Lower Bounds for the Numbers of Antipodal Pairs and Strictly Antipodal Pairs of Vertices in a Convex Polytope

Mahn Hung Nguyễn and Valeriu Soltan

Mathematical Institute of the Academy of Sciences,
Academic str. 5, Kishinev 277028, Republica Moldova

Abstract. The greatest lower bounds for the numbers of antipodal pairs and strictly antipodal pairs of vertices in a convex d -polytope with n vertices are determined.

1. Introduction

Let X be a finite set of points in d -dimensional Euclidean space E^d . Following Klee [4], points $x, z \in X$ are called *antipodal* provided there are two (distinct) parallel hyperplanes H_x, H_z supporting the polytope $P = \text{conv } X$ such that $x \in H_x$ and $z \in H_z$. If, additionally, $P \cap H_x = \{x\}$ and $P \cap H_z = \{z\}$, then the points x, z are called *strictly antipodal* (see [3]).

Klee [4] posed the problem of determining the maximum number m of points on the surface of a convex body in E^d , every pair of which is antipodal. Danzer and Grünbaum [1] solved this problem showing that $m \leq 2^d$ with $m = 2^d$ if and only if the points are the vertices of a d -parallelotope. Grünbaum [3] proved the inequality $k \geq 2d - 1$, where k is the maximum number of points on the surface of a d -dimensional convex body, every pair of which is strictly antipodal. It has been conjectured that $k = 2d - 1$, but Erdős and Füredi [2] proved the inequality $k > (1.15)^d$.

Another problem was originated by Grünbaum [3], who asked about the numbers of antipodal pairs and strictly antipodal pairs of points in an n -pointed set in E^d . Some upper bounds for these numbers were obtained by Makai and Martini [5].

In this paper we determine the greatest lower bounds for the numbers of antipodal pairs and strictly antipodal pairs of points in a finite set in E^d . In order to avoid degenerate situations, we consider the set to be the vertex-set of a convex polytope.

2. Main Results

In what follows P_n denotes a convex d -polytope with n vertices $v_1, \dots, v_n$ in E^d . Let $r(P_n)$ and $s(P_n)$ be, respectively, the numbers of antipodal pairs and strictly antipodal pairs of vertices in P_n .

The following result gives exact values of $r(P_n)$ and $s(P_n)$ in the case $d = 2$.

Theorem 1. *For a convex polygon $P_n \subset E^2$,*

$$r(P_n) = n + k \quad \text{and} \quad s(P_n) = n - k,$$

where k ($0 \leq k \leq \lfloor n/2 \rfloor$) is the number of pairs of parallel sides in P_n .

The inequalities

$$n \leq r(P_n) \leq 3n/2 \quad \text{and} \quad \lceil n/2 \rceil \leq s(P_n) \leq n$$

mentioned by Grünbaum [3] for $d = 2$, follow from Theorem 1.

Theorem 2. *For a convex polytope $P_n \subset E^d$, $n \geq d + 1$, $d \geq 3$,*

$$r(P_n) \geq n - 1 + d(d - 1)/2.$$

The equality

$$r(P_n) = n - 1 + d(d - 1)/2$$

holds if and only if P_n has $d + 1$ pairwise antipodal vertices, say $v_1, \dots, v_{d+1}$, and every other vertex v_k (in the case $n > d + 1$) is antipodal to a unique vertex v_i , $1 \leq i \leq d + 1 < k \leq n$.

For any vertex v_i of P_n , denote the supporting cone of P_n at v_i by C_i :

$$C_i = \bigcup \{v_i + \lambda(P_n - v_i) : \lambda \geq 0\}.$$

Theorem 3. *For a convex polytope $P_n \subset E^d$, $n \geq d + 1$, $d \geq 3$,*

$$s(P_n) \geq \lceil n/2 \rceil.$$

For an even n , the equality

$$s(P_n) = \lceil n/2 \rceil \tag{1}$$

holds if and only if $n \geq 2d$ and the vertex-set $\{v_1, \dots, v_n\}$ can be divided into $n/2$ pairwise disjoint subsets of the form $\{v_i, v_j\}$ such that

$$C_i - v_i = v_j - C_j$$

for each of them. For an odd n , equality (1) holds if and only if $n \geq 4d - 1$ and some $(n - 3)/2$ pairwise disjoint subsets of the form $\{v_i, v_j\}$ can be chosen in $\{v_1, \dots, v_n\}$ such that

$$C_i - v_i = v_j - C_j$$

for each of them, and the remaining three vertices, say v_k, v_l, v_m , satisfy the relation

$$(C_k - v_k) \cap (C_l - v_l) = v_m - C_m.$$

Observation. For every even $n \geq 2d$ and for every odd $n \geq 4d - 1$ there is a d -polytope $P_n \subset E^d$ satisfying (1).

Note that, for a convex polytope $P_n \subset E^d$ with a small number of vertices, lower bounds for $s(P_n)$ are considered by Nguyễn [7] and Makai and Martini [6].

It is shown later that Theorem 2 is a direct consequence of the following assertion. Let $\mathcal{M} = \{K_1, \dots, K_p\}$ be a family of convex cones with common zero apex. Two cones $K_i, K_j \in \mathcal{M}$ are called *opposite* provided $K_i \cap (-K_j) \neq \{0\}$ (or $(-K_i) \cap K_j \neq \{0\}$, which is the same). Denote by $t(\mathcal{M})$ the number of all opposite pairs of cones in $\mathcal{M}$.

