Approximation of Zonoids by Zonotopes in Fixed Directions

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Abstract. A construction is given which, for any zonoid $Z \subset \mathbf{R}^d$ and any unit vectors $u_1, \dots, u_k$, produces a zonotope Z' which is a sum of at most k segments and has the same support values as Z in the directions $u_1, \dots, u_k$. Applications to an estimation problem for fiber processes are also given as well as some related results for convex bodies.

1. Introduction

A *zonotope* in d -dimensional euclidean space $\mathbf{R}^d$ is a finite vector sum of line segments. A *zonoid* is a convex body which is the limit of zonotopes in the Hausdorff metric. Zonotopes and zonoids have a center of symmetry and, without loss of generality, we assume throughout that this center is at the origin. Then the *support function* $h(Z, \cdot) := \max_{y \in Z} \langle \cdot, y \rangle$ of the zonoid Z has the form

$$h(Z, \cdot) = \int_{\Omega} |\langle \cdot, x \rangle| \rho_Z(dx) \quad (1)$$

with a (uniquely determined) even Borel measure ρ_Z on the unit sphere $\Omega \subset \mathbf{R}^d$.

Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbf{R}^d$. The measure ρ_Z is called the *generating measure* of Z . For a zonotope Z , the generating measure ρ_Z is supported by k pairs of antipodal points and hence the integral in (1) reduces to a sum

$$h(Z, \cdot) = \sum_{i=1}^k |\langle \cdot, x_i \rangle|$$

with $x_i \in \mathbf{R}^d$.

The approximation of a given zonoid by zonotopes (e.g., with a fixed number of segments) has been discussed by a number of authors (Figiel *et al.*, 1977; Betke and McMullen, 1983; Gordon, 1985; Schechtman, 1987; Bourgain and Lindenstrauss, 1988; Bourgain *et al.*, 1989; Linhart, 1989). Zonoids and zonotopes have also found some interest in a number of practical problems in stochastic geometry. In particular, in the analysis of spatial fiber systems, observations are often of the form $h_1 := h(Z, \pm u_1), \dots, h_k := h(Z, \pm u_k)$ for some zonoid $Z \subset \mathbf{R}^d$ which is in general not explicitly known. The statistical problem is to give an estimate for the measure ρ_Z on the basis of the observed values $h_1, \dots, h_k$. A discrete solution to this problem is suggested by the following result.

Theorem 1. *For a zonoid $Z \subset \mathbf{R}^d$ and unit vectors $u_1, \dots, u_k$ there always exists a zonotope Z' which is the sum of at most k segments and fulfills*

$$h(Z, \pm u_1) = h(Z', \pm u_1), \dots, h(Z, \pm u_k) = h(Z', \pm u_k).$$

As J. Bourgain pointed out to us, Theorem 1 can be obtained as a simple consequence of Carathéodory's theorem. For the above-mentioned application, however, a constructive approach is necessary which is not provided using Carathéodory's result. In the next section we therefore present an alternative proof of Theorem 1, which allows us to construct the zonotope Z' explicitly as the solution of a linear program. In Section 3 some background information on the stereological questions is given and the method of Section 2 is applied to the statistical problem mentioned above. The final section contains some related results which are consequences of either Theorem 1 or our method of proof.

Throughout the following we use some properties of zonoids and zonotopes without giving all the relevant details. Here we refer to the survey articles of Schneider and Weil (1983) and Goodey and Weil (1993). As references for basic notions from the theory of convex bodies, we mention Bonnesen and Fenchel (1934).

2. Proofs of Theorem 1

We first mention the following short and elegant approach to Theorem 1, which is due to J. Bourgain.

We may assume that the generating measure ρ_Z of the given zonoid Z is normalized, $\rho_Z(\Omega) = 1$. For fixed directions $u_1, \dots, u_k \in \Omega$ let $A \subset \mathbf{R}^k$ be the set of all vectors $(h(Z, u_1), \dots, h(Z, u_k))$ where Z runs through the zonoids in $\mathbf{R}^d$ with normalized generating measure. A is convex and compact. The subset $B \subset A$ consisting of those vectors coming from line segments Z is compact and connected (it is a continuous image of Ω). Obviously, A is the closed convex hull of B , but because B is closed, $A = \text{conv } B$. By Carathéodory's theorem and since B is connected, any point in A is a convex combination of (at most) k points from B . For our given vector $(h(Z, u_1), \dots, h(Z, u_k))$ this implies the existence of vectors $x_1, \dots, x_k \in \mathbf{R}^d$ (with length $\frac{1}{2}$) and positive weights $\alpha_1, \dots, \alpha_k$ with $\alpha_1 + \dots + \alpha_k = 1$ and such that

$$h(Z, u_j) = \sum_{i=1}^k \alpha_i \langle u_j, x_i \rangle, \quad j = 1, \dots, k.$$

