Gaps in the Numbers of Vertices of Cubical Polytopes, I

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Abstract. A cubical polytope is a convex polytope of which every facet is a combinatorial cube. We ask for the numbers which occur as vertex numbers of d -dimensional cubical polytopes, and we show, as a first step, that every cubical d -polytope for even $d \geq 4$ has an even number of vertices.

1. Introduction

Let P be a convex d -polytope in $\mathbb{R}^d$ and let $f_j(P)$ be the number of j -dimensional faces of P ($0 \leq j \leq d-1$). A basic problem is to determine the possible $f_j(P)$ for all convex polytopes P or for interesting classes of convex polytopes P . For example, any 3-polytope P with $f_1(P) = 7$ does not exist [5, p. 187].

In the case of simple polytopes P , McMullen's conditions (see, e.g., p. 131 of [2]) give a complete characterization of the f -vector $(f_0(P), \dots, f_{d-1}(P))$ of P . Using these conditions Perles (lecture in Oberwolfach, 1990) studied the possible numbers of vertices of simple polytopes. It is easily seen that in odd dimensions only even numbers of vertices occur, and he asked for the smallest number past which all (resp. all even) numbers are realizable as vertex numbers of simple polytopes.

Analogously, we ask here for the possible numbers of vertices of *cubical* polytopes, that is, convex polytopes of which every facet is a combinatorial cube. It is known [1] that a cubical polytope has at least 2^d vertices whenever $d > 2$. Jockusch [6] conjectured that there will be some gaps for numbers slightly larger than 2^d but that all numbers of vertices will be possible past a certain point. In contrast to this we show:

Theorem. *Let P be a cubical d -polytope in $\mathbb{R}^d$ for $d \geq 4$, and let d be even. Then the number $f_0(P)$ of vertices of P is even.*

In dimension $d = 3$ cubical polytopes with odd numbers of vertices exist, namely, with $2^d = 8, 10, 11, 12, \dots$ vertices. We do not know whether cubical polytopes with an odd number of vertices in odd dimensions $d \geq 5$ exist.

We regard combinatorially isomorphic polytopes as identical. For example, an n -cube here means any combinatorial n -cube.

We need the following properties of the cube and some related terminology. Let F be any facet of an n -cube C_n , then one and only one facet F' disjoint from F exists. F' is said to be the *facet opposite to F in C_n* . Let K be a j -face of C_n ($0 \leq j \leq n - 1$). Let $F_1, \dots, F_{n-j}$ be the facets containing K , and let $F'_1, \dots, F'_{n-j}$ be the corresponding opposite facets. Then $K' := F'_1 \cap \dots \cap F'_{n-j}$ is also a j -face and is called the *j -face opposite to K in C_n* . Let F and F' be opposite facets of C_n . Every edge not contained in F or F' intersects F in a vertex v and F' in a vertex v' . Vertices v and v' are called *related vertices of the facets F and F'* .

Following the usual notation, $\text{vert } P$ is the set of vertices of P . For a set $S \subset P$, a *vertex of S* means a vertex of P lying in S . Thus $\text{vert } S := \text{vert } P \cap S$. If M is a finite set, $|M|$ is the number of its elements. A *subfacet* (of a facet F) of P is a $(d - 2)$ -face of P (contained in F).

The proof of the theorem uses shellings and colorings of cubical polytopes.

2. Shellings of Cubical Polytopes

Let P be a convex polytope and let ∂P be the boundary complex of P . A *shelling* of ∂P is a labeling $\{F_1, \dots, F_m\}$ of the $m = f_{d-1}(P)$ facets of P so that, for $s = 2, \dots, m - 1$, the set $B_s := F_s \cap \bigcup_{i=1}^{s-1} F_i$ is homeomorphic to a $(d - 2)$ -ball. This implies that $\bigcup_{i=1}^s F_i$ is homeomorphic to a $(d - 1)$ -ball for $s = 1, \dots, m - 1$. It is well known that ∂P is shellable (see, e.g., [3] or p. 173 of [7]).

For $s = 1, \dots, m$, define $M_s := \bigcup_{i=1}^s F_i$, and put $M_0 := \emptyset$. A vertex v of M_s is called an *inner vertex of M_s* if all the facets of P containing v lie in M_s , and is called an *outer vertex of M_s* otherwise. We have $M_s = F_s \cup M_{s-1}$, and this union is the *s*th step of the shelling.

We are interested in $\text{vert } M_s \setminus \text{vert } M_{s-1}$, the set of vertices added in the *s*th step. These are outer vertices of M_s . We are also interested in how the *s*th step changes the set of outer vertices; thus besides $\text{vert } M_s \setminus \text{vert } M_{s-1}$ we are interested in the set of outer vertices of M_{s-1} becoming inner vertices of M_s .

For $s = m$ we have $\text{vert } P = \text{vert } M_m = \text{vert } M_{m-1}$, so

$$\text{vert } M_m \setminus \text{vert } M_{m-1} = \emptyset,$$

and the outer vertices of M_{m-1} are just the vertices of F_m . For $s = 1$ we have

$$\text{vert } M_1 = \text{vert } F_1,$$

and all the vertices of $M_1 = F_1$ are outer vertices of M_1 .