Theorem 4. Let $\mathcal{L} = \{K_1, \dots, K_n\}$ be a family of closed d -dimensional convex acute polyhedral cones with a common zero apex in E^d , $d \geq 2$, such that

$$\bigcup_{i=1}^n K_i = E^d, \quad \text{int } K_i \cap \text{int } K_j = \emptyset, \quad i \neq j, \quad (2)$$

and the intersection $K_i \cap K_j$ of the two cones K_i, K_j is a face for both of them. Then

$$t(\mathcal{L}) \geq n - 1 + d(d - 1)/2. \quad (3)$$

In the case $d = 2$, the equality

$$t(\mathcal{L}) = n - 1 + d(d - 1)/2 \quad (4)$$

holds if and only if no two boundary rays of $K_1, \dots, K_n$ are opposite, and in the case $d \geq 3$ equality (4) holds if and only if $\mathcal{L}$ satisfies the following property:

- (S) There are $d + 1$ pairwise opposite cones, say $K_1, \dots, K_{d+1}$, and every other cone K_m (in the case $n > d + 1$) is opposite to a unique cone K_i , $1 \leq i \leq d + 1 < m \leq n$.

3. Proof of Theorem 4

From now on we consider $\mathcal{L}$ to be a family of cones satisfying the hypothesis of Theorem 4. We have divided proof of this theorem into a sequence of lemmas.

For any cone $K \in \mathcal{L}$, denote by $\gamma(K)$ the number of cones in $\mathcal{L}$ opposite to K . It is easily seen that

$$t(\mathcal{L}) = (\gamma(K_1) + \cdots + \gamma(K_n))/2.$$

A cone $K \in \mathcal{L}$ is called *special* provided $\gamma(K) \geq d$.

Lemma 1. *Every cone $K \in \mathcal{L}$ is opposite to at least one special cone.*

Proof. Let K be a cone in $\mathcal{L}$. K , being acute, has at least one extremal ray, say l . It is possible to prove (see, for instance, [8]) that l lies in the boundary of at least $d - 1$ cones in $\mathcal{L}$ distinct from K . The opposite ray $-l$ belongs to no cone in $\mathcal{L}$ containing l . Due to (2), $-l$ lies in a new cone $K' \in \mathcal{L}$. Hence K' is opposite to at least d cones in $\mathcal{L}$, i.e., K' is special. $\square$

Lemma 2. *Every cone $K \in \mathcal{L}$ with $\gamma(K) \geq 2$ is opposite to at least two special cones.*

Proof. Let $\gamma(K) \geq 2$ for a cone $K \in \mathcal{L}$. By Lemma 1 there is a special cone $K' \in \mathcal{L}$ opposite to K . Since $\gamma(K) \geq 2$, we have $-K \not\subset \text{int } K'$. Hence there is an extremal ray l of $-K$ which is not in the interior of K' . In this case l belongs to a new cone $K'' \in \mathcal{L}$ and K'' is opposite to K . As in Lemma 1, the ray $-l$, being extremal for K , belongs to at least d cones in $\mathcal{L}$. Hence K'' is opposite to at least d cones in $\mathcal{L}$, i.e., K'' is special. $\square$

Corollary 1. *$\mathcal{L}$ contains at least three special cones.*

Lemma 3. *$\mathcal{L}$ has at least four special cones in the case $d = 3$.*

Proof. Assume, in order to obtain a contradiction, that $\mathcal{L}$ contains exactly three special cones, say K_1, K_2, K_3 (see Corollary 1). We distinguish two cases.

Case 1: $\dim(K_1 \cap K_2) \leq 1$. Since $K_1 \cap K_2$ is a face for both K_1, K_2 , a plane H separating K_1, K_2 can be chosen such that

$$\dim(H \cap K_1) \leq 1 \quad \text{and} \quad \dim(H \cap K_2) \leq 1.$$

We may assume that $H \cap \text{int } K_i \neq \emptyset$ for every cone K_i satisfying the condition $\dim(H \cap K_i) = 2$, since otherwise it is possible to move H slightly in order to obtain the desired property. Let $K'_1, \dots, K'_m$ be the cones in $\mathcal{L}$ whose intersection with H is two-dimensional. It is easily seen that the family

$$\mathcal{M} = \{H \cap K'_1, \dots, H \cap K'_m\}$$

satisfies the hypothesis of Theorem 4 (with H instead of E^3). Due to Corollary 1, there are at least three special cones, say $L_i = K'_i \cap H$, $i = 1, 2, 3$, in $\mathcal{M}$. In other words, $\gamma_H(L_i) \geq 2$, $i = 1, 2, 3$, where $\gamma_H(L)$ means the number of cones in $\mathcal{M}$ opposite to a cone $L \in \mathcal{M}$. At least two of the cones K'_1, K'_2, K'_3 , say K'_1, K'_2 , are different from K_3 . Due to the choice of H , cones K'_1, K'_2 are different from K_1, K_2 . Since $\gamma(K'_i) \geq \gamma_H(L_i) \geq 2$, $i = 1, 2$, and since there are exactly three special cones in $\mathcal{L}$, each of K'_1, K'_2 is opposite to at least one of the cones K_1, K_2 (see Lemma 2). Hence each of K'_1, K'_2 is opposite to at least two cones in $\{K'_1, \dots, K'_m\}$ and to at least one cone in $\{K'_1, K'_2\}$, i.e., both K'_1, K'_2 are special in $\mathcal{L}$. In this case $\mathcal{L}$ has at least five special cones, a contradiction.