If we denote by $[-x, x]$ the segment from $-x$ to x , the zonotope

$$Z' := \sum_{i=1}^k \alpha_i [-x_i, x_i]$$

therefore fulfills the assertion.

In order to describe our alternative and constructive approach we consider, for given $u_1, \dots, u_k \in \Omega$, the corresponding orthogonal hyperplanes $E_1, \dots, E_k$ through the origin. They divide $\mathbf{R}^d$ into $m \leq 2^k$ polyhedral cones $C_1, \dots, C_m$. Let

$$\omega_1 := C_1 \cap \Omega, \quad \omega_2 := (C_2 \setminus C_1) \cap \Omega, \quad \omega_3 := (C_3 \setminus (C_1 \cup C_2)) \cap \Omega, \quad \dots,$$

thus $\omega_1, \dots, \omega_m$ are spherically convex, pairwise disjoint Borel sets which cover Ω .

For an arbitrary finite Borel measure ρ on Ω , let

$$s_i := \int_{\omega_i} x \rho(dx)$$

be the centroid of the measure ρ restricted to ω_i and define the zonotope Z'' as

$$Z'' := \sum_{i=1}^m [-s_i, s_i].$$

Then

$$h(Z'', u_j) = \sum_{i=1}^m |\langle u_j, s_i \rangle| = \sum_{i=1}^m \left| \left\langle u_j, \int_{\omega_i} x \rho(dx) \right\rangle \right| = \sum_{i=1}^m \left| \int_{\omega_i} \langle u_j, x \rangle \rho(dx) \right|.$$

On each ω_i , the sign of $\langle u_j, \cdot \rangle$ does not change. Hence

$$\left| \int_{\omega_i} \langle u_j, x \rangle \rho(dx) \right| = \int_{\omega_i} |\langle u_j, x \rangle| \rho(dx), \quad i = 1, \dots, m,$$

and thus

$$h(Z'', u_j) = \sum_{i=1}^m \int_{\omega_i} |\langle u_j, x \rangle| \rho(dx) = \int_{\Omega} |\langle u_j, x \rangle| \rho(dx)$$

for $j = 1, \dots, k$.

In particular, for $\rho = \rho_Z$, we get $h(Z'', u_1) = h(Z, u_1), \dots, h(Z'', u_k) = h(Z, u_k)$. The corresponding equations $h(Z'', -u_1) = h(Z, -u_1), \dots, h(Z'', -u_k) = h(Z, -u_k)$ follow immediately from the symmetry of Z and Z'' . The number m of segments of the zonotope Z'' is bounded above by Schläfli's lemma,

$$m \leq 2 \sum_{r=0}^{d-1} \binom{k-1}{r},$$

but in general m is greater than k . Therefore we now consider, for $\alpha_1 \geq 0, \dots, \alpha_m \geq 0$, the zonotope

$$Z_\alpha := \sum_{i=1}^m \alpha_i [-s_i, s_i].$$

Let

$$c_{ij} := |\langle u_j, s_i \rangle|, \quad i = 1, \dots, m, \quad j = 1, \dots, k,$$

then

$$h(Z_\alpha, u_j) = \sum_{i=1}^m \alpha_i c_{ij}, \quad j = 1, \dots, k.$$

The optimization problem of choosing $\alpha_1, \dots, \alpha_m$ so as to minimize the distance between Z_α and Z in the given directions $u_1, \dots, u_k$ can be formulated as a linear program:

$$\begin{aligned} &\text{Minimize} \quad f(\alpha_1, \dots, \alpha_m) := \sum_{j=1}^k (h_j - \sum_{i=1}^m \alpha_i c_{ij}) \\ &\text{subject to} \quad \sum_{i=1}^m \alpha_i c_{ij} \leq h_j, \quad j = 1, \dots, k; \quad \alpha_i \geq 0, \quad i = 1, \dots, m. \end{aligned}$$