In order to examine the *s*th step of the shelling for $s = 2, \dots, m - 1$, consider F_s and $B_s = F_s \cap M_{s-1}$. Clearly, $\text{vert } M_s \setminus \text{vert } M_{s-1}$ consists of the vertices of F_s

not lying in B_s . Furthermore, the vertices of M_{s-1} becoming inner vertices of M_s lie in B_s . Since M_{s-1} is homeomorphic to a $(d-1)$ -ball, a vertex v of B_s becomes an inner vertex of M_s if and only if all the subfacets of F_s containing v are contained in B_s .

Now let P be a cubical polytope. Then we can describe B_s more precisely: Let K be a k -face of F_s ($0 \leq k \leq d-2$). Then the cube F_s has the following subfacets:

- the subfacets containing K ,
- the subfacets disjoint from K , that is, the subfacets opposite to subfacets containing K , and
- if $k > 0$, the subfacets intersecting but not containing K ; they intersect K in $(k-1)$ -faces.

B_s is homeomorphic to a $(d-2)$ -ball, so B_s is the union of subfacets of F_s . Thus, induction on the number of subfacets of F_s contained in B_s shows: if B_s does not contain pairs of opposite subfacets of F_s , then B_s is the union of all the subfacets of F_s containing a certain k -face of F_s ($0 \leq k \leq d-2$).

Hence, according to the type of B_s , we have to distinguish three possibilities:

Type 1: B_s contains a pair S_1 and S_2 of opposite subfacets of F_s . Then all the vertices of F_s lie in B_s , so that

$$\text{vert } M_s \setminus \text{vert } M_{s-1} = \emptyset.$$

We claim that the set of outer vertices of M_{s-1} becoming inner vertices of M_s consists of pairs of related vertices of S_1 and S_2 . To see this let $v_1 \in S_1$ be an inner vertex of M_s . Then, by the above, all the subfacets of F_s passing through v_1 are contained in B_s . Let v_2 be the vertex of S_2 related to $v_1 \in S_1$. The subfacets of F_s containing v_2 are S_2 and some subfacets containing v_1 , and so they are all contained in B_s . Hence, by the above, v_2 is also an inner vertex of M_s , which completes the proof.

Type 2: B_s is the union of the subfacets of F_s containing a certain k -face K of F_s , with $1 \leq k \leq d-2$. Let K' be the k -face of F_s opposite to K in F_s . Then the vertices of F_s not lying in B_s are exactly the vertices of K' , so that

$$\text{vert } M_s \setminus \text{vert } M_{s-1} = \text{vert } K'.$$

We claim that no outer vertices of M_{s-1} become inner vertices of M_s . To see this let v be a vertex of B_s or, more generally, of F_s . Since $k \geq 1$, a subfacet S_v of F_s containing v and not containing K exists. Hence $S_v \not\subset B_s$. Thus, by the above, v is not an inner vertex of M_s , which completes the proof.

Type 3: B_s is the union of the subfacets of F_s passing through a certain vertex v_0 of F_s . Let v'_0 be the vertex of F_s opposite to v_0 in F_s . Then v'_0 is the only vertex of F_s not lying in B_s , so that

$$\text{vert } M_s \setminus \text{vert } M_{s-1} = \{v'_0\}.$$

We claim that *the set of outer vertices of M_{s-1} becoming inner vertices of M_s is $\{v_0\}$* . In fact, v_0 is an outer vertex of M_{s-1} , and since all the subfacets of F_s passing through v_0 are contained in B_s , v_0 is an inner vertex of M_s . Conversely, let $v \neq v_0$ be a vertex of B_s . Then v is contained in a subfacet through v'_0 and so is an outer vertex of M_s , which completes the proof.

3. Colorings of the Set of Vertices of a Polytope

If P is a polytope, vert P is said to be *well-colored* if the vertices of P are colored with two colors so that every two adjacent vertices have different colors.

It is easy to see that vert P is well-colorable if and only if the length of each circuit on P is even (a circuit on P is a sequence $\{x_0, x_1, \dots, x_k = x_0\}$ of successively adjacent vertices of P , and k is its length).

Lemma. *Let P be a d -polytope, $d \geq 2$, so that every 2-face of P has an even number of edges. Then vert P is well-colorable.*

This lemma is known (see p. 154 of [4], but the proof given there has a gap, since a circuit on P need not be homeomorphic to the boundary of a circle). However, the lemma follows from the shellability of ∂P :

Proof. We show that each circuit on P has even length. This is true for $d = 2$ and we prove it generally by induction on d . We assume, therefore, that it is true for $d - 1 \geq 2$. Let P be a d -polytope so that every 2-face of P has an even number of edges. Let $\{F_1, \dots, F_m\}$ ($m = f_{d-1}(P)$) be a shelling of ∂P . Then, by the induction hypothesis, every circuit on F_s has even length ($s = 1, \dots, m$). Define $M_s := \bigcup_{i=1}^s F_i$ for $s = 1, \dots, m$. Since vert $P = \text{vert } M_{m-1}$ and since all the edges of P lie in M_{m-1} , the assertion is true if all the circuits on M_s have even length for $s = 1, \dots, m - 1$. Because $M_1 = F_1$ this is true for $s = 1$, and we prove it generally by induction on s .