Case 2: $\dim(K_1 \cap K_2) = 2$. In this case the unique hyperplane, say H , separating K_1 and K_2 exists. As above, let $K'_1, \dots, K'_m$ be the cones in $\mathcal{L}$ whose intersection with H is two-dimensional. Consider the family $\mathcal{M} = \{H \cap K'_1, \dots, H \cap K'_m\}$. Unlike case 1, some of the cones $H \cap K'_1, \dots, H \cap K'_m$ may coincide. Choose in $\mathcal{M}$ the subfamily $\mathcal{F}$ of all pairwise distinct cones. As above, $\mathcal{F}$ satisfies the hypothesis of Theorem 4. By Corollary 1, in $\mathcal{F}$ a special cone of the form $K' \cap H$ can be chosen which is different from the cones $K_1 \cap K_2, K_3 \cap H$. If $K' \cap H$ is opposite to both cones $K_1 \cap K_2, K_3 \cap H$, then K' is opposite to K_1, K_2, K_3 , i.e., $\gamma(K') \geq 3$. If $K' \cap H$ is opposite to at most one of $K_1 \cap K_2, K_3 \cap H$, then, similarly to case 1, $\gamma(K') \geq 3$. In both cases K' is a new special cone in $\mathcal{L}$, a contradiction. $\square$

Lemma 4. *If for a cone $K \in \mathcal{L}$ there are m cones in $\mathcal{L}$ contained in the cone $-K$, then $\gamma(K) \geq d + m$.*

Proof. Assume that cones $K_{i_1}, \dots, K_{i_m} \in \mathcal{L}$ are contained in $-K_1$. Denote by H a $(d-1)$ -subspace satisfying the condition $H \cap K_1 = \{0\}$. (The last is possible, since K_1 is acute.) Let H' be a translate of H whose intersection with $-K_1$ is $(d-1)$ -dimensional. Put

$$L_0 = H' \cap (-K_1), \quad L_i = H' \cap K_{i_j}, \quad j = 1, \dots, n.$$

The set $M = L_{i_1} \cup \dots \cup L_{i_m}$ is compact and lies in L_0 . Besides $K_{i_1}, \dots, K_{i_m}$, the hyperplane H' intersects some other cones, say $K_{p_1}, \dots, K_{p_q}$, which have common points with M .

We claim that $q \geq d$. To prove this, choose a point x in the relative interior of M , and let $B_r(x)$ be a $(d-1)$ -dimensional ball of radius $r > 0$ and center x , contained in $H' \cap \text{rint } M$. Denote by G_j a $(d-2)$ -plane in H' supporting $B_r(x)$ and separating $B_r(x)$ from L_{p_j} , $j = 1, \dots, q$. Let W_j be the closed half-plane in H' determined by G_j and containing $B_r(x)$. Clearly, $L_{p_1}, \dots, L_{p_q}$ surround M . Hence the polyhedral set $W = W_1 \cap \dots \cap W_q$ is compact and has at most q facets. Since the number of facets of a $(d-1)$ -polytope is at least d , we have $q \geq d$.

By the choice of $K_{p_1}, \dots, K_{p_q}$, each of these cones is opposite to K_1 . Hence $\gamma(K_1) \geq q + m \geq d + m$. $\square$

Lemma 5. *If $\mathcal{L}$ contains m special cones, then $t(\mathcal{L}) \geq md/2 + n - m$.*

Proof. We can certainly assume that $\mathcal{L}_1 = \{K_1, \dots, K_m\}$, $m \leq n$, is the family of all the special cones in $\mathcal{L}$. Put $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$ and divide $\mathcal{L}_2$ into subfamilies

$$\mathcal{V} = \{K \in \mathcal{L}_2: \gamma(K) = 1\} \quad \text{and} \quad \mathcal{W} = \{K \in \mathcal{L}_2: \gamma(K) \geq 2\}.$$

$\mathcal{V}$ represents the union of pairwise disjoint subfamilies $\mathcal{V}_1, \dots, \mathcal{V}_p$ such that all members of $\mathcal{V}_j$ are opposite to the same cone, say $K_{i_j} \in \mathcal{L}$, and cones $K_{i_1}, \dots, K_{i_p}$ are pairwise distinct. By Lemma 2, $\gamma(K_{i_j}) \geq d + \text{card } \mathcal{V}_j$. Hence every cone K_{i_j} is special.

For any cone $K \in \mathcal{L}$, denote by $\gamma'(K)$ the number of cones in $\mathcal{L} \setminus \mathcal{V}$ opposite to K . By the above, $\gamma'(K) = \gamma(K)$ if K is not of the form K_{i_j} , and $\gamma'(K) \geq d$ if $K = K_{i_j}$ for $j = 1, \dots, p$. Then

$$\begin{aligned} 2t(\mathcal{L}) &= \sum_{i=1}^n \gamma(K_i) \\ &= \sum_{i=1}^m \gamma'(K_i) + \sum_{j=1}^p (\gamma(K_{i_j}) - \gamma'(K_{i_j}) + \sum \{\gamma(K): K \in \mathcal{V}_{j_j}\}) \\ &\quad + \sum \{\gamma(K): K \in \mathcal{W}\} \\ &\geq md + 2 \sum_{j=1}^p \text{card } \mathcal{V}_j + 2 \text{card } \mathcal{W} \\ &= md + 2(n - m). \end{aligned} \quad \square$$

Corollaries 2 and 3 follow from Lemmas 3 and 5.