Introducing slack variables $y_1, \dots, y_k$, the set of feasible solutions is transformed into the region $M \subset \mathbf{R}^m \times \mathbf{R}^k$ given by

$$\sum_{i=1}^m \alpha_i c_{ij} + y_j = h_j, \quad j = 1, \dots, k,$$

$$\alpha_i \geq 0, \quad y_j \geq 0, \quad i = 1, \dots, m, \quad j = 1, \dots, k.$$

Since the objective function $\tilde{f}, \tilde{f}(\alpha_1, \dots, \alpha_m, y_1, \dots, y_k) := f(\alpha_1, \dots, \alpha_m)$, is bounded below by 0 on M , the transformed linear program is solvable, and in fact a solution is given by $\alpha_1 = \dots = \alpha_m = 1, y_1 = \dots = y_k = 0$ with minimal value $\tilde{f} = 0$, as we have seen above. A standard result from optimization theory (see, e.g., Karlin, 1959) says that there is also a vertex $(\alpha_1, \dots, \alpha_m, y_1, \dots, y_k) \in M$ which solves the problem, hence has $\tilde{f}(\alpha_1, \dots, \alpha_m, y_1, \dots, y_k) = 0$. Moreover, if we write the constraints

$$\sum_{i=1}^m \alpha_i c_{ij} + y_j = h_j, \quad j = 1, \dots, k,$$

in matrix form,

$$A \begin{pmatrix} \alpha \\ y \end{pmatrix} = h$$

with $\alpha = (\alpha_1, \dots, \alpha_m)^T, y = (y_1, \dots, y_k)^T, h = (h_1, \dots, h_k)^T$, and

$$A = (C | E_k)$$

(where $C = ((c_{ij}))$ and E_k is the k by k identity matrix), then any vertex v of M has the property that the columns of A which belong to the nonzero coordinates of v are linearly independent. Since a vertex solution of our linear program must have $y = 0$ and since the rank of C is no more than k , we get that at most k values of $\alpha_1, \dots, \alpha_m$ in our solution are nonzero. The corresponding zonotope $Z' := Z_x$ therefore consists of at most k segments and satisfies

$$\sum_{j=1}^k \left(h_j - \sum_{i=1}^m \alpha_i c_{ij} \right) = 0$$

(as well as $h_j - \sum_{i=1}^m \alpha_i c_{ij} \geq 0, j = 1, \dots, k$), hence

$$h(Z', u_j) = h_j, \quad j = 1, \dots, k.$$

3. Fiber Processes

In the following stereological applications we concentrate on random collections of fibers (so-called *fiber processes*; mathematically these are point processes of curves). Random collections of lines, hyperplanes, or sheets (surfaces) can be treated in a similar way. We assume that the fiber process is *stationary*, i.e., its distribution is translation invariant. We cannot give all the relevant background here and refer to Stoyan *et al.* (1987) or Weil (1987) for more details.

The mean behavior of a stationary fiber process X in $\mathbf{R}^d$ (with locally finite intensity measure) is described by two quantities, the *length density* L_V and the *directional distribution* P . The first is a positive number which describes the mean total length of the fibers per unit volume; the second is an even probability measure on Ω which gives the distribution of the tangent direction in a typical point of the fibers. We remark that L_V and P do not depend on the individual fibers but only on their union set. Therefore, the fiber process X may be also considered as a random one-dimensional closed set in $\mathbf{R}^d$ (this is, e.g., the point of view in Stoyan *et al.* (1987)). If $E \subset \mathbf{R}^d$ is a hyperplane (with a randomly chosen direction, independent of X), then the intersection field $X \cap E$ is almost surely a stationary process of ordinary points in E , the mean behavior of which is described by a single quantity, the intensity γ_E . γ_E gives the mean number of (intersection) points per unit $(d-1)$ -dimensional volume in E . Again by stationarity, γ_E depends only on the direction $\pm u$ of E , hence it gives rise to an even function h on Ω , $h(u) := \gamma_{u^\perp}$. The following integral equation connects h , L_V , and P :

$$h = L_V \int_{\Omega} |\langle \cdot, x \rangle| P(dx).$$

Thus, there is a unique zonoid $Z = Z(X)$ associated with the fiber process X . The generating measure of Z is given by $\rho_Z := L_V \cdot P$, and the support function $h(Z, \cdot)$ of Z describes the intersection intensities of X .

