}_n(DK)$ (see (1.1)) we have $h_1 \geq \bar{r}$. Suppose we have chosen distances h_i and linear spaces L_i , $1 \leq i < m$. Let $h_{i+1} = \max\{\operatorname{dist}(x^k, L_i) : 1 \leq k \leq n\}$, where $\operatorname{dist}(x^k, L_i)$ denotes the Euclidean distance of x^k to L_i . We may assume $h_{i+1} = \operatorname{dist}(x^{i+1}, L_i)$ and we set $L_{i+1} = \operatorname{lin}(L_i \cup \{x^{i+1}\})$. We continue up to $i = m$ and obtain a monotone sequence of distances:

$$2h_1 \geq h_2 \geq h_3 \geq \dots \geq h_m \geq 0.$$

For $m = 1$ we have $\dim(\operatorname{conv}(C_n)) = 1$ and thus

$$V(\operatorname{conv}(C_n) + \rho K) \geq V(\operatorname{conv}(S_n(K)) + \rho K).$$

So let $m > 1$, $h_{m+1} = 0$, and $l \in \{1, \dots, m\}$ such that

$$h_l > \frac{\bar{r}}{\sqrt{2}} \quad \text{and} \quad h_{l+1} \leq \frac{\bar{r}}{\sqrt{2}}.$$

(2) For $1 \leq j \leq m$ let $C^j = L_j \cap \operatorname{conv}(C_n)$. Then $\dim(C^j) = j$ and we have, for the mixed volumes (see [Sc]),

$$\binom{d}{j} \rho^{d-j} V_j(\operatorname{conv}(C_n), K) \geq \binom{d}{j} \rho^{d-j} V_j(C^j, K) = \rho^{d-j} V(C^j) \cdot V(K|L_j^\perp).$$

Now let $v \in L_j \cap S^{d-1}$. Since $K \subset z_R + RB^d$ we have

$$V(K|v^\perp) \leq \kappa_{j-1} R^{j-1} V(K|L_j^\perp)$$

and so

$$\binom{d}{j} \rho^{d-j} V_j(\operatorname{conv}(C_n), K) \geq \rho^{d-j} V(C^j) \cdot V(K|v^\perp) \kappa_{j-1}^{-1} R^{1-j}. \quad (3.4)$$

In the last two steps of the proof we consider the two cases $l = 1$ and $l > 1$.

(3) Let $l = 1$. From (3.1), (3.2), and (3.4) we get, for $v \in L_1 \cap S^{d-1}$,

$$\begin{aligned} & V(\text{conv}(C_n) + \rho K) - V(\text{conv}(S_n(v, K)) + \rho K) \\ & \geq \sum_{j=1}^2 \binom{d}{j} \rho^{d-j} V_j(\text{conv}(C_n)) - \frac{2(n-1)\rho^{d-1}V(K|v^\perp)}{f_{DK}(v)} \\ & \geq \rho^{d-2}V(C^2) \cdot V(K|v^\perp)^{\frac{1}{2}}R^{-1} + \rho^{d-1} \left(V(C^1) - \frac{2(n-1)}{f_{DK}(v)} \right) V(K|v^\perp). \end{aligned}$$

Since $h_j \leq h_2 \leq \bar{r}/\sqrt{2}$ for $j > 1$, $\bar{r}B^d \subset DK$, and $(1/f_{DK}(v))B^d \cap L_1 \subset DK$, the intersection of $x^i + DK$ with L_1 contains a line segment of length $2(1 - h_2/\bar{r})/f_{DK}(v)$, $1 \leq i \leq n$. Furthermore, since C_n is a packing set with respect to DK , any two of these segments have disjoint interiors and from the definition of L_1 we get $V(C^1) \geq 2(n-1)(1 - h_2/\bar{r})/f_{DK}(v)$ and thus

$$V(C^2) \geq V(\text{conv}(C_1 \cup \{x^2\})) \geq h_2(n-1) \frac{1 - h_2/\bar{r}}{f_{DK}(v)}.$$

Now $h_2/\bar{r} \leq 1/\sqrt{2}$ and hence

$$\begin{aligned} & V(\text{conv}(C_n) + \rho K) - V(\text{conv}(S_n(v, K)) + \rho K) \\ & \geq \rho^{d-2} \frac{2(n-1)V(K|v^\perp)}{f_{DK}(v)\bar{r}} h_2 \left(\frac{1}{4} \left(1 - \frac{1}{\sqrt{2}} \right) \frac{\bar{r}}{R} - \rho \right). \end{aligned}$$

By the choice of ρ we obtain $V(\text{conv}(C_n) + \rho K) - V(\text{conv}(S_n(v, K)) + \rho K) \geq 0$.