Corollary 2. *Inequality (3) holds in the case $d \leq 3$.*

Corollary 3. *In the case $d \geq 3$, equality (4) holds only if $\mathcal{L}$ has at most $d + 1$ special cones.*

Indeed, inequality

$$n - 1 + d(d - 1)/2 = t(\mathcal{L}) \geq md/2 + n - m$$

is equivalent to $d(d - 2) \geq (m - 1)(d - 2)$.

Lemma 6. *In the case $d = 2$, we have $t(\mathcal{L}) \geq n$ with $t(\mathcal{L}) = n$ if and only if no two boundary rays of $K_1, \dots, K_n$ are opposite.*

Proof. The inequality $t(\mathcal{L}) \geq n$ is mentioned in Corollary 2. Repeating the

proof of Lemma 5 for $d = 2$, we obtain that $t(\mathcal{L}) = n$ holds if and only if

$$\gamma(K_{i_j}) = 2 + \text{card } \mathcal{V}_j, \quad j = 1, \dots, p,$$

and $\gamma(K_q) = 2$ for all $q \neq i_1, \dots, i_p$ such that $\gamma(K_q) \geq 2$. A trivial verification shows that both of the last conditions hold if and only if no two boundary rays of the cones $K_1, \dots, K_n$ are opposite. $\square$

The proof of Theorem 4 is organized by induction on d . The case $d = 2$ is considered in Lemma 6. Assume that the hypothesis of Theorem 4 holds for all $m \leq d - 1$ ($d \geq 3$) and let cones $K_1, \dots, K_n$ in E^d satisfy the hypothesis of Theorem 4.

First we prove the first part of the assertion, namely, inequality (3).

Lemma 7. *Let $\mathcal{L}$ contain a pair of nonopposite cones K_i, K_j with:*

- (i) $\gamma(K_i) + \gamma(K_j) \geq 2d$ in the case $d \geq 3$.
- (ii) $\gamma(K_i) + \gamma(K_j) \geq 2d - 1$ in the case $d \geq 4$.

Then $t(\mathcal{L}) > n - 1 + d(d - 1)/2$.

Proof. Since K_i, K_j are nonopposite (i.e., $K_i \cap (-K_j) = \{0\}$), there is a $(d - 1)$ -subspace H separating K_i and $-K_j$ such that $H \cap (K_i \cup K_j) = \{0\}$. Without loss of generality, we may put $i = 1, j = 2$.

It can be assumed that $H \cap \text{int } K_i \neq \emptyset$ for every cone K_i satisfying $H \cap K_i \neq \{0\}$, since otherwise it is possible to move H slightly in order to obtain the desired property. Let $K_3, \dots, K_l$ be the cones in $\mathcal{L}$ whose interiors intersect H . It is easily seen that the family $\mathcal{M} = \{H \cap K_3, \dots, H \cap K_l\}$ satisfies the hypothesis of Theorem 4 such that (2) holds (with H instead of E^d). By inductive hypothesis,

$$t(\mathcal{M}) \geq l - 3 + (d - 1)(d - 2)/2.$$

Put $\mathcal{L}_1 = \{K_3, \dots, K_l\}$. Denote by $\mathcal{L}_2$ the family of all special cones in $\mathcal{L} \setminus (\mathcal{L}_1 \cup \{K_1, K_2\})$, and let

$$\mathcal{L}_3 = \mathcal{L} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2 \cup \{K_1, K_2\}).$$

Then

$$\sum_{i=1}^n \gamma(K_i) = \gamma(K_1) + \gamma(K_2) + \sum_{i=1}^3 \left(\sum \{ \gamma(K) : K \in \mathcal{L}_i \} \right).$$

As in Lemma 5, divide $\mathcal{L}_3$ into the subfamilies

$$\mathcal{V} = \{K \in \mathcal{L}_3 : \gamma(K) = 1\} \quad \text{and} \quad \mathcal{W} = \{K \in \mathcal{L}_3 : \gamma(K) \geq 2\},$$

and introduce the function $\gamma'(K)$. Then

$$\begin{aligned}
 2t(\mathcal{L}) &= \sum_{i=1}^n \gamma(K_i) \\
 &= \gamma'(K_1) + \gamma'(K_2) + \sum \{\gamma'(K): K \in \mathcal{L}_1\} + \sum \{\gamma'(K): K \in \mathcal{L}_2\} \\
 &\quad + \sum_{j=1}^p (\gamma(K_j) - \gamma'(K_j) + \text{card } \mathcal{V}_j) + \sum \{\gamma(K_j): K \in \mathcal{W}\} \\
 &\geq (2d - 1) + 2t(\mathcal{L}_1) + d \cdot \text{card } \mathcal{L}_2 + 2 \text{ card } \mathcal{V} + 2 \text{ card } \mathcal{W} \\
 &= (2d - 1) + 2t(\mathcal{L}_1) + d \cdot \text{card } \mathcal{L}_2 + 2 \text{ card } \mathcal{L}_3.
 \end{aligned}$$

Due to Lemma 2, K_1 is opposite to at least two special cones $K'_1, K''_1 \in \mathcal{L}$, and K_2 is opposite to at least two special cones $K'_2, K''_2 \in \mathcal{L}$. Since $K_1 \setminus \{0\}$ and $K_2 \setminus \{0\}$ lie in the same open half-space determined by H , cones K_1 and K_2 are nonopposite and any of them differs from each of K'_1, K''_1, K'_2, K''_2 .