The estimation problem mentioned in the Introduction occurs in applications in biology, medicine, or geology in the following way: The (three-dimensional) fiber process X can only be observed in finitely many planar sections $X \cap E_1, \dots, X \cap E_k$ and the mean number of intersection points (per unit area) can be counted in each sectioning plane. Thus, (unbiased) estimators for the intensities $\gamma_{E_1}, \dots, \gamma_{E_k}$ can be given. The problem is then to estimate the measure ρ_Z from these values (which in turn gives estimators for $L_V = \rho_Z(\Omega)$ and $P = \rho_Z/\rho_Z(\Omega)$). A discrete approximation (and estimation) of ρ_Z is given by the generating measure $\rho_{Z'}$ of a zonotope which best fits the values $\gamma_{E_i} = h(Z, u_i), \dots, \gamma_{E_k} = h(Z, u_k)$ (with $u_i \perp E_i$). The existence of such a zonotope Z' is expressed by Theorem 1 and the method of the last section shows how to construct Z' . However, the procedure has to be modified slightly, since up to now the construction of Z' requires the knowledge of ρ_Z .

As before, let $C_1, \dots, C_m$ be the closed polyhedral cones which are generated by the hyperplanes $u_1^\perp, \dots, u_k^\perp$. For an interior point $y \in C_i$ and $j \in \{1, \dots, k\}$, let ε_{ij}

be the sign of $\langle u_j, y \rangle$. We consider the zonotope

$$Z(x_1, \dots, x_m) := \sum_{i=1}^m [-x_i, x_i],$$

where $x_1, \dots, x_m \in \mathbf{R}^d$ are variable. Then, by the results of Section 2, it follows that the linear program,

$$\begin{aligned} &\text{Minimize} \quad f(x_1, \dots, x_m) := \sum_{j=1}^k \left(h_j - \sum_{i=1}^m \varepsilon_{ij} \langle u_j, x_i \rangle \right) \\ &\text{subject to} \quad \sum_{i=1}^m \varepsilon_{ij} \langle u_j, x_i \rangle \leq h_j, \quad j = 1, \dots, k, \\ &\quad \text{and} \quad x_i \in C_i, \quad i = 1, \dots, m, \end{aligned}$$

is solvable, and a solution is given by

$$x_1 = s_1, \dots, x_m = s_m,$$

with optimal value

$$f(x_1, \dots, x_m) = 0.$$

This linear program has md variables and $(m+1)k$ inequality constraints, since the condition $x_i \in C_i$ can be replaced by k inequalities

$$\varepsilon_{ij} \langle u_j, x_i \rangle \geq 0, \quad j = 1, \dots, k.$$

The method described here is valid for each dimension $d \geq 2$, but only of practical interest in the three-dimensional case. For planar fiber fields X , the estimation of the directional distribution is quite often also not done directly, but with the help of the intersection point fields on random lines. The determination of the intensity of the latter is much less problematic than the direct measurement of tangential directions. The fitting of a zonotope Z' to the support values $h_1, \dots, h_k$ of the underlying unknown zonoid Z is however much simpler in the planar case since in $\mathbf{R}^2$ every centrally symmetric polygon is a zonotope. Hence the corresponding optimization problem can be easily solved by the zonotope

$$Z' = \bigcap_{i=1}^k \{x \in \mathbf{R}^2: \langle u_i, x \rangle \leq h_i\},$$

and the corresponding estimator for ρ_Z is obtained from the edge lengths of Z' . This procedure was proposed in Weil (1987) and has been discussed in more detail in Rataj and Saxl (1989).

We finally mention that the estimation procedures discussed here are consistent, if the directions in Theorem 1 are chosen randomly (i.e., independently and with uniform distribution). To be more precise, let $Z \subset \mathbb{R}^d$ be a zonoid, let $u_1, u_2, \dots$ be a sequence of independent and uniformly distributed random vectors in Ω , and let $Z^{(k)}$ be the zonotope which exists by Theorem 1 for the first k elements $u_1, \dots, u_k$ of this sequence (to be more precise, we should say one polytope, because in general $Z^{(k)}$ may not be uniquely determined). Then the random zonotope $Z^{(k)}$ converges almost surely to Z , as $k \rightarrow \infty$.

