On Subdivisions of Simplicial Complexes: Characterizing Local h -Vectors*

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Abstract. A well-known combinatorial invariant of simplicial complexes is the h -vector, which has been the subject of much combinatorial research. This paper deals with *local* h -vectors, recently defined by Stanley as a tool for studying h -vectors of simplicial subdivisions. The face-vector of any simplicial complex can only increase when the complex is subdivided; how does the h -vector change? Motivated by this question, Stanley derived certain useful properties of local h -vectors. In this paper we use mainly geometric arguments to show that these properties characterize local h -vectors, and *regular* local h -vectors.

1. Preliminaries

For general background and terminology on combinatorics of simplicial complexes, see Chapter 2 of [S2] or Sections 1.2 and 2.2 of [C].

Before we begin, we need some notation:

For any $n \in \mathbb{Z}$, $[n]$ denotes the set $\{1, 2, \dots, n\}$.

For any finite set V , $\#V$ denotes the number of elements in V , and 2^V denotes the *simplex* on vertex set V , $\{W: W \subseteq V\}$.

For any simplicial complex Δ , $|\Delta|$ denotes the *geometric realization* of Δ , and if F is any face of Δ , then $|F|$ denotes the image of F in $|\Delta|$. (See Section 0.3 of [S2]. The *link* of F in Δ , denoted by $lk_\Delta F$, is the subcomplex

$$\{G \in \Delta: G \cup F \in \Delta, G \cap F = \emptyset\}.$$

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For any (disjoint) simplicial complexes Δ and Γ , $\Delta * \Gamma$ denotes the simplicial complex $\{F \cup G : F \in \Delta, G \in \Gamma\}$, which is called their *simplicial join*.

Now let Δ be a finite $(d - 1)$ -dimensional simplicial complex with f_i i -dimensional faces (i -faces) for each i . Then the d -tuple $(f_0, \dots, f_{d-1})$ is called the f -vector of Δ , denoted $f(\Delta)$. By convention, Δ has one (-1) -dimensional face (the *empty face*), so $f_{-1} = 1$, unless $\Delta = \emptyset$ (in which case $f_{-1} = 0$).

An algebraically desirable alternative to the f -vector is the h -vector $h(\Delta) = (h_0, \dots, h_d)$, defined by

$$\sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{i=0}^d h_i x^{d-i}.$$

In particular, the h -polynomial $h(\Delta, x) = \sum_{i=0}^d h_i x^i$ is algebraically convenient to work with. For example, $h(2^V, x) = 1$ for any simplex 2^V . Also, for any disjoint simplicial complexes Δ and Γ , we can easily compute the h -vector of their simplicial join

$$h(\Delta * \Gamma, x) = h(\Delta, x) \cdot h(\Gamma, x).$$

From the definition, it is easy to derive the useful identity

$$h(\Delta, x) = \sum_{F \in \Delta} x^{\#F} (1-x)^{d-\#F}.$$

See [S3] for background on the algebraic significance of h -vectors, and what has been done to characterize h -vectors of large classes of simplicial complexes. Some of this characterization utilizes results from homological algebra and intersection homology theory.

Of particular interest are the *Cohen–Macaulay complexes*, as defined, for example, in [S1]. These simplicial complexes have been the subject of much research in both algebraic and topological contexts [S2, Section 2.4], [Re]. For example, all simplicial spheres and balls are Cohen–Macaulay. It is known that h -vectors of Cohen–Macaulay complexes are nonnegative, and in fact they have been completely characterized [S2, Section 2.3].

2. Subdivisions of Simplicial Complexes

A natural question to ask about $h(\Delta)$ is whether or not it increases, as $f(\Delta)$ does, when some faces of Δ are subdivided, i.e., if $h(\Delta) = (h_0, h_1, \dots, h_d)$ and $h(\Delta') = (h'_0, h'_1, \dots, h'_d)$, where Δ' is a subdivision of Δ , then must $h'_i \geq h_i$ hold for each i ?

For example, if Δ is the boundary of a tetrahedron, and Δ' is the subdivision of Δ pictured in Fig. 1, then the answer is yes. In fact, if Δ is *any* Cohen–Macaulay complex, the answer is yes, at least if the subdivision is *quasi-geometric* (defined

Fig. 1. (a) $h(\Delta) = (1, 1, 1, 1)$; (b) $h(\Delta') = (1, 2, 2, 1)$.

below). The proof uses certain properties of *local h -vectors* of the subdivision restricted to each face of Δ [S4].