We consider separately the following three cases.

Case 1. If K'_1, K''_1, K'_2, K''_2 are pairwise distinct, and p ($0 \leq p \leq 4$) of them belong to $\mathcal{L}_1$, then

$$2t(\mathcal{L}_1) \geq 2t(\mathcal{M}) + p, \quad \text{card } \mathcal{L}_2 \geq 4 - p,$$

and

$$\begin{aligned}
 2t(\mathcal{L}) &\geq 2d - 1 + 2t(\mathcal{M}) + p + d(4 - p) + 2(n - l - 4 + p) \\
 &\geq 2n - 1 + d(d - 1).
 \end{aligned}$$

Case 2. Let, for example, $K'_1 = K'_2$ and K'_1, K''_1, K''_2 be pairwise distinct. Since $\mathcal{L}$ has at least five special cones, we have $t(\mathcal{L}) > n + 2$ in the case $d = 3$ (see Corollary 2). Suppose that $d \geq 4$. If all cones K'_1, K''_1, K''_2 belong to $\mathcal{L}_1$, then

$$2t(\mathcal{L}) \geq 2d - 1 + 2t(\mathcal{M}) + 4 + 2(n - l) \geq 2n - 1 + d(d - 1).$$

Assume that only some q ($0 \leq q \leq 2$) of K'_1, K''_1, K''_2 belong to $\mathcal{L}_1$. Then $2t(\mathcal{L}_1) \geq 2t(\mathcal{M}) + q$ and

$$\begin{aligned}
 2t(\mathcal{L}) &\geq 2d - 1 + 2t(\mathcal{M}) + q + d(3 - q) + 2(n - l - 3 + q) \\
 &\geq 2n - 1 + d(d - 1).
 \end{aligned}$$

Case 3. Let $K'_1 = K'_2$ and $K''_1 = K''_2$. As above,

$$\begin{aligned}
 2t(\mathcal{L}) &\geq 2d - 1 + 2t(\mathcal{M}) + q + d(2 - v) + 2(n - l - 2 + v) \\
 &\geq 2n - 1 + d(d - 1) + (d - 4)(2 - v),
 \end{aligned}$$

where v ($0 \leq v \leq 2$) is the number of cones in $\{K'_1, K''_1\}$ contained in $\mathcal{L}_1$. We have

$$2n - 1 + d(d - 1) + (d - 4)(2 - v) \geq 2n - 1 + d(d - 1)$$

if $d \geq 4$ or $d = 3$ and $v = 2$. Hence it remains to consider the case $d = 3$ and $v \leq 1$. By the hypothesis, $\gamma'(K_1) + \gamma'(K_2) \geq 2d$. If $v = 1$, then

$$2t(\mathcal{L}) \geq 2d + 2t(\mathcal{M}) + 2 + d + 2(n - l - 1) \geq 2n - 1 + d(d - 1).$$

Let $v = 0$, i.e., $K'_1, K''_1 \in \mathcal{L}_2$. Due to Corollary 2, it is sufficient to consider the case when K_1, K_2, K'_1, K''_1 are the only special cones in $\mathcal{L}$. Since $\mathcal{M}$ satisfies the hypothesis of Theorem 4 (with H instead of E^d), it contains at least three special cones (see Corollary 2). Hence there are cones $L_1, L_2, L_3 \in \mathcal{L}_1$ with $\gamma(L_i) \geq 2$ for all $i = 1, 2, 3$. Due to Lemma 2, every L_i is opposite to two special cones in $\mathcal{L}$. By the assumption, these cones are of the form K_1, K_2, K'_1, K''_1 . In this case $2t(\mathcal{L}_1) \geq 2t(\mathcal{M}) + 6$ and

$$2t(\mathcal{L}) \geq 2d + 2t(\mathcal{M}) + 6 + 2d + 2(n - l - 2) > 2n + d(d - 1).$$

In all the cases above, $t(\mathcal{L}) > n - 1 + d(d - 1)/2$. □

Lemma 8. *Let $d \geq 4$ and let $\mathcal{L}$ be such that every two cones $K_i, K_j \in \mathcal{L}$ are opposite if $\gamma(K_i) + \gamma(K_j) \geq 2d - 1$. Then (3) holds.*

Proof. Due to Corollary 2, there are at least three special cones, say $K_1, K_2, K_3 \in \mathcal{L}$. We distinguish two cases.

Case 1: $\dim(K_1 \cap K_2) \leq d - 2$. In this case a hyperplane H separating K_1, K_2 can be chosen such that

$$\dim(H \cap K_1) \leq d - 2 \quad \text{and} \quad \dim(H \cap K_2) \leq d - 2.$$

Let $\mathcal{H}$ be the family of cones in $\mathcal{L}$ intersected by H . If K_3 were not in $\mathcal{H}$, then K_3 would not be antipodal to one of K_1, K_2 . Hence $K_3 \in \mathcal{H}$. Choose in $\mathcal{H}$ the subfamily $\mathcal{F}$ of all pairwise distinct cones whose intersection with H is $(d - 1)$ -dimensional. As above, $K_3 \in \mathcal{F}$. Without loss of generality, it can be assumed that $\mathcal{F} = \mathcal{H} = \{K_3, \dots, K_l\}$.