This follows easily, since the sequence $Z^{(1)}, Z^{(2)}, \dots$ is bounded, and we may therefore assume, by the Blaschke Selection Theorem, that $Z^{(1)}, Z^{(2)}, \dots$ converge to a zonoid $\tilde{Z}$. Then $\tilde{Z}$ satisfies

$$h(\tilde{Z}, u_i) = h(Z, u_i), \quad i = 1, 2, \dots$$

Since the sequence $u_1, u_2, \dots$ is almost surely dense in Ω , the continuous functions $h(\tilde{Z}, \cdot)$ and $h(Z, \cdot)$ coincide on a dense subset of Ω . Therefore, $h(\tilde{Z}, \cdot) = h(Z, \cdot)$ and thus $\tilde{Z} = Z$.

The convergence of $Z^{(k)}$ to Z implies the weak convergence of $\rho_{Z^{(k)}}$ to ρ_Z . Therefore, if $Z = Z(X)$ is the associated zonoid of a fiber process X , the estimated (discrete) directional distributions $P^{(k)} := \rho_{Z^{(k)}} / \rho_{Z^{(k)}}(\Omega)$ converge almost surely (and weakly) to the directional distribution P of X .

This result holds similarly for sequences $u_1, u_2, \dots$ on Ω which are uniformly distributed in the sense of number theory (for more details, see Linhart, 1989).

4. Related Results

In the following we collect some further results which are related to Theorem 1 or follow from the construction in Section 2. First, we consider similar results for projections, which stem from the interpretation of zonoids as projection bodies.

The class of zonoids in $\mathbb{R}^d$ with inner points coincides with the class of *projection bodies* (of bodies with inner points). For a convex body $K \subset \mathbb{R}^d$ without inner points, the projection body is either a segment (if K is $(d-1)$ -dimensional) or a point (if K is at most $(d-2)$ -dimensional). To be more precise, let $v(K, u)$ denote the $(d-1)$ -dimensional content of the orthogonal projection of the convex body K onto the hyperplane orthogonal to u , $\|u\| = 1$. Then

$$v(K, u) = \frac{1}{2} \int_{\Omega} |\langle u, x \rangle| S_{d-1}(K, dx), \quad (2)$$

where $S_{d-1}(K, \cdot)$ denotes the $(d-1)$ st *surface area measure* of K . The projection body ΠK of K is then defined by $h(\Pi K, \cdot) := v(K, \cdot)$. ΠK is a zonoid with generating measure $\frac{1}{4}(S_{d-1}(K, \cdot) + S_{d-1}(K^*, \cdot))$ (here, K^* denotes the reflection of K with respect to the origin).

In the following proof we make use of the *mixed volume*

$$V_1(K, L) := V(K, \dots, K, L) = \frac{1}{d} \int_{\Omega} h(L, x) S_{d-1}(K, dx), \quad (3)$$

for convex bodies K, L , and of the *Minkowski inequality*

$$V^{d-1}(K)V(L) \leq V_1^d(K, L) \quad (4)$$

(here V denotes the volume), where equality holds for bodies K, L with inner points, if and only if K and L are homothetic.

Proposition 2. *Let K be a convex body with inner points, which is not a polytope, and let $u_1, \dots, u_k$ be arbitrary directions. Then there exists a polytope P with*

$$v(K, \pm u_1) = v(P, \pm u_1), \dots, v(K, \pm u_k) = v(P, \pm u_k) \quad (5)$$

and such that $V(P) < V(K)$.

Proof. Let $Z := \Pi K$ and Z'' be the zonotope constructed in Section 2 for the measure $\rho = S_{d-1}(K, \cdot)$. Let ρ' be the discrete measure on Ω , defined by $\rho'(s_i/\|s_i\|) := \|s_i\|$, for all $i \in \{1, \dots, m\}$ with $s_i \neq 0$. ρ' has centroid 0 since

$$\int_{\Omega} x \rho'(dx) = \sum_{i=1}^m s_i = \sum_{i=1}^m \int_{\omega_i} x \rho(dx) = \int_{\Omega} x S_{d-1}(K, dx) = 0.$$

If Z'' has inner points, then, by Minkowski's existence theorem, ρ' is the surface area measure of a polytope P which fulfills $Z'' = \Pi P$ and therefore (5) (in view of (2)).