Before moving on, let us formally define *subdivision* and describe three basic kinds of subdivisions (first defined in [S4]).

Definition. Let Γ be a simplicial complex and let σ be a map from Γ to the simplex 2^V such that:

- (i) For each $W \subseteq V$, the *restriction* $\Gamma_W = \sigma^{-1}(2^W)$ is a subcomplex of Γ with geometric realization homeomorphic to the ball B^{*W-1} .
- (ii) For each $F \in \Gamma$, $\sigma(F) = W$ if and only if F is an *interior* face of Γ_W , i.e., $|F|$ is not contained in the boundary of $|\Gamma_W|$.

Then Γ is called a *subdivision of 2^V with subdivision map σ* . For each $F \in \Gamma$, we call $\sigma(F)$ the *carrier* of F , and say F *lies on* W if $\sigma(F) \subseteq W$.

Let Δ and Δ' be simplicial complexes and let σ be a map from Δ' to Δ . For each face $F \in \Delta$, let Δ'_F denote $\sigma^{-1}(2^F)$. Then if, for each F , σ restricts to a subdivision map for Δ'_F as a subdivision of 2^F , we say Δ' is a *subdivision of Δ with subdivision map σ* .

It is convenient to be able to picture a subdivision of 2^V as a *realization* in $\mathbf{R}^d$ in the following sense.

Definition. If Γ is a subdivision of 2^V with subdivision map σ , then a *realization* of (Γ, σ) is a geometric realization of $|\Gamma|$ in $\mathbf{R}^d$ such that, for each $W \subseteq V$, $|\Gamma_W|$ is the convex hull of $\{|\sigma^{-1}(\{x\})| : x \in W\}$.

Example. Let Γ have maximal faces $\{1, 2, 4\}$, $\{2, 3, 4\}$ and let $\sigma: \Gamma \rightarrow 2^{[3]}$ be given by $\sigma(\{4\}) = \sigma(\{1, 4\}) = \sigma(\{3, 4\}) = \{1, 3\}$ and $\sigma(\{2, 4\}) = \sigma(\{1, 2, 4\}) = \sigma(\{2, 3, 4\}) = \{1, 2, 3\}$, with $\sigma(F) = F$ for all other $F \in \Gamma$. Then a realization of Γ is shown in Fig. 2.

Basic Kinds of Subdivisions

A subdivision Γ of 2^V is *geometric* if it can be realized in $\mathbf{R}^d$ with all convex faces. The subdivision in Fig. 2 is geometric, for example.

Fig. 2.

A subdivision Γ is *quasi-geometric* if no face $F \in \Gamma$ has all its vertices lying on a face of 2^V of dimension less than $\dim(F)$. Every geometric subdivision is clearly quasi-geometric, but the converse is not true. Doug Jungreis has suggested a simple example:

Start with a tetrahedron and cut away small convex tetrahedra leaving only a knotted tube, as in Fig. 3(a). Finally, put a new vertex inside the tube and join it with the boundary faces of the tube. This is quasi-geometric, but not geometric. On the other hand, not every subdivision is quasi-geometric. See Fig. 3(b).

A geometric subdivision Γ of 2^V is *regular* if it is the projection of a strictly convex polyhedral surface (with boundary) in $\mathbf{R}^{d+1}$. Formally, Γ is regular if a height function $\omega: |\Gamma| \rightarrow \mathbf{R}$ which is *piecewise linear and strictly convex* exists, i.e.:

- (i) ω is *piecewise linear*: For all $F \in \Gamma$, ω restricts to a linear function ω_F on $|F|$.
- (ii) ω is *convex*: For all $x, y \in |\Gamma|$, and all $\mu \in [0, 1]$,

$$\omega(\mu x + (1 - \mu)y) \geq \mu\omega(x) + (1 - \mu)\omega(y).$$

- (iii) For all F, G distinct maximal faces of Γ , the functions ω_F, ω_G are distinct (as linear functions).