By inductive hypothesis (see Lemma 7),

$$t(\mathcal{H}) \geq l - 3 + (d - 1)(d - 2)/2.$$

Due to Corollary 1, there are at least three special cones, say $L_i = K'_i \cap H$, $i = 1, 2, 3$, in the family $\mathcal{M} = \{H \cap K_3, \dots, H \cap K_l\}$. In other words, $\gamma_H(L_i) \geq d - 1$, $i = 1, 2, 3$, where $\gamma_H(L)$ means the number of cones in $\mathcal{M}$ opposite to L . Since

$K_3 \in \mathcal{H}$, at least two of the cones K'_1, K'_2, K'_3 , say K'_1, K'_2 , are distinct from K_3 . Due to

$$\gamma(K'_i) \geq \gamma_H(L_i) \geq d - 1, \quad i = 1, 2,$$

each of K'_1, K'_2 is opposite to both K_1, K_2 . Hence

$$2t(\mathcal{L}) \geq 2d + 2t(\mathcal{H}) + 4 + 2(n - l) \geq 2n + d(d - 1).$$

Case 2: $\dim(K_1 \cap K_2) = d - 1$. Let H be the hyperplane separating K_1, K_2 . As above, assume $\mathcal{H} = \{K_2, \dots, K_l\}$ is the family of all cones in $\mathcal{L} \setminus \{K_1\}$ whose intersection with H is $(d - 1)$ -dimensional. A special cone of the form $K' \cap H \in \mathcal{M}$ distinct from the cones $K_1 \cap K_2, K_3 \cap H$ can be chosen. Since $\gamma(K') \geq d - 1$, K' is antipodal to K_1 . By the antipodality of K_1 and K_3 ,

$$2t(\mathcal{L}) \geq 2d + 2t(\mathcal{H}) + 2 + 2(n - l) \geq 2n - 2 + d(d - 1). \quad \square$$

Summing up the assertions of Corollary 2 and Lemmas 7 and 8, we obtain that inequality (3) holds for all $d \geq 3$. To finish the proof of Theorem 4, it remains to consider the conditions under which equality (4) holds when $d \geq 3$.

Lemma 9. *In the case $d \geq 3$, equality (4) holds if and only if $\mathcal{L}$ has property (S).*

Proof. First we show that $\mathcal{L}$ has $d + 1$ special cones. The case $d = 3$ follows from Lemma 3 and Corollary 3. Let $d \geq 4$. Choose in $\mathcal{L}$ two special cones, say K_1, K_2 , and let H be a hyperplane separating K_1, K_2 . As in Lemma 8, consider the family $\mathcal{H}$, say $\{K_3, \dots, K_l\}$, of cones in $\mathcal{L}$ whose intersection with H is $(d - 1)$ -dimensional. Analysis of the proof of Lemma 8 shows that (4) holds only if $\dim(K_1 \cap K_2) = d - 1$. Then

$$\begin{aligned} 2n - 2 + d(d - 1) = 2t(\mathcal{L}) &\geq 2d + 2t(\mathcal{H}) + 2 + 2(n - l) \\ &\geq 2n - 2 + d(d - 1). \end{aligned}$$

Hence

$$t(\mathcal{M}) = l - 3 + (d - 1)(d - 2)/2.$$

By inductive hypothesis, $\mathcal{M}$ has d special cones. Since $K_1 \cap K_2$ belongs to $\mathcal{M}$, at least $d - 1$ special cones in $\mathcal{M}$ of the form $K'_1 \cap H, \dots, K'_d \cap H$ are different from $K_1 \cap K_2$, i.e., they satisfy $\gamma_H(K'_i) \geq d$, where $\gamma_H(K'_i)$ means the number of cones in $\mathcal{M}$ opposite to $K'_i \cap H$. Due to Lemma 7, each of $K'_1, \dots, K'_d$ is opposite to both K_1, K_2 . Therefore we have found $d + 1$ special cones in $\mathcal{L}$.

Since property (S) trivially implies (4), it remains to prove that (4) implies (S). From the proof of Lemma 5 it follows that $\gamma(K) = d$ for every special cone $K \in \mathcal{L}$.

and the number of special cones in $\mathcal{L}$ is at most $d + 1$. By the above, $\mathcal{L}$ has exactly $d + 1$ special cones. Due to Lemma 7, each of the two special cones in $\mathcal{L}$ are opposite. Hence all special cones in $\mathcal{L}$ generate $d(d + 1)/2$ pairs of opposite cones. In this situation, every nonspecial cone $K \in \mathcal{L}$ determines exactly $\gamma(K)$ new pairs of opposite cones. Since the number of these new pairs equals $n - d - 1$ (see the proof of Lemma 5), the last is possible only if $\mathcal{W} = \emptyset$, i.e., if $\mathcal{L}$ has property (S).