Moreover, we have, from (3) and the convexity of $h(K, \cdot)$,

$$\begin{aligned} V_1(P, K) &= \frac{1}{d} \int_{\Omega} h(K, x) S_{d-1}(P, dx) = \frac{1}{d} \int_{\Omega} h(K, x) \rho'(dx) \\ &= \frac{1}{d} \sum_{i=1}^m h(K, s_i) = \frac{1}{d} \sum_{i=1}^m h\left(K, \int_{\omega_i} x \rho(dx)\right) \\ &\leq \frac{1}{d} \sum_{i=1}^m \int_{\omega_i} h(K, x) S_{d-1}(K, dx) = \frac{1}{d} \int_{\Omega} h(K, x) S_{d-1}(K, dx) = V(K). \end{aligned}$$

Thus, (4) implies

$$V^{d-1}(P)V(K) \leq V^d(K)$$

and hence

$$V(P) \leq V(K)$$

with equality, if and only if P and K are translates.

If Z'' has no inner points, we choose additional directions $u_{k+1}, \dots, u_r$ and consider the corresponding zonotope Z' which satisfies

$$h(Z, u_1) = h(Z', u_1), \dots, h(Z, u_r) = h(Z', u_r).$$

Since Z has inner points, we can make this choice in such a way that Z' also has inner points. By the above arguments, the corresponding polytope P' then fulfills the assertion and the proof is completed. $\square$

The result has an interesting consequence for the following problem. For a given body K and directions $u_1, \dots, u_k$ let $\mathcal{A}$ be the set of all convex bodies L with

$$v(K, \pm u_1) = v(L, \pm u_1), \dots, v(K, \pm u_k) = v(L, \pm u_k).$$

If there is an element $L_{\min} \in \mathcal{A}$ which has minimal volume $V(L_{\min}) > 0$, then $L_{\min}$ is a polytope. In general, it seems to be an open problem to determine whether or under which conditions minimal or maximal elements in $\mathcal{A}$ do exist and to characterize them.

It is more complicated to formulate a result, analogous to Proposition 2, where the volume condition is replaced by a condition on the number of facets of P . The reason is that the zonotope Z' from Section 2 need not be a projection body, even if the zonoid $Z = \Pi K$ is smooth and the directions $u_1, \dots, u_k$ are in general position. In case $2 \leq \dim Z' \leq d-1$ we have to proceed as in the proof of Proposition 1 and add $r := d - \dim Z'$ suitable directions $u_{k+1}, \dots, u_{k+r}$. As a result we get, for example, a centrally symmetric polytope P with at most $k + d - 2$ pairs of opposite facets and such that

$$v(K, \pm u_1) = v(P, \pm u_1), \dots, v(K, \pm u_k) = v(P, \pm u_k).$$

Another direct consequence of Theorem 1 concerns an illumination problem posed by Martini (1987). For the case that Z is the unit ball and arbitrary directions $u_1, \dots, u_k$, the theorem implies that there is a zonotope Z' which is the sum of (at most) k segments and fulfills

$$h(Z', u_1) = 1, \dots, h(Z', u_k) = 1.$$

In terms of parallel illumination this implies that *the whole boundary of an arbitrary convex polytope $P \subset \mathbf{R}^d$ with k facets can be isophotically illuminated by k parallel illuminations* (see Martini, 1987, for more details).

Finally, we get a solution to a problem on zonoids, which was posed in a slightly less general form by the last author at the Oberwolfach meeting on “Convex Geometry” in 1982.

Proposition 3. *Let $Z \subset \mathbf{R}^d$ be a zonoid and let $P, Q \subset \mathbf{R}^d$ be polytopes with*

$$P \subset Z \subset Q.$$

Then there is a sequence $(Z_k)_{k \in \mathbf{N}}$ of zonotopes Z_k with $Z_k \rightarrow Z$ (as $k \rightarrow \infty$) and

$$P \subset Z_k \subset Q, \quad k \in \mathbf{N}.$$

Proof. Without loss of generality, we may assume that P, Z , and Q have interior points and that all vertices $x_1, \dots, x_r$ of P lie on the boundary of Z (the latter follows if we replace P by an appropriate larger polytope).