For example, let Γ be the subdivision of 2^V with one interior vertex z and maximal faces of the form $F \cup \{z\}$, where F is any $(d - 1)$ -dimensional face of 2^V (see Fig. 4). Then Γ is a regular subdivision of 2^V .

Notes. (1) Every regular subdivision is *shellable* [BM]. Thus geometric subdivisions are not necessarily regular [Ru].

(2) Let A be a finite affine point configuration, and let $\mathcal{P}$ be the polytope $\text{conv}(A)$, whose vertices are contained in A . Then the regular subdivisions of $\mathcal{P}$ are in one-to-one correspondence with the vertices of the *secondary polytope* $\Sigma(A)$ [BFS].

Fig. 3

Fig. 4. Γ is a projection of the tent.

3. Local h -Vectors

Let $V = [d]$ for some nonnegative integer d . We now give the definition and key properties of the *local h -vector* of any subdivision of the simplex 2^V . See [S4] for elaboration and proofs.

Definition. Let Γ be a subdivision of 2^V . Its *local h -polynomial* is defined by

$$l_V(\Gamma, x) = \sum_{W \subseteq V} (-1)^{\#V - \#W} h(\Gamma_W, x),$$

and its *local h -vector* $l_V(\Gamma)$ is $(l_0, \dots, l_d)$, where $l_V(\Gamma, x) = \sum_{i=0}^d l_i x^i$. See Lemmas 4.2–4.4 for examples.

The following result validates the terminology *local h -vector*.

Theorem 3.1 [S4, Theorem 3.2]. *If Δ' is any subdivision of a pure simplicial complex Δ , we have*

$$h(\Delta', x) = \sum_{F \in \Delta} l_F(\Delta'_F, x) \cdot h(\text{lk}_\Delta F, x).$$

Theorem 3.2 [S4, Corollary 4.7]. *If Γ is quasi-geometric, then $l_V(\Gamma)$ is nonnegative, i.e., $l_i \geq 0$, for all $0 \leq i \leq d$.*

Theorems 3.1 and 3.2 result in a partial answer to the motivating question behind local h -vectors.

Corollary 3.3 [S4, Theorem 4.10]. *If Δ' is a quasi-geometric subdivision of a Cohen–Macaulay simplicial complex Δ , then $h(\Delta') \geq h(\Delta)$.*

We are particularly interested in the following properties of local h -vectors.

Theorem 3.4 [S4, Theorems 2.3, 3.3, 5.2]. *The local h -vector of any subdivision is symmetric, i.e., $l_i = l_{d-i}$ for $0 \leq i \leq d$. Furthermore, $l_0 = 0$ and $l_1 > 0$. If the subdivision is regular, then $l_V(\Gamma)$ is unimodal, i.e.,*

$$l_0 \leq l_1 \leq \dots \leq l_{\lfloor d/2 \rfloor} \geq \dots \geq l_d.$$

4. Characterization of Local h -Vectors

Let $V = [d]$, where d is a positive integer. We now show that local h -vectors of arbitrary subdivisions of 2^V can be completely characterized as follows:

Theorem 4.1 [C, Theorem 2.4.1]. *Let $l = (l_0, \dots, l_d) \in \mathbf{Z}^{d+1}$. Then $l = l_V(\Gamma)$ for some subdivision Γ of 2^V if and only if l is symmetric and $l_0 = 0$, $l_1 \geq 0$.*

As mentioned in Section 3, the “only if” direction has already been proved. To prove the “if” direction, we construct a subdivision Γ , given any l satisfying the above requirements, so that $l_V(\Gamma) = l$. We can do so with the help of three lemmas.

Let Γ be a subdivision of 2^V . See Figs. 5–7 for illustrations of the subdivisions described in the lemmas below.