4. Proofs of Theorems 1–3

Proof of Theorem 1. Let $v_1, \dots, v_n$ be all the vertices of P_n given in the clockwise order. Two pairs of vertices in P_n of the form

$$\{v_i, v_{j+1}\}, \{v_{i+1}, v_j\}, \quad i \neq j + 1, \quad j \neq i + 1,$$

are called adjacent. It is easy to check that this relation of adjacency divides the family of all pairs of vertices in P_n into n pairwise disjoint classes, say $\mathcal{A}_1, \dots, \mathcal{A}_n$. Simple geometric considerations show that:

1. Every class $\mathcal{A}_k$ contains at least one but at most two antipodal pairs of vertices in P_n .
2. A class $\mathcal{A}_k$ contains two antipodal pairs, say $\{v_i, v_{j+1}\}, \{v_{i+1}, v_j\}$, if and only if the sides $[v_i, v_{i+1}]$, $[v_j, v_{j+1}]$ of P_n are parallel.

Similarly, every class $\mathcal{A}_k$ contains at most one strictly antipodal pair of vertices in P_n and $\mathcal{A}_k$ contains no strictly antipodal pair if and only if there are two adjacent pairs, say $\{v_i, v_{j+1}\}, \{v_{i+1}, v_j\}$, in $\mathcal{A}_k$ such that the sides $[v_i, v_{i+1}]$, $[v_j, v_{j+1}]$ are parallel. Now the assertion of Theorem 1 readily follows. $\square$

Proof of Theorem 2. Let $P_n \subset E^d$ be a convex polytope with n vertices. For every vertex v_i of P_n , denote by K'_i the cone generated by all extremal normals to P_n at v_i . Consider the cones

$$K_i = K'_i - v_i, \quad i = 1, \dots, n.$$

We see at once that the family $\mathcal{L} = \{K_1, \dots, K_n\}$ satisfies the hypothesis of Theorem 4. Moreover, vertices v_i, v_j of P_n are antipodal if and only if the respective cones K_i, K_j are opposite. Now Theorem 2 trivially follows from Theorem 4. $\square$

Proof of Theorem 3. The first assertion of Theorem 3 is trivial. Indeed, every vertex v_i in P_n belongs to at least one strictly antipodal pair of vertices (see, for instance, [8]). Hence $s(P_n) \geq \lceil n/2 \rceil$.

Below we determine the conditions under which (1) holds.

1. Let $n = 2k$. It is proved in [8] that $s(B) \geq d$ for every convex body $B \subset E^d$ with $s(B) = d$ if and only if B is a cross-polytope. Hence (1) holds for $n \geq 2d$ only.

If $s(P_{2k}) = k$, then every vertex of P_{2k} belongs to exactly one strictly antipodal pair, i.e., the vertices of P_{2k} can be uniquely divided into disjoint pairs, say $\{v_i, v_{i+k}\}$, $i = 1, \dots, k$, describing all antipodal pairs of vertices in P_{2k} .

Assume, in order to obtain a contradiction, that

$$C_i - v_i \not\subset v_{i+k} - C_{i+k}$$

for an index $i = 1, \dots, k$. Then there is an edge $[v_i, v_j]$ of P_{2k} such that v_j is not in the cone $v_i + v_{i+k} - C_{i+k}$. Denote by H a hyperplane passing through v_i and strictly separating v_j from $v_i + v_{i+k} - C_{i+k}$. If H' is the hyperplane containing v_{i+k} and parallel to H , then $H' \cap P_{2k} = \{v_{i+k}\}$. The last relation implies that $\{v_j, v_{i+k}\}$ is a new pair of strictly antipodal vertices different from $\{v_i, v_{i+k}\}$. The contradiction obtained shows that $C_i - v_i \subset v_{i+k} - C_{i+k}$. Similarly, $v_{i+k} - C_{i+k} \subset C_i - v_i$.

Conversely, if the vertex-set $v_1, \dots, v_{2k}$ of P_{2k} is divided into disjoint pairs, say $\{v_i, v_{i+k}\}$, $i = 1, \dots, k$, such that

$$C_i - v_i = v_{i+k} - C_{i+k} \quad \text{for all } i = 1, \dots, k,$$

then, clearly, $\{v_i, v_{i+k}\}$, $i = 1, \dots, k$, describe all pairs of strictly antipodal vertices in P_{2k} .

2. Let $n = 2k + 1$. The relation $s(P_{2k+1}) = k + 1$ is possible if and only if exactly one of the vertices of P_{2k+1} , say v_1 , belongs to two different strictly antipodal pairs and every other vertex belongs to exactly one such a pair. Assume, without loss of generality, that pairs

$$\{v_1, v_2\}, \{v_1, v_3\}, \{v_i, v_{i+k-1}\}, \quad i = 4, \dots, k + 2, \quad (5)$$

determine all strictly antipodal pairs of vertices in P_{2k+1} . By analogy to condition 1, we have

$$\left. \begin{aligned} v_1 - C_1 &= (C_2 - v_2) \cap (C_3 - v_3), \\ C_i - v_i &= v_{i+k-1} - C_{i+k-1}, \quad i = 4, \dots, k + 2. \end{aligned} \right\} \quad (6)$$

Conversely, if the vertices $v_1, \dots, v_{2k}$ of P_{2k} are divided into subsets, say (5), such that (6) holds, then (5) describes all strictly antipodal pairs of vertices in P_{2k} .