(4) Let $l > 1$. The construction of C^l implies $V(C^l) \geq (2/l!) \prod_{i=1}^l h_i$. Let Q^l be the box with edge lengths $2h_1, 2h_2, \dots, 2h_l$ circumscribed to C^l such that its facets have distance h_i from L_{i-1} , $2 \leq i \leq l$. Clearly,

$$V(Q^l) = 2^l \prod_{i=1}^l h_i \leq l! 2^{l-1} V(C^l). \quad (3.5)$$

Let $\bar{x}^j$, $1 \leq j \leq n$, denote the projection of x^j onto L_l and let $v \in L_l \cap S^{d-1}$ be chosen such that $f_{DK}(v) = \max\{f_{DK}(u) : u \in L_l \cap S^{d-1}\}$. Since $h_{l+1} \leq \bar{r}/\sqrt{2}$, the intersection of $x^i + DK$ with L_l contains an l -dimensional ball of radius $\gamma > 0$, say, and center $\bar{x}^i$, $1 \leq i \leq n$. Any two of these balls have disjoint interiors and are contained in a box with the same center as Q^l and edge lengths $2(h_1 + \gamma), 2(h_2 + \gamma), \dots, 2(h_l + \gamma)$. Thus $n\kappa_l \gamma^l < \prod_{i=1}^l (1 + \gamma/h_i) V(Q^l)$ or, with (3.5),

$$2n\kappa_l \leq V(C^l) l! 2^l \cdot \prod_{i=1}^l \left(\frac{1}{\gamma} + \frac{1}{h_i} \right).$$

By the choice of v we have

$$(1/f_{DK}(v))(B^d \cap L_l) \subset DK$$

and hence $\gamma \geq (1 - h_{l+1}/\bar{r})/f_{DK}(v)$. Together with $h_1 \geq 1/f_{DK}(v)$ we obtain

$$\frac{1}{\gamma} + \frac{1}{h_1} \leq \frac{2\sqrt{2} - 1}{\sqrt{2} - 1} f_{DK}(v)$$

or

$$\frac{2n}{f_{DK}(v)} \kappa_l \leq V(C^l) l! 2^l \cdot \left(\frac{2\sqrt{2} - 1}{\sqrt{2} - 1} \right) \prod_{i=2}^l \left(\frac{1}{\gamma} + \frac{1}{h_i} \right).$$

Since $\bar{r}B^d \subset DK$ we also have $\gamma \geq (\bar{r}^2 - h_{l+1}^2)^{1/2}$ and on account of $h_i > \bar{r}/\sqrt{2}$, $1 \leq i \leq l$, we get

$$\begin{aligned} \frac{2n}{f_{DK}(v)} \kappa_l &\leq V(C^l) l! 2^l \cdot \left(\frac{2\sqrt{2} - 1}{\sqrt{2} - 1} \right) (2\sqrt{2})^{l-1} \bar{r}^{1-l} \\ &= V(C^l) l! 2^{(5l-3)/2} \cdot \left(\frac{2\sqrt{2} - 1}{\sqrt{2} - 1} \right). \end{aligned}$$

Now we use this lower bound of $V(C^l)$ in formula (3.4) and by (3.1)–(3.3) and the choice of ρ we find

$$\begin{aligned} &V(\text{conv}(C_n) + \rho K) - V(\text{conv}(S_n(v, K)) + \rho K) \\ &\geq \frac{2(n-1)V(K|v^\perp)}{f_{DK}(v)} \rho^{d-l} \left(\frac{r(K)}{R} \right)^{l-1} \\ &\quad \times \left(\frac{\kappa_l}{\kappa_{l-1}} (l!)^{-1} 2^{(3-5l)/2} \left(\frac{2\sqrt{2} - 1}{\sqrt{2} - 1} \right)^{-1} - (32d)^{1-l} \right). \end{aligned}$$

Let the term in the last brackets be denoted by $g(l, d)$. It remains to show that $g(l, d) \geq 0$ for $2 \leq l \leq d$. Obviously, $g(l, d) \geq g(l, l)$ and for $l = 2$ it can easily be checked that $g(l, l) > 0$. So let $l \geq 3$. With $(2\sqrt{2} - 1)/(\sqrt{2} - 1) < \frac{9}{2}$, $\kappa_l/\kappa_{l-1} > (2\pi/(l+1))^{1/2}$ [BGW] and Stirling's formula $l! < (l/e)^l (2\pi(l+1))^{1/2}$ we get

$$g(l, d) \geq g(l, l) > \left(\frac{1}{l} \frac{e}{4\sqrt{2}} \right)^{l-1} \left(\frac{e}{9(l+1)l} - \left(\frac{\sqrt{2}}{8e} \right)^{l-1} \right).$$

From this the assertion follows, since $h(l) := (l(l+1))^{-1/(l-1)}$ is monotonely increasing and the difference in the large brackets is positive for $l = 3$. $\square$

Let us remark that the constant factor in the upper bound of ρ can easily be improved, but clearly this factor has no influence on the asymptotic behavior of the bound as d tends to infinity.

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