Lemma 4.2. *Let F be a maximal face of Γ , and let Γ' be the subdivision of Γ with one new vertex z in $\text{Int}(F)$. So the face F of Γ is replaced by the faces $f \cup \{z\}$ in Γ' . Then $h(\Gamma', x) = h(\Gamma, x) + x + x^2 + \dots + x^{d-1}$, and*

$$l_V(\Gamma', x) = l_V(\Gamma, x) + x + x^2 + \dots + x^{d-1}.$$

Proof. We first show that $h(\Gamma', x) = h(\Gamma, x) + x + x^2 + \dots + x^{d-1}$:

$$\begin{aligned} h(\Gamma', x) &= \sum_{G \in \Gamma'} x^{\#G} (1-x)^{d-\#G} \\ &= \sum_{G \in \Gamma, G \neq F} x^{\#G} (1-x)^{d-\#G} + \sum_{G \subset F} x^{\#(G \cup \{z\})} (1-x)^{d-\#(G \cup \{z\})} \\ &= h(\Gamma, x) - x^{\#F} (1-x)^{d-\#F} + \frac{x}{1-x} \sum_{G \subset F} x^{\#G} (1-x)^{d-\#G} \\ &= h(\Gamma, x) - x^{\#F} (1-x)^{d-\#F} + \frac{x}{1-x} (h(2^F, x) - x^{\#F} (1-x)^{d-\#F}) \\ &= h(\Gamma, x) - x^d + \frac{x}{1-x} (1-x^d) \\ &= h(\Gamma, x) + x + x^2 + \dots + x^{d-1}. \end{aligned}$$

Fig. 5

Fig. 6. A copy of G is pushed into the interior of Γ , as in the two-dimensional case of Fig. 3(b).

Now we show that $l_V(\Gamma', x) = l_V(\Gamma, x) + x + x^2 + \cdots + x^{d-1}$. Since $\Gamma'_W = \Gamma_W$ for all $W \neq V$,

$$\begin{aligned}
 l_V(\Gamma', x) &= \sum_{W \subseteq V} (-1)^{d-\#W} h(\Gamma'_W, x) \\
 &= h(\Gamma', x) + \sum_{W \subsetneq V} (-1)^{d-\#W} h(\Gamma_W, x) \\
 &= h(\Gamma', x) + l_V(\Gamma, x) - h(\Gamma, x) \\
 &= l_V(\Gamma, x) + x + x^2 + \cdots + x^{d-1}.
 \end{aligned}
 \quad \square$$

Lemma 4.3. *Let $d \geq 4$. Let G be a $(d-2)$ -dimensional face of Γ with $(d-2)$ -dimensional carrier W , and let Γ' be the subdivision of Γ with one new vertex w (with carrier W) and one new maximal face $G \cup \{w\}$. So in Γ' , G has carrier V . Then $h(\Gamma', x) = h(\Gamma, x) + x$, and $l_V(\Gamma', x) = l_V(\Gamma, x) - x^2 - \cdots - x^{d-2}$.*

Proof. We first show that $h(\Gamma', x) = h(\Gamma, x) + x$:

$$\begin{aligned}
 h(\Gamma', x) &= \sum_{F \in \Gamma'} x^{\#F} (1-x)^{d-\#F} \\
 &= \sum_{F \in \Gamma} x^{\#F} (1-x)^{d-\#F} + \sum_{F \subseteq G} x^{\#F \cup \{w\}} (1-x)^{d-\#F \cup \{w\}} \\
 &= h(\Gamma, x) + x \cdot \sum_{F \in 2^G} x^{\#F} (1-x)^{d-1-\#F} \\
 &= h(\Gamma, x) + x \cdot h(2^G, x) \\
 &= h(\Gamma, x) + x.
 \end{aligned}$$

Fig. 7. $\hat{\Gamma}$ is six tetrahedra glued together.

Now we show that $l_V(\Gamma', x) = l_V(\Gamma, x) - x^2 - \cdots - x^{d-2}$. Since $\Gamma'_U = \Gamma_U$ if $U \not\subseteq W$,

$$\begin{aligned} l_V(\Gamma', x) &= \sum_{U \subseteq V} (-1)^{d-|U|} h(\Gamma'_U, x) \\ &= h(\Gamma', x) - h(\Gamma'_W, x) + \sum_{U \not\subseteq W} (-1)^{d-|U|} h(\Gamma_U, x) \end{aligned}$$

(applying Lemma 4.2 to Γ'_W)

$$\begin{aligned} &= h(\Gamma', x) - h(\Gamma_W, x) - (x + \cdots + x^{d-2}) + l_V(\Gamma, x) - h(\Gamma, x) + h(\Gamma_W, x) \\ &= h(\Gamma', x) - (x + x^2 + \cdots + x^{d-2}) + l_V(\Gamma, x) - h(\Gamma, x) \\ &= l_V(\Gamma, x) - x^2 - \cdots - x^{d-2}. \end{aligned} \quad \square$$