It remains to prove that relation $s(P_{2k+1}) = k + 1$ is possible for $2k + 1 \geq 4d - 1$ only. Let P_{2k+1} be a polytope with $2k + 1$ vertices, satisfying $s(P_{2k+1}) = k + 1$. By the above, there are three vertices, say v_1, v_2, v_3 , such that (6) holds. In this case the line segment $[v_2, v_3]$ is an edge of P_{2k+1} . From (6) it follows that the centrally symmetric convex polytope $Q = P_{2k+1} - P_{2k+1}$ has $2k + 2$ vertices. Since P_{2k+1} is not centrally symmetric, Q is reducible (i.e., by definition it can be represented as $Q = P - P$ for a nonsymmetric convex polytope P). Due to Yost [9], any centrally symmetric reducible polytope has at least $4d$ vertices. In the case of Q , we have $2k + 2 \geq 4d$, i.e., $2k + 1 \geq 4d - 1$. $\square$

Proof of the Observation. Let $n = 2k$ ($\geq 2d$) and let $P_n \subset E^d$ be a convex d -polytope symmetric with respect to the origin 0. If $v_1, \dots, v_{2k}$ are the vertices of P_n such that $v_{i+k} = -v_i$ for all $i = 1, \dots, k$, then $\{v_i, v_{i+k}\}$, $i = 1, \dots, k$, describe all strictly antipodal pairs of vertices in P_n . Hence $s(P_n) = \lfloor n/2 \rfloor$.

Now we show that, for every natural number $m = 2k + 1 \geq 4d - 1$, a d -polytope $P_{2k+1} \subset E^d$ with $s(P_{2k+1}) = k + 1$ exists.

For this purpose, choose in E^d a convex polytope B with center 0 and $2k - 2d + 2$ ($\geq 2d$) vertices $v_1, \dots, v_{2k-2d+2}$, having a pair of symmetric facets of the form $S = \text{conv}(v_1, \dots, v_d)$, $T = -S$. Without loss of generality, we can put $v_{k-d+i+1} = -v_i$ for all $i = 1, \dots, k - d + 1$. Choose a point $v_0 \notin B$ sufficiently close to the center of S such that the polytope $Q = \text{conv}(v_0 \cup -v_0 \cup B)$ has new (relative to B) facets

$$S_i = \text{conv}(v_1, \dots, v_{i-1}, v_0, v_{i+1}, \dots, v_d), \quad T_i = -S_i, \quad i = 1, \dots, d.$$

Denote by H a hyperplane satisfying the conditions:

1. $v_1 \in H$.
2. H strictly separates v_0 from B .
3. $H \cap Q$ is a $(d - 1)$ -simplex of the form $\text{conv}(v_1, w_2, \dots, w_d)$, where $w_i \in]v_0, v_i[$, $i = 2, \dots, d$.

Similarly, let G be a hyperplane strictly separating $-v_0$ and B such that $G \cap Q$ is a $(d - 1)$ -simplex of the form $\text{conv}(z_1, \dots, z_d)$, where $z_i \in]-v_0, -v_i[$, $i = 1, \dots, d$.

Denote by P that part of Q which lies in the closed strip between H and G . It follows easily that P is a convex d -polytope with $2k + 1$ vertices, namely,

$$v_1, \dots, v_{2k-2d+2}, w_2, \dots, w_d, z_1, \dots, z_d,$$

and

$$\{v_i, -v_i\}, \quad i = 1, \dots, k - d + 1, \quad \{w_i, z_i\}, \quad i = 2, \dots, d, \quad \{v_1, z_1\}$$

describe all $k - d + 1 + d - 1 + 1 = k + 1$ strictly antipodal pairs of vertices in P . Hence $s(P) = k + 1$. $\square$

Acknowledgment

The authors thank the referees for their helpful comments on an earlier draft of the paper.

References

1. L. Danzer, B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee, *Math. Z.* **79** (1962), 95–99.

2. P. Erdős, Z. Füredi, The greatest angle among n points in the d -dimensional euclidean space, *Ann. Discrete Math.* **17** (1983), 275–283.
3. B. Grünbaum, Strictly antipodal sets, *Israel J. Math.* **1** (1963), 5–10.
4. V. Klee. Unsolved problems in intuitive geometry, Mimeographed notes, Seattle, 1960.
5. E. Makai, Jr., H. Martini, On the number of antipodal or strictly antipodal pairs of points in finite subsets of R^d , in: *Applied Geometry and Discrete Mathematics. The V. Klee Festschrift*, DIMACS Series in Discrete Mathematics and Theoretic Computer Science, Vol. 4, 1991, pp. 457–470.
6. E. Makai, Jr., H. Martini, On the number of antipodal or strictly antipodal pairs of points in finite subsets of R^d . II, Preprint No. 7/1992, Mathematical Institute of the Hungarian Academy of Science, 1992, 24 pp.
7. M. N. Nguyễn, Lower estimate of the number of exposed diameters of a convex polytope with a small number of vertices, Manuscript, Kishinev State University, Kishinev, 1990, 27 pp. (Deposited in Moldavian NIINTI 22.01.90, No. 1154-M90, in Russian.)
8. V. Soltan, M. N. Nguyễn, Lower bounds for the number of extremal and exposed diameters of a convex body, *Studia Sci. Math. Hungar.* **28** (1993), 95–100.
9. D. Yost, Irreducible convex sets, *Mathematika* **38** (1991), 134–155.

Received June 10, 1992, and in revised form May 14, 1993.