We first consider the case, where Z is strictly convex and smooth (i.e., each boundary point has a unique supporting hyperplane) and Z lies in the interior, int Q , of Q . Let $u_1, \dots, u_r$ be the (uniquely determined) outer normals of Z at the points $x_1, \dots, x_r$. We choose $\varepsilon > 0$ such that, for $1 \leq i \leq r$, the ball B_i of radius ε centered at u_i lies in the interior of the normal cone of P at the vertex x_i . Then there are (linearly independent) unit vectors $u_{ij}, j = 1, \dots, d$, in B_i such that u_i lies in the interior of the cone spanned by $u_{i1}, \dots, u_{id}$. Let $v_1, \dots, v_s$ be the facet normals of Q and define $h_{ij} := h(Z, u_{ij}), i = 1, \dots, r, j = 1, \dots, d, h_{i0} := h(Z, u_i), i = 1, \dots, r$, and $\tilde{h}_k := h(Z, v_k), k = 1, \dots, s$. We define the polytope

$$R_\varepsilon := \bigcap_{i=1}^r \bigcap_{j=1}^d \{x \in \mathbf{R}^d: \langle u_{ij}, x \rangle \leq h_{ij}\} \\ \cap \bigcap_{i=1}^r \{x \in \mathbf{R}^d: \langle u_i, x \rangle \leq h_{i0}\} \cap \bigcap_{k=1}^s \{x \in \mathbf{R}^d: \langle v_k, x \rangle \leq \tilde{h}_k\}.$$

Then, obviously, $Z \subset R_\varepsilon \subset Q$. Moreover, the polytope R_ε is circumscribed to Z and since Z is strictly convex and smooth, the unit vectors $u_i, u_{ij}, v_k, i = 1, \dots, r, j = 1, \dots, d, k = 1, \dots, s$, are exactly the facet normals of R_ε . The facet $R_\varepsilon(u_i)$ in direction u_i contains x_i and, for $\varepsilon \rightarrow 0$, $R_\varepsilon(u_i)$ shrinks to the point x_i . By Theorem 1, there is a zonotope Z_ε with at most $m := r(d+1) + s$ segments which fulfills

$$h(Z_\varepsilon, u_{ij}) = h_{ij}, \quad i = 1, \dots, r, \quad j = 0, \dots, d,$$

and

$$h(Z_\varepsilon, v_k) = \tilde{h}_k, \quad k = 1, \dots, s.$$

By construction, we have $Z_\varepsilon \subset R_\varepsilon \subset Q$ and Z_ε has a vertex in every facet of R_ε . In particular, there is a point of Z_ε in $R_\varepsilon(u_i)$, $i = 1, \dots, r$. Blaschke's Selection Theorem implies that we can choose a sequence $\varepsilon_n \rightarrow 0$ such that the zonotopes Z_{ε_n} converge, as $n \rightarrow \infty$, to a convex body Z' . Since all zonotopes Z_{ε_n} have at most m segments, Z' has to be a zonotope with the same property. Moreover, since the facets $R_{\varepsilon_n}(u_i)$ shrink to x_i , all vertices $x_1, \dots, x_r$ of P must lie in Z' . Therefore, $P \subset Z' \subset Q$.

Now we consider the general case. First we still assume that Z is in the interior of Q , but not necessarily strictly convex and smooth. By a familiar smoothing procedure, Z can be approximated from outside by strictly convex and smooth zonoids Z_η , $Z_\eta \rightarrow Z$ as $\eta \rightarrow 0$, and such that $Z_\eta \subset Q$. From the argument above, we get a zonotope Z'_η with $P \subset Z'_\eta \subset Q$ for each $\eta > 0$. Since all Z'_η have at most m segments we again get, for $\eta \rightarrow 0$, a zonotope Z' with the property $P \subset Z' \subset Q$. The restriction $Z \subset \text{int } Q$ can now be removed in a similar manner, e.g., by approximating Q from outside by ρQ , $\rho > 1$.

We have now shown that there is a zonotope Z' with

$$P \subset Z' \subset Q.$$

The proof of Proposition 3 is thus completed if we replace P by an increasing sequence of polytopes and Q by a decreasing sequence, both converging to Z . $\square$

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