Lemma 4.4. *Let Ω be the subdivision of $2^{(d+1, d+2)}$ with one interior vertex. Then if $\hat{V} = [d+2]$ and $\hat{\Gamma} = \Gamma * \Omega$, we have $l_{\hat{V}}(\hat{\Gamma}, x) = x \cdot l_V(\Gamma, x)$.*

Proof.

$$\begin{aligned} l_{\hat{V}}(\hat{\Gamma}, x) &= \sum_{\hat{W} \subseteq \hat{V}} (-1)^{d+2-|\hat{W}|} h(\hat{\Gamma}_{\hat{W}}, x) \\ &= (-1)^d \sum_{\hat{W} \subseteq \hat{V}} (-1)^{|\hat{W}|} h(\hat{\Gamma}_{\hat{W}}, x) \\ &= (-1)^d \sum_{W \subseteq V} \sum_{U \subseteq \{d+1, d+2\}} (-1)^{|W \cup U|} h(\hat{\Gamma}_{W \cup U}, x). \end{aligned}$$

For each $W \subseteq V$ and $U \subseteq \{d+1, d+2\}$, we have

$$h(\hat{\Gamma}_{W \cup U}, x) = h((\Gamma_W * \Omega_U), x) = h(\Gamma_W, x) \cdot h(\Omega_U, x).$$

It is easy to compute

$$h(\Omega_{\emptyset}, x) = h(\Omega_{\{d+1\}}, x) = h(\Omega_{\{d+2\}}, x) = 1, \quad h(\Omega_{\{d+1, d+2\}}, x) = 1 + x.$$

So

$$\begin{aligned} l_{\hat{V}}(\hat{\Gamma}, x) &= (-1)^d \sum_{W \subseteq V} (-1)^{|W|} \{h(\Gamma_W, x) - h(\Gamma_W, x) - h(\Gamma_W, x) + (1+x) \cdot h(\Gamma_W, x)\} \\ &= x \cdot \sum_{W \subseteq V} (-1)^{d-|W|} h(\Gamma_W, x) \\ &= x \cdot l_V(\Gamma, x). \end{aligned} \quad \square$$

Now we can put it all together.

Fig. 8

Proof of Theorem 4.1. Given $l_0 = 0$, $l_1 \geq 0$, and $l_i = l_{d-i}$ for all i , we want a subdivision Γ of 2^V such that

$$l_V(\Gamma, x) = l(x) = \sum_{i=0}^d l_i x^i.$$

If $d \leq 3$, we can construct Γ directly from 2^V by l_1 applications of Lemma 4.2. For example, we can get $l = (0, 2, 2, 0)$ by putting a new vertex x in the face $\{1, 2, 3\}$ of $2^{[3]}$, and then putting a new vertex y in the face $\{1, 2, x\}$ of the resulting subdivision (see Fig. 8).

Analogously, for $d \geq 4$, if we have a subdivision Γ_0 of 2^V with

$$l_V(\Gamma_0, x) = l(x) - l_1 \cdot (x + x^2 + \cdots + x^{d-1}),$$

then we can get Γ from Γ_0 by l_1 applications of Lemma 4.2.

We now show that such a subdivision Γ_0 can be constructed. Let $s = l_2 - l_1$. We can assume by induction that a subdivision Γ' of $2^{[d-2]}$ exists with

$$l_{[d-2]}(\Gamma', x) = x^{-1} \cdot (l(x) - l_1 \cdot (x + x^2 + \cdots + x^{d-1}) - s \cdot (x^2 + \cdots + x^{d-2})).$$

If $s \leq 0$, apply Lemma 4.4 to get a subdivision of 2^V from Γ' , and then get Γ_0 by $|s|$ applications of Lemma 4.3. If $s > 0$, start with Γ' , apply Lemma 4.2 s times, and then apply Lemma 4.4 to the resulting subdivision to get Γ_0 .