We assume, therefore, that every circuit on M_{s-1} has even length for $1 \leq s - 1 \leq m - 2$, and we recall that every circuit on F_s has even length. Now let $C = \{x_0, \dots, x_k = x_0\}$ be a circuit on $M_s = F_s \cup M_{s-1}$ with points in both F_s and M_{s-1} . Then C intersects $B_s := F_s \cap M_{s-1}$ in some successive points $x_{i_0}, \dots, x_{i_j} = x_{i_0}$. Since B_s is homeomorphic to a $(d - 2)$ -ball, B_s is the union of subfacets of F_s and is connected. Thus a circuit C_0 on B_s passing $x_{i_0}, \dots, x_{i_j} = x_{i_0}$ successively exists. Then the two circuits C and C_0 split into two circuits lying entirely in F_s (resp. M_{s-1}). Since these two circuits and C_0 have even length, so has C , which completes the proof. $\square$

4. Proof of the Theorem

Let P be a cubical d -polytope, with $d \geq 4$. The number of vertices of P is

$$\begin{aligned} |\text{vert } P| = |\text{vert } M_m| &= (|\text{vert } M_m| - |\text{vert } M_{m-1}|) + \dots + (|\text{vert } M_s| - |\text{vert } M_{s-1}|) \\ &\quad + \dots + (|\text{vert } M_2| - |\text{vert } M_1|) + |\text{vert } M_1|. \end{aligned}$$

According to Section 2 we have $|\text{vert } M_m| - |\text{vert } M_{m-1}| = 0$ and $|\text{vert } M_1| = 2^{d-1}$. Furthermore, $|\text{vert } M_s| - |\text{vert } M_{s-1}|$ is odd if and only if the s th step of the shelling is of type 3 ($2 \leq s \leq m-1$). Hence, $|\text{vert } P|$ is even if and only if the number of steps of type 3 is even.

In order to study this, let $\text{vert } P$ be well-colored with the two colors black and white. Let $B(M_s)$ be the number of black *outer* vertices of M_s , let $W(M_s)$ be the number of white *outer* vertices of M_s , and put $BW(M_s) := B(M_s) - W(M_s)$, $s = 1, \dots, m$.

Clearly,

$$\begin{aligned} BW(M_{m-1}) &= (BW(M_{m-1}) - BW(M_{m-2})) + \cdots + (BW(M_s) - BW(M_{s-1})) \\ &\quad + \cdots + (BW(M_2) - BW(M_1)) + BW(M_1). \end{aligned}$$

According to Section 2, the outer vertices of M_{m-1} are just the vertices of F_m , so that $BW(M_{m-1}) = 0$. All the vertices of $M_1 = F_1$ are outer vertices, so that $BW(M_1) = 0$. Thus

$$\begin{aligned} (BW(M_{m-1}) - BW(M_{m-2})) + \cdots + (BW(M_s) - BW(M_{s-1})) \\ + \cdots + (BW(M_2) - BW(M_1)) = 0. \quad (*) \end{aligned}$$

According to Section 2, the s th step in the shelling ($2 \leq s \leq m-1$) changes the set of outer vertices as follows. It lets $\text{vert } M_s \setminus \text{vert } M_{s-1}$ appear as new outer vertices, and it lets the outer vertices of M_{s-1} becoming inner vertices of M_s disappear. Thus, if the s th step is of type 1 or 2, $BW(M_s) - BW(M_{s-1}) = 0$. However, if the s th step is of type 3, precisely one new outer vertex v'_0 appears, and precisely one outer vertex v_0 disappears, and v_0 and v'_0 are opposite vertices of F_s . Thus if the dimension d of P is even, that is, if the dimension $(d-1)$ of F_s is odd, then v_0 and v'_0 have different colors. Thus $BW(M_s) - BW(M_{s-1}) = \pm 2$ if the s th step in the shelling is of type 3, and if dimension d is even.

Together with (*) this implies that the number of steps of type 3 is even whenever d is even, which completes the proof of the theorem.

5. Remarks

Pursuing the total number of black vertices and the total number of white vertices of M_s ($s = 1, \dots, m$) in the shelling we obtain:

Corollary 1. *Let P be a cubical d -polytope in $\mathbb{R}^d$ for $d \geq 4$, and let d be even. If the vertices of P are well-colored with the two colors black and white, then P has as many white vertices as black ones.*

The f -vector $(f_0(P), \dots, f_{d-1}(P))$ of a cubical polytope satisfies a certain set of linear equations, see, e.g., p. 156 of [5]. It is easy to deduce from these

equations that, for even-dimensional cubical polytopes, $f_0(P) - f_1(P)$ is even. So we have:

Corollary 2. *Let P be a cubical d -polytope in $\mathbb{R}^d$ for $d \geq 4$, and let d be even. Then the number $f_1(P)$ of edges of P is even.*

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