For example, let $d = 4$ and $l = (0, 1, -1, 1, 0)$, see Fig. 9. First we form the simplicial join $2^{[2]} * \Omega$, as in Lemma 4.4. Then we put a vertex x in a 2-face on the boundary, and push the original 2-face into the interior, as in Lemma 4.3. Since $s = l_2 - l_1 = 2$, we do this operation again (with another vertex y) to get Γ_0 . Finally we put a vertex in a maximal face of Γ_0 (as in Lemma 4.2) to get a subdivision Γ of 2^V with the desired local h -vector. $\square$

Fig. 9

Fig. 10

5. Local h -Vectors of Regular Subdivisions

Using the same construction as in the previous section, we now show that local h -vectors of regular subdivisions of 2^V can be characterized as follows:

Theorem 5.1 [C, Theorem 2.5.1]. *If $l = (l_0, l_1, \dots, l_d) \in \mathbf{Z}^n$, then $l = l_V(\Gamma)$ for some regular subdivision Γ of 2^V if and only if $l_0 = 0$ and l is unimodal and symmetric.*

Proof. Stanley proved the “only if” direction in [S4]. To prove the “if” direction, it suffices to show that the subdivision Γ constructed in the proof of Theorem 4.1 is regular when $l_0 = 0$ and l is unimodal and symmetric. For $V = [d]$, let Γ be a regular subdivision of 2^V and let $\omega: |\Gamma| \rightarrow \mathbf{R}$ be piecewise linear and strictly convex.

Lemma 5.2. $\Gamma' = \Gamma * \{x\}$ is a regular subdivision of $2^{V \cup \{x\}}$ (see Fig. 10).

Proof. Let ω' be the unique piecewise linear extension of ω to Γ' such that $\omega'(x) = 0$. It is easy to verify that ω' is strictly convex. So ω' is a height function for Γ' . $\square$

Lemma 5.3. *Let G be an i -face of Γ with $i \geq 2$, and let Γ' be the subdivision of Γ which results from putting a new vertex v in the interior of G and joining v with all faces of the form $G' - \{z\}$, where $G' \supseteq G$ and $z \in G$. Then Γ' is regular (see Fig. 11).*

Proof. Based on the corresponding stellar subdivision of the polyhedron bounded by the graph of ω , it is easy to define a height function $\omega': \Gamma' \rightarrow \mathbf{R}$. $\square$

Fig. 11. (a) Two tetrahedra intersecting on G . (b) Six tetrahedra.

Fig. 12

Proof of Theorem 5.1. As mentioned earlier, we only need to show the “if” direction. Given $l \in \mathbb{Z}^{d+1}$ symmetric and unimodal with $l_0 = 0$, let Γ be the subdivision of 2^V with $l_V(\Gamma) = l$, constructed as in the proof of Theorem 4.1. Since l is unimodal, $s \geq 0$ at every step of the construction. So Γ is built up only from subdivisions described in Lemmas 4.2 and 4.4.

For example, let $d = 4$ and $l = (0, 1, 2, 2, 1, 0)$, see Fig. 12. As in the example at the end of Section 4, we start with $2^{[2]}$ and do the operations described in Lemma 4.2 and then in Lemma 4.4 (once each) to get Γ_0 . Then we do the operation described in Lemma 4.2 once more to get Γ with the desired local h -vector.

The subdivision in Lemma 4.2 is achieved by putting one new vertex in a $(d - 1)$ -face as in Lemma 5.3. The subdivision in Lemma 4.4 is achieved by joining a given simplicial complex with a vertex x as in Lemma 5.2, and joining the resulting complex with another vertex y as in Lemma 5.2, and then putting a new vertex z in the face $\{x, y\}$ as in Lemma 5.3. Thus, since we begin the whole process with a trivial subdivision 2^W for some W , our construction remains regular at each step (by Lemmas 5.2 and 5.3), so the resulting subdivision Γ is regular. $\square$

6. Conclusion.

Thus we have completely characterized local h -vectors of subdivisions, and of regular subdivisions. An obvious next goal would be to characterize local h -vectors of quasi-geometric subdivisions.

Conjecture [S4, Conjecture 5.4]. *If $l = l_V(\Gamma)$ for some quasi-geometric subdivision Γ of 2^V , then l is unimodal.*

If this conjecture is valid then our work in Section 5 shows that $l_0 = 0$ and l is symmetric and unimodal *if and only if* l is the local h -vector of some quasi-geometric subdivision.

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